1.5. Deligne and Cheeger-Simons cohomologies

Deligne cohomology was invented by Deligne around 1972 for the purpose of having one cohomology theory for algebraic varieties which includes ordinary singular (or Čech) cohomology and the intermediate jacobians of Griffiths. We will mostly deal with a smooth analog of this theory, which we call smooth Deligne cohomology. It is called Cheeger-Simons cohomology in [B-K]. We will see the relation with the Cheeger-Simons groups of differential characters later on.

For M a smooth paracompact manifold which is finite-dimensional (or, more generally, modeled on an ILH space), we saw in §4 that the Čech cohomology groups $\check{H}^p(M, \mathbb{R}_M)$ are canonically isomorphic to the de Rham cohomology groups. We gave an explicit isomorphism in Proposition 1.4.17. Essentially, the construction of this isomorphism amounted to moving upwards on a "staircase" oriented southeast to northwest. This staircase was inside the Čech double complex for the complex of sheaves

$$0 \rightarrow \mathbb{R} \rightarrow \underline{A}_M^0 \rightarrow \underline{A}_M^1 \cdots \rightarrow \underline{A}_M^{p-1} \qquad (1-27)$$

where $\mathbb{R}$ is put in degree 0. We truncate the de Rham complex at $\underline{A}_M^{p-1}$ if we are interested in degree p cohomology classes.

The complex of sheaves (1-27) is essentially the smooth Deligne complex of index p, except for some twist which we will now try to motivate. The motivation comes from the theory of characteristic classes, which produces cohomology classes in $H^{2p}(M, \mathbb{Z})$ associated to a bundle over M. However, these classes only become integral after some natural expression in the curvature of the bundle is divided by a factor $(2\pi\sqrt{-1})^p$. Except for this arbitrary factor, the expression is purely algebraic over $\mathbb{Q}$, of the type $P(\Omega)$, where Ω is the curvature and P is some invariant polynomial with rational coefficients. Deligne's viewpoint is to eliminate this unnatural division by a power of $2\pi\sqrt{-1}$. So he introduces the cyclic subgroup $\mathbb{Z}(p) = (2\pi\sqrt{-1})^p \cdot \mathbb{Z}$ of $\mathbb{C}$ and views a characteristic class as lying in $H^{2p}(M, \mathbb{Z}(p))$ instead of $H^{2p}(M, \mathbb{Z})$. Similarly, for a subring B of $\mathbb{R}$ (the most interesting examples being $B = \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$), he introduces the free cyclic B-module $B(p) = (2\pi\sqrt{-1})^p \cdot B \subset \mathbb{C}$. Note that product in $\mathbb{C}$ induces an isomorphism $B(p) \otimes_B B(q) \xrightarrow{\sim} B(p+q)$ of B-modules. In particular, $\mathbb{R}(p)$ is equal to $\mathbb{R}$ for p even and to $\sqrt{-1} \cdot \mathbb{R}$ for p odd.

Recall that $A^p(M)_{\mathbb{C}} = A^p(M) \otimes \mathbb{C}$ denotes the space of complex-valued p-forms on M and $\underline{A}_{M,\mathbb{C}}^p$ the sheaf of complex-valued differential forms. Let $i : B(p)_M \rightarrow \underline{A}_{M,\mathbb{C}}^0$ denote the inclusion of constant into smooth functions.

1.5.1. Definition. *Let M be a smooth manifold. For B a subring of* $\mathbb{R}$

and for $p \geq 0$, the smooth Deligne complex $B(p)^\infty_D$ is the complex of sheaves:

$$B(p)_M \xrightarrow{i} \underline{A}^0_{M,\mathbb{C}} \xrightarrow{d} \underline{A}^1_{M,\mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^{p-1}_{M,\mathbb{C}} \qquad (1-28)$$

The hypercohomology groups $H^q(M, B(p)^\infty_D)$ are called the smooth Deligne cohomology groups of M, and are sometimes denoted by $H^q_D(M, B(p)^\infty)$.

