Relating categorical approaches to type dependency

Trabajo de grado

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Chapter 1

Categorical description of type dependency

A type system is described by a setting, which determines the dependencies that may arise, and some additional features, like products, constants or axioms [Jac 91]. A number of categorical descriptions exists for the setting of type dependency (where terms may occur in types). In this chapter we will establish some relations between them. The structures we will study are comprehension categories [Jac 90, 91], categories with attributes [Mog], display categories [Tay] and contextual categories [Str]. The relations are summarized in a diagram at the end of the chapter.

1 Comprehension Categories

In this paper knowledge of the concepts of fibred category theory is assumed. The reader is referred to [Jac 90, 91] for background information. For a fibration $p:\mathbf{E}\to\mathbf{B}$ we will denote a cartesian lifting of a morphism $u:A\to pE$ in $\mathbf{B}$ by $u^{-}(E):u^{*}(E)\to E$. For a category $\mathbf{B}$ we write $\mathbf{B}^{\to}$ for the category of arrows of $\mathbf{B}$.

1.1 DEFINITION. (Jacobs) (i) A *comprehension category* is a functor $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ such that

(a) the functor $\mathrm{cod}\circ\mathcal{P}:\mathbf{E}\to\mathbf{B}$ is a fibration;

(b) a morphism f is cartesian in $\mathbf{E}$ implies that $\mathcal{P}f$ is a pullback in $\mathbf{B}$;

(c) the base category $\mathbf{B}$ has a terminal object $\mathbf{t}$.

(ii) A category Comp($\mathbf{B}$) is defined having comprehension categories with base $\mathbf{B}$ as objects. Morphisms from $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ to $\mathcal{Q}:\mathbf{D}\to\mathbf{B}^{\to}$ are functors $H:\mathbf{E}\to\mathbf{D}$, such that $\mathcal{Q}\circ H=\mathcal{P}$ and f is cartesian in $\mathbf{E}$ implies that Hf is cartesian in $\mathbf{D}$.

(iii) A category Comp, a bit more general, is defined having comprehension categories as objects and as arrows $\langle K,H\rangle:(\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to})\to(\mathcal{Q}:\mathbf{D}\to A^{\to})$ pairs of functors $H:\mathbf{E}\to\mathbf{D}$ and $K:\mathbf{B}\to\mathbf{A}$ such that $\mathcal{Q}\circ H=K^{\to}\circ\mathcal{P}$, f is cartesian in $\mathbf{E}$ implies that Hf is cartesian in $\mathbf{D}$; and such that K preserves terminal object.

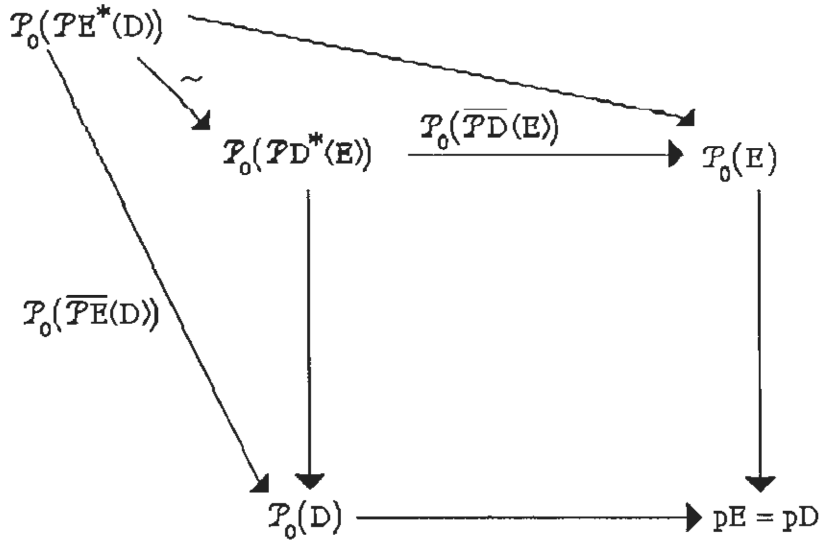
(iv) A comprehension category $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ is called *full* when $\mathcal{P}$ is a full and faithful functor.

(v) A comprehension category $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ is called *split* when the fibration involved is split.

1.1.1 NOTATION. Let $\mathcal{P}$ be a comprehension category, we write p for the fibration $\mathrm{cod}\circ\mathcal{P}$ and $\mathcal{P}_0$ for the composition $\mathrm{dom}\circ\mathcal{P}$. With this notation $\mathcal{P}$ can be seen as a natural transformation from $\mathcal{P}_0$ to p.

In the definition of a comprehension category $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ the existence certain pullbacks in the base category $\mathbf{B}$, is prescribed (viz. those of the form $\mathcal{P}f$, for a cartesian arrow f). Given an object E in $\mathbf{E}$, a pullback functor $\mathcal{P}E^{\#}:\mathbf{B}/_{pE}\to\mathbf{B}/_{\mathcal{P}_0E}$ can be defined by sending an arrow $u:A\to pE$ in $\mathbf{B}$ (i.e. an object in $\mathbf{B}/_{pE}$) to $\mathcal{P}_0(u^{-}(E))$; condition (b) in the definition above ensures that one obtains a pullback. The following technical result will be heavily used.

1.1.2 LEMMA. Let $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ be a comprehension category. For all objects $E,D\in\mathbf{E}$ with $pE=pD$, there exist a unique isomorphism $\mathcal{P}_0(\mathcal{P}D^{*}(E))\cong\mathcal{P}_0(\mathcal{P}E^{*}(D))$ in $\mathbf{B}$ making the following diagram commute.



Proof. Easy, because both squares are pullbacks. □

1.1.3 REMARK. The requirement in (i)(c) in definition 1.1 does not occur in [Jac 90, Jac 91], but here it's more convenient to add it.

1.2 SOME CONSTRUCTIONS ON COMPREHENSION CATEGORIES.

In the rest of this section some subcategories of Comp are described and some properties are established which will be useful later.

1.2.1 DEFINITION. (i) A *discrete* comprehension category is a comprehension category where the fibres are discrete. The (full) subcategory of Comp of discrete comprehension categories is denoted by |Comp|.

(ii) A category $\text{Comp}_{\text{split}}$ is defined as the subcategory of Comp having split comprehension categories as objects and functors which preserve the splitting "on the nose" as arrows.

(iii) $\text{Comp}_{\text{full}}$ is defined as the (full) subcategory of Comp of full comprehension categories. The obvious restriction of $\text{Comp}_{\text{full}}$ to the split case will be denoted by $\text{Comp}_{\text{full,split}}$.

1.2.2 LEMMA. The (full) inclusion functor $|\text{Comp}| \hookrightarrow \text{Comp}_{\text{split}}$ has a right adjoint denoted by | _ |.

Proof. Given a split comprehension category $\mathcal{P}$, a discrete comprehension category $|\mathcal{P}|:\text{Split}(\mathbf{E}) \rightarrow \mathbf{B}^{\rightarrow}$ can be defined as follows. The total category Split(**E**) has all the objects of **E** but only the arrows of the splitting. The functor $|\mathcal{P}|$ is the restriction of

$\mathcal{P}$ to Split(E). One obtains a counit $|\mathcal{P}| \to \mathcal{P}$ by inclusion. The unit is the identity since $|\mathcal{P}| = \mathcal{P}$ for discrete $\mathcal{P}$. □

1.2.3 LEMMA. The inclusion functor $\text{Comp}_{\text{full}} \hookrightarrow \text{Comp}$ has a left adjoint denoted by $(\_)^{\blacktriangledown}$. This adjunction restricts to the corresponding split categories.

Proof. Given a comprehension category $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ a full comprehension category $\mathcal{P}^{\blacktriangledown}:\mathbf{E}^{\blacktriangledown} \to \mathbf{B}^{\to}$ called the *heart* of $\mathcal{P}$ is defined as follows. The category $\mathbf{E}^{\blacktriangledown}$ has objects $E \in \mathbf{E}$. Arrows from E to E' in $\mathbf{E}^{\blacktriangledown}$ are arrows in $\mathbf{B}^{\to}$ from $\mathcal{P}E$ to $\mathcal{P}E'$. The functor $\mathcal{P}^{\blacktriangledown}$ is equal to $\mathcal{P}$ on objects and the identity on arrows. The unit of the adjunction is the identity on objects and sends an arrow f in $\mathbf{E}$ to the arrow $(pf, \mathcal{P}_0 f)$ in $\mathbf{E}^{\blacktriangledown}$. The counit on arrows is the inverse of $\mathcal{P}$, i.e. the isomorphism given by the fulness of $\mathcal{P}$. □

1.2.4 PROPOSITION. There is an equivalence of categories between |Comp| and $\text{Comp}_{\text{full,split}}$. This equivalence is determined by $(\_)^{\blacktriangledown}$ and $|\_|$.

Proof. By composing the adjunctions of 1.2.2 and 1.2.3, see e.g. [Mac]. One easily verifies that that the resulting unit and counit are isomorphisms. □

This result is not very surprising because for full comprehension categories $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$, morphism in $\mathbf{E}$ are superfluous.

1.2.5 DEFINITION. Let $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category. The comprehension category $\mathcal{P}[\mathbf{t}]:\mathbf{E}[\mathbf{t}] \to \mathbf{B}[\mathbf{t}]^{\to}$ is defined as follows. The base category $\mathbf{B}[\mathbf{t}]$ has as objects chains $\langle E_1,\dots,E_n \rangle$ of objects of $\mathbf{E}$ with $n \geq 0$, $pE_1 = \mathbf{t}$ and for $i \in \{1,\dots,n-1\}$, $pE_{i+1} = \mathcal{P}_0 E_i$. Arrows from $\langle C_1,\dots,C_m \rangle$ to $\langle E_1,\dots,E_n \rangle$ in $\mathbf{B}[\mathbf{t}]$ are arrows in $\mathbf{B}$ from $\mathcal{P}_0 C_m$ to $\mathcal{P}_0 E_n$; if m (resp. n) is equal to 0 then the terminal object plays the role of $\mathcal{P}_0 C_m$ (resp. $\mathcal{P}_0 E_n$). The category $\mathbf{E}[\mathbf{t}]$ has *non-empty* chains of objects in $\mathbf{E}$. Arrows in $\mathbf{E}[\mathbf{t}]$ from $\langle C_1,\dots,C_m \rangle$ to $\langle E_1,\dots,E_n \rangle$ are arrows in $\mathbf{E}$ from C_m to E_n. Finally the functor $\mathcal{P}[t]$ applied to an object $\langle E_1,\dots,E_n \rangle$ is defined as $\mathcal{P}E_n$. This construction is a particular case of the one in [Jac 90] (Def. 3.4.5).

