Sector theory for Levin-Wen models.

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12 / 11 / 2025

GPT Seminar

Outline:

I. Review of superselection theory.

II. Sector theory for Levin-Wen models.
Ongoing work with Boris Kjaer.

I. Sector Theory.

Naaijkens 11'
Ogata 22'.

Let $\mathcal{A} = \overline{\bigotimes_{v \in \mathbb{Z}^2} \mathcal{A}_v}^{\|\cdot\|}$ be a 2D quantum spin system, and

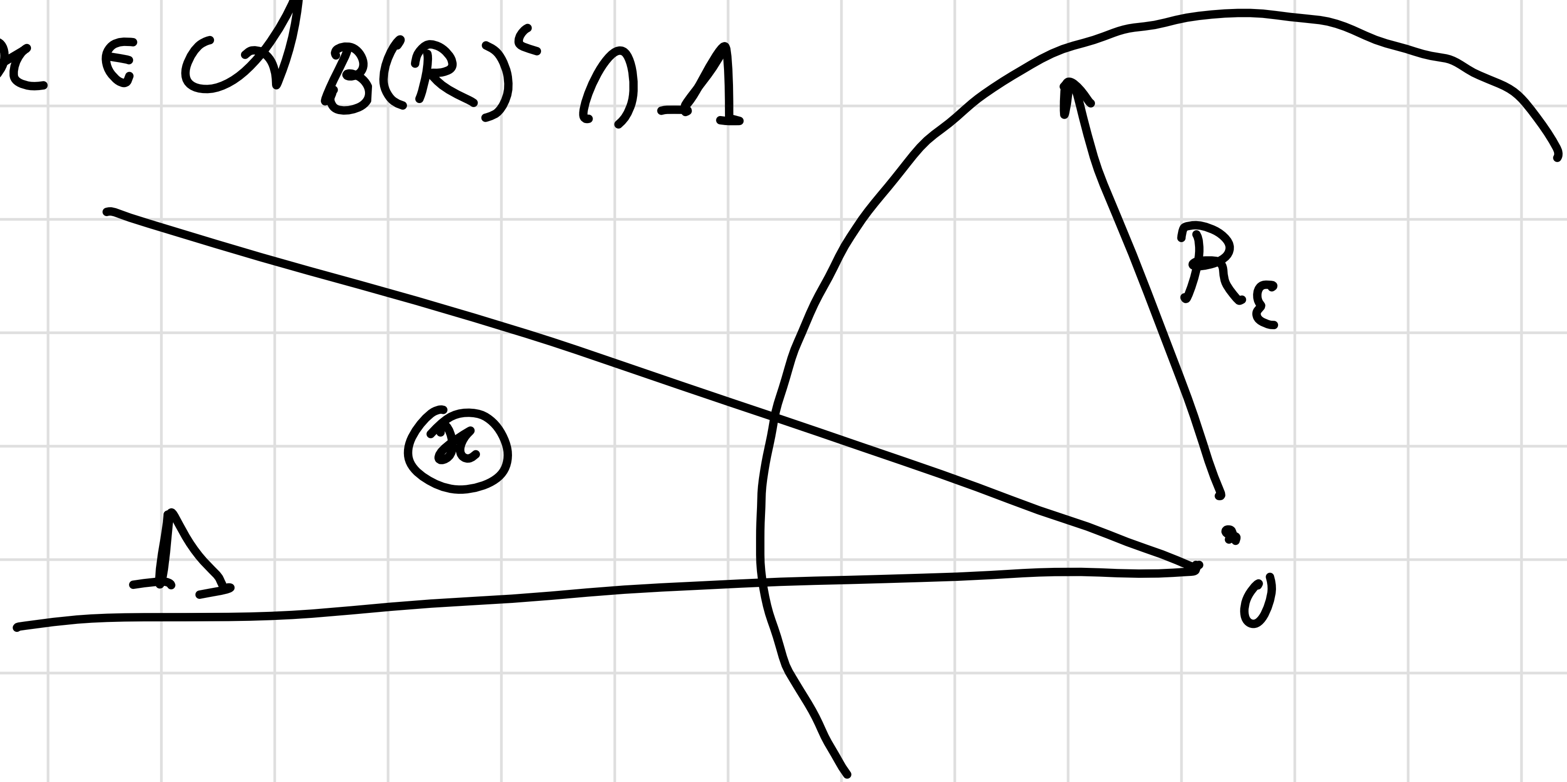
$$\omega^0 : \mathcal{A} \longrightarrow \mathbb{C}$$

a gapped ground state.

A pure state $\psi : \mathcal{A} \to \mathbb{C}$ is an <u>anyonic excitation</u> of ω^{Λ} if for any cone Λ and any $\varepsilon > 0$, $\exists R > 0$ s.t.

$$|\omega^{\Lambda}(x) - \psi(x)| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{A}_{B(R)^c \cap \Lambda}$



Let now $(\pi^{\mathbb{1}}, \mathcal{H}^{\mathbb{1}}, \Omega)$ be the GNS of $\omega^{\mathbb{1}}$.

— " — $(\pi_\psi, \mathcal{H}_\psi, \Psi)$ — " — ψ

Then $\pi_\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\psi)$ satisfies the <u>superselection criterion</u> w.r.t. $\pi^{\mathbb{1}}$. i.e

$$\pi_\psi |_{\mathcal{A}_\Lambda} \simeq_{u.e} \pi^{\mathbb{1}} |_{\mathcal{A}_\Lambda}$$

for any cone Λ.

Such π are called <u>anyon representations</u>

If **Haag duality** holds: $\pi^{1}(\mathcal{A}_\Lambda)' = \pi^{1}(\mathcal{A}_{\Lambda^c})''$
then for each anyon rep π and each cone Λ
$\exists$ an endomorphism ρ_Λ^π such that

$$\rho_\Lambda^\pi(x) = x \quad \text{if} \quad \mathrm{supp}(x) \subset \Lambda^c.$$

$$\rho_\Lambda^\pi \circ \pi^{1} \simeq_{u.e.} \pi .$$

i.e. anyons can be produced by "string operators".

The endomorphisms ρ^{π}_{Λ} are the objects of a braided C^*-tensor category.

morphisms: $(\rho \to \sigma) := \{ T \in B(\mathcal{H}^{\pi}) :$

$T\rho(x) = \sigma(x) T \quad \forall x \in \pi^{\pi}(\mathcal{A}) \}$

tensor product: $\rho \otimes \sigma = \rho \circ \sigma$

for $S \in (\rho \to \sigma)$, $T \in (\rho' \to \sigma')$,

$S \otimes T = S\rho(T) \in (\rho \otimes \rho' \to \sigma \otimes \sigma')$

braiding: $b(\rho, \sigma) \in (\rho \otimes \sigma \to \sigma \otimes \rho)$

II. Levin-Wen Models.

We use the version of L. Kong 14'

Learned most of this from

- D. Penneys' lecture notes
- Green, Huston, Kawagoe, Penneys, Poudel, Sanford 24'
- Kawagoe, Jones, Green, Penneys 24'
- Jones, Naaijkens, Penneys, Wallick 23'

- Kirillov Jr., Balsam 10'
- Kirillov Jr. 11'

Unitary fusion category & graphical calculus

Let $\mathcal{C}$ be a UFC.

Prime example: $\mathcal{C} = \mathrm{Rep}_f^{\dagger}(G)$ for a finite group G.

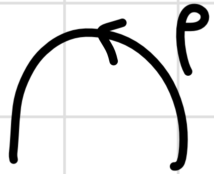
<u>objects</u>: f.d. unitary reps (ρ, V_ρ)

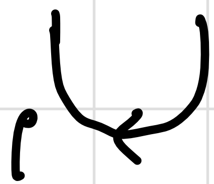
<u>intertwiners</u>: $\mathcal{C}(\rho \to \sigma) = \{ t : V_\rho \to V_\sigma : t\rho(g) = \sigma(g)t \text{ for all } g \in G \}$

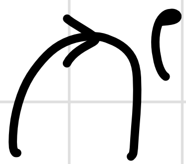
usual <u>tensor product</u> $\otimes$ of representations & linear maps. <u>Tensor unit</u> $\mathbb{1} = (1, \mathbb{C})$.