We now compute Deligne cohomology groups and show that Deligne cohomology is a refinement of ordinary cohomology. First we see that $B(0)^\infty_D$ is simply equal to B_M, so that $H^q(M, B(0)^\infty_D)$ is just sheaf cohomology $H^q(M, B_M)$. The complex $\mathbb{Z}(1)^\infty_D$ is much more interesting. We have

$$\mathbb{Z}(1)^\infty_D = \quad [\mathbb{Z}(1) \rightarrow \underline{\mathbb{C}}_M] \qquad (1-29)$$

We will need the notion of translation of a complex of groups (or of sheaves). For $K^\bullet$ a complex and $p \in \mathbb{Z}$, the complex $K^\bullet[p]$ is the complex obtained by translating $K^\bullet$ by p steps to the left. Thus $(K^\bullet[p])^q = K^{p+q}$, and $H^q(K^\bullet[p]) = H^{p+q}(K^\bullet)$.

1.5.2. Proposition. *The complex of sheaves $\mathbb{Z}(1)^\infty_D$ is quasi-isomorphic to $\underline{\mathbb{C}}^*_M[-1]$. Hence we have $H^q(M, \mathbb{Z}(1)^\infty_D) = H^{q-1}(M, \underline{\mathbb{C}}^*_M)$ for all q.*

Proof. Consider the following morphism of complexes of sheaves *exp* : $\mathbb{Z}(1)^\infty_D \rightarrow \underline{\mathbb{C}}^*_M[-1]$.

$$\begin{tikzcd}
\mathbb{Z}(1) \arrow[r] \arrow[d, "0"] & \underline{\mathbb{C}}_M \arrow[d, "exp"] \\
0 \arrow[r] & \underline{\mathbb{C}}^*_M
\end{tikzcd}$$

ϕ is a quasi-isomorphism of complex of sheaves, since the sequence

$$0 \rightarrow \mathbb{Z}(1) \rightarrow \underline{\mathbb{C}}_M \xrightarrow{exp} \underline{\mathbb{C}}^*_M \rightarrow 0$$

is exact by Corollary 1.1.9. ■

Note that this result already gives some justification for the decision we made to start this complex of sheaves with $\mathbb{Z}(1)$ instead of $\mathbb{Z}$. One can generalize Proposition 1.5.2 to any subring B of $\mathbb{R}$, but we will not get into this here, except to say that $H^1(M, B(1)^\infty_D)$ has to do with multi-valued smooth functions f for which two branches of f at some point differ by a constant in $B(1) \subset \mathbb{C}$.

We now study smooth Deligne cohomology in general. There is a natural homomorphism $\kappa : H^q(M, B(p)^\infty_D) \to H^q(M, B(p))$ since the complex $B(p)^\infty_D$ projects to the constant sheaf $B(p)_M$. We have an exact sequence of complexes of sheaves

$$0 \to \sigma_{\leq p-1}(\underline{A}^\bullet_{M,\mathbb{C}})[-1] \to B(p)^\infty_D \to B(p) \to 0, \qquad (1-30)$$

where $\sigma_{\leq p-1}(\underline{A}^\bullet_{M,\mathbb{C}})$ denotes the complex $\underline{A}^0_{M,\mathbb{C}} \to \cdots \to \underline{A}^{p-1}_{M,\mathbb{C}}$ obtained by chopping off the part of the complex $\underline{A}^\bullet_{M,\mathbb{C}}$ in degrees $\geq p$, giving a long exact sequence of hypercohomology groups. We assume now that the sheaves $\underline{A}^p_M$ are soft, hence so are their complexifications $\underline{A}^p_{M,\mathbb{C}}$. Then the hypercohomology groups of the complex of sheaves $\sigma_{\leq p-1}\underline{A}^p_{M,\mathbb{C}}$ are simply

$$H^q(M, \sigma_{\leq p-1}\underline{A}^p_{M,\mathbb{C}}) = \left\{ \begin{array}{l} H^q_{DR}(M) \otimes \mathbb{C} \text{ if } \mathrm{q} \leq \mathrm{p} - 2 \\ A^{p-1}_{DR}(M)_{\mathbb{C}}/d\, A^{p-2}(M)_{\mathbb{C}} \text{ if } \mathrm{q} = \mathrm{p} - 1 \\ 0 \text{ if } \mathrm{q} \geq \mathrm{p} \end{array} \right\}. \qquad (1-31)$$