1.2.6 LEMMA. The assignment $\mathcal{P} \mapsto \mathcal{P}[\mathbf{t}]$ determines a functor Comp→Comp. This endofunctor restricts both to categories of full and split comprehension categories. □

2 Categories With Attributes

2.1 DEFINITION. (i) An indexed category is a functor of the form $\Psi:\mathbf{B}^{op}\rightarrow\mathrm{Cat}$.

(ii) Let $\Psi:\mathbf{B}^{op}\rightarrow\mathrm{Cat}$ and $\Phi:\mathbf{A}^{op}\rightarrow\mathrm{Cat}$ be indexed categories. A morphism from Ψ to Φ of indexed categories is a pair (K,α), where $K:\mathbf{B}\rightarrow\mathbf{A}$ is a functor and $\alpha:\Psi\rightarrow\Phi K$ is a natural transformation. This determines a category ICat.

The "Grothendieck construction" yields a functor $\mathrm{ICat}\rightarrow\mathrm{Fib}_{split}$ (the category of split fibrations). From an indexed category $\Psi:\mathbf{B}^{op}\rightarrow\mathrm{Cat}$ a split fibration is constructed as follows. The total category $\mathcal{G}\Psi$ has pairs (A,X), with $A\in\mathbf{B}$, $X\in\Psi(A)$ as objects; arrows from (A,X) to (B,Y) are pairs of arrows (u,f), where $u:A\rightarrow B$ in $\mathbf{B}$ and $f:X\rightarrow\Psi(u)(Y)$ in $\Psi(A)$. The first projection $\Pi_\Psi:\mathcal{G}\Psi\rightarrow\mathbf{B}$ is the split fibration required. On arrows this functor is defined by $(K,\alpha)\mapsto(K,\mathcal{G}\alpha)$ where $\mathcal{G}\alpha(A,X) = (KA,\alpha_A(X))$ and $\mathcal{G}\alpha(u,f) = (Ku,\alpha_A(f))$.The functor $\mathrm{ICat}\rightarrow\mathrm{Fib}_{split}$ is part of an equivalence of categories between ICat and Fib_{split}.

2.2 DEFINITION. ([Car], [Mog]) (i) A *category with attributes* is a quadruple $(\mathbf{B},\Psi,\Psi_0,\psi)$, such that

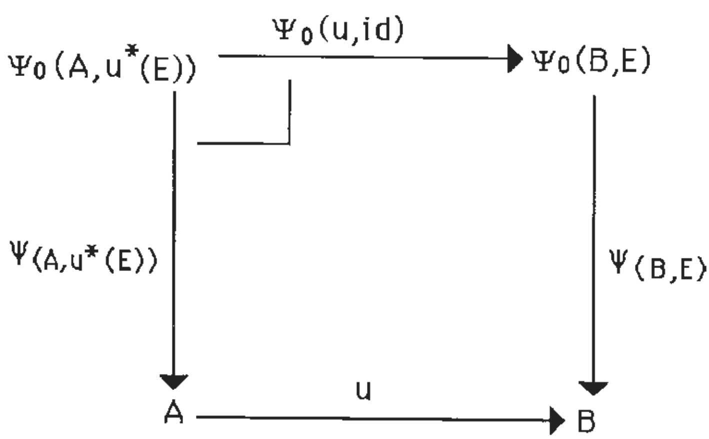
-$\mathbf{B}$ is a category with terminal object $\mathbf{t}$;

-Ψ is a discrete $\mathbf{B}$-indexed category, i.e. for every object A in $\mathbf{B}$, ΨA is discrete;

-Ψ_0: $\mathcal{G}\Psi\rightarrow\mathbf{B}$ is a functor;

-ψ: $\Psi_0\rightarrow\Pi_\Psi$ is a natural transformation;

- for every morphism $u:A\rightarrow B$ in $\mathbf{B}$ and object E in $\Psi(B)$, the following diagram is a pullback.



The following abbreviations are used.

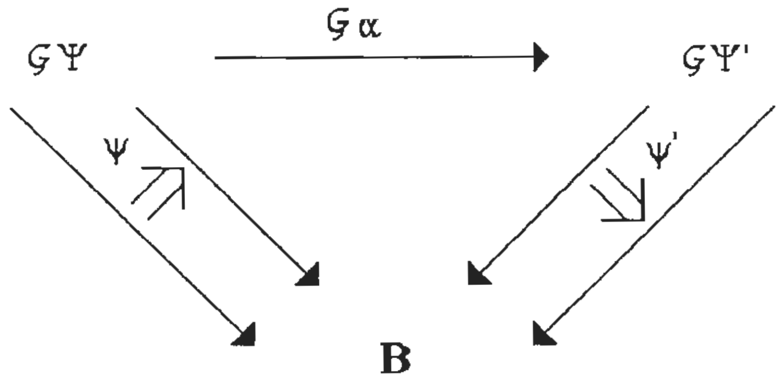
$u^*(E)$ for $\Psi(u)(E)$,

$u^-(E)$ for (u,id_E),

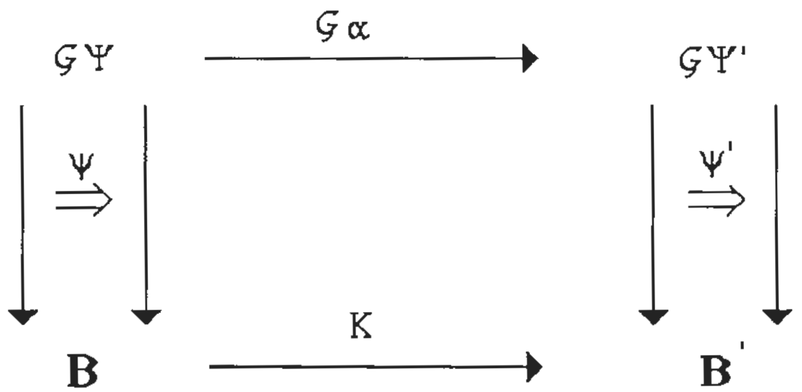
E for (A,E),

when no confusion arises.

(ii) Let **B** be a category, a category Cwa(**B**) is defined having categories with attributes with base **B** as objects. Morphisms from $(\mathbf{B},\Psi,\Psi_0,\psi)$ to $(\mathbf{B},\Psi',\Psi'_0,\psi')$ are morphism of indexed categories $(\mathrm{Id}, \alpha):\Psi\to\Psi'$ such $\Psi_0 = \Psi'_0 \circ \mathcal{G}\alpha$ and $\psi'(\mathcal{G}\alpha) = \psi$, as in the following diagram.

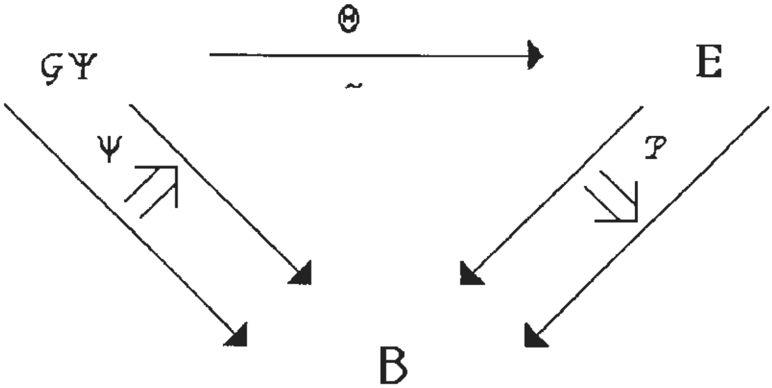


(iii) As for comprehension categories, the previous definition can be made more general: we write Cwa for the category of categories with attributes and morphisms of indexed categories (K,α): $\Psi\to\Psi'$ making the following diagram commute.



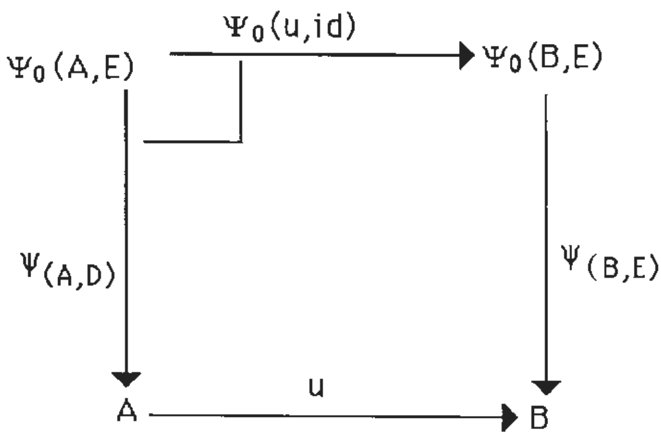
2.3 THEOREM. There exist an equivalence between the categories |Comp| and Cwa. This equivalence restricts to the fixed base case, i.e. between |Comp(**B**)| and Cwa(**B**).

Proof. A functor $\mathcal{U}$:|Comp| $\to$ Cwa is defined as follows. Given a discrete comprehension category $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$, the fibration p is split. An indexed category $\Psi:\mathbf{B}^{op}\to\mathrm{Cat}$ can be obtained as usual, mapping an object A in **B** to the fibre $\mathbf{E}_A$ and an arrow $u:A\to B$ to the functor $u^*:\mathbf{E}_B\to\mathbf{E}_A$. As the fibres are discrete, Ψ is discrete. The functor $\Psi_0:\mathcal{G}\Psi\to\mathbf{B}$ is constructed easily from $\mathcal{P}_0$ and the natural transformation ψ from $\mathcal{P}$. The naturality of ψ follows from the naturality of $\mathcal{P}$ and the following diagram, where Θ is the obvious isomorphism given by $(A,E) \mapsto E$ and $(u,f):(A,E) \to (B,D) \mapsto u^{-}(D) \circ f$. It is determined by the equivalence between ICat and $\text{Fib}_{\text{split}}$



If <K,H> is a morphism in |Comp| then $\mathcal{U}$(<K,H>) is defined as <K,α> where α_B is the restriction of the functor H to $\mathbf{E}_B$. Obviously, this definition restricts to |Comp(**B**)|→Cwa(**B**), the fixed base case.