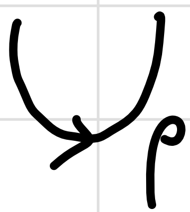
<u>dagger</u> : If $t : \rho \to \sigma$ then $t^\dagger : \sigma \to \rho$

<u>duals</u> : To each (ρ, V_ρ) its dual $(\rho^*, V_\rho{}^*)$
and intertwiners

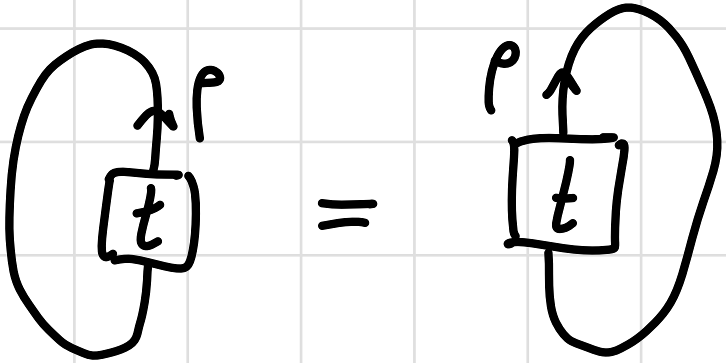
$ev_\rho =$  $: \rho^* \otimes \rho \to \mathbb{1} : f \otimes v \to f(v)$

$coev_\rho =$  $: \mathbb{1} \to \rho \otimes \rho^* : \lambda \mapsto \lambda \sum_i v^i \otimes v_i$

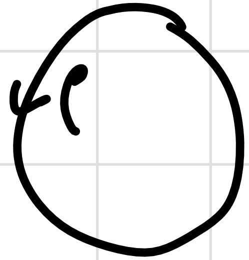
$coev_\rho^\dagger =$ 

$ev_\rho^\dagger =$ 

trace : For $t : \rho \to \rho$, define

$tr(t) :=$ 

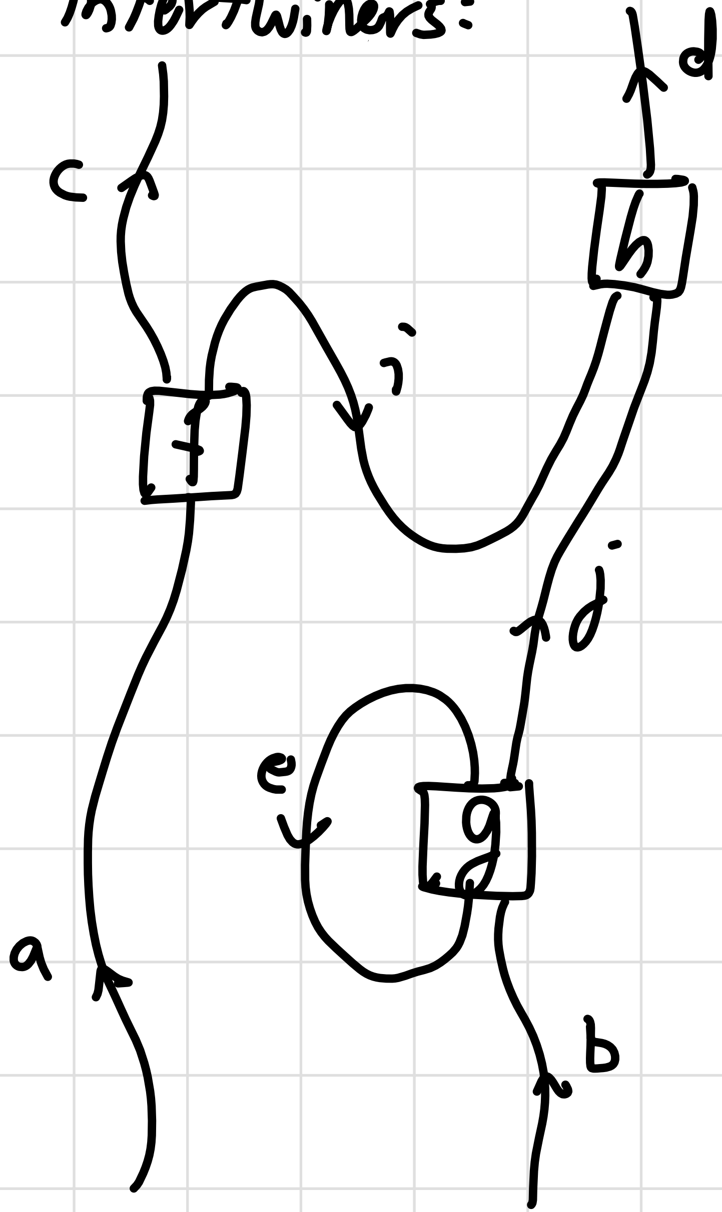
(for $Rep(G)$ this is the usual trace of a linear map)

dimension : $d_\rho = tr(id_\rho) =$  ≥ 0.

finiteness : $\exists$ finite set of irreps. $Irr(\mathcal{C})$ such that

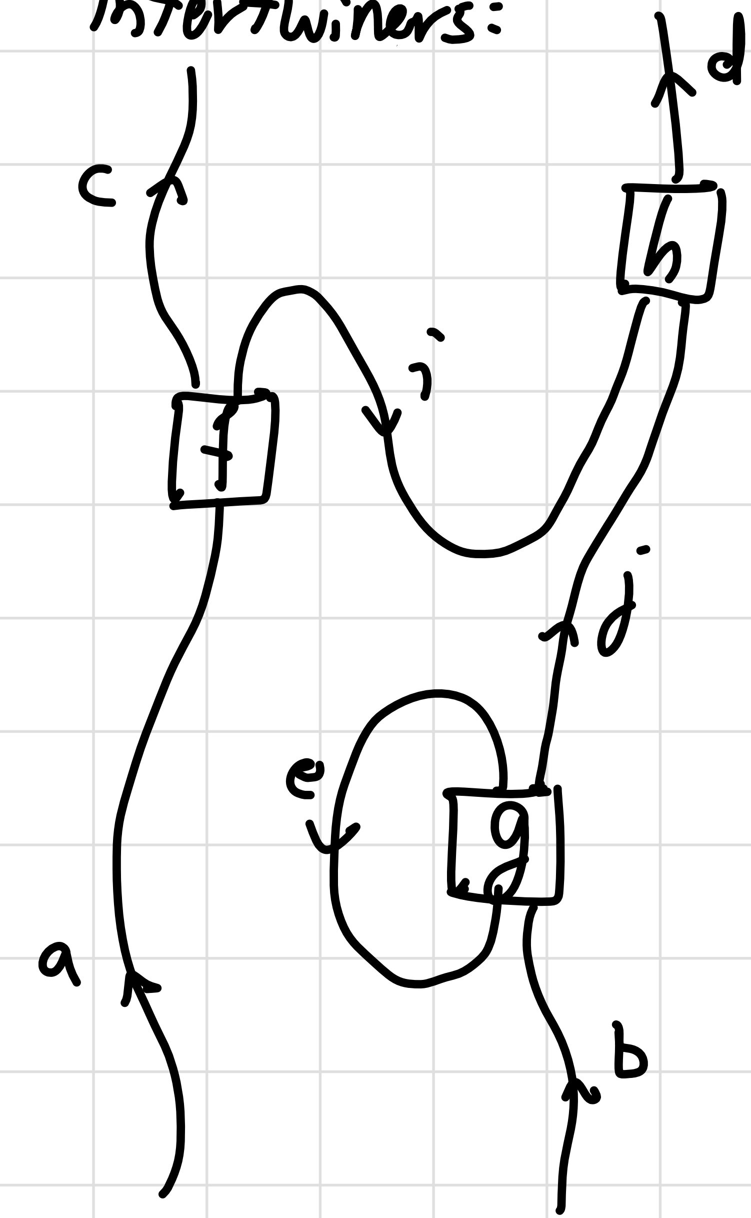
$$\rho \cong \bigoplus_{a \in Irr(\mathcal{C})} a \otimes \mathbb{C}^{m_a} \quad \text{for all reps } \rho.$$

<u>Graphical Calculus</u>: pictures represent intertwiners:



$\in \mathcal{C}(a \otimes b \to c \otimes d)$.

Graphical Calculus: pictures represent intertwiners:



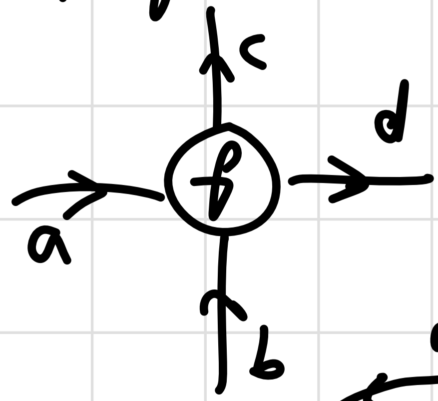
intertwiner is invariant under

- Isotopy
- "Local moves".

