Lattices

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Abstract

Some lattice stuff for the current EIDMA course.

1 Intro

For many applications one needs dense arrangements of balls in Euclidean space. Parameters to be optimized are e.g. packing density, covering density, kissing number (the number of balls that touch a given ball). Just as in coding theory, where the best codes sometimes are nonlinear but most of the theory concerns the linear case, here the best arrangements sometimes are non-lattice, but most of the theory concerns lattice arrangements.

2 Lattices

Lattice

A *lattice* Λ is a discrete additive subgroup of $\mathbf{R}^n$. Equivalently, it is a finitely generated free $\mathbf{Z}$-module with positive definite symmetric bilinear form.

Basis

Assume that our lattice Λ has dimension n, i.e., spans $\mathbf{R}^n$. Let $\{a_1, ..., a_n\}$ be a $\mathbf{Z}$-basis of Λ. Let A be the matrix with the vectors a_i as rows. If we choose a different $\mathbf{Z}$-basis $\{b_1, ..., b_n\}$, so that $b_i = \sum s_{ij} a_j$, and B is the matrix with the vectors b_i as rows, then $B = SA$, with $S = (s_{ij})$. Since S is integral and invertible, it has determinant ± 1. It follows that $|\det A|$ is uniquely determined by Λ, independent of the choice of basis.

Volume

$\mathbf{R}^n/\Lambda$ is an n-dimensional torus, compact with finite volume. Its volume is the volume of the fundamental domain, which equals $|\det A|$.

If Λ' is a sublattice of Λ, then $\mathrm{vol}(\mathbf{R}^n/\Lambda') = \mathrm{vol}(\mathbf{R}^n/\Lambda).|\Lambda/\Lambda'|$.

Gram matrix

Let G be the matrix (a_i, a_j) of inner products of basis vectors for a given basis. Then $G = AA^\top$, so $\mathrm{vol}(\mathbf{R}^n/\Lambda) = \sqrt{\det G}$.

Dual lattice

The *dual* Λ^* of a lattice Λ is the lattice of vectors having integral inner products with all vectors in Λ: $\Lambda^* = \{x \in \mathbf{R}^n | (x, r) \in \mathbf{Z} \text{ for all } r \in \Lambda\}$.

It has a basis $\{a_1^*, ..., a_n^*\}$ defined by $(a_i^*, a_j) = \delta_{ij}$.

Now $A^*A^\top = I$, so $A^* = (A^{-1})^\top$ and Λ^* has Gram matrix $G^* = G^{-1}$.

It follows that $\mathrm{vol}(\mathbf{R}^n/\Lambda^*) = 1/\mathrm{vol}(\mathbf{R}^n/\Lambda)$.

We have $\Lambda^{**} = \Lambda$.

Integral lattice

The lattice Λ is called *integral* when the inner products of lattice vectors are all integral.

For an integral lattice Λ one has $\Lambda \subseteq \Lambda^*$.

The lattice Λ is called *even* when (x, x) is an even integer for each $x \in \Lambda$. An even lattice is integral. An integral lattice that is not even is called *odd*.

Discriminant

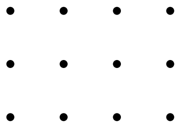
The *determinant*, or *discriminant*, $\mathrm{disc}\,\Lambda$ of a lattice Λ is defined by $\mathrm{disc}\,\Lambda = \det G$. When Λ is integral, we have $\mathrm{disc}\,\Lambda = |\Lambda^*/\Lambda|$.

A lattice is called *self-dual* or *unimodular* when $\Lambda = \Lambda^*$, i.e., when it is integral with discriminant 1. An even unimodular lattice is called *Type II*, the remaining unimodular lattices are called *Type I*.

If there is an even unimodular lattice in $\mathbf{R}^n$, then n is divisible by 8. (This follows by studying the associated theta series and modular forms. See also below under Leech lattice.)