Given a category with attributes $(\mathbf{B},\Psi,\Psi_0,\psi)$ a discrete comprehension category can be obtained using the Grothendieck construction. The functor $\mathcal{P}$ is defined by the natural transformation ψ on objects. Given and arrow (u,id) in $\mathcal{G}\Psi$, one takes $\mathcal{P}$(u,id) to be the pair $\langle u,\Psi_0(u,id)\rangle$ as in the following diagram.



This construction can be extended to a functor $\mathcal{F}$: Cwa→|Comp| using the Grothendieck construction on arrows.

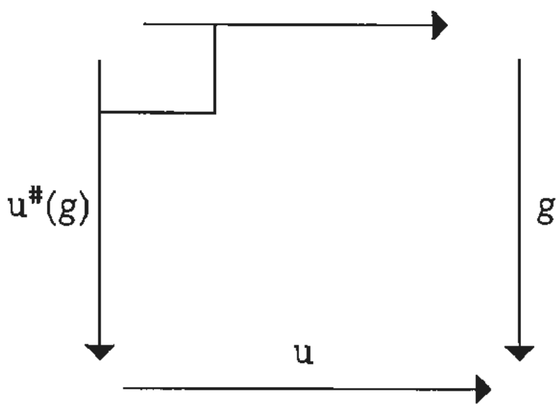
The components of the unit Id→ $\mathcal{UF}$ of the equivalence are the natural isomorphisms α defined by $\alpha_B(B,E) = (B,(B,E))$ and $\alpha_B\,(u,id) = (u,(u,id))$. The components of the counit send (A,E) to E and (u,id_D) to $u^{-}(D)$; these components are isomorphisms by uniqueness of cartesian arrows in discrete comprehension categories. □

2.4 COROLLARY. There exists an equivalence of categories between Cwa and $\mathrm{Comp}_{\mathrm{full,split}}$

Proof. Immediate by the previous result and Lemma 1.2.4. □

3 Display categories

3.1 DEFINITION. A *display category* $(\mathbf{B},\mathcal{D})$ consists of a base category $\mathbf{B}$ and a class $\mathcal{D}$ of arrows of $\mathbf{B}$ with the following properties. Given arrows g in $\mathcal{D}$ and u in $\mathbf{B}$ with cod(g) = cod(u), one can choose an arrow $u^{\#}(g)$ in $\mathcal{D}$ which determines a pullback in $\mathbf{B}$ as in the diagram below. Further, the class $\mathcal{D}$ is closed under isomorphism, i.e. if $f \cong g$ in $\mathbf{B}/A$ and $g \in \mathcal{D}$ then $f \in \mathcal{D}$.



3.2 DEFINITION: (i) A category Disp(**B**) is defined having display categories with base **B** as objects. An arrow from $(\mathbf{B},\mathcal{D})$ to $(\mathbf{B},\mathcal{D}')$ exists iff $\mathcal{D} \subseteq \mathcal{D}'$.

(ii) A category Disp is defined as having display categories as objects. Arrows $F:(\mathbf{B},\mathcal{D})\to(\mathbf{B}',\mathcal{D}')$ in Disp are given by functors $F:\mathbf{B}\to\mathbf{B}'$, sending $\mathcal{D}$ inside $\mathcal{D}'$ and preserving pullbacks.

3.3 THEOREM. There is a functor from Comp to Disp with full and faithful right adjoint. This reflection restricts to the fixed base case.

Proof. Given a comprehension category $\mathcal{P}.\mathbf{E}\to\mathbf{B}^{\to}$ a display category $(\mathbf{B},\mathcal{D}_{\mathcal{P}})$ can be formed, taking the set of display maps $\mathcal{D}_{\mathcal{P}}$ to be the set $\{f \mid f \cong \mathcal{P}E$ in $\mathbf{B}/pE$, for

some E ∈ **E**}. This can be extended to a functor Comp→Disp sending a morphism in Comp to its base functor.
The functor above has a full and faithful right adjoint, which maps each display category to the inclusion $\mathbf{B}^{\rightarrow}(\mathcal{D})\hookrightarrow\mathbf{B}^{\rightarrow}$, where $\mathbf{B}^{\rightarrow}(\mathcal{D})$ is the full subcategory of $\mathbf{B}^{\rightarrow}$ determined by $\mathcal{D}$. The counit of this adjunction is the identity, and the unit in $\mathcal{P}$ is $\mathcal{P}$ itself considered as a functor from **E** to $\mathbf{B}^{\rightarrow}(\mathcal{D})$. □

4 Contextual categories

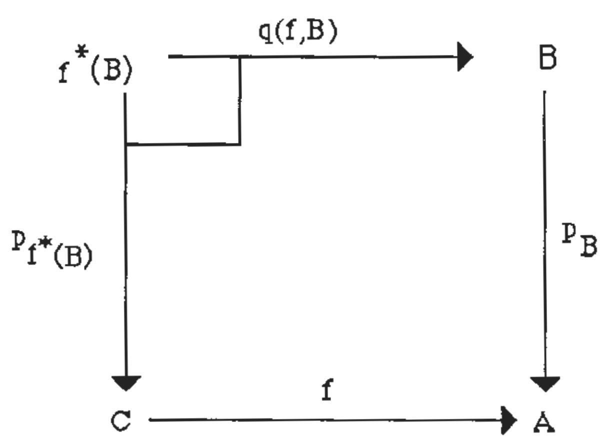
4.1 DEFINITION. A *tree* is a triple <**A**, level, father>, where **A** is a non-empty set, level and father are mappings, level:**A** →**N**, father: **A** →**A**, such that

(a) for every object x in **A**, level(x) = n+1 implies level(father(x)) = n;

(b) for all x,y ∈ **A**, level(x) = level(y) = 0 implies x=y and father(x) = x.

4.2 DEFINITION. ([Car], [Str]) (i) A *contextual category* is a 7-tuple <**C**,level,father,**1**,*p*,*,q> where

- **C** is a category;
- <Obj(**C**),level,father> is a tree;
- **1** is a terminal object in **C** such that level(**1**) = 0;
- *p* is a mapping which associates to each object A of **B** different from **1**, an arrow pA:A→father(A) in **C** called the *canonical projection* ;
- given objects A,B,C in **C** such that father(B) = A and a morphism f:C → A, there exists an object f^*B in **C** such that father(f^*B) = C and level(f^*B) = level(C) + 1; further, there is
- a morphism q(f,B):f^*B → B in **C** such that the following diagram is a pullback.



Additionally the following functoriality conditions are required.

- $id^*B = B$ and $q(id_A,B) = id_B$
- $(f \circ g)^*B = g^*(f^*B)$ and $q(f \circ g, B) = q(f,B) \circ q(g,f^*B)$.

(ii) Let <**C**,level,father,**1**,*p*,*,q> and <**C**',level',father,'**1**',*p*',*',q'> be contextual categories. A *contextual functor* is a functor F:**C** → **C**' such that

- father(B) = A ⇒ father'(F(B)) = F(A)
- level(B) = n ⇒ level'(F(B)) = n
- F(**1**) = **1**'
- F(*p*(B)) = *p*'(F(B))
- $F(f^*B) = F(f)^{*'}F(B)$
- F(q(f,B)) = q'(Ff,FB).

(iii) Cont is the category of contextual categories with contextual functors as arrows.

4.3 THEOREM. The construction from def. 1.2.6 yields a functor from $Comp_{full,split}$ to Cont which has a full and faithful left adjoint.

Proof. The functor $\mathcal{U}$: $Comp_{full,split}$→Cont forgets the objects and arrows which are not reachable from the terminal object; i.e. given a comprehension category $\mathcal{P}$:**E**→**B**→ it takes the largest tree structure in **B** defined by $\mathcal{P}$. This goal is achieved taking the category **B[t]** as basis and defining a contextual category on top of it in the following way.

- level $<E_1,...,E_n>$ = n;
- father $<E_1,...,E_n>$ = $<E_1,...,E_{n-1}>$;
- $p<E_1,...,E_n>$ = $\mathcal{P}E_n$.

For each object $<E_1,...,E_n>$ and arrow $u:<C_1,...,C_m> \to <E_1,...,E_{n-1}>$

- $u^*<E_1,...,E_n> = <C_1,...,C_m,u^*E_n>$;
- $q\ (u,<E_1,...,E_n>) = \mathcal{P}_0(u^-(E_n))$.

Given a morphism <K,H> of $Comp_{full,split}$, $\mathcal{U}$<K,H> is defined as H applied componentwise to objects and as K on arrows.

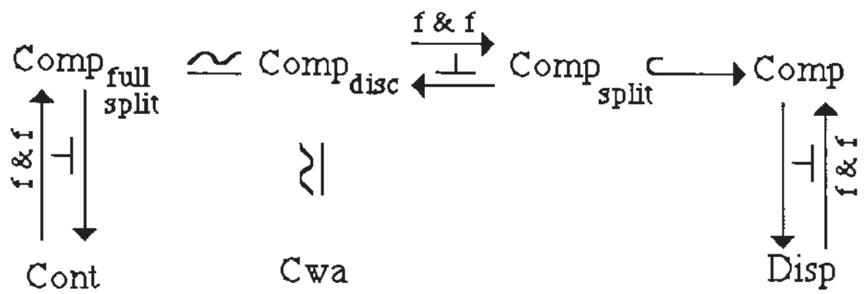
Given a contextual category <**B**, level,father,**1**,p,*,q> a full split comprehension category $\mathcal{P}$:**E**→**B**→ is obtained as follows. Objects in **E** are the objects of **B** with the exception of **1**; arrows from E to D in **E** are arrows from *p*E to *p*D in **B**→. Note that **E** is isomorphic to the full subcategory of **B**→ defined by the projections. Hence any contextual category can be seen as a display category. The functor $\mathcal{P}$ is obviously defined as *p* on objects and as the identity on arrows. In this way one obtains a functor $\mathcal{F}$:Cont→$Comp_{full,split}$.

The unit η:Id → $\mathcal{UF}$ is defined as follows. Let E be an object of the contextual category **B** with level(E) = n. For i ∈ {1,...,n-1} we put $E_i = father^i(E)$, then $father(E_{n-1}) = \mathbf{1}$. The component $\eta_{\mathbf{B}}(E)$ is defined by $<E_{n-1},...,E_1,E>$. The components of the counit send a chain $<E_1,...,E_n>$ to E_n in the total category and to

$\mathcal{P}_0(E_n)$ in the base. These components are isomorphisms, hence the functor $\mathcal{F}$ is full and faithful. □

5 Summary

The following picture summarizes the relationships established in this chapter.



