# Lecture 4

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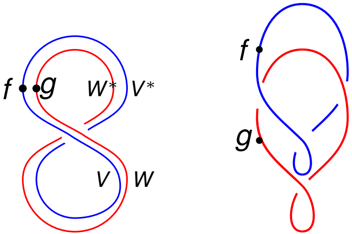


## Tangles

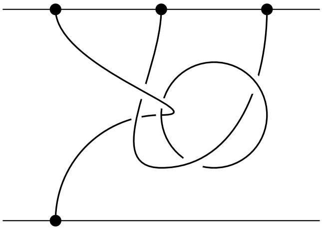
Tangles generalize braids, knots and links. A tangle is a collection of circles and arcs piecewise smoothly immersed in $\mathbb{R}^2 \times [0, 1]$ with endpoints on the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$. Specifically let $m$ and $n$ be given nonnegative integers; we will define a tangle of type $(m, n)$. Let us fix $m$ points in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$: for definiteness

$$m = \{(k, 0, 0) | 1 \leqslant k \leqslant m\}, \qquad \{n = (\ell, 0, 0) | 1 \leqslant \ell \leqslant n\}.$$

These are to be the endpoints of the arcs. We identify two tangles if they are equivalent by an ambient isotopy that fixes the endpoints on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$.

## Tangles form a category

The objects in the tangle category are the nonnegative integers $\mathbb{N}$. We think of an $(m, n)$ tangle as a morphism $m \to n$. We will draw this upside down with the $m$ at the top. Here is a $(3, 1)$ tangle represented by its projection onto the plane.

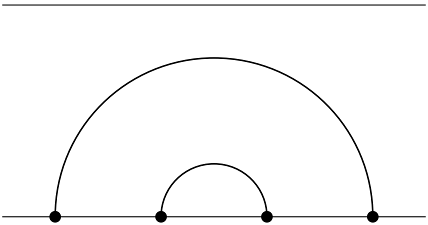


Morphisms may be composed by gluing $(k, 0, 1)$ to $(k, 0, 0)$, then rescaling to fit between the planes $z = 0$ and $z = 1$.

## Tangles form a rigid monoidal category

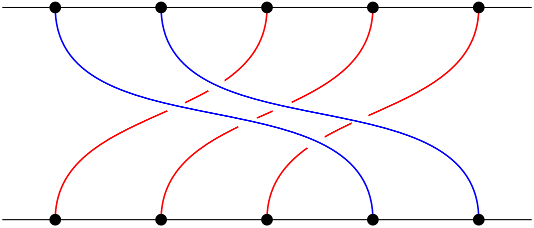
The monoidal structure identifies $m_1$ and $m_2$ with $m_1 + m_2$. Given tangles $T_1$ in $\mathrm{Hom}(m_1, n_1)$ and $T_2$ in $\mathrm{Hom}(m_2, n_2)$, we may juxtapose them to get a tangle in $\mathrm{Hom}(m_1 + m_2, n_1 + n_2)$.

We may even define $m^* = m$ and make the tangle category into a rigid category. Here is the coevaluation map for $m = 2$. It is an object in $\mathrm{Hom}(0, 4) = \mathrm{Hom}(0, 2 \otimes 2^*)$.



# Tangles form a braided category

We may introduce a braiding by specifying morphisms in $\mathrm{Hom}(m \otimes n, n \otimes m)$. Here is the braiding for $m = 2, n = 3$.



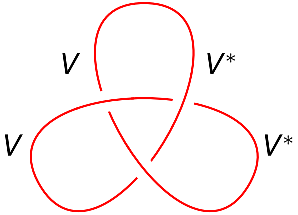
## Framed Tangles

A framed tangle associates to each strand a family of normal vectors. Fattening up the strand in the direction of these normal vectors produces a ribbon.

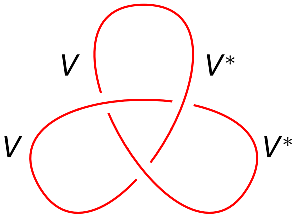
Framed tangles again form a braided monoidal category.

