Heisenberg homology of surface configurations

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based on joint work with Martin Palmer and Awais Shaukat,
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Lawrence representations of braid groups

- Lawrence (1990): Family of representations of the classical braid groups B_m

$$L_n : B_m \to GL(H_n(\widetilde{\mathcal{C}}_n(D^2_m)),\ n \geq 2\ .$$

- $\widetilde{\mathcal{C}}_n$ is a $\mathbb{Z}^2$-cover of the unordered configuration space $\mathcal{C}_n(D^2_m)$ of n distinct points in the m-punctured disc.
- Theorem (Bigelow, Krammer, 2001-2002): L_2 is **faithful**.
- Kohno: Lawrence (LKB) representations are equivalent to $sl(2)$ **quantum** representations on heighest weight spaces.
- $B_m = \mathfrak{M}(D^2_m)$ is a mapping class group.

Surface configurations and Heisenberg group

- Goal: LKB type representations for $\mathfrak{M}(\Sigma = \Sigma_{g,1})$, $g \geq 1$, from homology groups on the configuration spaces $\mathcal{C}_n(\Sigma)$.
- The Heisenberg group $\mathcal{H}(H_1(\Sigma, \mathbb{Z}))$ is the central extension of $H = H_1(\Sigma, \mathbb{Z})$ defined by the intersection cocycle $(x, y) \mapsto x.y$

$$\mathcal{H}(H) = \mathbb{Z} \times H \text{ with } (k, x)(l, y) = (k + l + x.y, x + y).$$

Our results: Homological representations of MCG

- There is a quotient homomorphism $\phi : B_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma)) \to \mathcal{H}(H)$.
- A representation $\rho : \mathcal{H}(H) \to GL(V)$ defines a local system on the configuration space $\mathcal{C}_n(\Sigma)$, so that we have homology groups $H_*(\mathcal{C}_n(\Sigma), V)$, $H_*^{BM}(\mathcal{C}_n(\Sigma), V)$, ...
- We study these groups and the twisted MCG action.
- For q a root of unity of odd order $p \geq 3$, we specialise to a Shrödinger representation $L^2(\mathbb{Z}_p^g)$ and obtain projective unitary representations of the MCG.

Surface braid groups

- $B_n(\Sigma) = \pi_1(\mathcal{C}_n(\Sigma), *)$, $\Sigma = \Sigma_{g,1}$, $g \geq 1$.
- Bellingeri presentation, revisited by Bellingeri-Godelle: classical generators $\sigma_1, \ldots, \sigma_{n-1}$, π_1 generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ (only the first point is moving), and relations:

$$\begin{cases}
(\mathbf{BR1})\ [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\
(\mathbf{BR2})\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i-j| = 1, \\
(\mathbf{CR1})\ [\alpha_r, \sigma_i] = [\beta_r, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } r, \\
(\mathbf{CR2})\ [\alpha_r, \sigma_1 \alpha_r \sigma_1] = [\beta_r, \sigma_1 \beta_r \sigma_1] = 1 & \text{for all } r, \\
(\mathbf{CR3})\ [\alpha_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\alpha_r, \sigma_1^{-1} \beta_s \sigma_1] & \\
\quad = [\beta_r, \sigma_1^{-1} \alpha_s \sigma_1] = [\beta_r, \sigma_1^{-1} \beta_s \sigma_1] = 1 & \text{for all } r < s, \\
(\mathbf{SCR})\ \sigma_1 \beta_r \sigma_1 \alpha_r \sigma_1 = \alpha_r \sigma_1 \beta_r & \text{for all } r.
\end{cases}$$

We compose from the right.

Heisenberg group

- The Heisenberg group $\mathcal{H}(H)$ is the central extension of $H = H_1(\Sigma, \mathbb{Z})$ defined with the intersection cocycle.
- $\mathcal{H}(H) = \mathbb{Z} \times H$ with $(k, x)(l, y) = (k + l + x.y, x + y)$.
- Theorem: $B_n(\Sigma)/[\sigma_1, B_n(\Sigma)]^N$ is isomorphic to the Heisenberg group $\mathcal{H}(H)$; σ_1 becomes central.
- An isomorphism ϕ is defined by

$$\sigma_i \mapsto u = (1, 0) \ , \ \alpha_i \mapsto \tilde{a}_i = (0, a_i) \ , \ \beta_i \mapsto \tilde{b}_i = (0, b_i) \ ,$$

$a_i = [\alpha_i]$, $b_i = [\beta_i]$ in $H_1(\Sigma, \mathbb{Z})$.

