MULTIPLICATION OF DISTRIBUTIONS

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Feynman diagram

Feynman amplitude

\[ G(x_1, x_2) \Delta(x_2, x_3)^2 G(x_3, x_4) \Delta(x_1, x_4) \Delta(x_4, x_5) \Delta(x_5, x_6) \Delta(x_6, x_7) G(x_5, x_7) \]
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**Quantum Field Theory**

- Feynman diagram

- Feynman amplitude

\[
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\]
- Feynman diagram

- Feynman amplitude

\[ G(x_1, x_2) \Delta(x_2, x_3)^2 G(x_3, x_4) \Delta(x_1, x_4) \Delta(x_4, x_5) \Delta(x_5, x_6) \Delta(x_6, x_7) G(x_5, x_7) \]

- Multiply distributions on the largest domain where this is well defined \( \mathcal{D}(\mathbb{R}^{7d} \setminus \{x_i = x_j\}) \)

- Renormalization: extend the result to \( \mathcal{D}(\mathbb{R}^{7d}) \)
Multiplication of distributions
  • Motivation
  • The wave front set of a distribution
  • Application and topology

Extension of distributions (Viet)
  • Renormalization as the solution of a functional equation
  • The scaling of a distribution
  • Extension theorem

Renormalization on curved spacetimes (Kasia)
  • Epstein-Glaser renormalization
  • Algebraic structures (Batalin-Vilkovisky, Hopf algebra)
  • Functional analytic aspects
Joint work with Yoann Dabrowski, Nguyen Viet Dang and Frédéric Hélein
- Trying to multiply distributions
  - Singular support
  - Fourier transform
- The wave front set
  - Examples
  - Characteristic functions
  - Hörmander’s theorem for distribution products
- Examples in quantum field theory
- Topology
- **Heaviside step function**
  \[ \theta(x) = 0 \text{ for } x < 0, \]
  \[ \theta(x) = 1 \text{ for } x \geq 0. \]

- **As a function** \( \theta^n = \theta \)

- **Heaviside distribution**
  \[
  \langle \theta, f \rangle = \int_{-\infty}^{\infty} \theta(x)f(x)dx = \int_{0}^{\infty} f(x)dx
  \]

- If \( \theta^n = \theta \) then \( n\theta^{n-1}\delta = \delta \) and \( n\theta\delta = \delta \) for \( n \geq 2 \)
**REGULARIZATION**

- **Mollifier** \( \varphi \) such that \( \int \varphi(x) \, dx = 1 \)
- Distributions are *mollified* by convolution with \( \delta_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi \left( \frac{x}{\varepsilon} \right) \)
- Mollified Heaviside distribution
  \[
  \theta_\varepsilon(x) = \int_{-\infty}^{x} \delta_\varepsilon(y) \, dy
  \]
- Then,
  \[
  \theta \delta = \lim_{\varepsilon \to 0} \theta_\varepsilon \delta_\varepsilon = \frac{1}{2} \delta
  \]
- But \( \delta^2 = \lim_{\varepsilon \to 0} \delta^2_\varepsilon \) diverges
- Very heavy calculations (Colombeau generalized functions)
- How detect a singular point in a distribution \( u \)?

- Multiply by a smooth function \( g \in \mathcal{D}(M) \) around \( x \in M \)

- Look whether \( gu \) is smooth or not
Let $u$ be a distribution on $M = \mathbb{R}^d$ and $g \in \mathcal{D}(M)$ such that $gu$ is a smooth function. For $e_\xi(x) = e^{i\xi \cdot x}$

$$g(x)u(x) = \langle gu, \delta_x \rangle = \int \frac{d\xi}{(2\pi)^d} \langle gu, e_\xi \rangle e^{-i\xi \cdot x}$$

All the derivatives of $gu$ exist:

$$\forall N, \exists C_N, s.t. \forall \xi, \quad |\langle gu, e_\xi \rangle| \leq C_N (1 + |\xi|)^{-N}$$

The **singular support** of $u$ is the complement of the set of points $x \in M$ such that there is a $g \in \mathcal{D}(M)$ with $gu$ a smooth function and $g(x) \neq 0$
EASY PRODUCTS