# Knot invariants

We may try to model a knot or link in a rigid braided category. Let us pick a module $V$ in the category. Let $K$ be the unit object in the category. Assume that $V^{**} \cong V$ so that the evaluation morphism $\mathrm{coev}_{V^*} : V^* \otimes V^{**} \to K$ can be regarded as a morphism $V^* \otimes V \to K$. Now we label the strands of a 2-dimensional projection as follow:



# Knot invariants, continued

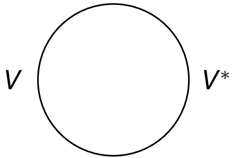



Interpreting the caps and cups as coevaluation and evaluation, this is a morphism $K \to K$. If $K$ happens to be a field, it is a scalar. This approach to knot invariants has some problems, but ultimately can be made to succeed.

## The simplest knot

The simplest knot is an unknotted circle.

$$K \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} K.$$

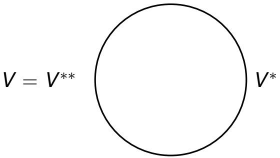



Applying the above mentioned heuristic will expose some of the problems with this plan.

# Dimension

The braided category of finite-dimensional vector spaces over a field $K$ is symmetric: the maps $c_{U,V}$ and $c_{V,U}^{-1} : U \to V$ are equal. We may identify $V$ with its double dual $V^{**}$ and so we a linear map

$$K \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} K.$$

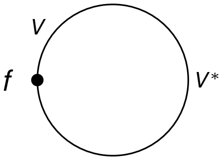



Remember that if $v_i$ and $v_i^*$ are dual bases of $V$ and $V^*$ then $\mathrm{coev}_V(1) = \sum v_i^* \otimes v_i$. From this, this endomorphism of $K$ is the scalar $\dim(V)$.

# Trace

More generally we may include an endomorphism of $V$ and compute its trace.

$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{f \otimes 1_{V^*}} V \otimes V^* \xrightarrow{\text{ev}_{V^*}} K.$$



## The trace is multiplicative

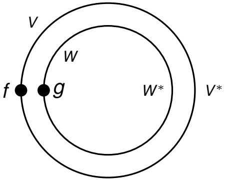
Still working in the symmetric category of vector spaces, if $f : V \to V$ and $g : W \to W$ are endomorphisms then

$$\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g).$$

Here is a graphical proof. Remember,

$$\mathrm{coev}_{V \otimes W} = (1_V \otimes \mathrm{coev}_W \otimes 1_{V^*})\,\mathrm{coev}_W,$$

$$\mathrm{ev}_{(V \otimes W)^*} = \mathrm{ev}_{W^* \otimes V^*} = \mathrm{ev}_{V^*}(1_V \otimes \mathrm{ev}_{W^*} \otimes 1_{V^*})$$

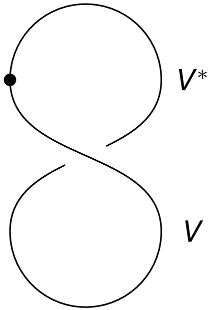



The evaluations $\mathrm{ev}_{V^*}$, $\mathrm{ev}_{W^*}$ may be carried out separately, then multiplied together.

# The trace in a braided rigid category

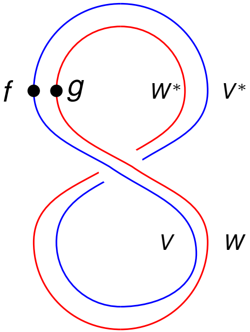
We can try to make a trace in a braided rigid category. We create $V \otimes V^*$ with coev. We have to interchange them before we evaluate:

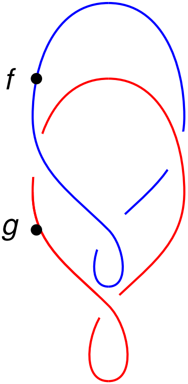
$$K \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{\text{ev}_V} K$$



# This trace is not multiplicative

$f : V \to V, g : W \to W$



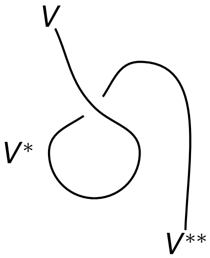




If we try to prove multiplicativity for $\mathrm{tr}(f \otimes g)$ we cannot because the two paths are linked and cannot be separated. This is a sign that we need a new ingredient to make a satisfactory theory.