MCG action on Heisenberg group

- $\mathfrak{M}(\Sigma) = \mathrm{Diff}(\Sigma, \partial\Sigma)/\mathrm{Diff}_0(\Sigma, \partial\Sigma)$.
- For $f \in \mathfrak{M}(\Sigma)$, $\mathcal{C}_n(f)$ induces an automorphism $f_{\mathcal{H}} \in \mathrm{Aut}^+(\mathcal{H})$ (identity on center).
- $\mathrm{Aut}^+(\mathcal{H}) \simeq Sp(H) \ltimes H^*$ is the affine symplectic group.
- $f_{\mathcal{H}} : (k, x) \mapsto (k + \mathfrak{d}_f(x), f_*(x))$, with $\mathfrak{d}_f \in H^* = H^1(\Sigma, \mathbb{Z})$.
- $f \mapsto \mathfrak{d}_f$ is a crossed homomorphism, i.e.

$$\mathfrak{d}_{g \circ f}(x) = \mathfrak{d}_f(x) + f^*(\mathfrak{d}_g)(x) \ .$$

- Morita: $\mathfrak{d}$ generates $H^1(\mathfrak{M}(\Sigma), H^*) \cong \mathbb{Z}$.

Local system from an Heisenberg group representation

- We denote by $\widetilde{\mathcal{C}}_n(\Sigma))$ the regular cover associated to the kernel of $\phi : B_n(\Sigma) \to \mathcal{H}(\Sigma)$ and call it the Heisenberg cover of surface configurations.
- The (singular or cellular) chain complex of the Heisenberg cover, denoted by $S_*(\widetilde{\mathcal{C}}_n(\Sigma))$, is a right $\mathbb{Z}[\mathcal{H}]$-module.
- Given a representation $\rho : \mathcal{H} \to GL(V)$, the corresponding local homology is that of the complex

$$S_*(\mathcal{C}_n(\Sigma), V) := S_*(\widetilde{\mathcal{C}}_n(\Sigma)) \otimes_{\mathbb{Z}[\mathcal{H}]} V \ .$$

Action of mapping classes

- For $f = [g] \in \mathfrak{M}(\Sigma)$, the map $\mathcal{C}_n(g)$ lifts to the Heisenberg cover and the lift $\widetilde{\mathcal{C}}_n(g)$ induces a chain map $S_*(\widetilde{\mathcal{C}}_n(g))$ which is twisted linear: $S_*(\widetilde{\mathcal{C}}_n(g))(z.h) = S_*(\widetilde{\mathcal{C}}_n(g))(z).f_{\mathcal{H}}(h)$.
- We get homology maps

$$\mathcal{C}_n(f)_* : H_*(\mathcal{C}_n(\Sigma), {}_{f_{\mathcal{H}}}V) \to H_*(\mathcal{C}_n(\Sigma), V) \ ,$$

Here ${}_{f_{\mathcal{H}}}V$ is the vector space V with twisted action $\rho \circ f_{\mathcal{H}}$.

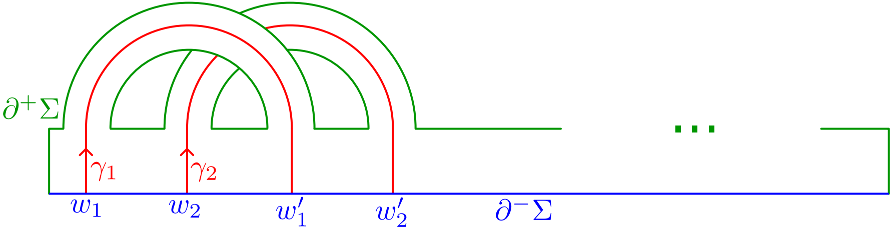
Notation

- H_*^{BM} denotes the Borel-Moore homology,

$$H_n^{BM}(\mathcal{C}_n(\Sigma);V)=\varprojlim_{T} H_n(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma)\setminus T;V),$$

  the inverse limit is taken over all compact subsets $T\subset\mathcal{C}_n(\Sigma)$.
- Borel-Moore homology is functorial with respect to proper maps and for a proper embedding $B\subset A$, the relative homology $H_*^{BM}(A,B)$ is defined.
- $\mathcal{C}_n(\Sigma,\partial^-(\Sigma))$ is the properly embedded subspace of $\mathcal{C}_n(\Sigma)$ consisting of all configurations intersecting a given arc $\partial^-\Sigma\subset\partial\Sigma$.

Model surface



- A lift of $\gamma_1 \times \gamma_2$ in the Heisenberg cover represents a relative cycle,
  $[\widetilde{\gamma_1 \times \gamma_2} \otimes v] \in H_2(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); V)$.
- A lift of $\mathcal{C}_2(\gamma_1)$ represents a relative Borel-Moore cycle,
  $[\widetilde{\mathcal{C}_2(\gamma_1)} \otimes v] \in H_2^{BM}(\mathcal{C}_2(\Sigma), \mathcal{C}_2(\Sigma, \partial^-(\Sigma)); V)$.

Computation

Theorem

Let $n \geq 2$, $g \geq 1$, V a representation of the discrete Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma = \Sigma_{g,1})$ over a ring R.
The module $H_n^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$ is isomorphic to the direct sum of $\begin{pmatrix} 2g+n-1 \\ n \end{pmatrix}$ copies of V. Furthermore, it is the only non-vanishing module in $H_*^{BM}(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma)); V)$.