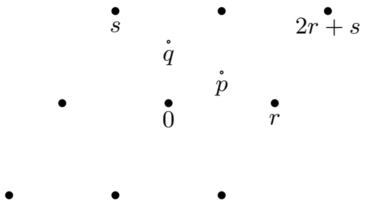
3 Examples

3.1 $\mathbf{Z}^n$



Unimodular, type I.

3.2 A_2



Basis $\{r, s\}$. Gram matrix $G = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, so that $\det G = 3$. A fundamental region for A_2 is the parallelogram on $0, r, s$. A fundamental region for A_2^* is the parallelogram on $0, p, q$. Note that the area of the former $(\sqrt{3})$ is thrice that of the latter $(1/\sqrt{3})$.

The representation of this lattice in $\mathbf{R}^2$ has nonintegral coordinates. It is easier to work in $\mathbf{R}^3$, on the hyperplane $\sum x_i = 0$, and choose $r = e_1 - e_2 = (1, -1, 0)$, $s = e_2 - e_3 = (0, 1, -1)$. Then A_2 consists of the points (x_1, x_2, x_3) with $x_i \in \mathbf{Z}$ and $\sum x_i = 0$. The dual lattice A_2^* consists of the points (x_1, x_2, x_3) with $x_1 \equiv x_2 \equiv x_3 \pmod 1$ and $\sum x_i = 0$ (so that $3x_1 \in \mathbf{Z}$). It contains for example $p = \frac{1}{3}(2r + s) = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$.

4 Constructions

Let $\rho : \mathbf{Z}^n \to 2^n$ be coordinatewise reduction mod 2. Given a binary linear code C, the lattice $\rho^{-1}(C)$ is integral, since it is contained in $\mathbf{Z}^n$, but never unimodular, unless it is all of $\mathbf{Z}^n$, a boring situation.

It turns out to be more useful to look at $\frac{1}{\sqrt{2}}\rho^{-1}(C)$. This is an integral lattice when inner products of code words are even, that is, when C is self-orthogonal. If $\dim C = k$ then $\mathrm{vol}(\mathbf{R}^n/\rho^{-1}(C)) = 2^{n-k}$ and hence $\mathrm{vol}(\mathbf{R}^n/\frac{1}{\sqrt{2}}\rho^{-1}(C)) = 2^{\frac{1}{2}n-k}$. In particular, $\frac{1}{\sqrt{2}}\rho^{-1}(C)$ will be unimodular when C is self-dual, and even when C is "doubly even".

Example

Let C be the [8,4,4] extended Hamming code. Then $\frac{1}{\sqrt{2}}\rho^{-1}(C)$ is a unimodular 8-dimensional lattice known as E_8.

The code C has weight enumerator $1+14X^4+X^8$ (that is, has one word of weight 0, 14 words of weight 4, and one word of weight 8). It follows that the *roots* (vectors r with $(r,r)=2$) in this incarnation of E_8 are the 16 vectors $\pm\frac{1}{\sqrt{2}}(2,0,0,0,0,0,0,0)$ (with 2 in any position), and the $16.14=224$ vectors $\frac{1}{\sqrt{2}}(\pm1,\pm1,\pm1,\pm1,0,0,0,0)$ with ±1 in the nonzero positions of a weight 4 vector. Thus, there are 240 roots.

5 Root lattices

A *root lattice* is an integral lattice generated by roots (vectors r with $(r,r)=2$). The set of roots in a root lattice is a (reduced) *root system* Φ, i.e., satisfies

(i) If $r\in\Phi$ and $\lambda r\in\Phi$, then $\lambda=\pm1$.

(ii) Φ is closed under the reflection w_r that sends s to $s-2\frac{(r,s)}{(r,r)}r$ for each $r\in\Phi$.

(iii) $2\frac{(r,s)}{(r,r)}\in\mathbf{Z}$.

For example, A_2 and E_8 are root lattices.

Since Φ generates Λ and Φ is invariant under $W=\langle w_r|r\in\Phi\rangle$, the same holds for Λ, so root lattices have a large group of automorphisms.