The symbol "≃" denotes equivalence of categories and "f & f" means full and faithful functor.

Is interesting to note that if we restrict Disp to the split case, there is a linear order relation among the different structures. A category with attributes can be identified with a full split comprehension category. A (split) display category is a split comprehension category $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ where $\mathcal{P}$ is a full embedding. Finally, a contextual category can be seen as a full embedding $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ where $\mathcal{P}$ defines a tree structure in the base category **B**.

Chapter 2

Closed categorical type systems

Some important features of type systems are unit types, products and sums. In categorical type systems these features are usually defined via suitable adjunctions. We call the structures from the previous chapter *closed* in case they have these three features (additionally, fulness is required for comprehension categories).The aim of this chapter is to establish relations, analogous to the ones in the previous chapter, for the closed case. A diagram, which summarizes this result, can be found at the end of the chapter.

1 Closed Comprehension Categories

1.1 DEFINITION. (i) A comprehension category with *unit* is given by a fibration p provided with a terminal object functor $\mathbf{1}:\mathbf{B}\to\mathbf{E}$, which has a left adjoint $\mathcal{P}_0$. The functor $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ defined as $\mathcal{P}(E) = p(\varepsilon_E)$, where ε is the counit of the adjunction $\mathbf{1} \dashv \mathcal{P}_0$, forms a comprehension category (see [Jac 90]). These categories were introduces in [Ehr] under the name *D-Categories*.
(ii) A morphism $<K,H>:\mathcal{P}\to\mathcal{P}'$ of comprehension category with unit is a morphism of fibrations $p\to p'$ which preserves the terminal object functor; further, the canonical map $K\mathcal{P}_0\to \mathcal{P}_0'H$ is the identity (see lemma 1.1.2.(i) below for more details).
(iii) A split comprehension category with unit is defined by a split fibration with a split terminal object functor as in (i). Morphisms of split comprehension categories with unit should preserve the terminal object on the nose.

1.1.1 LEMMA. (i) Let $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ be a comprehension category with unit. For every object $E \in \mathbf{E}$ the arrow $\mathcal{P}\mathbf{1}E$ is an isomorphism.

(ii) Suppose $\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$ is a full comprehension category and $\mathbf{1}:\mathbf{B}\to\mathbf{E}$ a functor satisfying $p\mathbf{1} = id$ and $\mathcal{P}\mathbf{1}$ is an isomorphism. Then $\mathcal{P}$ is a comprehension category with unit.

Proof. (i) The unit $\eta:Id\to\mathcal{P}_0\mathbf{1}$ is an iso, since $\mathbf{1}$ is full and faithful. But $\mathcal{P}\mathbf{1} \circ \eta = p\varepsilon\mathbf{1} \circ p\mathbf{1}\eta = id$.

(ii) Let $A \in \mathbf{B}$, $E \in \mathbf{E}_A$.

	$\mathbf{E}(\mathbf{1}A, E)$	
$\cong$	$\mathbf{B}^{\to}(\mathcal{P}\mathbf{1}A, \mathcal{P}E)$	using the fulness of $\mathcal{P}$.
$\cong$	$\mathbf{B}^{\to}(id_A, \mathcal{P}E)$	because $\mathcal{P}\mathbf{1}$ is an isomorphism.
$\cong$	$\mathbf{B}(A, dom(\mathcal{P}E))$	because $id_{(-)}$ is left adjoint of dom.

The counit of this adjunction is given by $\varepsilon_E = \mathcal{P}^{-1}(\mathcal{P}E, \mathcal{P}\mathbf{1}\mathcal{P}_0E)$; the unit by $\eta_A = (\mathcal{P}\mathbf{1}A)^{-1}$. Notice that $p\varepsilon = \mathcal{P}$. □

1.1.2 LEMMA. (i) Definition 1.1 (ii) above indeed yields a map of comprehension categories.

(ii) If $\mathcal{P}$ and $\mathcal{P}'$ are full comprehension categories with unit, then a morphism $<K,H>:\mathcal{P}\to\mathcal{P}'$ of comprehension categories which preserves the terminal object functor, is a morphism of comprehension categories with unit.

(iii) If $\mathcal{P}$ and $\mathcal{P}'$ are split full comprehension categories with unit, then a morphism $<K,H>:\mathcal{P}\to\mathcal{P}'$ of split comprehension categories which preserves the terminal object functor, induces a map of adjunctions from $\mathbf{1} \dashv \mathcal{P}_0$ to $\mathbf{1}' \dashv \mathcal{P}_0'$.

Proof. (i) Let $\Phi: 1'K \to H1$ be the natural isomorphism required (because <K,H> preserves the terminal object). The canonical morphism $K(\mathcal{P}_0E) \to \mathcal{P}_0'(HE)$ is the transpose of the arrow $H\varepsilon_E \circ \Phi_{\mathcal{P}_0E}: 1'K(\mathcal{P}_0E) \to HE$, i.e. $\mathcal{P}_0'(H\varepsilon_E \circ \Phi_{\mathcal{P}_0E}) \circ \eta'_{K(\mathcal{P}_0E)}$, where ε is the counit of $1 \dashv \mathcal{P}_0$ and η' the unit of $1' \dashv \mathcal{P}_0'$. Then

$$
\begin{aligned}
\mathcal{P}'(HE) &= p'(\varepsilon'_{HE}) \circ \mathcal{P}_0'(H\varepsilon_E \circ \Phi_{\mathcal{P}_0E}) \circ \eta'_{K(\mathcal{P}_0E)} \\
&= p'(\varepsilon'_{HE} \circ 1'\mathcal{P}_0'(H\varepsilon_E \circ \Phi_{\mathcal{P}_0E}) \circ 1'\eta'_{K(\mathcal{P}_0E)}) \\
&= p'(H\varepsilon_E \circ \Phi_{\mathcal{P}_0E} \circ \varepsilon'_{1'K\mathcal{P}_0E} \circ 1'\eta'_{K(\mathcal{P}_0E)}) \quad \text{by naturality of } \varepsilon \\
&= p'H\varepsilon_E \\
&= Kp(\varepsilon_E) \\
&= K\mathcal{P}E
\end{aligned}
$$

(ii) In the full case, it is easy to check that the canonical arrow is the identity using the unit and counit as described in the proof of 1.1.1.(ii).

(iii) Easy, it is enough to prove that H commutes with the counit. □

1.2 DEFINITION. Let $\mathcal{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category,

(i) $\mathcal{P}$ admits *products* if the following two conditions hold.

(a) for every $E \in \mathbf{E}$, the weakening functor $\mathcal{P}E^*: \mathbf{E}_{pE} \to \mathbf{E}_{\mathcal{P}_0E}$ has a right adjoint Π_E;

(b) for every cartesian arrow $f: D \to E$ in $\mathbf{E}$, the canonical natural transformation

$$(pf)^*\Pi_E \to \Pi_D\, (\mathcal{P}_0 f)^* \text{ is an isomorphism.}$$

(ii) $\mathcal{P}$ admits *sums* if the following two conditions hold.

(a) for every $E \in \mathbf{E}$, the weakening functor $\mathcal{P}E^*: \mathbf{E}_{pE} \to \mathbf{E}_{\mathcal{P}_0E}$ has a left adjoint Σ_E;

(b) for every cartesian arrow $f: D \to E$ in $\mathbf{E}$, the canonical natural transformation

$$\Sigma_D\, (\mathcal{P}_0 f)^* \to (pf)^*\Sigma_E \text{ is an isomorphism.}$$

One says that $\mathcal{P}$ admits *strong* sums if it has sums in such a way that for all objects $D, E \in \mathbf{E}$ with $pD = \mathcal{P}_0E$, the canonical morphism $\mathcal{P}_0D \to \mathcal{P}_0\Sigma_E(D)$ is an isomorphism.

Condition (b) for products and sums is called in the literature the Beck-Chevalley condition.

1.3 DEFINITION. (i) A *closed* comprehension category is a full comprehension category with unit which has products and strong sums. A category CCompC is defined having closed comprehension categories as objects. Arrows from $(\mathcal{P}: \mathbf{E} \to \mathbf{B}^{\to})$ to $(\mathcal{P}': \mathbf{E}' \to \mathbf{B}'^{\to})$ are morphisms <K,H> of comprehension categories with unit, which preserve products and sums, i.e. for every object $E \in \mathbf{E}$,

$$H \circ \Pi_E \cong \Pi'_{HE} \circ H \quad \text{and} \quad H \circ \Sigma_E \cong \Sigma'_{HE} \circ H \quad \text{canonically.}$$

(ii) A split closed comprehension category is a split full comprehension category with unit which, additionally, admits products and strong sums in such a way that the isomorphisms given in 1.2.(i).(a) and 1.2.(ii).(a) are equalities. The last requirement is equivalent to say that the functors $(pf)^*$ and $(\mathcal{P}_0 f)^*$ induce a map of adjunctions. A category CCompC_{split} is defined having split closed comprehension categories as objects. Arrows are morphisms of split comprehension categories with unit, which preserve products and sums on the nose.