1.5.3. Theorem. *Let M be a smooth paracompact manifold such that the sheaves $\underline{A}^p_M$ are soft (for example, M is finite-dimensional or satisfies the assumptions of Theorem 1.4.15). The smooth Deligne cohomology groups $H^\bullet(M, B(p)^\infty_D)$ are as follows:*

(1) *For $q \leq p - 1$, the group $H^q(M, B(p)^\infty_D)$ fits in the exact sequence*

$$\begin{aligned} &0 \to H^{q-1}(M, B(p)) \to \\ &\quad \to H^{q-1}(M, \mathbb{C}) \to H^q(M, B(p)^\infty_D) \xrightarrow{\kappa} \mathrm{Tors}\, \mathrm{H}^{\mathrm{q}}(\mathrm{M}, \mathrm{B}(\mathrm{p})) \to 0, \end{aligned} \qquad (1-32)$$

where Tors $\mathrm{H}^{\mathrm{q}}(\mathrm{M}, \mathrm{B}(\mathrm{p}))$ *denotes the torsion subgroup of $H^q(M, B(p))$.*

(2) *The group $H^p(M, B(p)^\infty_D)$ fits in the exact sequence*

$$0 \to A^{p-1}(M)_{\mathbb{C}}/A^{p-1}(M)_{\mathbb{C},0} \to H^p(M, B(p)^\infty_D) \xrightarrow{\kappa} H^p(M, B(p)) \to 0, \qquad (1-33)$$

where $A^{p-1}(M)_{\mathbb{C},0}$ denotes the group of closed complex-valued $(p-1)$-forms on M whose cohomology class belongs to the image of $H^{p-1}(M, B(p)) \to H^{p-1}(M, \mathbb{C})$.

(3) *For $q \geq p + 1$, we have $H^q(M, B(p)^\infty_D) \xrightarrow{\sim} H^q(M, B(p))$.*

Proof. We use the long exact sequence associated to the exact sequence

(1-30) of complex of sheaves. This gives an exact sequence

$$H^{q-1}(M,B(p)) \xrightarrow{f_{q-1}} H^{q-1}(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}}) \to H^q(M,B(p)_D^\infty) \to$$
$$\to H^q(M,B(p)) \xrightarrow{f_q} H^q(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}}).$$

The groups $H^\bullet(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}})$ were determined in (1-31). The map $f_q : H^q(M,B(p)) \to H^q(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}})$ is the composite of the map $H^q(M,B(p)) \to H^q(M,\mathbb{C})$, of the isomorphism

$$H^q(M,\mathbb{C}) \simeq H^q_{DR}(M) \otimes \mathbb{C},$$

and of the projection map

$$H^q_{DR}(M) \otimes \mathbb{C} = H^q(A^\bullet(M)_{\mathbb{C}}) \to H^q(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}}).$$

For $q \leq p-1$, we have $H^{q-1}(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}}) = H^{q-1}_{DR}(M) \otimes \mathbb{C} = H^{q-1}(M,\mathbb{C})$. For these values of q, the map $H^q(M,\mathbb{C}) \to H^q(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}})$ is injective, so the kernel of f_q is the torsion subgroup of $H^q(M,B(p))$. This proves (1). For $q = p$, the map f_{p-1} is the map from $H^{p-1}(M,B(p))$ to $A^{p-1}(M)_{\mathbb{C}}/dA^{p-2}(M)_{\mathbb{C}}$. Its image consists of those degree $p-1$ complex-valued de Rham cohomology classes which belong to the image of $H^{p-1}(M,B(p))$ in $H^{p-1}(M,\mathbb{C})$. Hence the cokernel of f_{p-1} is equal to $A^{p-1}(M)_{\mathbb{C}}/A^{p-1}(M)_{\mathbb{C},0}$. On the other hand, $H^p(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}}) = 0$. This proves (2).