A *fundamental system* of roots Π in a root lattice Λ is a set of roots generating Λ and such that $(r,s)\le 0$ for distinct $r,s\in\Pi$. A *reduced fundamental system* of roots is a fundamental system that is linearly independent. A non-reduced fundamental system is called *extended.*

For example, in A_2 the set $\{r,s\}$ is a reduced fundamental system, and $\{r,s,-r-s\}$ is an extended fundamental system.

The Dynkin diagram of a fundamental system Π such that $(r,s)\ne-2$ for $r,s\in\Pi$, is the graph with vertex set Π where r and s are joined by an edge when $(r,s)=-1$. (The case $(r,s)=-2$ happens only for a non-reduced system with A_1 component. In that case we do not define the Dynkin diagram.)

Every root lattice has a reduced fundamental system: Fix some vector u, not orthogonal to any root. Put $\Phi^+(u)=\{r\in\Phi|(r,u)>0\}$ and $\Pi(u)=\{r\in\Phi^+(u)|r$ cannot be written as $s+t$ with $s,t\in\Phi^+(u)\}$. Then $\Pi(u)$ is a reduced fundamental system of roots, and written on this basis each root has only positive or only negative coefficients.

(Indeed, if $r, s \in \Pi(u)$ and $(r,s) = 1$, then say $r - s \in \Phi^+(u)$ and $r = (r-s) + s$, contradiction. This shows that $\Pi(u)$ is a fundamental system. If $\sum \gamma_r r = 0$, then separate the γ_r into positive and negative ones to get $\sum \alpha_r r = \sum \beta_s s = x \neq 0$ where all coefficients α_r, β_s are positive. Now $0 < (x,x) = \sum \alpha_r \beta_s (r,s) \leq 0$, contradiction. This shows that $\Pi(u)$ is reduced. Each root in $\Phi^+(u)$ has an expression over $\Pi(u)$ with only positive coefficients.)

Proposition 5.1 *Let Π be a reduced fundamental system.*

(i) For all $x \in \mathbf{R}^n$ there is a $w \in W$ such that $(w(x), r) \geq 0$ for all $r \in \Pi$.

(ii) $\Pi = \Pi(u)$ for some u. (That is, W is transitive on reduced fundamental systems.)

(iii) If Λ is irreducible (not an orthogonal direct sum), then there is a unique $\tilde{r} \in \Phi$ such that $\Pi \cup \{\tilde{r}\}$ is an extended fundamental system.

Proof: (i) Let G be the Gram matrix of Π, and write $A = 2I - G$. Since G is positive definite, A has largest eigenvalue less than 2. Using Perron-Frobenius, let $\gamma = (\gamma_r)_{r \in \Pi}$ be a positive eigenvector of A. If $(x,s) < 0$ for some $s \in \Pi$, then put $x' = w_s(x) = x - (x,s)s$. Now

$$(x', \sum_r \gamma_r r) = (x, \sum_r \gamma_r r) - (G\gamma)_s (x,s) > (x, \sum_r \gamma_r r).$$

But W is finite, so after finitely many steps we reach the desired conclusion.

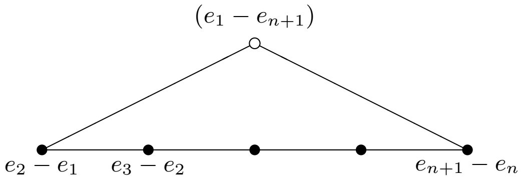
(ii) Induction on $|\Pi|$. Fix x with $(x,r) \geq 0$ for all $r \in \Pi$. Then $\Pi_0 = \Pi \cap x^\perp$ is a fundamental system of a lattice in a lower-dimensional space, so of the form $\Pi_0 = \Pi_0(u_0)$. Take $u = x + \epsilon u_0$ for small $\epsilon > 0$. Then $\Pi = \Pi(u)$.