- You can multiply a distribution $u$ and a smooth function $f$
  \[ \langle fu, g \rangle = \langle u, fg \rangle \]

- You can multiply two distributions $u$ and $v$ with disjoint singular supports
  \[ \langle uv, g \rangle = \langle u, vfg \rangle + \langle v, u(1 - f)g \rangle \]

  where
  - $f = 0$ on a neighborhood of the singular support of $v$
  - $f = 1$ on a neighborhood of the singular support of $u$
Product of distributions with common singular support

Consider

\[ u_+(x) = \frac{1}{x - i0^+} = i \int_{0}^{\infty} e^{-ik\xi} d\xi \]

More precisely

\[ \langle u_+, g \rangle = i \int_{0}^{\infty} \hat{g}(-\xi) d\xi \]

Its singular support is \( \Sigma(u_+) = \{0\} \)
Product of distributions with common singular support

Consider also

\[ u_-(x) = \frac{1}{x + i0^+} = -i \int_0^\infty e^{ik\xi} d\xi \]

More precisely

\[ \langle u_-, g \rangle = -i \int_0^\infty \hat{g}(\xi) d\xi \]

Its singular support is \( \Sigma(u_-) = \{0\} \)
Fourier transform

- Convolution theorem \( \hat{uv} = \hat{u} \ast \hat{v} \)
- Define the product by \( uv = \mathcal{F}^{-1}(\hat{u} \ast \hat{v}) \)
- Example
  \[
  u_+(x) = \frac{1}{x - i0^+} \quad \quad \hat{u}_+(\xi) = 2i\pi \theta(\xi)
  \]
- Square of \( u_+ \)
  \[
  \hat{u}_+^2(\xi) = -2\pi \int_{\mathbb{R}} \theta(\eta)\theta(\xi - \eta) \, d\eta = -2\pi \xi \theta(\xi)
  \]
Example

\[ u_+(x) = \frac{1}{x - i0^+} \quad \hat{u}_+(\xi) = 2i\pi \theta(\xi) \]

\[ u_-(x) = \frac{1}{x + i0^+} \quad \hat{u}_-(\xi) = -2i\pi \theta(-\xi) \]

Product \( u_+ u_- \)

\[ \hat{u}_+ u_-(\xi) = 2\pi \int_{\mathbb{R}} \theta(\eta)\theta(\eta - \xi) d\eta \quad \text{diverges} \]
Fourier transform

- Interpretation

\[ \hat{u}_+(\eta) \]

\[ \hat{u}_+(\xi - \eta) \]

\[ \hat{u}_-(\xi - \eta) \]

- \( \hat{u}(\eta) \) can be integrable in some direction

- The non-integrable directions of \( \hat{u}(\eta) \) can be compensated for by the integrable directions of \( \hat{v}(\xi - \eta) \)
Interpretation

\[ \hat{u}_+(\eta) \]

\[ \hat{u}_+(\xi - \eta) \]

\[ \hat{u}_+(\eta) \hat{u}_+(\xi - \eta) \]

Integrable: \( u_+^2 \) is well-defined
Interpretation

\( \hat{u}_+(\eta) \)

\( \hat{u}_-(\xi - \eta) \)

\( \hat{u}_+(\eta) \hat{u}_-(\xi - \eta) \)

Not integrable: \( u_+ u_- \) is not well-defined
Define the product by \( uv = F^{-1}(\hat{u} \ast \hat{v}) \)

What if the distributions have no Fourier transform?