(3) is clear, since both $H^{q-1}(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}})$ and $H^q(\sigma_{\leq p-1}\, A^\bullet(M)_{\mathbb{C}})$ are 0 for $q \geq p$. ■

The theorem shows that the Deligne cohomology group is most interesting for $q = p$, when it maps onto $H^p(M,B(p))$ with a large kernel, consisting of all $(p-1)$-forms modulo those which are closed and have periods in $B(p)$. There is a morphism d of complexes

$$\begin{tikzcd} B(p)_M \arrow[r, "i"] & \underline{A}^0_{M,\mathbb{C}} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \underline{A}^{p-1}_{M,\mathbb{C}} \arrow[d, "d"] \\ & & & \underline{A}^p_{M,\mathbb{C}} \end{tikzcd} \tag{1-34}$$

This induces a homomorphism $d : H^p(M,B(p)_D^\infty) \to A^p(M)_{\mathbb{C}}{}^{cl}$, where $A^p(M)_{\mathbb{C}}{}^{cl}$ is the space of closed complex-valued p-forms. The following result shows that classes in smooth Deligne cohomology can be used to compare classes in Čech cohomology with classes in de Rham cohomology. This will be used in Chapter 2; see [Br-ML1] for applications to formulas for characteristic classes.

1.5.4. Proposition. *Let $a \in H^p(M, \mathbb{R}(p)^\infty_D)$. Then the Čech cohomology class $\kappa(a) \in H^p(M, \mathbb{R}(p))$ corresponds to the de Rham cohomology class of $(-1)^p \cdot da$ in $H^p(A^\bullet(M)_\mathbb{C})$ under the isomorphism between Čech and de Rham cohomologies.*

Proof. Let $(a_0, \dots, a_p)$ be a representative for a in the Čech double complex of some open covering with coefficients in the complex of sheaves $\mathbb{R}(p)^\infty_D$. So a_0 is a degree p Čech cocycle with coefficients in $\mathbb{R}(p)$, which represents $\kappa(a)$. For $i \geq 1$, a_i is a Čech $(p-i)$-cocycle with coefficients in $\underline{A}^{i-1}_{M,\mathbb{C}}$. We have $\delta(a_1) = (-1)^{p-1} i(a_0)$ in $C^p(\mathcal{U}, \underline{A}^0_{M,\mathbb{C}})$. It follows that the total boundary of $(a_1, \dots, a_p)$ in the Čech double complex is equal to $(-1)^{p-1} i(a_0) + da_p$. This means that the cohomology classes of $(-1)^p \cdot a_0$ and of the p-form da_p correspond to each other under the Čech-de Rham isomorphism of §4. ■

One can define characteristic classes with values in this Deligne cohomology group; for instance, the k-th Chern class of a complex vector bundle equipped with a connection may be defined as an element of $H^{2k}(M, \mathbb{Z}(k)_D)$. An explicit formula for such a class is given in [**Br-ML1**]. In case the usual Chern class is 0, one obtains a complex-valued $2k-1$-form defined modulo $A^{2k-1}_{\mathbb{C},0}$, which can be identified with the differential form of Chern and Simons [**Cher-S**].

In fact, the cohomology group $H^p(M, \mathbb{Z}(p)^\infty_D)$ is very closely related to the group of *differential characters* in the sense of Cheeger and Simons [**Chee-S**]. The definition we give here is slightly different from [**Chee-S**], in that we consider cochains with complex (as opposed to real) coefficients. We let $S_p(M)$ be the group of smooth singular chains on M, and $Z_p(M)$ the group of smooth singular p-cycles. We have the boundary map $\partial : S_p(M) \to S_{p-1}(M)$. For $S_p(M)^*$ the dual group of smooth singular cocycles, we have the transpose coboundary map $\partial^* : S_p(M)^* \to S_{p+1}(M)^*$.