1.4 LEMMA. Let $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category with unit which has products. Products are then preserved by $\mathcal{P}$, i.e. for every $u:A \to pE$ in $\mathbf{B}$, there is a natural isomorphism

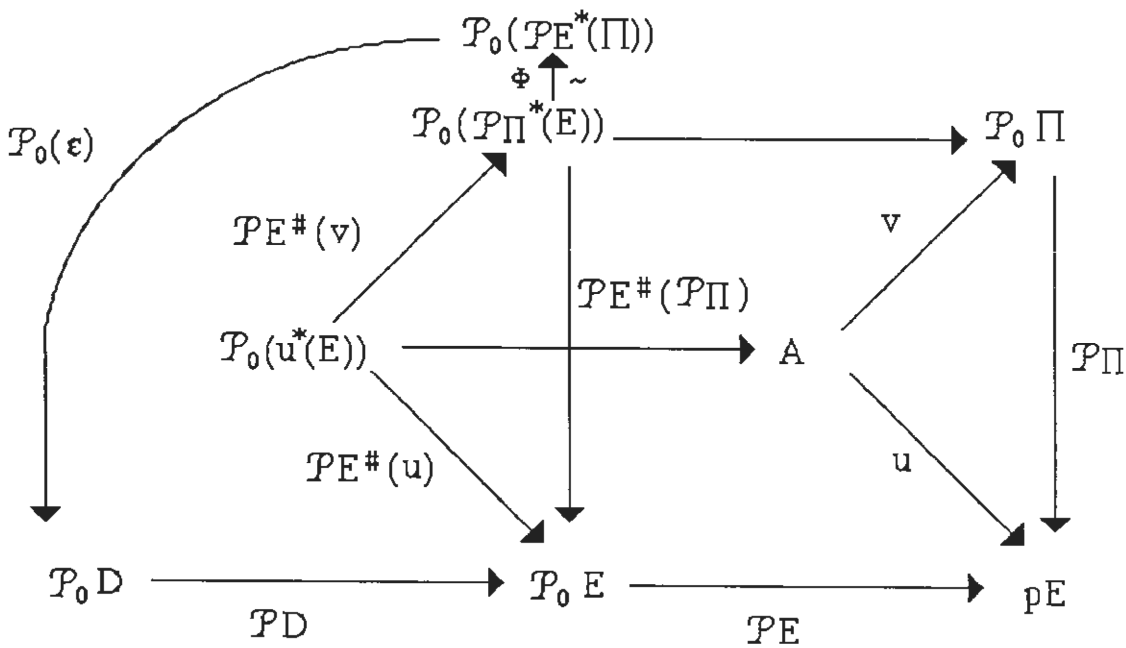
$$\mathbf{B}/pE(u, \mathcal{P}(\Pi_E(D))) \cong \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}E^{\#}(u), \mathcal{P}D)$$

where $\mathcal{P}E^{\#}(u)$ is the pullback functor defined in the first chapter 1.1.2.

Proof.

$$\begin{aligned}
& \mathbf{B}/pE(u, \mathcal{P}(\Pi_E(D))) \\
\cong\ & \mathbf{E}_u(\mathbf{1}A, \Pi_E(D)) && \text{because } \mathbf{1} \text{ is left adjoint of } \mathcal{P}_0 \\
\cong\ & \mathbf{E}_A(\mathbf{1}A, u^*(\Pi_E(D))) \\
\cong\ & \mathbf{E}_A(\mathbf{1}A, \Pi_{u^*(E)}(\mathcal{P}E^{\#}(u)^*(D)) && \text{Beck-Chevalley condition} \\
\cong\ & \mathbf{E}_{\mathcal{P}_0(u^*(E))}((\mathcal{P}u^*(E))^*(\mathbf{1}A), (\mathcal{P}E^{\#}(u)^*(D)) && \text{by the adjunction for products} \\
\cong\ & \mathbf{E}_{\mathcal{P}_0(u^*(E))}(\mathbf{1}\mathcal{P}_0(u^*(E)), (\mathcal{P}E^{\#}(u)^*(D)) && \mathbf{1} \text{ commutes with reindexing} \\
\cong\ & \mathbf{E}_{\mathcal{P}_0(u^*(E))}(\mathbf{1}\mathcal{P}_0(u^*(E)), D) \\
\cong\ & \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}E^{\#}(u), \mathcal{P}D)
\end{aligned}$$

This isomorphism sends v to $\mathcal{P}_0(\varepsilon_E(D)) \circ \Phi \circ \mathcal{P}E^{\#}(v)$, where Φ is the isomorphism defined in I.1.1.2 in the following diagram.



We have written ε for $\varepsilon_E(D)$ and Π for $\Pi_E(D)$. □

1.5 LEMMA. Let $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ be a full comprehension category with the following property. For every pair of objects D, E in $\mathbf{E}$ with $pD = \mathcal{P}_0E$ there exist an object

$\Sigma_E(D)$ in $\mathbf{E}_{pE}$ such that $\mathcal{P}\Sigma_E(D) \cong \mathcal{P}E \circ \mathcal{P}D$ in $\mathbf{B}/pE$. Then there exist an isomorphism between the hom-sets $\mathbf{E}_{pE}(\Sigma_E(D),A)$ and $\mathbf{E}_{\mathcal{P}_0 E}(D,\mathcal{P}E^*(A))$.

Proof.

$$\begin{aligned} & \mathbf{E}_{pE}(\Sigma_E(D), A) & \\ \cong\ & \mathbf{B}/pE(\mathcal{P}\Sigma_E(D), \mathcal{P}A) & \text{because } \mathcal{P} \text{ is full} \\ \cong\ & \mathbf{B}/pE(\mathcal{P}E \circ \mathcal{P}D, \mathcal{P}A) & \\ \cong\ & \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}D, \mathcal{P}E^{\#}(\mathcal{P}A))) & \text{because the composition functor } \mathcal{P}E\circ\_ \\ & & \text{is left adjoint to the pullback functor } \mathcal{P}E^{\#} \\ \cong\ & \mathbf{E}_{\mathcal{P}_0 E}(D, \mathcal{P}E^*(A)). & \end{aligned}$$

In case we have that the Beck-Chevalley condition holds for the resulting sums, then the comprehension category $\mathcal{P}$ has strong sums. The isomorphism $\mathcal{P}\Sigma_E(D) \cong \mathcal{P}E \circ \mathcal{P}D$ given above is then the canonical one. □

1.6 LEMMA. The endofunctor (-)[t]:Comp→Comp defined in lemma I.1.2.6 restricts to the closed case.

Proof. Straightforward, given the fact that for a comprehension category $\mathcal{P}[\mathbf{t}]:\mathbf{E}[\mathbf{t}]\to\mathbf{B}[\mathbf{t}]^{\to}$, the hom-set $\mathbf{E}[\mathbf{t}](\langle E_1,\ldots,E_n\rangle, \langle D_1,\ldots,D_m\rangle)$ is equal to $\mathbf{E}(E_n, D_m)$. □

2 Closed Categories with Attributes

In this section the definitions of unit products and sums are given. These features can only be defined in the base category because the fibres are discrete. It is for this reason that the constructions are quite complicated.

In the rest of this section, the following notational conventions introduced in [Mog] will be used. Given a category with attributes $(\mathbf{B},\Psi,\Psi_0,\psi)$, we write

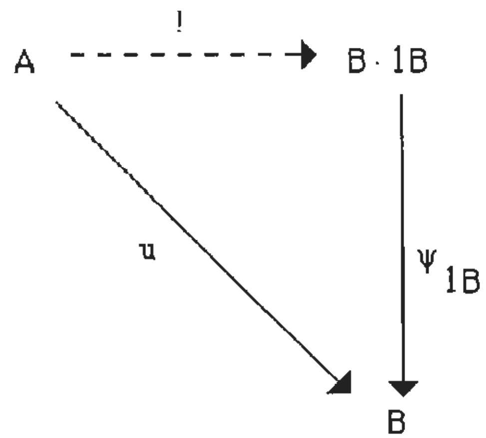
$A{\cdot}E$ for $\Psi_0(A,E)$

$u{\cdot}E$ for $\Psi_0(u^-(E)) = \Psi_0(u,id_{u^*E})$

2.1 DEFINITION. (i) A category with attributes $(\mathbf{B},\Psi,\Psi_0,\psi)$ has *unit* if for every object $A \in \mathbf{B}$ there exist an object $1(A) \in \Psi(A)$ such that the following two conditions hold.

(a) For every morphism $u:A\to B$ in $\mathbf{B}$, $u^*(1(B)) = 1(A)$

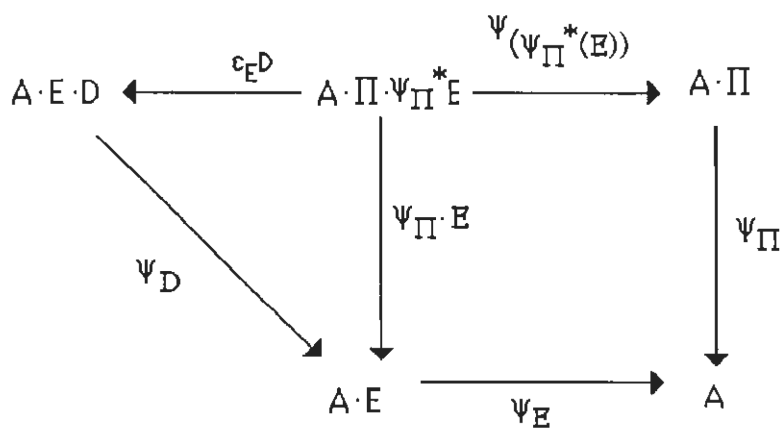
(b) For every morphism $u:A\to B$ there exist a unique arrow $!_A:A\to B{\cdot}1B$ such that the following diagram commutes



Condition (b) can be formulated equivalently as

(b') For every object A in **B**, there exist an isomorphism $!_A{:}A \cong A{\cdot}1A$, from id_A to $\psi_{1(A)}$ in **B**/A.

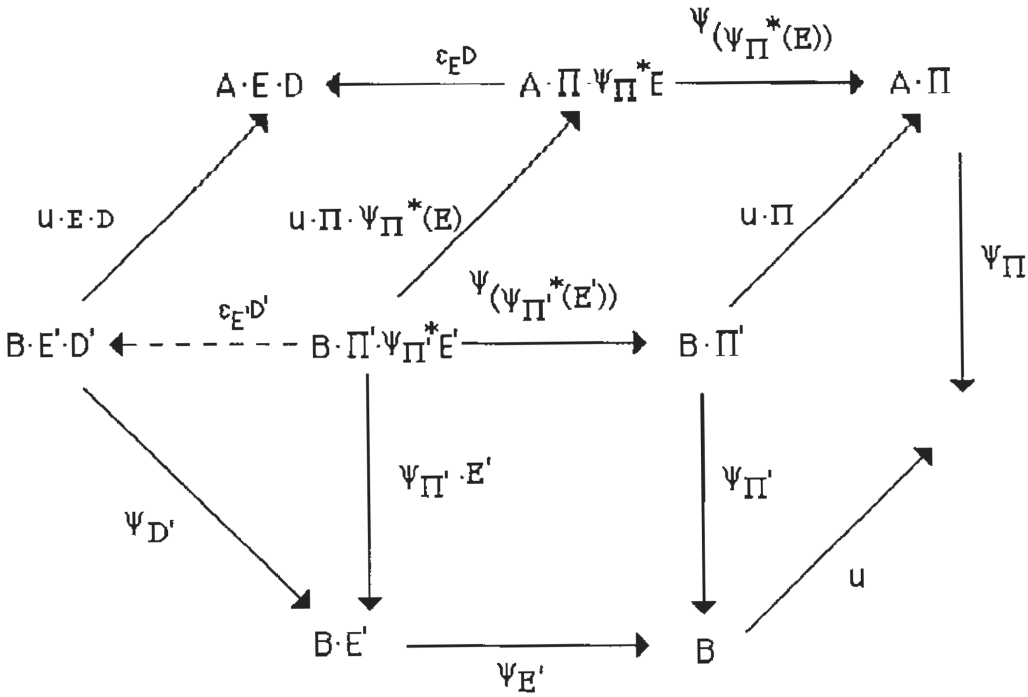
(ii) A category with attributes $(\mathbf{B},\Psi,\Psi_0,\psi)$ has *products* if given objects $E \in \Psi(A)$ and $D \in \Psi(A{\cdot}E)$, there exist an object $\Pi_E(D) \in \Psi(A)$ and an arrow $\varepsilon_E(D)$ in **B** as in the following diagram[1]; such that conditions (a) and (b) stated below hold.