Twisted action

Theorem

There is a natural twisted representation of the mapping class group $\mathfrak{M}(\Sigma)$ on the R-modules

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma,\partial^-(\Sigma));{}_\tau V\big)\ ,\ \tau\in\mathrm{Aut}(\mathcal{H})\ ,$$

where the action of $f\in\mathfrak{M}(\Sigma)$ is $\mathcal{C}_n(f)_*$:

$$H_n^{BM}\big(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma,\partial^-(\Sigma));{}_{\tau\circ f_{\mathcal{H}}}V\big)\to H_n^{BM}\big(\mathcal{C}_n(\Sigma),\mathcal{C}_n(\Sigma,\partial^-(\Sigma));{}_\tau V\big)$$

- For a representation $\rho:\mathcal{H}\to GL(V)$ and $\tau\in\mathrm{Aut}(\mathcal{H})$, the τ-twisted representation $\rho\circ\tau$ is denoted by ${}_\tau V$.
- This theorem can be formulated as a functor on a groupoid whose objects are elements $\tau\in\mathrm{Aut}(\mathcal{H})$ and morphisms are compatible mapping classes.

Finite dimensional Schrödinger representations, odd case

- Let p be an odd integer. The Heisenberg algebra $\mathcal{H}(\Sigma)$ has a finite dimensional quotient $\mathcal{H}_p(\Sigma)$.
  $\mathcal{H}_p(\Sigma) = \mathbb{Z}_p \times H_1(\Sigma, \mathbb{Z}_p)$ with Heisenberg product.
- For q a p-th root of unity $\mathcal{H}_p(\Sigma)$ acts on a f.d. Hilbert space.

$$\rho_q : \mathcal{H}_p(\Sigma) \to U(W_q \cong \mathbb{C}^{pg}) \ .$$

- Case $g = 1$, $H_1(\Sigma, \mathbb{Z})$ has basis (m, l), $m.l = 1$.
  $\mathcal{H}_p(\Sigma) = \{(k, xm + yl),\ k, x, y \in \mathbb{Z}_p\}$.
  W_q has basis b_i, $i \in \mathbb{Z}_p$ with

$$\rho_q(1,0)(b_i) = qb_i \ ,\ \rho_q(0,l)(b_i) = b_{i+1} \ ,\ \rho_q(0,m)(b_i) = q^2 b_i \ .$$

  Similar formula in higher genus.

Stone von-Neuman theorem

Theorem (Stone-von Neumann)

W_q is an irreducible representation of $\mathcal{H}_p(\Sigma)$ and up to unitary isomorphism is the unique irreducible unitary representation of $\mathcal{H}_p(\Sigma)$ whose character on the center is $(k,0) \mapsto q^k$.

- Case $p \equiv 0 \mod 4$: Gelca-Uribe.
  Case $p \equiv 2 \mod 4$, more subtilities.
- For $\tau \in \mathrm{Aut}(\mathcal{H})$, the Stone-von Neumann theorem provides a unitary isomorphism ${}_\tau W_q \cong W_q$ defined up to $\lambda \in S^1$.

Untwisted representation of MCG

- We deduce projective representations of the MCG

$$\mathfrak{M}(\Sigma) \to PU(\mathcal{V}_{q,n}) \ , \ \mathcal{V}_{q,n} = H_n^{BM}\left(\mathcal{C}_n(\Sigma), \mathcal{C}_n(\Sigma, \partial^-(\Sigma); W_q\right)$$

- We get a unitary representation of a central extension

$$\widetilde{\mathfrak{M}}(\Sigma) \to U(\mathcal{V}_{q,n}) \ .$$

Recovering Quantum representation

Marco de Renzi and Jules Martel recently proved the followings

- There is an action of the small $sl(2)$ quantum group $\mathcal{U}_q$ on the sum $\oplus_n \mathcal{V}_{q,n}$.
- The troncated sum $\oplus_{0\leq n<p} \mathcal{V}_{q,n}$ contains a subrepresentation isomorphic to $(\mathcal{U}_q^{ad})^{\otimes g}$ as a quantum $sl(2)$ module and as a MCG projective representation (Lyubashenko non semisimple quantum representations of MCG).

Action of cobordisms ?

- The Schrödinger representation can be defined from a lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$: $W_q = W_q(L)$.
- A cobordism C, $\partial C = -\Sigma \cup_{S^1} \Sigma'$ provides a *lagrangian correspondence* $L_C \in H_1(\partial C, \mathbb{Z})$ (the kernel of inclusion map).

Theorem (Beliakova-B)

$$W_q(L_C) \otimes_{\mathbb{Z}[\mathcal{H}(\Sigma)]} W_q(L) \cong W_q(L') \ ,$$

where $L' = L_C.L \in H_1(\Sigma', \mathbb{Z})$ *is the matching lagrangian subspace.*

- We get an action of cobordisms on the Schrödinger local systems.
- Natural action of cobordisms on Shrödinger homologies ?? Hilbert case ??