1.5.5. Definition. [**Chee-S**] *Let M be a smooth finite-dimensional manifold. A differential character of degree p on M is a homomorphism $c : Z_{p-1}(M) \to \mathbb{C}/\mathbb{Z}(p)$ such that there exists a complex-valued p-form α for which*

$$c(\partial\gamma) = \int_\gamma \alpha \mod \mathbb{Z}(p) \qquad (1-35)$$

for any smooth singular p-chain γ, with boundary $\partial\gamma$. The group of degree p differential characters is denoted by $\hat{H}^p(M, \mathbb{Z}(p))$.

Note that our notation for the group of differential characters is different from the notation in [**Chee-S**].

1.5.6. Lemma. [Chee-S] *We have an exact sequence*

$$0 \to A^{p-1}(M)_{\mathbb{C}}/A^{p-1}(M)_{\mathbb{C},0} \to \hat{H}^p(M,\mathbb{Z}(p)) \to H^p(M,\mathbb{Z}(p)) \to 0.$$

Proof. Let $q : \mathbb{C} \to \mathbb{C}/\mathbb{Z}(p)$ be the projection. One defines a map $\phi : \hat{H}^p(M,\mathbb{Z}(p)) \to H^p(M,\mathbb{Z}(p))$ as follows. Let c be a differential character, and α a corresponding p-form satisfying (1-34). Since $\mathbb{C}/\mathbb{Z}(p)$ is a divisible group (hence an injective abelian group), and $S_{p-1}(M)$ is a free abelian group, there exists a group homomorphism $c' : S_{p-1}(M) \to \mathbb{C}$ such that $q(c'(\gamma)) = c(\gamma)$ for $\gamma \in Z_{p-1}(M)$. The difference $\sigma = \alpha - \partial^*(c')$ is a $\mathbb{Z}(p)$-valued p-cochain, according to (1-34). We have $\partial^*\sigma = \partial^*\alpha$, hence $\partial^*\sigma = d\alpha$ is a differential form which gives rise to a cochain with values in $\mathbb{Z}(p)$. This implies $d\alpha = 0$, hence $\partial^*\sigma = 0$. Thus σ is a smooth p-cocycle with values in $\mathbb{Z}(p)$. We claim that the cohomology class of σ is independent of the choices. If we change c' to $c' + a$, where a takes values in $\mathbb{Z}(p)$, we change σ to $\sigma - \partial^*(a)$, which has the same cohomology class. If we change c' to $c' + b$, where b vanishes on $p-1$-cycles, then σ does not change. This proves that the cohomology class of σ is well-defined. We define $\phi(c) = [\sigma]$.

We show next that ϕ is onto. Let $u \in H^p(M,\mathbb{Z}(p))$, and let α be a closed p-form representing u. Let σ be a $\mathbb{Z}$-valued smooth p-cocycle in the class of u. Then the difference cochain $\alpha - \sigma$ vanishes on $Z_p(M)$, so it is of the form $\partial^* c'$ or some smooth $p-1$-cochain c'. Let c be the restriction to $Z_{p-1}(M)$ of $q \circ c'$. Now c and α satisfy (1-35), and we see that $\phi(c) = u$.

Let $c \in \hat{H}^p(M,\mathbb{Z}(p))$ be in the kernel of ϕ. This means that one can choose the extension c' of c to all $(p-1)$-chains so that $\partial^* c' = \alpha$. Since singular cohomology is isomorphic to de Rham cohomology, we also have $\alpha = d\beta$ for some $(p-1)$-form β. $c' - \beta$ is a cocycle, so its restriction to $Z_{p-1}(M)$ is represented by a closed $(p-1)$-form β_1. c' and $\beta + \beta_1$ agree on $Z_{p-1}(M)$. Thus we have shown that c is given by some $(p-1)$-form. This $(p-1)$-form is determined precisely up to the ambiguity of a $(p-1)$-form ω such that $\int_\gamma \omega \in \mathbb{Z}(p)$ for all $(p-1)$-cycles γ, i.e., up to an element of $A^p(M)_{\mathbb{C},0}$. ■

Given this result, the following proposition is not surprising.