# Isomorphisms $V \to V^{**}$

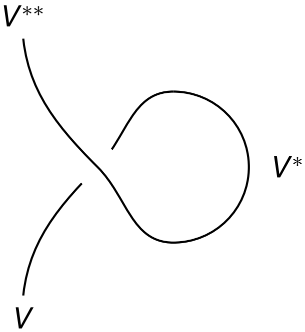
In a rigid braided category $V$ and $V^{**}$ are naturally isomorphic, but there are potentially an infinite number of such natural isomorphisms corresponding to increasingly twisted tangles. The following morphism will be denoted $u_V$:

$$V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$



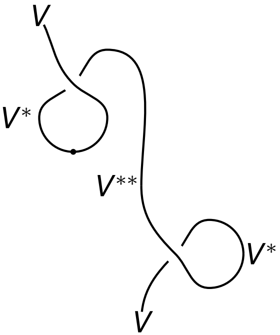
# The inverse of $u_V$

$$V^{**} \xrightarrow{1_{V^{**}} \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^{*} \xrightarrow{c_{V^{**},V} \otimes 1_{V^*}} V \otimes V^{**} \otimes V^{*} \xrightarrow{1_V \otimes \mathrm{ev}_V^{*}} V$$



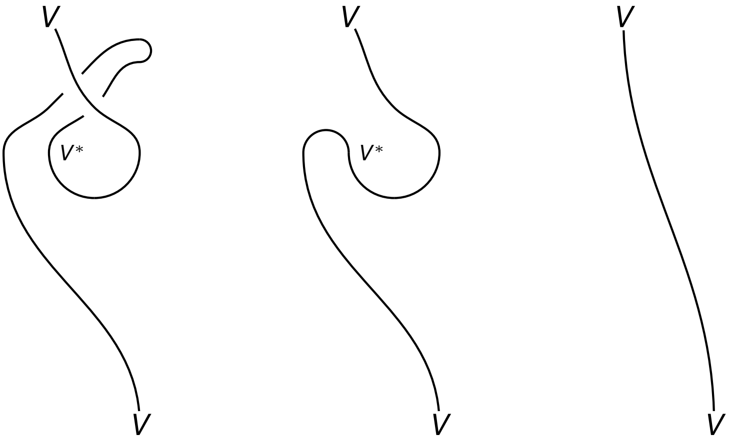
## Checking the inverse

Let us check that with $u_V : V \to V^{**}$ and $u_V^{-1}$ that indeed $u_V^{-1} u_V = 1_V$.




We use the naturality of the second (lower) crossing to move it before (above) the first crossing.

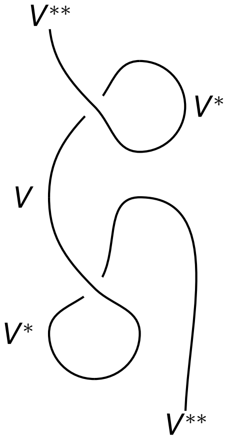
## Checking the inverse (continued)

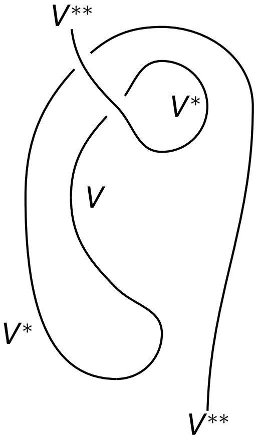



This shows that $u_V^{-1} u_V = 1_V$.