Local Degrees of Freedom

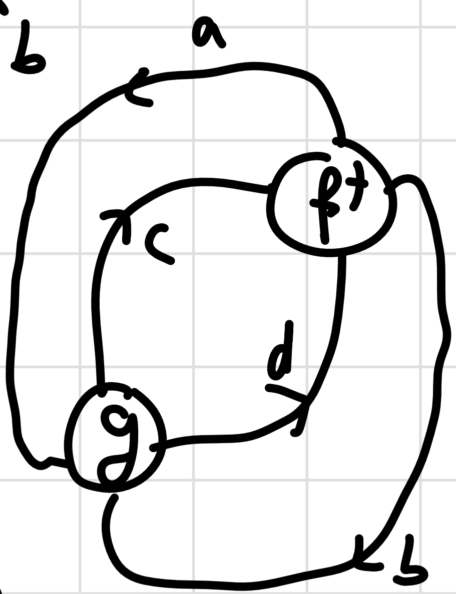
For $a, b, c, d \in \mathrm{Irr}(\mathcal{C})$, the space

$$\mathcal{C}(a \otimes b \to c \otimes d) \ni$$



has an inner product

$$\langle f, g \rangle := \frac{\mathrm{tr}(f^\dagger \circ g)}{\sqrt{d_a d_b d_c d_d}} = \frac{1}{\sqrt{d_a d_b d_c d_d}}$$



For each $v \in \mathbb{Z}^2$, put

$$\mathcal{H}_v \simeq \bigoplus_{\substack{a,b,c,d \\ \in \mathrm{Irr}(\mathcal{C})}} \mathcal{C}(a \otimes b \to c \otimes d).$$

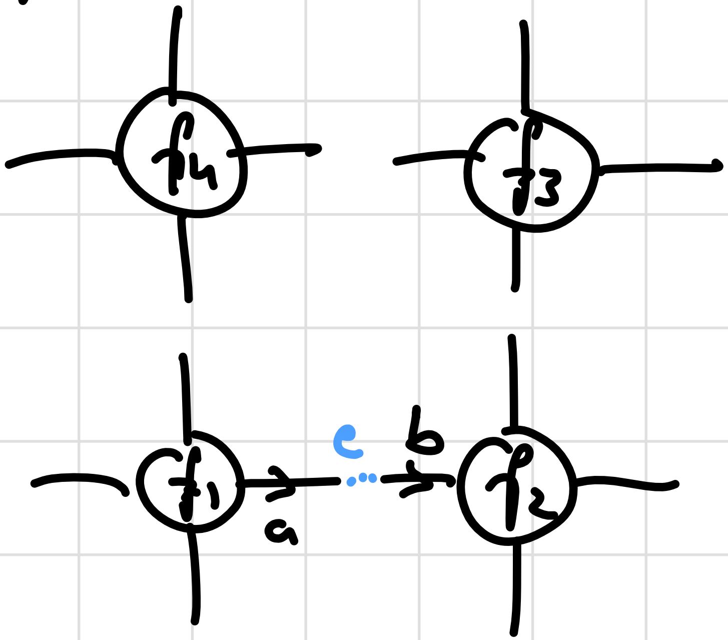
For finite $X \subset\subset \mathbb{Z}^2$, put $\mathcal{H}_X = \bigotimes_{v \in X} \mathcal{H}_v$ and

$$\mathcal{A}_v := \mathrm{End}(\mathcal{H}_v), \quad \mathcal{A}_X = \bigotimes_{v \in X} \mathcal{A}_v$$

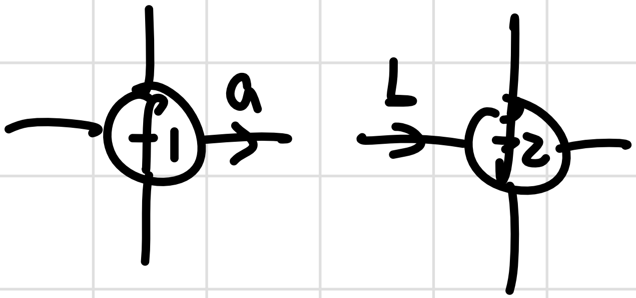
which determines $\mathcal{A}^{loc}_Y$ & $\mathcal{A}_Y$ for infinite Y. $\mathcal{A}^{loc} := \mathcal{A}^{loc}_{\mathbb{Z}^2}$, $\mathcal{A} := \mathcal{A}_{\mathbb{Z}^2}$.

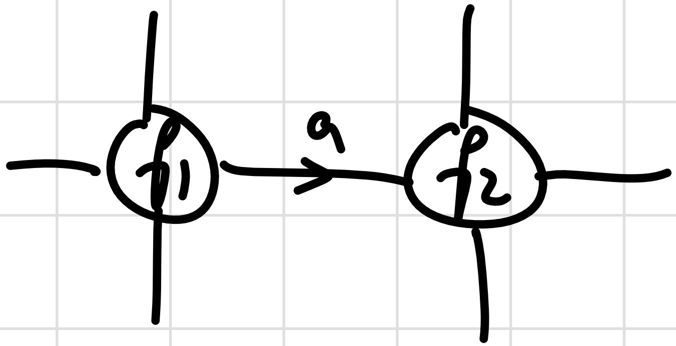
String net subspaces.

A typical product state on 4 vertices:



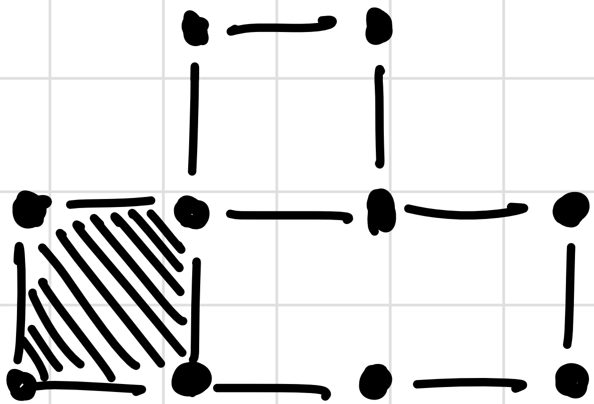
Define projectors

A_e : 

$= \delta_{a,b}$ 

$C^{\mathbb{Z}^2}$: cell complex w. vertices, edges & faces of $\mathbb{Z}^2$.

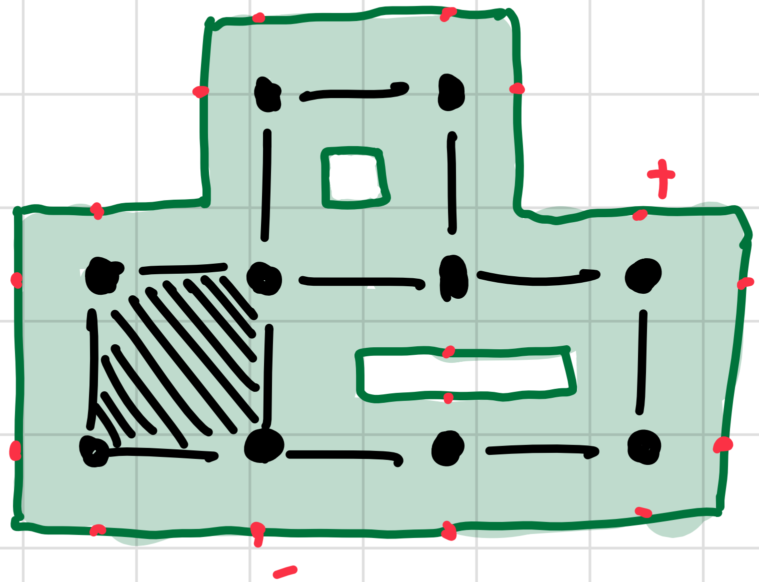
A <u>region</u> C is a subcomplex of $C^{\mathbb{Z}^2}$.

C 

$C^{\mathbb{Z}^2}$: cell complex w. vertices, edges & faces of $\mathbb{Z}^2$.

A <u>region</u> C is a subcomplex of $C^{\mathbb{Z}^2}$.