(iii) If $r \in \Phi^+(u)$ has maximal (r,u), then $\tilde{r} = -r$ is the unique root that can be added. It can be added, since $(\tilde{r}, s) \geq 0$ means $(r,s) < 0$, so that $r + s$ is a root, contradicting maximality of r. And it is unique because linear dependencies of an extended system correspond to an eigenvector with eigenvalue 2 of the extended Dynkin diagram, and by Perron-Frobenius up to a constant there is a unique such eigenvector when the diagram is connected, that is, when Λ is irreducible. □

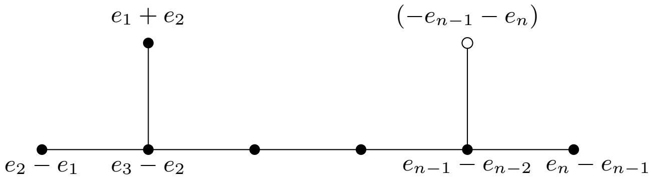
The examples

The irreducible root lattices one finds are A_n $(n \geq 0)$, D_n $(n \geq 4)$, E_6, E_7, E_8. Each is defined by its Dynkin diagram.

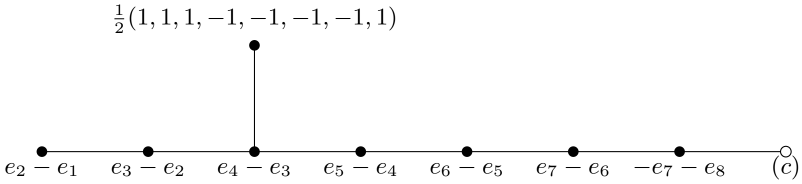
(1) A_n: The lattice vectors are: $x \in \mathbf{Z}^{n+1}$ with $\sum x_i = 0$. There are $n(n+1)$ roots: $e_i - e_j$ $(i \neq j)$. The discriminant is $n+1$, and $A_n^*/A_n \simeq \mathbf{Z}_{n+1}$, with the quotient generated by $\frac{1}{n+1}(e_1 + ... + e_n - ne_{n+1}) \in A_n^*$.



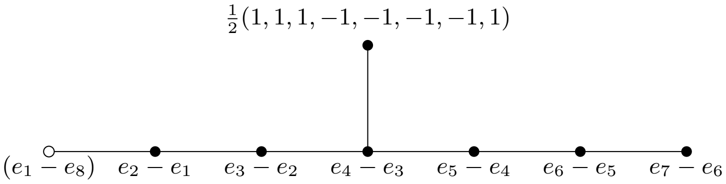
(2) D_n: The lattice vectors are: $x \in \mathbf{Z}^n$ with $\sum x_i \equiv 0 \pmod 2$. There are $2n(n-1)$ roots $\pm e_i \pm e_j$ $(i \neq j)$. The discriminant is 4, and D_n^*/D_n is isomorphic to $\mathbf{Z}_4$ when n is odd, and to $\mathbf{Z}_2 \times \mathbf{Z}_2$ when n is even. D_n^* contains e_1 and $\frac{1}{2}(e_1 + ... + e_n)$. Note that $D_3 \simeq A_3$.



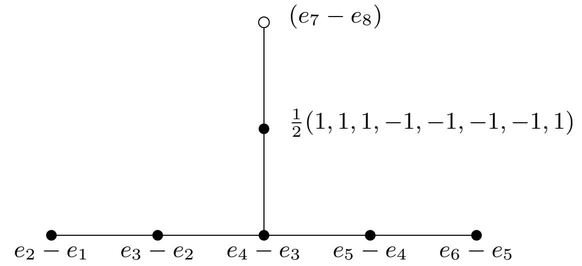
(3) E_8: (Recall that we already gave a construction of E_8 from the Hamming code.) The lattice is the span of D_8 and $c := \frac{1}{2}(e_1 + ... + e_8)$. There are $240 = 112 + 128$ roots, of the forms $\pm e_i \pm e_j$ $(i \neq j)$ and $\frac{1}{2}(\pm e_1 \pm ... \pm e_8)$ with an even number of minus signs. The discriminant is 1, and $E_8^* = E_8$.