The product of distributions is local: \( w = uv \) near \( x \) if

\[
\hat{f^2w} = \hat{fu} \ast \hat{fv} \quad \text{for} \quad f = 1 \quad \text{in a neighborhood of} \quad x
\]

How should the integral converge?

\[
\hat{f^2uv}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{fu}(\eta) \hat{fv}(\xi - \eta) d\eta
\]

Absolute convergence is not enough if we want the Leibniz rule to hold.
How can the integral converge?

\[
\hat{f^2uv}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}u(\eta) \hat{f}v(\xi - \eta) d\eta
\]

The order of \( f u \) is finite: \( |\hat{f}u(\eta)| \leq C(1 + |\eta|)^m \)

If \( \hat{f}u(\eta) \) does not decrease along direction \( \eta \), then \( \hat{f}v(\xi - \eta) \) must decrease faster than any inverse polynomial

Conversely, \( \hat{f}u(\eta) \) must compensate for the directions along which \( \hat{f}v(\xi - \eta) \) does not decrease fast
OUTLINE

- Trying to multiply distributions
  - Singular support
  - Fourier transform

- The wave front set
  - Examples
  - Characteristic functions
  - Hörmander’s theorem for distribution products

- Examples in quantum field theory

- Topology
THE WAVE FRONT SET

Mikio Sato
1928-

Lars Valter Hörmander
1931-2012
A point \((x_0, \xi_0) \in T^*\mathbb{R}^d\) **does not belong** to the **wave front set** of a distribution \(u\) if there is a test function \(f\) with \(f(x_0) \neq 0\) and a conical neighborhood \(V \subset \mathbb{R}^d\) of \(\xi_0\) such that, for every integer \(N\) there is a constant \(C_N\) for which

\[
|\hat{f} u(\xi)| \leq C_N (1 + |\xi|)^{-N}
\]

for every \(\xi \in V\)
The wave front set is a cone: if \((x, \xi) \in \text{WF}(u)\), then \((x, \lambda \xi) \in \text{WF}(u)\) for every \(\lambda > 0\).

The wave front set is closed.

\(\text{WF}(u + v) \subset \text{WF}(u) \cup \text{WF}(v)\)

The singular support of \(u\) is the projection of \(\text{WF}(u)\) on the first variable.
The wavefront set describes in which direction the distribution is singular above each point of the singular support.

The Dirac $\delta$ function is singular at $x = 0$ and its Fourier transform is $\hat{\delta}(\xi) = 1$.

Its wave front set is $WF(\delta) = \{(0, \xi); \xi \neq 0\}$.

The distribution $u_+(x) = (x - i0^+)^{-1}$ is also singular at $x = 0$ but its Fourier transform is $\hat{u}_+(\xi) = 2i\pi\theta(\xi)$.

Its wave front set is $WF(u_+) = \{(0, \xi); \xi > 0\}$. 
• Relation to the Radon transform
CHARACTERISTIC FUNCTION

- Characteristic function of a disk: the wave front set is perpendicular to the edge

- The wave front set is used in edge detection for machine vision and image processing
- Shape and wave front set detection by counting intersections
Product of distributions

\[ \hat{f^2 uv}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}_u(\eta) \hat{f}_v(\xi - \eta) d\eta \]

Hörmander thm: The product of two distributions \( u \) and \( v \) is well defined if there is not point \((x, \xi) \in WF(u)\) such that \((x, -\xi) \in WF(v)\)

The wave front set of the product is

\[ WF(uv) \subset WF(u) \oplus WF(v) \cup WF(u) \cup WF(v) \]

WF\((u) \oplus WF(v) = \{(x, \xi + \eta); (x, \xi) \in WF(u) \text{ and } (x, \eta) \in WF(v)\} \]
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QFT: THE CAUSAL APPROACH

Stueckelberg
Bogoliubov
Radzikowski

Klaus Fredenhagen
Romeo Brunetti
Stefan Hollands
Robert Wald
Katarzyna Rejzner
\[ \Delta_+(x) = \langle 0 | \varphi(x) \varphi(0) | 0 \rangle \]

- **Product of fields**
- **Singular support**
  \[ \{(x, y, t); t^2 - x^2 - y^2 = 0\} \]
  - **Wavefront set**
- **Powers** \( \Delta^n_+ \) are allowed
- **Quantization does not need renormalization**

Wightman propagator
- Time-ordered product of fields
  \[ \Delta_F(x) = \langle 0|T(\varphi(x)\varphi(0))|0\rangle \]
- Singular support
  \[ \{(x, y, t); t^2 - x^2 - y^2 = 0\} \]
- Wavefront set
- Powers \( \Delta^n_F \) are allowed away from \( x = 0 \)
- Powers \( \Delta^n_F \) are forbidden at \( x = 0 \)
- Renormalize only at \( x = 0 \)
Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets and $f : U \rightarrow V$ a smooth map.