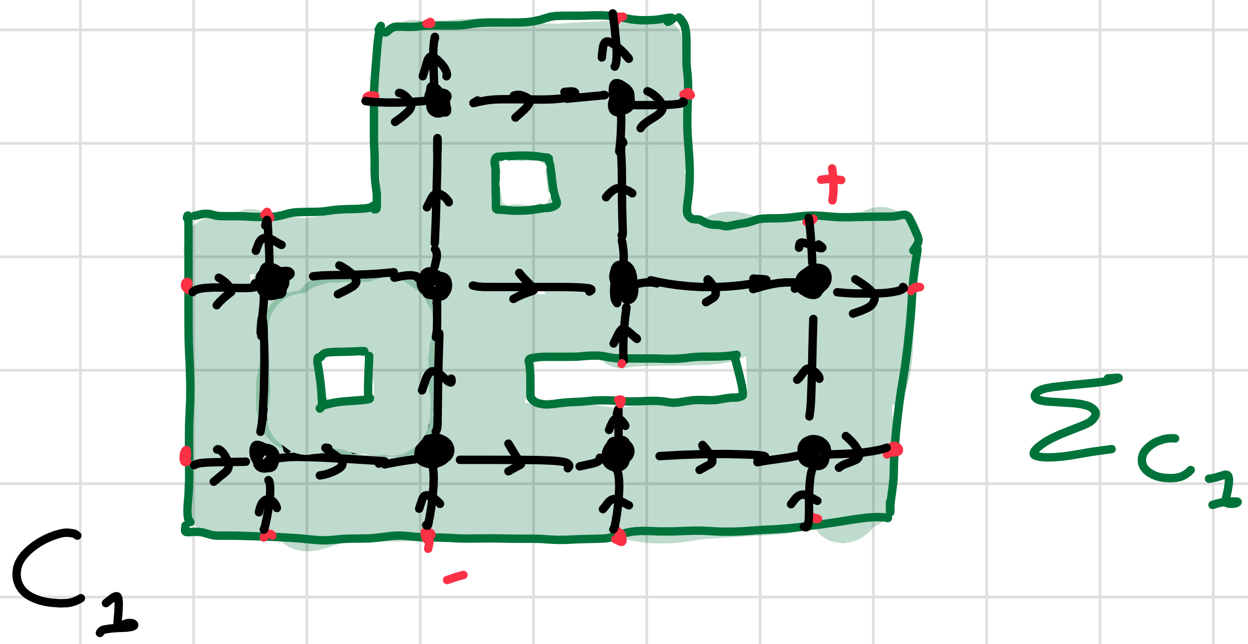
Decorated surface $\Sigma_C \subset \mathbb{R}^2$.



String-net subspace of C:

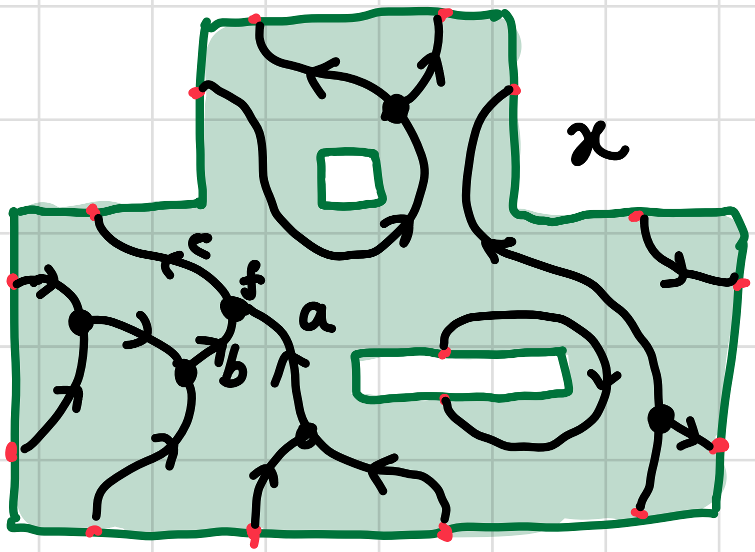
$$\mathcal{H}_{C_1} = \left(\prod_{e \in E(C)} A_e \right) \mathcal{H}_{V(C)}$$

is spanned by product states



More flexible pictures!

$S(\Sigma)$: string diagrams on a decorated surface Σ



Skein module : $A(\Sigma) := \mathbb{C}[S(\Sigma)]/\sim$

where $\sim$ means :
- isotopy
- local moves in disks.

<u>Fact</u>: (Kirillov 11) The maps

$$\pi_{c_1} : \mathcal{H}_{c_1} \longrightarrow A(\Sigma_{c_1})$$

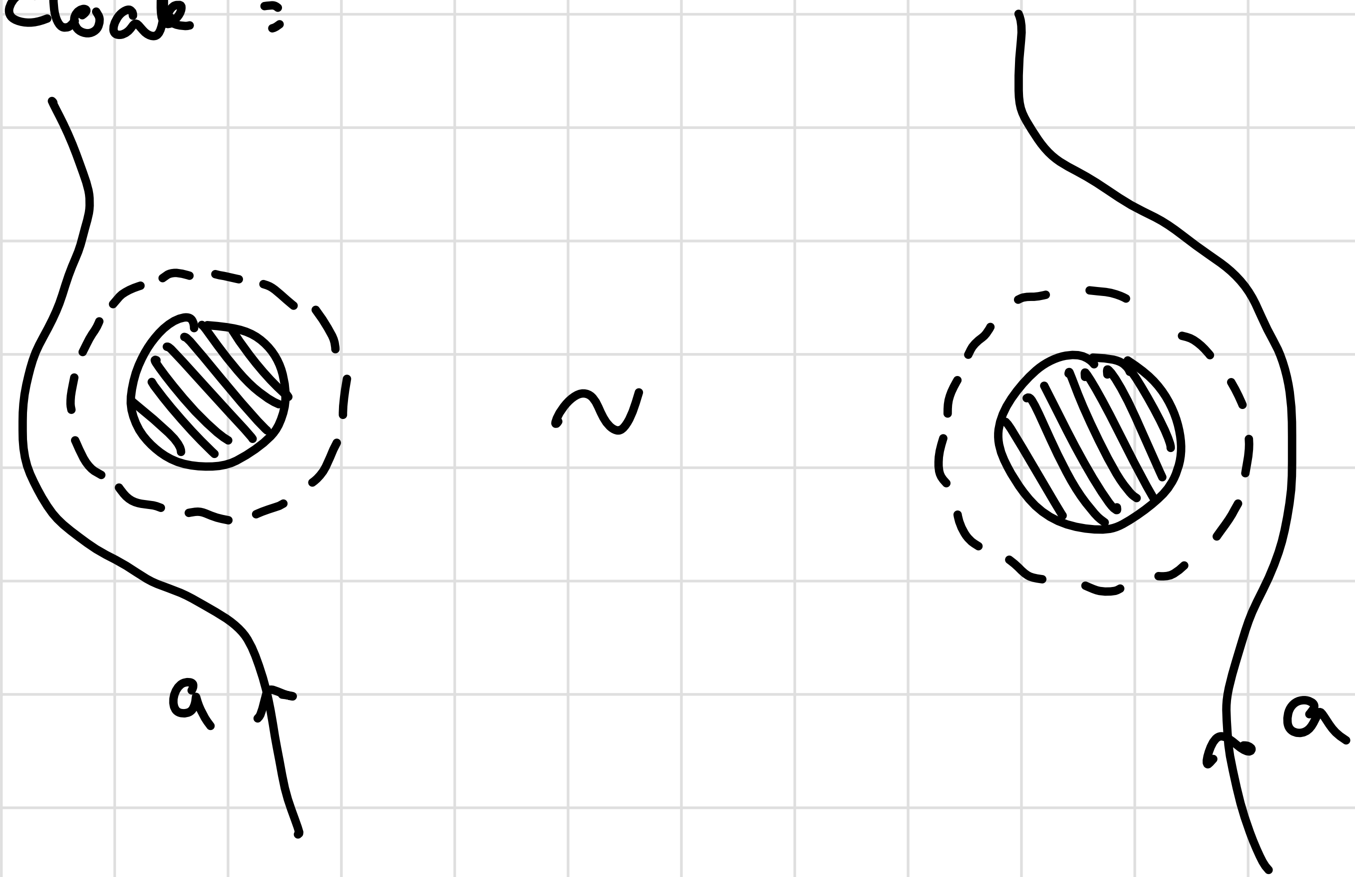
which interpret product states in $\mathcal{H}_{c_1}$ as representative string net-pictures in $A(\Sigma_{c_1})$, are <u>isomorphisms</u> of vector spaces.

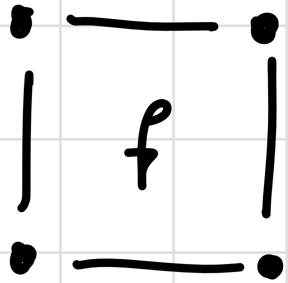
Use this to graphically define operators on $\mathcal{H}(G)$.

Masking punctures,

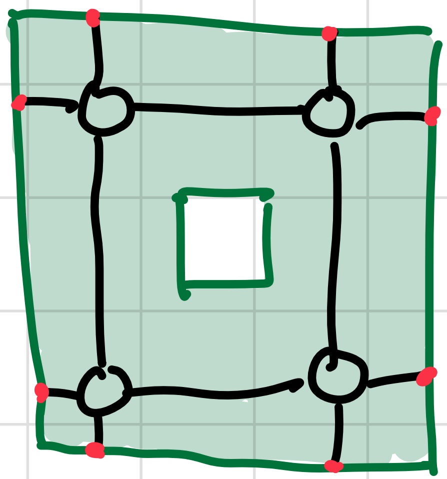
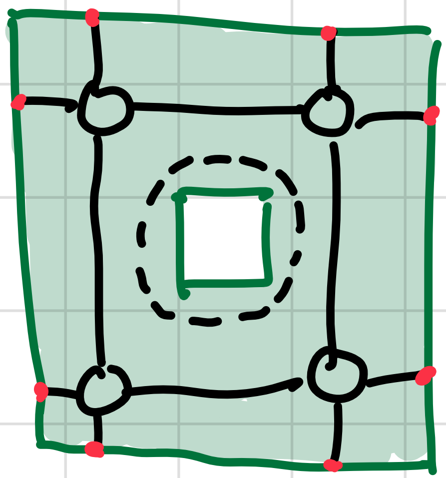
"Regular line" : $\quad := \frac{1}{D^2} \sum_{a \in \mathrm{Irr}(\mathcal{C})} d_a \quad a$

is a "cloak" :



Let $C^f =$ 

and define

B_f  $:=$ 

on $H_{c^f} \subseteq \mathcal{H}_{v(c^f)}$ & extend to act as 0 on $H_{c^f}^{\perp}$.