(4) E_7: Take $E_7 = E_8 \cap c^\perp$. There are $126 = 56 + 70$ roots. The discriminant is 2, and E_7^* contains $\frac{1}{4}(1, 1, 1, 1, 1, 1, -3, -3)$.



(5) E_6: For the vector $d = -e_7 - e_8$, take $E_6 = E_8 \cap \{c, d\}^\perp$. There are $72 = 32 + 40$ roots. The discriminant is 3, and E_6^* contains the vector $\frac{1}{3}(1, 1, 1, 1, -2, -2, 0, 0)$.



That this is all, is an easy consequence of the Perron-Frobenius theorem: $A = 2I - G$ is the adjacency matrix of a graph, namely the Dynkin diagram, and it is easy to classify the connected graphs with largest eigenvalue less than 2—they are the Dynkin diagrams of reduced fundamental systems of irreducible root systems—and the connected graphs with largest eigenvalue 2—they are the Dynkin diagrams of extended root systems.

6 The Leech lattice

Theorem 6.1 *There exists a unique even unimodular lattice without roots in* $\mathbf{R}^{24}$. *It has 196560 vectors of weight 4.*

Construction: a spanning set consists of the vectors $\frac{1}{\sqrt{8}}(\mp 3, \pm 1^{23})$ with ∓ 3 in any position, and the upper signs in a code word of the extended binary Golay code.

For the vectors of weight 4 one finds the shapes $4^2\,0^{22}$, $3\,1^{23}$, $2^8\,0^{16}$ (omitting the $\frac{1}{\sqrt{8}}$) with frequencies $2^2\binom{24}{2}$, $2^{12}.24$ and $2^7.759$, respectively.

Uniqueness is proved using θ-functions and the theory of modular forms.

Given a lattice Λ, define

$$\theta_\Lambda(z) = \sum_{x \in \Lambda} q^{\frac{1}{2}(x,x)}$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$.

One has

$$\theta_{\Lambda^*}(z) = \det(\Lambda)^{\frac{1}{2}} \left(\frac{i}{z}\right)^{\frac{n}{2}} \theta_\Lambda(-\frac{1}{z}).$$

Here the Leech lattice has $\Lambda = \Lambda^*$ and $\det(\Lambda) = 1$, so that $\theta_\Lambda(z)$ is a modular form of weight 12.

The space of modular forms of weight 12 has dimension 2, and the two conditions: unique vector of norm 0, no vectors of norm 2, determine $\theta_\Lambda(z)$

uniquely. Thus, any even unimodular lattice without roots in $\mathbf{R}^{24}$ must have the same weight enumerator as the Leech lattice.

Some more work gives the desired conclusion.

7 The Barnes-Wall lattice

Take the first order Reed-Muller code C of length 16. This is a binary linear [16,5,8] code, with weight enumerator $1 + 30X^8 + X^{16}$.

Define the Barnes-Wall lattice Λ_{16} by: $\frac{1}{\sqrt{2}}x \in \Lambda_{16}$ iff x is an integral vector, with x mod 2 in C and $\sum x_i \equiv 0 \pmod 4$.

This is a lattice of determinant 2^8 and minimum norm 4. The kissing number equals $\tau = 4320$ (namely, 480 vectors of shape $\frac{1}{\sqrt{2}}(\pm 2^2, 0^{14})$ and 3840 vectors of shape $\frac{1}{\sqrt{2}}(\pm 1^8, 0^8)$ with the 1's at the positions of a code word.

(This type of construction is especially useful when $d = 8$, since two shapes contribute to the kissing number. For larger n chains of binary codes are used to place the 1's, 2's, 4's, etc.)

8 The Coxeter-Todd lattice

Let $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ be a cube root of unity, and put $\theta = \omega - \bar{\omega} = \sqrt{-3}$.