1.5.7. Proposition. *The group* $\hat{H}^p(M,\mathbb{Z}(p))$ *is canonically isomorphic to the smooth Deligne cohomology group* $H^p(M,\mathbb{Z}(p)_D^\infty)$.

We refer to [E] [D-H-Z] for a construction of this isomorphism (see also [Br-ML1]).

The notion of differential character is very concrete and well-suited to differential-geometric applications. One advantage of Deligne cohomology is a very pleasant description of products

$$\cup : B(p)_D^\infty \otimes B(q)_D^\infty \to B(p+q)_D^\infty,$$

due to Deligne and Beilinson [**Be**]. To avoid confusion, we will use $\tilde{d}$ to denote the differential in a Deligne complex, which does not always coincide with exterior differentiation d. For $x \in B(p)_D^\infty$, we denote by $deg(x)$ its degree in this complex of sheaves. We then put

$$x \cup y = \left\{ \begin{array}{ll} x \cdot y & \text{if deg (x) = 0} \\ x \wedge dy & \text{if deg(x) > 0 and deg(y) = q} \\ 0 & \text{otherwise} \end{array} \right\} \qquad (1-36)$$

1.5.8. Proposition. (1) *Each* $\cup$ *is a morphism of complexes.*

(2) *The multiplication* $\cup$ *is associative and is commutative up to homotopy.*

(3) *The class* 1 *is a left identity for* $\cup$ *(hence a right identity up to homotopy).*

Proof. We have

$$\tilde{d}(x \cup y) = \left\{ \begin{array}{ll} x \cdot dy & \text{if deg(x) = 0, deg(y)} \leq \text{q} \\ dx \wedge dy & \text{if deg(x) > 0, deg(y) = q} \\ 0 & \text{otherwise} \end{array} \right\} = \tilde{d}x \cup y + (-1)^{deg(x)}\, x \cup \tilde{d}y,$$

which proves (1).

The associativity of $\cup$ is easily checked. For commutativity of $\cup$ up to homotopy, we want to compare the morphisms of complexes $\cup$ and $\cup \circ s$, where $s : B(p)_D^\infty \otimes B(q)_D^\infty \xrightarrow{\sim} B(q)_D^\infty \otimes B(p)_D^\infty$ is the transposition $s(x \otimes y) = (-1)^{deg(x)deg(y)} y \otimes x$. There is a homotopy between these two morphisms, given by

$$H(x \otimes y) = \left\{ \begin{array}{l} \text{0 if deg(x) = 0 or deg(y) = 0} \\ (-1)^{deg(x)}\, x \wedge y \text{ otherwise} \end{array} \right.$$

(3) is obvious. ■

An illustration of this cup-product will be given in Chapter 2, in connection with a method used to construct line bundles with connection.

So far we have studied smooth Deligne cohomology. Deligne cohomology itself was invented by Deligne for complex manifolds.

1.5.9. Definition. *Let X be a complex manifold. For a subring B of $\mathbb{R}$, the Deligne complex $B(p)_D$ is the complex of sheaves*

$$B(p)_M \xrightarrow{i} \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_M \qquad (1-37)$$

The hypercohomology groups $H^q(M, B(p)_D)$ are called the Deligne cohomology groups, and are sometimes denoted by $H^q_D(M, B(p))$.

In analogy with Proposition 1.5.2, we have

1.5.10. Proposition. *The complex of sheaves $\mathbb{Z}(1)_D$ is quasi-isomorphic to $\mathcal{O}^*_X\,[-1]$, hence we have $H^q(X, \mathbb{Z}(1)_D) = H^{q-1}(X, \mathcal{O}^*_X)$.*