$\{B_f\}_{f \in F(\mathbb{Z}^2)}$ are commuting projectors.

Define <u>skein subspaces</u>

$$H_c := \left(\prod_{f \in F(c)} B_f \right) H_{c_1}.$$

Let $\pi_c : H_{c_1} \to A(\Sigma_c)$ interpret product states as string diagrams on Σ_c.

<u>Fact</u>: (Kirillov '11)

$$\pi_c|_{H_c} : H_c \to A(\Sigma_c)$$

is an isomorphism.

The Levin-Wen Hamiltonian

$$H_{LW} = -\sum_{f \in F(\mathbb{Z}^2)} B_f .$$

<u>Fact</u>: (Qui, Wang 20' & Jones, Naaijkens, Penneys, Wallick 25')
H_{LW} has a unique frustration free ground state $\omega^{\mathbb{1}}$. i.e.

$$\omega^{\mathbb{1}}(B_f) = 1 \quad \text{for all } f \in F(\mathbb{Z}^2).$$

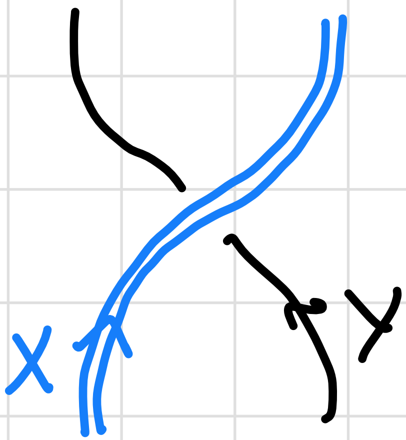
Let $(\pi^{\mathbb{1}}, \mathcal{H}^{\mathbb{1}}, \Omega)$ be the GNS of $\omega^{\mathbb{1}}$.

Anyon states.

The Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the category w. objects (X, σ^X) where $X \in \mathcal{C}$ and

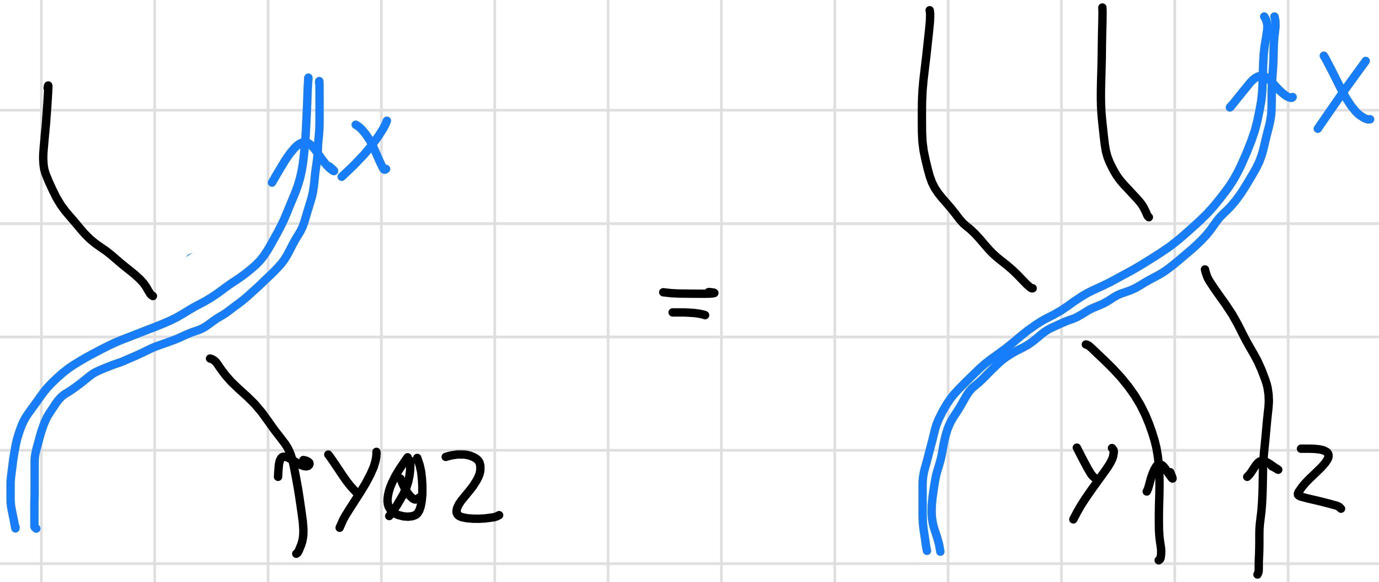
$$\sigma^X : X \otimes - \longrightarrow - \otimes X$$

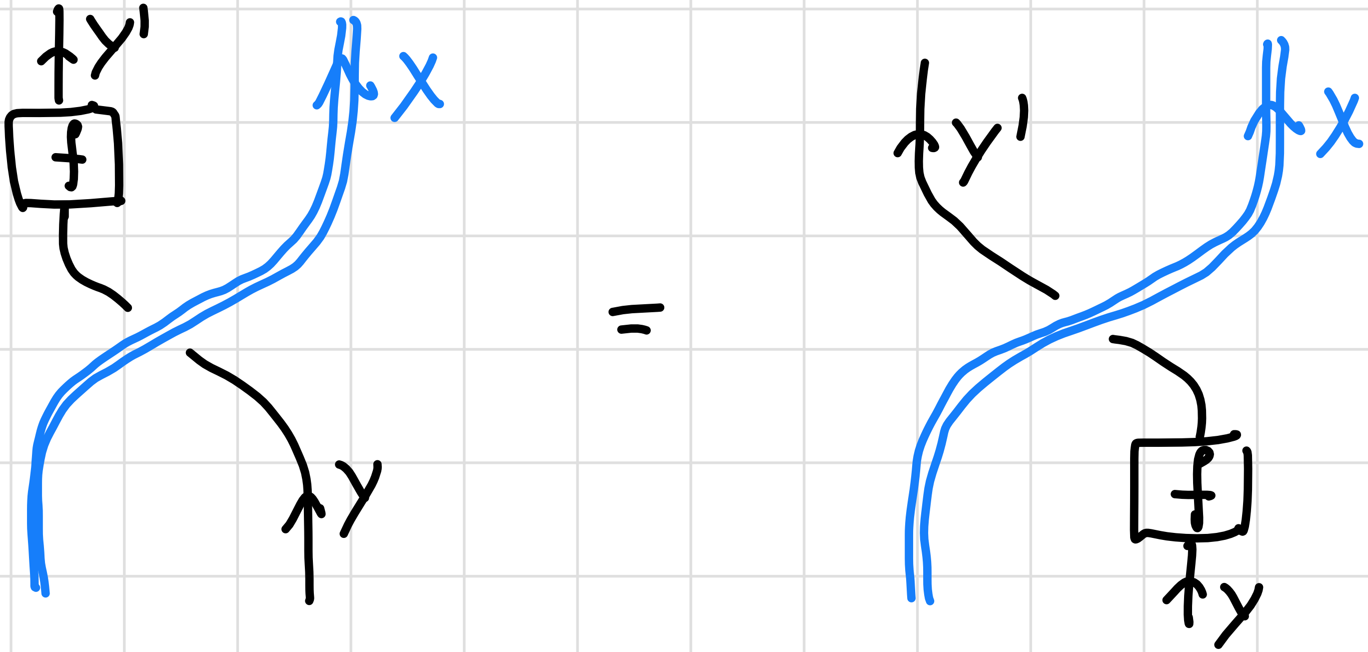
is a "half-braiding". i.e. a collection of unitary intertwiners

$\sigma^X_Y =$ 

$: X \otimes Y \longrightarrow Y \otimes X$ for all $Y \in \mathcal{C}$.