Let Λ be the 6-dimensional $\mathbf{Z}[\omega]$-lattice spanned by the six vectors $(4,0,0,0,0,0)$, $(2,2,0,0,0,0)$, $(2,0,2,0,0,0)$, $(2,0,0,2,0,0)$, $(2,0,0,0,2,0)$, $(\theta,1,1,1,1,1)$, provided with the positive definite inner product $(u,v) = u\bar{v}^\top$. Since $\theta = 2\omega + 1$ all inner products are divisible by 4, and we see that $\frac{1}{2}\Lambda$ is a unimodular lattice $\mathbf{Z}[\omega]$-lattice.

This definition looks asymmetric, but in fact the group is large: the automorphism group of the 6-dimensional $\mathbf{Z}[\omega]$-lattice is $6.U_4(3).2$ of order $2^9.3^7.5.7$, inducing the symmetric group Sym(6) on the coordinate positions.

This 6-dimensional $\mathbf{Z}[\omega]$-lattice Λ is a 12-dimensional $\mathbf{Z}$-lattice. If the complex lattice has coordinate positions e_h $(1 \leq h \leq 6)$, then we can take 12 coordinate positions e_h, ie_h for the real lattice, and given a $\mathbf{Z}[\omega]$-basis with vectors u_j, we find a $\mathbf{Z}$-basis with vectors u_j, $(\omega - 1)u_j$. A diagonal 2 now becomes $\begin{pmatrix} 2 & \\ -3 & \sqrt{3} \end{pmatrix}$. Consequently, $K_{12} := \frac{1}{\sqrt{2}}\Lambda$ is an integral lattice with determinant 3^6. (One checks that $(a+bi)(c-di) = ac+bd+(bc-ad)i$ and $((a,b),(c,d)) = ac + bd$ so that the real inner product is the real part of the complex inner product. Since $\mathrm{Re}(\omega) = -\frac{1}{2}$, we need an extra factor 2 to guarantee that the inner product is integral.)

The dual K_{12}^* is spanned by the six vectors $\frac{1}{\sqrt{2}}u_j$ and the six vectors $\frac{1}{3\sqrt{2}}(\omega - 1)u_j$. (Thus, K_{12}^*/K_{12} can be regarded as a 6-dimensional vector space over $GF(3)$.)

The minimum norm is 4, and the kissing number is 756. (Indeed, there are $32.6.3 = 576$ vectors of the form $\frac{1}{\sqrt{2}}\omega^j(\pm\theta, \pm 1, ..., \pm 1)$ with an even number of minus signs, the $\pm\theta$ in any position, and $j = 0, 1, 2$, and also $15.4.3 = 180$ vectors of the form $\frac{1}{\sqrt{2}}\omega^j(\pm 2, \pm 2, 0, 0, 0, 0)$ with the ± 2 in any positions.)

There are many equivalent descriptions. A short one: take the vectors $u + \omega\theta v$, with $u \in E_6$ and $v \in E_6^*$.

9 Records

We quote a table from Conway & Sloane giving the best lattices known in low dimensional spaces.

dimension	1	2	3	4	5	6	7	8	12	16	24
densest packing	$\mathbf{Z}$	A_2	A_3	D_4	D_5	E_6	E_7	E_8	K_{12}	Λ_{16}	Λ_{24}
highest kissing number	$\mathbf{Z}$ 2	A_2 6	A_3 12	D_4 24	D_5 40	E_6 72	E_7 126	E_8 240	P_{12a} 840	Λ_{16} 4320	Λ_{24} 196560
thinnest covering	$\mathbf{Z}$	A_2	A_3^*	A_4^*	A_5^*	A_6^*	A_7^*	A_8^*	A_{12}^*	A_{16}^*	Λ_{24}
best quantizer	$\mathbf{Z}$	A_2	A_3^*	D_4	D_5^*	E_6^*	E_7^*	E_8	K_{12}	Λ_{16}	Λ_{24}

Here K_{12} is the Coxeter-Todd lattice, Λ_{16} is the Barnes-Wall lattice, Λ_{24} is the Leech lattice, and P_{12a} is a certain non-lattice packing.