(a) Let $u{:}B \to A$ be a morphism in **B**, E' and D' the objects $u^*(E)$ and $(u{\cdot}E)^*(D)$ respectively. Then $\Pi_{E'}(D') = u^*(\Pi_E(D))$ and $\varepsilon_{E'}(D')$ is the unique arrow which makes the following diagram commute.

---

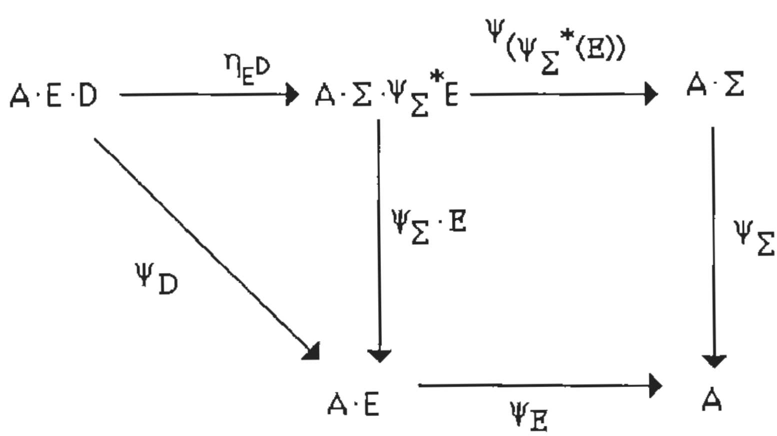
1 The reader may notice that the counit here has a different (but isomorphic) domain from the usual one, when products are defined as right adjoints. As a result we have to insert the isomorphism from I.1.1.2 in some proofs below.



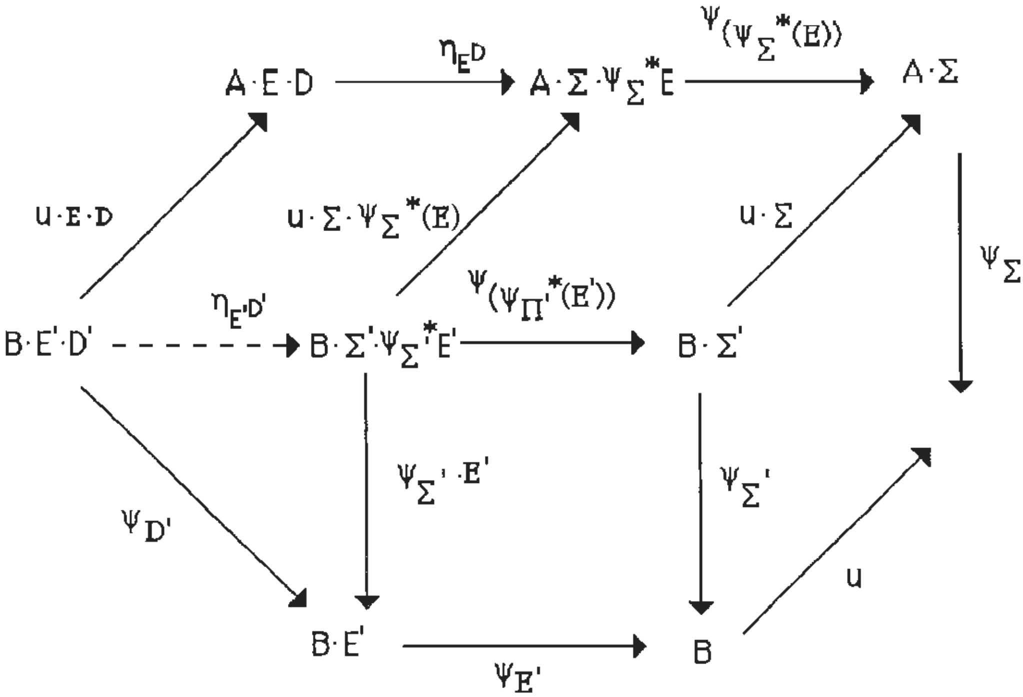
We have written Π for $\Pi_E(D)$ and Π' for $\Pi_{E'}(D')$

(b) There is a natural isomorphism $\mathbf{B}/A(u, \psi_{\Pi E(D)}) \cong \mathbf{B}/(A{\cdot}E)(u{\cdot}E, \psi_D)$ sending v to the composition $\varepsilon \circ v{\cdot}(\psi_{\Pi E(D)}{}^*(E))$.

(iii) A category with attributes $(\mathbf{B},\Psi,\Psi_0,\psi)$ has *sums* if given objects $E \in \Psi(A)$ and $D \in \Psi(A{\cdot}E)$, there exist an object $\Sigma_E(D) \in \Psi(A)$ and an arrow $\eta_E(D)$ in **B** as in the following diagram; additionally, they should satisfy requirements (a) and (b) stated below.



(a) Let $u:B \to A$ be a morphism in $\mathbf{B}$, E' and D' the objects $u^*(E)$ and $(u\cdot E)^*(D)$ respectively. Then $\Sigma_{E'}(D') = u^*(\Sigma_E(D))$ and $\eta_{E'}(D')$ is the unique arrow which makes the following diagram commute.



We have written Σ for $\Sigma_E(D)$ and Σ' for $\Sigma_{E'}(D')$.

(b) There is a natural isomorphism $\mathbf{B}/A(\psi_{\Sigma E(D)},u) \cong \mathbf{B}/(A\cdot E)(\psi_D,u\cdot E)$ which sends an arrow v to the composition $v\cdot(u^*(E)) \circ \eta$.
The following equivalent condition can be used

(b') The arrow $\psi \circ \eta$ from $\psi_E \circ \psi_D$ to $\psi_{\Sigma E(D)}$ in $\mathbf{B}/A$ is an isomorphism.

2.2 DEFINITION. Let $(K,\alpha):(\mathbf{B},\Psi,\Psi_0,\psi)\to(\mathbf{B}',\Psi',\Psi'_0,\psi')$ be a morphism of Cwa.

(a) (K,α) preserves unit if for every object $B \in \mathbf{B}$, $\alpha_B(1B) = 1(KB)$;

(b) (K,α) preserves products if for every $A \in \mathbf{B}$, $E \in \Psi(A)$ and $D \in \Psi(\Psi_0(E))$ the following two equations hold.

$$\alpha_A(\Pi_E(D)) = \Pi_{\alpha A(E)}(\alpha_{\Psi 0(E)}(D))$$

and $\quad K(\varepsilon_E(D)) = \varepsilon_{\alpha A(E)}(\alpha_{\Psi 0(E)}(D))$;

(c) (K,α) preserves sums if for every $A \in \mathbf{B}$, $E \in \Psi(A)$ and $D \in \Psi(\Psi_0(E))$ the following two equations hold.

$$\alpha_A(\Sigma_E(D)) = \Sigma_{\alpha A(E)}(\alpha_{\Psi 0(E)}(D))$$

and $\quad K(\eta_E(D)) = \eta_{\alpha A(E)}(\alpha_{\Psi 0(E)}(D))$

2.3 DEFINITION. A category CCwa is defined having closed categories with attributes as objects, i.e. categories with attributes with unit, products and sums.

Morphisms in CCwa are morphism of Cwa which preserve unit, products and sums.

Definition 2.1 is taken from |Mog|, whereas definitions 2.2 and 2.3 are our addition.

2.4 LEMMA. There is a functor $\mathcal{F}^{\heartsuit}$:Cwa→$\text{Comp}_{\text{full,split}}$ obtained by composition from $\mathcal{F}$:Cwa→|Comp| (see I.2.3) and $(\_)^{\heartsuit}$:|Comp|→$\text{Comp}_{\text{full,split}}$ (see I.1.2.3). This functor restricts to a functor from CCwa to $\text{CCompC}_{\text{split}}$ between the categories of corresponding closed structures.

Proof. Let $(\mathbf{B},\Psi,\Psi_0,\psi)$ be a closed category with attributes and $\mathcal{F}^{\heartsuit}(\mathbf{B},\Psi,\Psi_0,\psi) = \mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to}$. The category $\mathbf{E}$ has the same objects as $\mathcal{G}\Psi$; arrows from (A,E) to (B,D) in $\mathbf{E}$ are arrows from $\psi_{(A,E)}$ to $\psi_{(B,D)}$ in $\mathbf{B}^{\to}$. The functor $\mathcal{P}$ is given by ψ on objects and is the identity on arrows.
(i) $\mathcal{P}$ has units; for all $A \in \mathbf{B}$, 1A is terminal in $\mathbf{E}_A$, because

$$\mathbf{E}_A((A,E), 1A)$$
$$= \mathbf{B}/A(\mathcal{P}(A,E), \mathcal{P}(1A)),$$ and this hom-set is a singleton.

By definition of units in Cwa, $1B = u^*(1A)$ for all $u:B \to A$, hence terminal are preserved under reindexing. Since $\mathcal{P}(1A)$ is an isomorphism, see condition 2.1.(i).(b'), lemma 1.1.2 (i) applies.