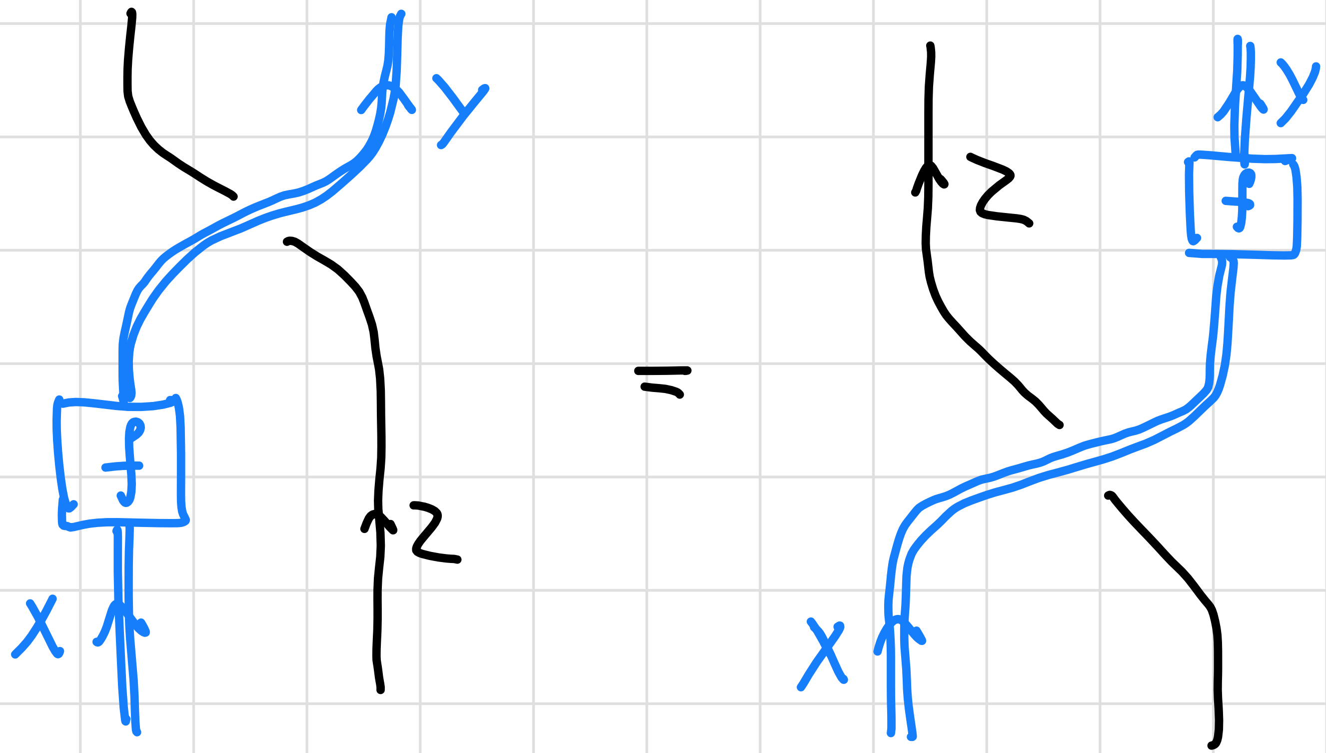
Such that





<u>morphisms</u>: $f \in Z(\mathcal{C})\big((X,\sigma^X) \to (Y,\sigma^Y)\big)$

$\subseteq \mathcal{C}(X \to Y)$ if



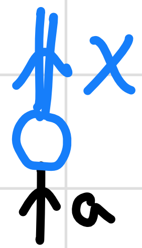
for all $Z \in \mathcal{C}$.

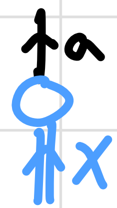
Lines & morphisms of $Z(\mathcal{C})$ are <u>transparent</u>.

Fact : (Müger 03', Izumi 00')

$\mathcal{Z}(\mathcal{C})$ is a UMTC. In particular, $\exists$ a finite representative set of irreducibles $\mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$.

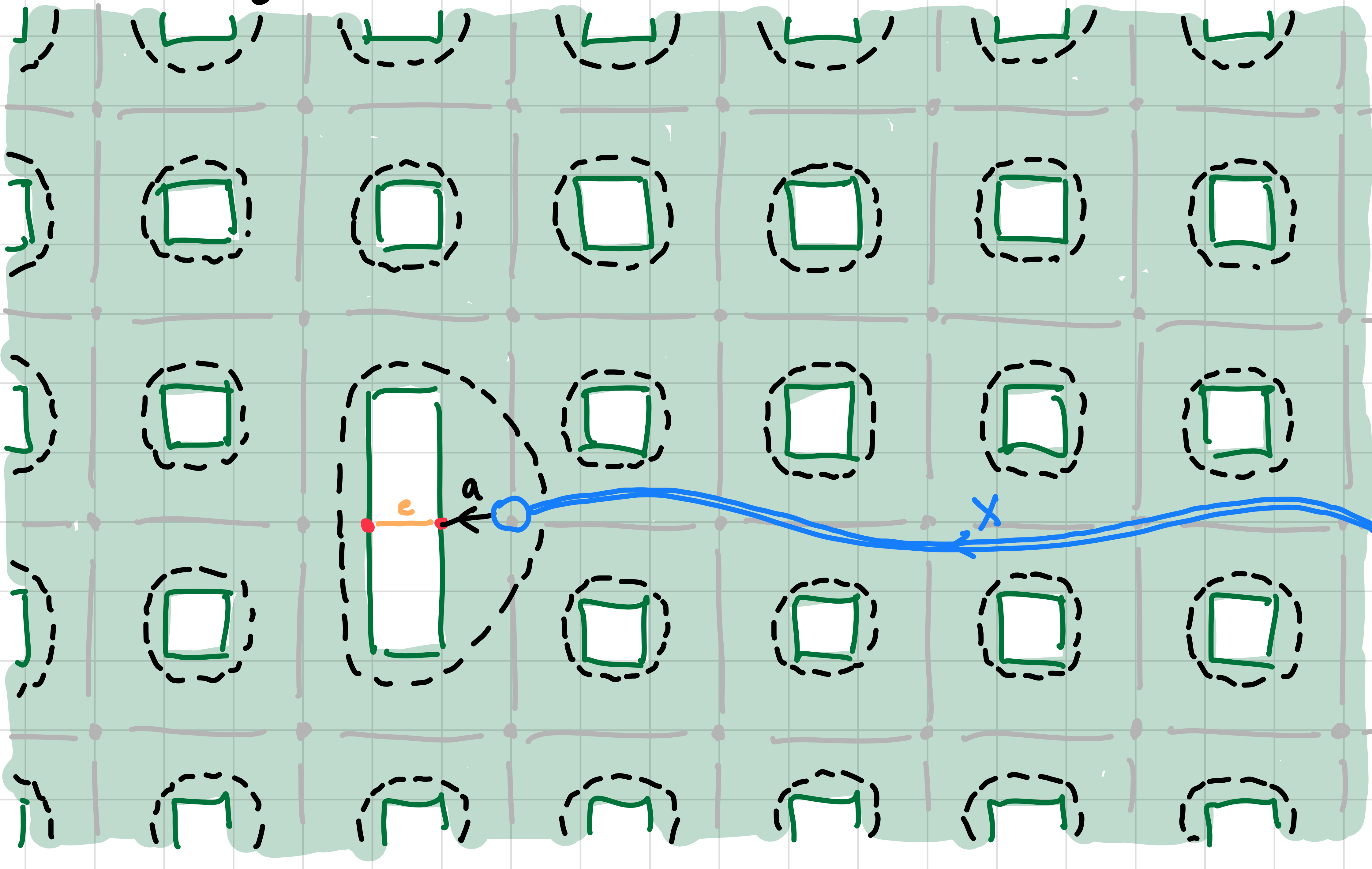
For each $X \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$, fix an isometry

$v_X =$  $: a \to X$

$v_X^\dagger =$  $: X \to a$

for some $a \in \mathrm{Irr}(\mathcal{C})$.

Anyon state :



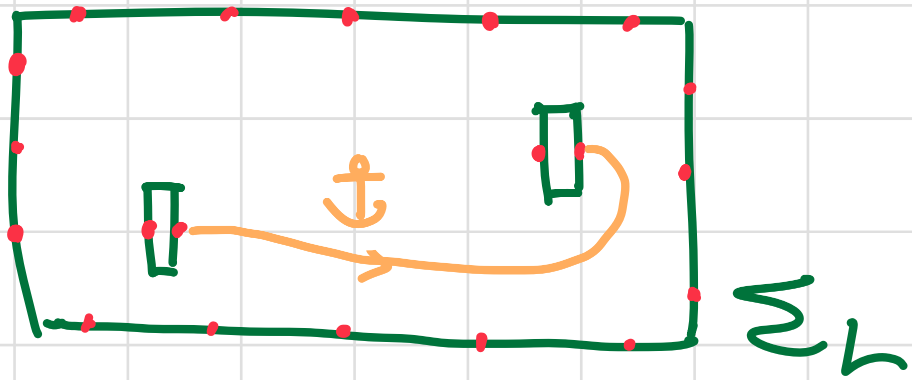
This define a <u>pure state</u> ω_e^X

Let $(\pi_{\omega_e^X}, \mathcal{H}^X, \Omega^X)$ be its GNS.

$$\boxed{\text{\underline{Fact}: } \bullet\ \pi_{\omega_e^X} \text{ is an irreducible anyon rep.}}$$

Construction of string operators.