The pull-back of a distribution $\nu \in \mathcal{D}'(V)$ by $f$ is determined by the wave front set.

The dual space of a distribution is determined by its wave front set.

The restriction of a distribution to a submanifold is determined by the wave front set.
The true propagator is $G(x, y) = \Delta_F(x - y)$

By pull-back by $f(x, y) = x - y$, its wave front set is

$$\text{WF}(G) = \{( (x, y), (\xi, -\xi) ) ; (x - y, \xi) \in \text{WF}(\Delta_F) \}$$

In curved space time, the wave front set of the propagator is obtained by pull-back:

- either $((x, x), (\xi, -\xi))$ for arbitrary $\xi \neq 0$
- or $((x, y), (\xi, -\eta))$ such that there is a null geodesic between $x$ and $y$, and $\eta$ is obtained by parallel transporting $\xi$ along the geodesic
- Feynman diagram

- Feynman amplitude

\[ G(x_1, x_2) \Delta(x_2, x_3)^2 G(x_3, x_4) \Delta(x_1, x_4) \Delta(x_4, x_5) \Delta(x_5, x_6) \Delta(x_6, x_7) G(x_5, x_7) \]

- The wave front set of the amplitude is obtained by a recursive construction

- The amplitude is well defined, except on the diagonals

- It remains to renormalize to define the product on the diagonals
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**TOPOLOGY**

- For a closed cone $\Gamma \subset T^* M$ we define
  $$\mathcal{D}'_{\Gamma}(U) = \{u \in \mathcal{D}'(U); \text{WF}(u) \subset \Gamma\}$$
- We furnish $\mathcal{D}'_{\Gamma}(U)$ with a locally convex topology.
- Let $E$ be a vector space over $\mathbb{C}$. A *semi-norm* on $E$ is a map $p : E \to \mathbb{R}$ such that
  - $p(\lambda x) = |\lambda|p(x)$ for all $\lambda \in \mathbb{C}$ and $x \in E$
  - $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$
- A locally convex space is a vector space $E$ equipped with a family $(p_i)_{i \in I}$ of semi-norms on $E$.
- The sets $V_{i, \epsilon} = \{x \in E; p_i(x) < \epsilon\}$ form a sub-base of the topology generated by the semi-norms.
The seminorms of $\mathcal{D}'_\Gamma(U)$ are:

- $p_B(u) = \sup_{f \in B} |\langle u, f \rangle|$ where $B$ is bounded in $\mathcal{D}(U)$ are the seminorms of the strong topology of $\mathcal{D}'(U)$

- $||u||_{N,V,\chi} = \sup_{k \in V} (1 + |k|)^N |\widehat{u\chi}(k)|$ for all integers $N$, closed cones $V$ and functions $\chi \in \mathcal{D}(U)$ s.t. $\text{supp}\chi \times V \cap \Gamma = \emptyset$

The second set of seminorms is used to ensure that the Fourier transform of $u \in \mathcal{D}'_\Gamma(U)$ around $x \in \text{supp}(\chi)$ decreases faster than any inverse polynomial: the wave front set of $u \in \mathcal{D}'_\Gamma(U)$ is in $\Gamma$
Thm.

- $\mathcal{D}'_\Gamma(U)$ is complete
- $\mathcal{D}'_\Gamma(U)$ is semi-Montel (its closed and bounded subsets are compact)
- $\mathcal{D}'_\Gamma(U)$ is semi-reflexive
- $\mathcal{D}'_\Gamma(U)$ is nuclear
- $\mathcal{D}'_\Gamma(U)$ is a normal space of distributions
**Thm.** With the topology of $\mathcal{D}_\Gamma'(U)$

- The pull-back is continuous
- The push-forward is continuous
- The multiplication of distributions is hypocontinuous
- The tensor product of distributions is hypocontinuous
- The duality pairing is hypocontinuous
FOR YOUR ATTENTION