(ii) $\mathcal{P}$ has products; for every object $(A, E) \in \mathbf{E}_A$ and $(\mathcal{P}_0(A,E), D) \in \mathbf{E}_{\mathcal{P}_0(A,E)}$ the object $\prod_{(A,E)}(\mathcal{P}_0(A,E), D)$ is defined as $(A, \prod_E(D))$.
Then, $\mathcal{P}(A,E)^*$ is left adjoint of $\prod_{(A,E)}$.

$$\begin{aligned}
&\mathbf{E}_A((A,C), \textstyle\prod_{(A,E)}(B,D)) \\
= \;& \mathbf{B}/A(\mathcal{P}(A,C), \mathcal{P}\textstyle\prod_{(A,E)}(B,D)) && \text{by definition of heart} \\
\cong \;& \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}_0(\mathcal{P}(A,C)^{-}(A,E)), \mathcal{P}(B,D)) && \text{by products in Cwa} \\
\cong \;& \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}(\mathcal{P}(A,E)^*(A,C)), \mathcal{P}(B,D)) \\
\cong \;& \mathbf{E}_{\mathcal{P}_0(E)}(\mathcal{P}(A,E)^*(A,C), (B,D)).
\end{aligned}$$

the counit of this adjunction in (B,D) is $\langle id, \varepsilon_E(D) \circ \Phi\rangle$, where Φ is the mediating arrow between relevant pullbacks and $\varepsilon_E(D)$ is the arrow defined in 2.1 (ii).

For each split arrow $\langle u, \mathcal{P}_0(u^{-}(A,E))\rangle$, the functors u^* and $\mathcal{P}_0(u^{-}(A,E))^*$ define a mapping of adjunctions (Beck-Chevalley condition)

(a) $\mathcal{P}(u^*(A,E))^* \circ u^* \quad = \quad \mathcal{P}_0(u^{-}(A,E))^* \circ \mathcal{P}(A,E)^*$
(b) $\prod_{u^*(A,E)} \circ \mathcal{P}_0(u^{-}(A,E))^* \quad = \quad u^* \circ \prod_{(A,E)}$
(c) $\mathcal{P}_0(u^{-}(A,E))^*(\langle id, \varepsilon_E(D) \circ \Phi\rangle) \quad = \quad \langle id, \varepsilon_{u^*E}(\mathcal{P}_0(u^{-}(A,E))^*(D)) \circ \Phi'\rangle$

The equality on (a) always holds; (b) is valid by definition of products in Cwa. The last equation can be proved using the uniqueness of the counit in Cwa, the details are left to the reader.

(iii) $\mathcal{P}$ has strong sums; given an object (A,E) in **E**, $\Sigma_{(A,E)}(\mathcal{P}_0(A,E),D)$ is defined as $(A,\Sigma_E(D))$. Then

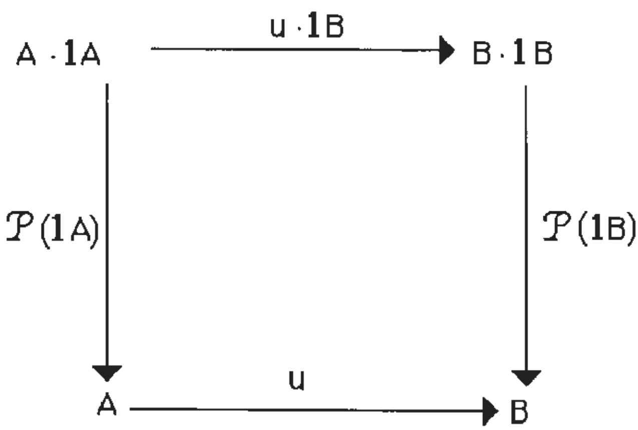
$$\begin{aligned} & \mathcal{P}_0(\mathcal{P}_0(A,E), D) \\ \cong\ & \mathcal{P}_0(A, \Sigma_E(D)) && \text{by definition of sums in Cwa} \\ =\ & \mathcal{P}_0(\Sigma_{(A,E)}(\mathcal{P}_0(A,E), D)) \end{aligned}$$

The Beck-Chevalley condition holds; the proof is analogous to the one for products. Then, lemma 1.5 gives the required adjunction.

Let <K,α> be a morphism from $(\mathbf{B},\Psi,\Psi_0,\psi)$ to $(\mathbf{B}',\Psi',\Psi'_0,\psi')$ of closed categories with attributes, $\mathcal{F}^{\blacktriangledown}$<K,α> is K in the base; in the total category it sends an object (A,E) ∈ **E** to $(KA,\alpha_A E)$, and an arrow (u,v) ∈ **E** to (Ku,Kv).
The morphism $\mathcal{F}^{\blacktriangledown}$<K,α> preserves the unit by lemma 1.1.1.(ii). Products and sums are preserved by definition. □

2.5 LEMMA. There is a forgetful functor $\mathcal{U}$:$\text{Comp}_{\text{full,split}}$→Cwa given by composition from (the restriction of) | _ |:$\text{Comp}_{\text{full,split}}$→|Comp| (see I.1.2.2) and from the equivalence between Cwa and |Comp| stated in theorem I.2.3. This functor restricts to a functor between $\text{CCompC}_{\text{split}}$ and CCwa.

Proof. Let $\mathcal{P}$:**E**→**B**$^{\to}$ be a closed comprehension category and $\mathcal{U}(\mathcal{P}:\mathbf{E}\to\mathbf{B}^{\to})$ = $(\mathbf{B},\Psi,\Psi_0,\psi)$.
(i) $(\mathbf{B},\Psi,\Psi_0,\psi)$ has unit via the isomorphism $\mathcal{P}\mathbf{1}$ (see condition 2.1.(i).(b')). Given an arrow u:A→B in **B** the following diagram commutes, because p**1** = id and terminal objects are preserved under reindexing.



(ii) $(\mathbf{B},\Psi,\Psi_0,\psi)$ has products. Using lemma 1.2 it follows that for every morphism u:A→pE in **B** there is a natural isomorphism

$$\mathbf{B}/pE(u, \mathcal{P}(\Pi_E(D))) \cong \mathbf{B}/\mathcal{P}_0(E)(\mathcal{P}_0(u^{-}(E)), \mathcal{P}D)$$

This isomorphism is the canonical one which sends an arrow v to

$$\mathcal{P}_0(\varepsilon_E(D)) \circ \Phi \circ \mathcal{PE}^{\#}(v) = \varepsilon \circ \mathcal{PE}^{\#}(v),$$ where ε is the arrow defined in 2.1.(ii).

Beck-Chevalley condition for products in Cwa (2.1.(ii).(a)) follows from the commutativity of the following diagram.

$$\begin{array}{ccccc}
A\cdot E\cdot D & \xleftarrow{\mathcal{P}_0\,\varepsilon_E D} & A\cdot\Pi\cdot\Psi^*E & \xleftarrow[\sim]{\Phi} & A\cdot E\cdot\Psi^*\Pi \\
\uparrow{\scriptstyle u\cdot E\cdot D} & & \uparrow{\scriptstyle u\cdot\Pi\cdot\Psi^*(E)} & & \uparrow{\scriptstyle u\cdot E\cdot\Psi^*(\Pi)} \\
B\cdot E'\cdot D' & \xleftarrow{\mathcal{P}_0\,\varepsilon_{E'}D'} & B\cdot\Pi'\cdot\Psi^*E' & \xleftarrow[\sim]{\Phi'} & B\cdot E'\cdot\Psi^*\Pi'
\end{array}$$

The left square commutes because the functors u^* and $\mathcal{P}_0(u^-(E))^*$ induce a mapping of adjunctions.
The right one commutes by a uniqueness diagram chase.
It is then easy to check that $\varepsilon_{E'}(D') \circ \Phi'$ is the unique arrow which make the diagram in 2.1.(ii).(a) commute.

(iii) $(\mathbf{B},\Psi,\Psi_0,\psi)$ has sums, given by the isomorphism

$$\mathcal{P}_0(\mathcal{PE}^-(\Sigma_E(D)) \circ \eta_E(D)) : \mathcal{P}_0(D) \to \mathcal{P}_0(\Sigma_E(D)).$$

which fulfils condition 2.1.(iii).(b).
The proof of the "Beck-Chevalley" condition is analogous to the one for products.
□

2.5. THEOREM. There is an equivalence of categories between $CCompC_{split}$ and CCwa.

Proof. Directly, from lemmas 2.3 and 2.4. and theorem I.2.3 □

3 Closed Display Categories

Given a display category $(\mathbf{B},\mathcal{D})$ we use the notation $\mathbf{B}(\mathcal{D})^{\rightarrow}$ for the full subcategory of $\mathbf{B}^{\rightarrow}$ with objects in $\mathcal{D}$ (introduced in the proof of lemma I.3.3). Given an object $A \in \mathbf{B}$, the fibre $\mathbf{B}(\mathcal{D})^{\rightarrow}{}_A$, determined by the codomain fibration $\mathbf{B}(\mathcal{D})^{\rightarrow} \rightarrow \mathbf{B}$, is identified with the "slice" category $\mathcal{D}/A$ of arrows with codomain A and commuting triangles.

3.1 DEFINITION. Let $(\mathbf{B},\mathcal{D})$ be a display map category

(i) $(\mathbf{B},\mathcal{D})$ has unit if for every object $A \in \mathbf{B}$ there is an iso $i_A \in \mathcal{D}$ with codomain A.

(ii) $(\mathbf{B},\mathcal{D})$ has products if for every display map $f:A\rightarrow B$ in $\mathcal{D}$, the pullback functor $f^{\#}:\mathcal{D}/B\rightarrow\mathcal{D}/A$ has a right adjoint $\prod_f$ in such a way that Beck-Chevalley condition holds (see theorem 3.3 below). A morphism $F:(\mathbf{B},\mathcal{D})\rightarrow(\mathbf{B}',\mathcal{D}')$ of display categories preserves products if for all $g:C\rightarrow A$, $f:A\rightarrow B$ in $\mathcal{D}$, the object $F(\prod_f(g))$ is isomorphic to $\prod_{Ff}(Fg)$ canonically.

(iii) $(\mathbf{B},\mathcal{D})$ has sums iff $\mathcal{D}$ is closed under composition, i.e. given two display maps $f,g \in \mathcal{D}$ then $f \circ g \in \mathcal{D}$.

3.2 DEFINITION: A category CDisp is defined having closed display categories as objects, i.e. display categories with unit, sums and products. Arrows in CDisp are product preserving morphisms of Disp. This is enough, since by functoriality every morphism of display categories preserves unit and sums.