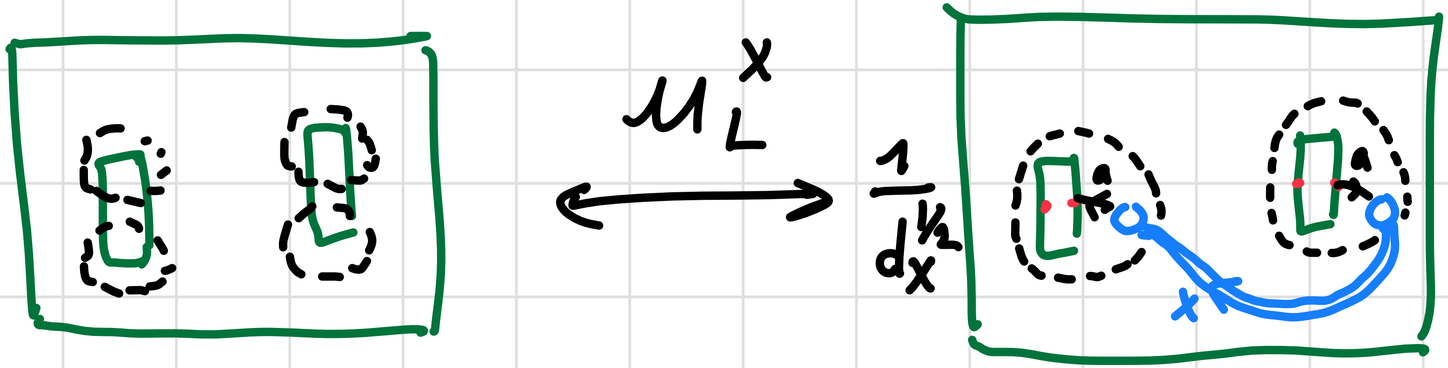
A link L is a region s.t



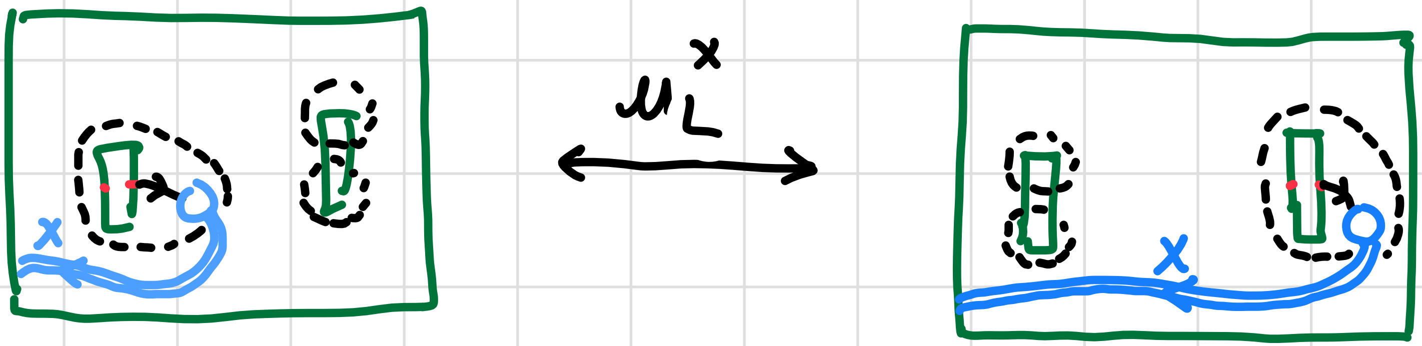
together w. an "anchor" ⚓

Given link L and $X \in \mathrm{Irr}(Z(\mathcal{C}))$, construct unitary $u_L^X \in \mathcal{A}_{V(L)}$ s.t.

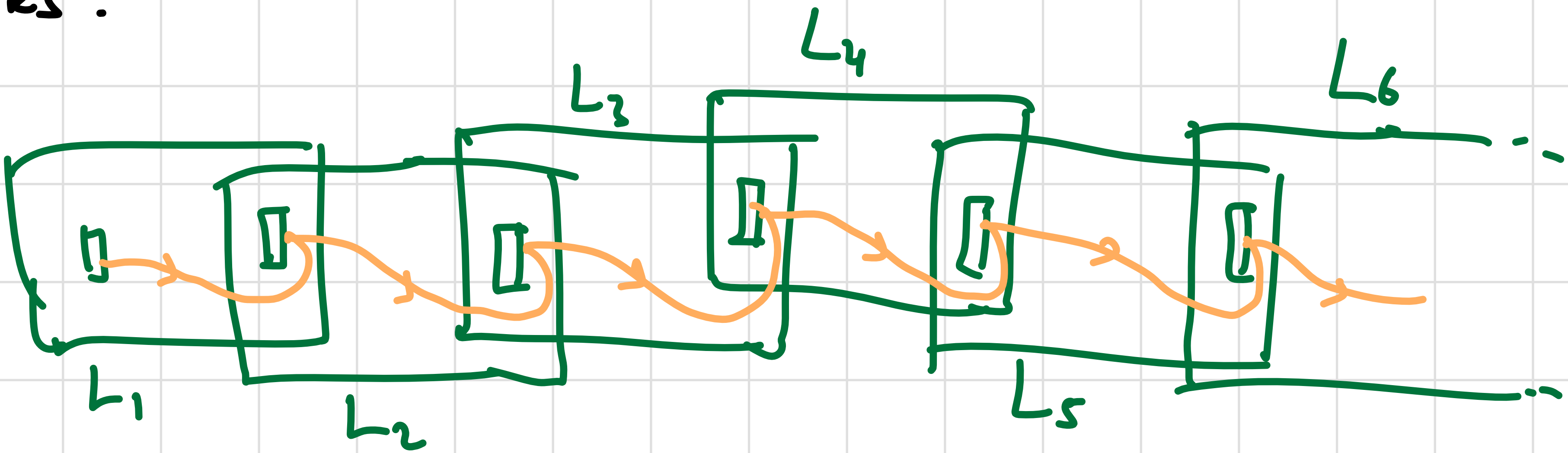
pair creation/annihilation :



hopping :



and acts as $\mathbb{1}$ on all other states, Let $\mathcal{C} = \{L_i\}_{i\in\mathbb{N}}$ be a <u>chain</u> of "composable" links:



For $n \geq 1$ define $\rho^X_{\mathcal{C}_n} := \mathrm{Ad}[(\mathcal{U}^X_{\mathcal{C}_n})^\dagger]$ with $\mathcal{U}^X_{\mathcal{C}_n} = u^X_{L_n} \cdots u^X_{L_1}$.

<u>Fact</u>: $\rho^X_{\mathcal{C}} := \lim\limits_{n\uparrow\infty} \rho^X_{\mathcal{C}_n}$ is a well-defined endomorphism of $\mathcal{A}$, supported on $\mathcal{C}$.

- $\omega^{\mathbb{1}} \circ \rho_e^X = \omega_{\partial_0 e}^X$

- $\pi_e^X := \pi^{\mathbb{1}} \circ \rho_e^X \simeq_{u.e} \pi_{\omega_e^X}$.

- Any irreducible anyon representation is isomorphic to all π_e^X for some $X \in \mathrm{Irr}(Z(\mathcal{C}))$.

Fusion rules & F-symbols

Fix $\mathcal{C} = \{L_i\}_{i \in \mathbb{N}}$ and write $\mu_i^X = \mu_{L_i}^X$, $\rho^X = \rho_e^X$, $\pi^X = \pi^{\mathbb{1}} \circ \rho^X$ etc.