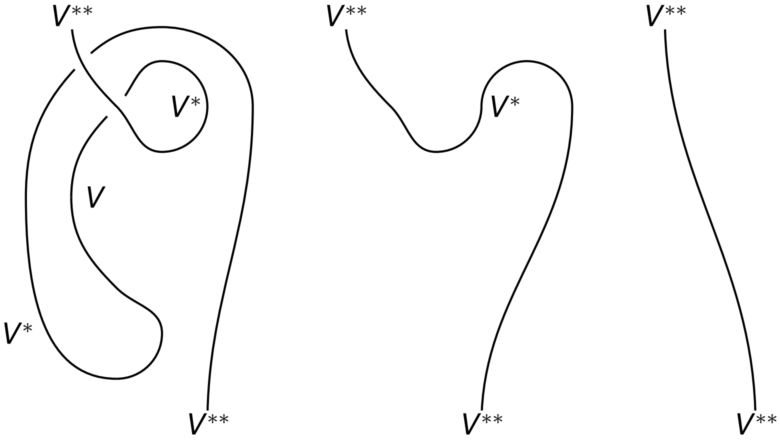
# Checking the inverse (continued)

Now let us show that $u_V u_V^{-1} = 1_V$.



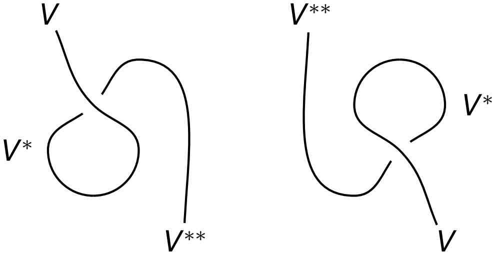
## Checking the inverse (continued)



## Another isomorphism

$$u_V : V \xrightarrow{1_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{c_{V,V^*} \otimes 1_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes 1_{V^{**}}} V^{**}$$

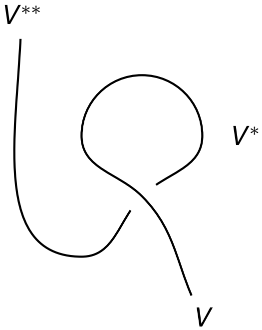
$$v_V : V^{**} \xrightarrow{1_V \otimes \mathrm{coev}_V} V^{**} \otimes V \otimes V^* \xrightarrow{1_{V^{**}} \otimes c_{V,V^*}} V^{**} \otimes V^* \otimes V \xrightarrow{\mathrm{ev}_{V^*} \otimes 1_V} V$$

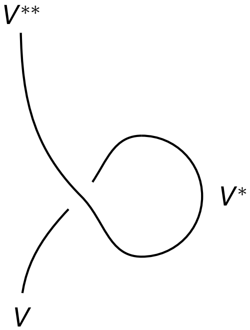



In addition to $u_V$, whose definition we repeat, we will need another isomorphism $v_V : V^{**} \to V$. This is not $u_V^{-1} : V^{**} \to V$ whose definition we have already considered.

# Why are there two isomorphisms

Let us compare $v_V : V^{**} \to V$ with $u_V^{-1} : V^{**} \to V$.



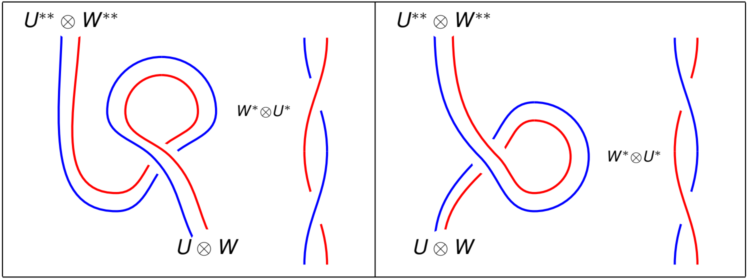




Left: $v_V$. Right: $u_V^{-1}$.

## Why are there two isomorphisms

Let us compare $v_V : V^{**} \to V$ with $u_V^{-1} : V^{**} \to V$.

Let $V = U \otimes W$.

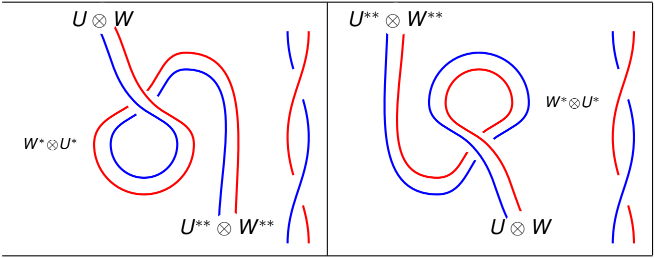



The difference between $v_V$ and $u_V^{-1}$ is made clear if $V = U \otimes W$: it is in the direction of twisting.