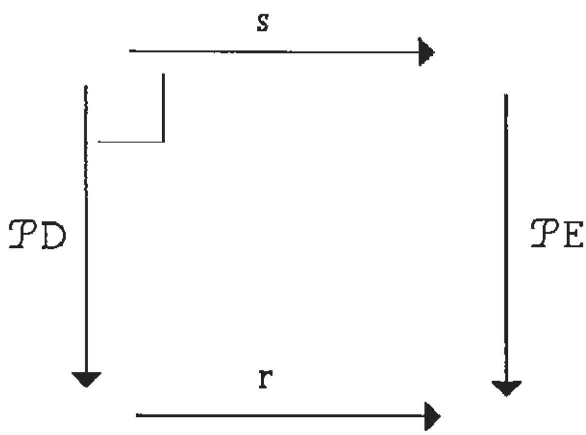
3.3 THEOREM. There is a full and faithful functor CDisp→CCompC which has a left adjoint (see theorem I.3.2).

Proof. The constructions are the same as in I.2.3. Given a closed display category $(\mathbf{B},\mathcal{D})$, the inclusion $\mathbf{B}(\mathcal{D})^{\rightarrow}\hookrightarrow\mathbf{B}^{\rightarrow}$ is a full comprehension category.

(i) This inclusion has unit. As $i_A \in \mathcal{D}$ is an iso, one has that it is terminal in $\mathcal{D}/A$, moreover terminal objects are preserved under reindexing. Hence we can define a terminal object functor sending A to i_A satisfying the hypothesis of lemma 1.1.1.

(ii) It is easy to see that it has products. As the inclusion is full, the Beck-Chevalley condition for comprehension categories is equivalent to the following.

For every pullback in $\mathbf{B}$ of the form



$r^*\Pi_E \cong \Pi_D s^*$ canonically, which is the Beck-Chevalley condition for display categories.

(iii) Given a composable pair of arrows f,g ∈ $\mathcal{D}$, the object $\Sigma_f(g)$ can be defined as f ∘ g. Then there are adjunctions $\Sigma_f \dashv f^\#$ satisfying the Beck-Chevalley condition and the corresponding comprehension category $\mathbf{B}(\mathcal{D})^{\rightarrow} \hookrightarrow \mathbf{B}^{\rightarrow}$ admits sums. Moreover dom(Σ_f(g)) = dom(f ∘ g) = dom(g), so these sums are strong.

The other way round, given a closed comprehension category $\mathcal{P}$:**E**→**B**$^{\rightarrow}$, one obtains a display category (**B**,$\mathcal{D}_{\mathcal{P}}$) taking $\mathcal{D}_{\mathcal{P}}$ to be the set {f| f ≅ pE in **B**/pE, for some E ∈ **E**} defined in I.3.2. Is easy to see that this display category is closed. Using theorem I.3.2 one obtains the required adjunction. □

4 Closed Contextual Categories

The definition for products and sums below is taken from [Str], whereas the definition for unit is our addition.

In the definition of products we use the following notation. Given an object E in a contextual category (**B**,level,father,**1**,p,*,q), the collection of |E| *sections* is defined as the set {f:father(E)→E| pE ∘ f = $id_{father(E)}$}.

4.1 DEFINITION Let $\mathcal{C}$ = (**B**,level,father,**1**,p,*,q) be a contextual category.

(i) $\mathcal{C}$ has *unit* if there exist an object **U** ∈ **B** with level(**U**) = 1 such that p**U** is an isomorphism.

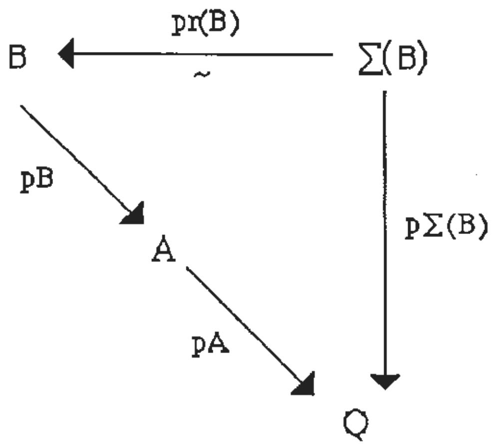
(ii) $\mathcal{C}$ has *products* if for every object B∈ **B** with level(B) ≥ 2, there is an object Π(B); put A = father(B) and Q = father(A), then father(Π(B)) = Q. Further if there is an arrow $eval_B$:pA*(Π(B))→B over A, i.e. pB ∘ $eval_B$ = p(pA*(Π(B))), such that the following properties hold. Given a section f ∈ |B| there exist a unique g ∈

$|\prod(B)|$ such that $eval_B \circ pA^*(g) = f$. For all $m:P\rightarrow Q$ in **B** the Beck-Chevalley condition holds, i.e:

$m^*(\prod(B)) = \prod(m^*(B))$

$q(m,A)^*(eval_B)) = eval_{q(m,A)^*(B)}$.

(iii) One says that a contextual category $\mathcal{C}$ has *sums* if for every object $B \in$ **B** with level(B) ≥ 2, there is an object $\Sigma(B)$ such that if A = father(B) and Q = father(A), then father(Σ(B)) = Q. Further if there is an isomorphism pr(B):Σ(B)$\rightarrow$B such that the following diagram commutes



Additionally for all $m:P\rightarrow Q$ one requires

$m^*(\Sigma(B)) = \Sigma(m^*(B))$

$m^*(pr(B))) = pr(m^*(B))$

(iv) CCont is defined as the category having closed contextual categories, i.e. contextual categories with unit, products and sums, as objects. Arrows are contextual functors which preserves unit, products and sums on the nose.

4.2 LEMMA. A contextual category (**B**,level,father,**1**,p,*,q) has unit if and only if for every object A in **B**, there exist an object 1A over A with p1A:1A$\rightarrow$A an isomorphism.

Proof. (if) One takes **U** = 1**1**.
(only if) Let $!_A$:A$\rightarrow$**1** be the unique morphism in **B**, then 1A is defined as $!_A^*$(**U**).
□

The definition given for unit is type theoretical, in the sense that **U** is a "closed type" (level(**U**)=1) and it contains exactly one element in any "context". The lemma above gives a more useful formulation.

4.3 LEMMA. The full and faithful functor $\mathcal{F}$:Cont $\to$ $\mathrm{Comp}_{\mathrm{full,split}}$, defined in theorem 4.3 in the previous chapter, restricts to the closed case.

Proof. Let $\mathcal{C}$ = (**B**,level,father,**1**,p,*,q) be a closed contextual category, and $\mathcal{F}(\mathcal{C})$ = $\mathcal{P}$:**E**$\to$**B**$^{\to}$. The category **E** has objects from **B**; arrows from E to D in **E** are arrows from pE to pD in **B**$^{\to}$.

(i) $\mathcal{P}$ has unit because p1A is an iso, and hence terminal in **B**/A. Pullbacks of isomorphisms are isomorphisms, so terminal objects are preserved under reindexing on the nose. Then lemma 1.1.2 (i) applies.

(ii) $\mathcal{P}$ has products: given objects E,D $\in$ **B** with father(D) = E, one takes $\prod$(D), the object defined in 4.1.(ii), as $\prod_E$(D). Then

$$
\begin{aligned}
& \mathbf{E}_{pE}(A, \textstyle\prod_E(D)) \\
= \ & \mathbf{B}/pE(\mathcal{P}A, \mathcal{P}(\textstyle\prod_E(D))) \\
= \ & \mathbf{B}/pE(pA, p(\textstyle\prod_E(D))) \\
\cong \ & \mathbf{B}/A(id_A, p(\textstyle\prod_E(pA^*(D)))) \\
\cong \ & \mathbf{B}/pA^*(D)(id_{pA^*(D)}, p(pA^*(D))) \\
\cong \ & \mathbf{B}/\mathcal{P}_0E(pE^*(pA), pD) \\
= \ & \mathbf{E}_{\mathcal{P}_0(E)}(\mathcal{P}E^*(A), D)
\end{aligned}
$$

Beck-Chevalley holds by definition of products in contextual categories

(iii) $\mathcal{P}$ has sums: given objects E,D $\in$ **B** with father(D) = E one takes Σ_E(D) to be the object Σ(D) defined in 4.1.(iii). The arrow $\mathrm{pr}(D)^{-1}$:$\mathcal{P}_0(D)\to\mathcal{P}_0(\Sigma_E(D))$ is an isomorphism. Hence by lemma 1.3 Σ_E is left adjoint of $\mathcal{P}E^*$. The Beck-Chevalley condition for this comprehension category can be proved easily. □

4.4 LEMMA. The functor $\mathcal{U}$:$\mathrm{Comp}_{\mathrm{full,split}}\to$Cont defined in 4.3 in the first chapter restricts to the closed case.

Proof. Let $\mathcal{P}$:**E**$\to$**B**$^{\to}$ be a closed comprehension category, an let $\mathcal{U}$($\mathcal{P}$:**E**$\to$**B**$^{\to}$) = $\mathcal{C}$, i.e. $\mathcal{C}$ = (**B**[**t**],level,father,<>,p,*,q).

(i) It has unit because for every object $<E_1,\ldots,E_n>$ in **B**[**t**] the morphism $\mathcal{P}(\mathbf{1}\mathcal{P}_0(E_n))$:$<E_1,\ldots,E_n,\mathbf{1}\mathcal{P}_0(E_n)>\to<E_1,\ldots,E_n>$ is an iso.

(ii) $\mathcal{C}$ also has products taking the particular case of lemma 1.4 when the arrow considered (called u in the formulation) is the identity.

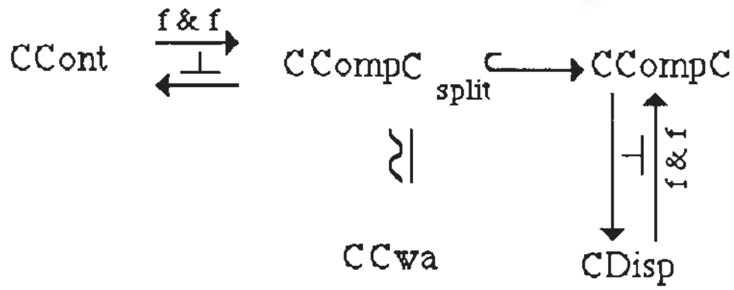
(iii) Finally, $\mathcal{C}$ has sums, because of the isomorphism of strong sums in $\mathcal{P}$. □

4.5 THEOREM. There is an adjunction $\mathcal{F} \dashv \mathcal{U}$ between $\mathrm{CCompC}_{\mathrm{split}}$ and CCont.

Proof. From 4.3, 4.4 and I.4.3. □

5 Summary

The following picture summarizes the relationships established in this chapter.



Conclusions

In this work some relations between different categorical versions of type dependent systems have been established. It shows that contextual categories, categories with attributes and display categories can all be studied inside suitable categories of comprehension categories

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