Deligne's motivation for introducing this cohomology theory was to have a rich theory of cohomology classes associated to algebraic subvarieties of a projective complex algebraic manifold X. To explain this, we note the exact sequence of complexes of sheaves

$$0 \to \Omega^\bullet_X / F^p\Omega^\bullet_X\,[-1] \to \mathbb{Z}(p)_D \to \mathbb{Z}(p) \to 0, \qquad (1-38)$$

where

$$F^p\Omega^\bullet_X = \begin{array}{cccccc} [& \Omega^p_X & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n_X] \\ & | & & & & | \\ & deg\ p & & & & deg\ n \end{array}$$

is the so-called truncated de Rham complex. This truncation gives a decreasing filtration of the holomorphic de Rham complex $\Omega^\bullet_X$, as well as a filtration of any injective resolution $I^{\bullet\bullet}$ of $\Omega^\bullet_X$. The corresponding spectral sequence is the second spectral sequence of the double complex $\Gamma(X, I^{\bullet\bullet})$. This is called the Hodge to de Rham spectral sequence, or the Fröhlicher spectral sequence. Hodge theory says that this spectral sequence degenerates at E_1 and that the corresponding filtration on the abutment $H^\bullet(X, \mathbb{C})$ is the Hodge filtration F^p (see [Del]). It follows that $H^q(X, \Omega^\bullet_X/F^p\Omega^\bullet_X) = H^q(X, \mathbb{C})/F^p\ H^q(X, \mathbb{C})$.

Therefore the exact sequence of complexes of sheaves (1-38) induces the long exact sequence of cohomology groups

$$\begin{aligned} \cdots \to H^{q-1}(X, \mathbb{Z}(p)) \to H^{q-1}(X, \mathbb{C})/F^p\ H^{q-1}(X, \mathbb{C}) \to \\ \to H^q(X, \mathbb{Z}(p)_D) \to H^q(X, \mathbb{Z}(p)) \to \cdots \end{aligned} \qquad (1-39)$$

For algebraic cycles, the crucial Deligne cohomology group is $H^{2p}(X, \mathbb{Z}(p)_D)$. Let $\mathrm{Hdg}^p(X)$ the group of Hodge cohomology classes, i.e., the group of classes γ in $H^{2p}(X, \mathbb{Z}(p))$ such that the image of γ in $H^{2p}(X, \mathbb{C})$ belongs to $F^p\ H^{2p}(X, \mathbb{C})$.

1.5.11. Theorem. *(Deligne)* (1) *There is an exact sequence*

$$0 \to J^p \to H^{2p}(X, \mathbb{Z}(p)_D) \to \mathrm{Hdg}^p(X) \to 0, \qquad (1-40)$$

where

$$J^p = H^{2p-1}(X, \mathbb{C})/F^p\ H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z}(p)) \qquad (1-41)$$

is the Griffiths intermediate jacobian of [**Gri1**].

(2) *Every algebraic subvariety Z of X of pure codimension p has a cohomology class in $H^{2p}(X, \mathbb{Z}(p)_D)$. The image in $H^{2p}(X, \mathbb{Z}(p)_D)$ of this class is the class of Z in Čech cohomology.*

We refer to [**R-S-S**] or [**E-V**] for a proof and for discussions of the geometric significance of this theorem, as well as for the relation with Griffiths' work.

The product $B(p)_D \otimes B(q)_D \to B(p+q)_D$ is constructed in exactly the same way as in the smooth case, so we will not need to repeat the description.

Beilinson [**Be1**] [**Be2**] has developed Deligne cohomology much further and has introduced a version with growth conditions, which is much better suited to non-compact algebraic manifolds. The theory of Beilinson is called *Deligne-Beilinson cohomology.* Beilinson has found regulator maps from algebraic K-theory of algebraic manifolds to their Beilinson-Deligne cohomology groups, and has given conjectures for varieties over $\mathbb{Q}$, expressing values of their Hasse-Weil L-functions in terms of these regulators. This is far from the topic of this book, and so for further information we will refer the reader to the aforementioned articles of Beilinson, the book [**R-S-S**] and the survey article [**Sou**].

1.6. The Leray spectral sequence

In this section we study the spectral sequence invented by J. Leray during the Second World War [**Le**]. Its purpose is to analyze the cohomology of a space Y, with the help of a mapping $f : Y \to X$, in terms of the cohomology of X and that of the fibers $Y_x = f^{-1}(x)$. The spectral sequence is mostly used for fibrations, but not exclusively.

Let A be a sheaf of abelian groups on Y. There is a corresponding