Fact: $\pi^{XY} := \pi^{\mathbb{1}} \circ \rho^X \circ \rho^Y$

$$\simeq_{u.e} \bigoplus_{Z \in \mathrm{Irr}(Z(\mathcal{C}))} \pi^Z \otimes \mathbb{C}^{N_{XY}^Z}.$$

where $N_{XY}^Z = \dim Z(\mathcal{C})(X \otimes Y \to Z)$

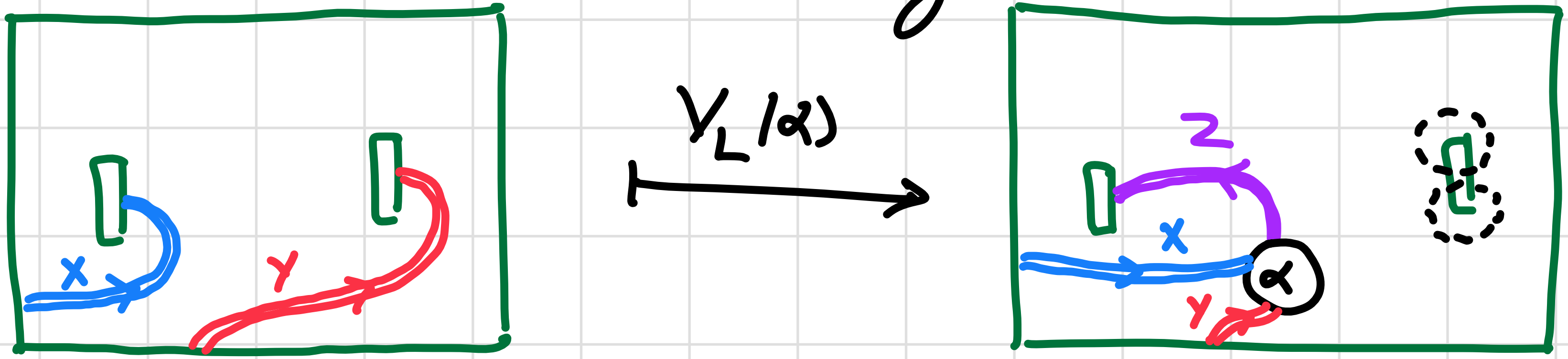
~We can construct a basis for

$$\mathrm{Hom}(\pi^{XY} \to \pi^{Z}).$$

For a link L and a morphism

$$\alpha \in Z(e)(X \otimes Y \to Z),$$

construct a partial isometry



~Write $V_n(\alpha) := V_{L_n}(\alpha)$ and put

$$\Phi_n(X,Y;Z,\alpha) := \pi^{1}\Big((u^Z_{n-1} \cdots u^Z_1)^* \times V_n(\alpha) \times (u^Y_{n-1} \cdots u^Y_1)(u^X_n \cdots u^X_2) \Big)$$

Fact: If $\{\alpha_i\}_{i=1}^{N_{XY}^{Z}}$ is an ONB of $\mathcal{Z}(\mathcal{C})(Z \to X \otimes Y)$ then

$$\Phi(X,Y;Z,\alpha_i) := \lim_{n \uparrow \infty} \Phi_n(X,Y;Z,\alpha_i)$$

is an ONB of $\mathrm{Hom}(\pi^{XY} \to \pi^{Z})$.

We can compute F & R symbols w.r.t. such a basis.

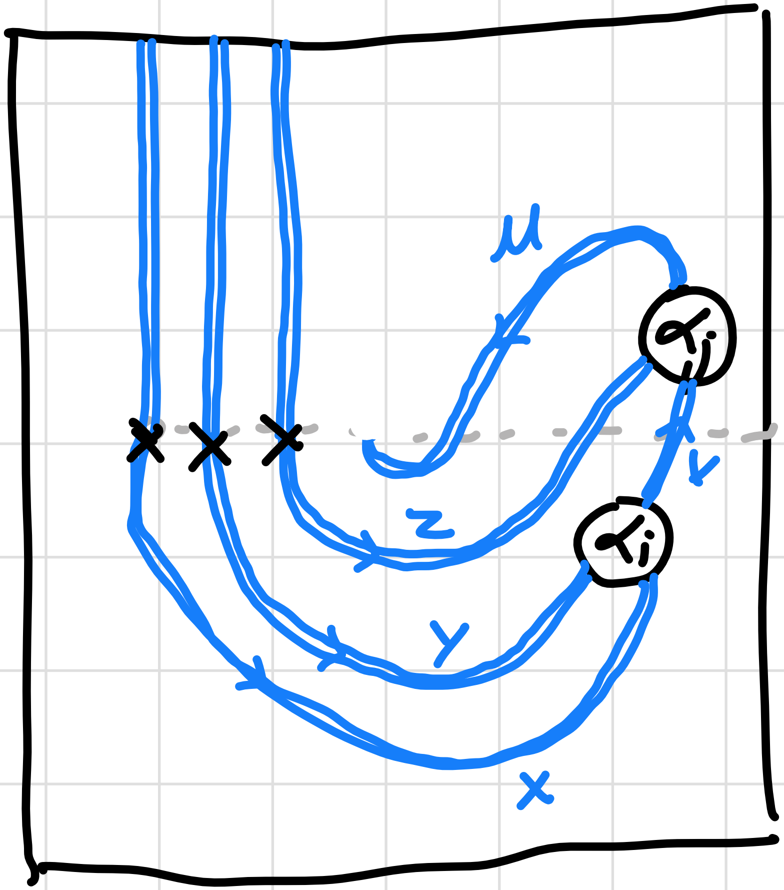
for $X, Y, Z, U \in \mathrm{Irr}(Z(\mathcal{C}))$,

$$\Phi(V, Z; U, \alpha_i) \cdot \left(\Phi(X, Y; V, \alpha_j) \otimes \mathbb{1}_{\pi Z} \right)$$

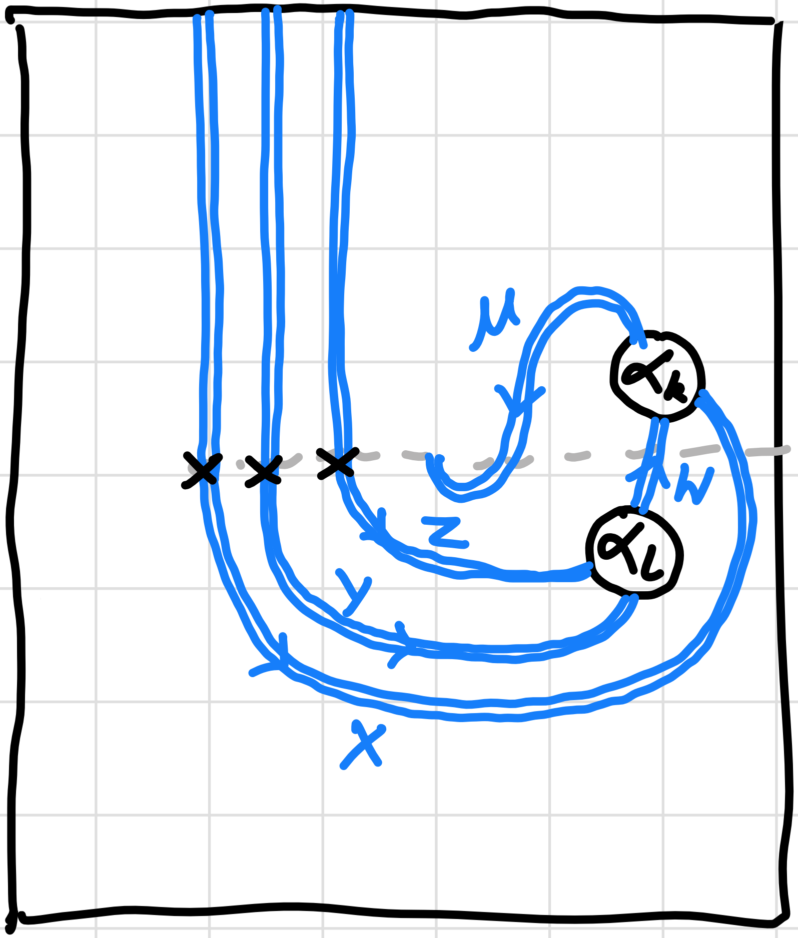
$$= \sum_{\substack{W \in \mathrm{Irr}(Z(\mathcal{C})) \\ k = 1, \dots, N_{YZ}^{W} \\ l = 1, \dots, N_{XW}^{U}}} \left(F_{XYZ}^{U; VW} \right)_{ij}^{kl} \, \Phi(X, W; U, \alpha_k) \cdot \left(\mathbb{1}_{\pi X} \otimes \Phi(Y, Z; W, \alpha_l) \right)$$

unitary transf. between 2 bases of $\mathrm{Hom}(\pi^{XYZ} \to \pi^{U})$.

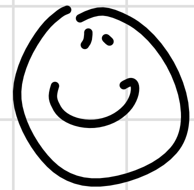
Acting on vector state $|\Psi\rangle$ of π^{xyz} which satisfies ground state constraints near @ yields



$$= \sum_{w,k,l} \left(F_{xyz}^{\mu;vw}\right)_{ij}^{kl}$$



LHS & RHS are related by an F-move of $\mathcal{Z}(\mathcal{C})$, R-symbols are computed similarly.

<u>Result</u>: Sector theory applied to Levin-Wen w. input category $\mathcal{C}$ yields anyon theory $Z(\mathcal{C})$. 

Thanks !