## $u_V$ and $v_V$

Both $u_V : V \to V^{**}$ and $v_V : V^{**} \to V$ are counter clockwise $2\pi$ twists. (Our $z$ axis points down and the $y$ axis points away from the viewer).

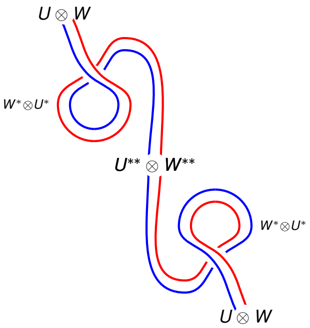
Left: $u_V$.
Right: $v_V$.

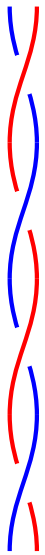



Composing them, $v_V \circ u_V : V \to V$ is a clockwise twist in $4\pi$. We could solve many problems such as the non-multiplicativity of the trace if we had a map $V \to V$ that is a twist in $2\pi$.

## Possible twistings

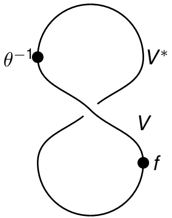
We have used the example $V = U \otimes W$ to show what kinds of twisting we can obtain with the tools we have so far. In a braided rigid category, we can construct morphisms $V \to V$ that twist a multiple of $4\pi$ times.



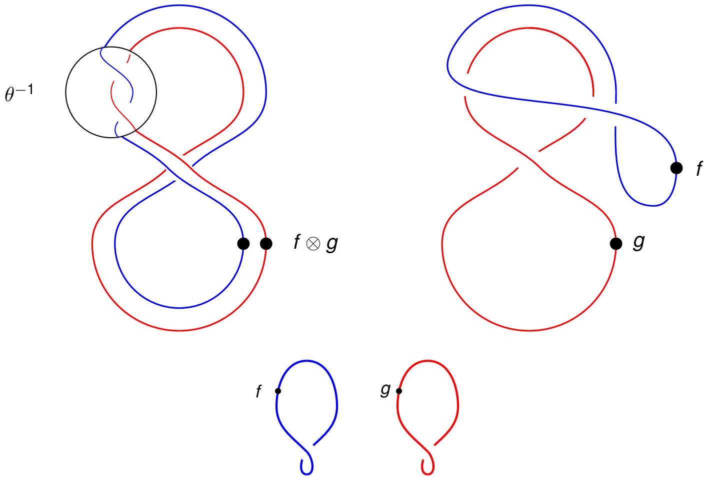
## Motivating the notion of a ribbon category

What we need, however, is a natural morphism $\theta : V \to V$ that twists by $2\pi$. We expect that $\theta^2 = v_V \circ u_V$. With such a morphism in hand, we can construct a multiplicative trace.



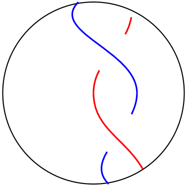

We have not yet give an proper definition of $\theta$ but heuristically show how it solves this problem.

# Multiplicativity of the ribbon trace (informal)

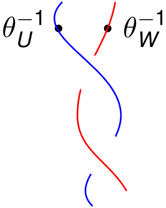


## What did we forget?

We glossed over the following point. The morphism $\theta^{-1}_{U \otimes W}$ isn't actually this:



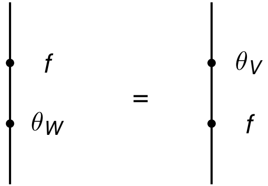
It's this, because $U$ and $W$ themselves are ribbons that can twist:



## Twists

We will now formulate the axioms that the twist $\theta_V$ in a rigid braided category must statisfy. We want a natural isomorphism $\theta_V : V \to V$ for every object in the category satisfying certain axioms. A braided rigid category with a twist is called a ribbon category.

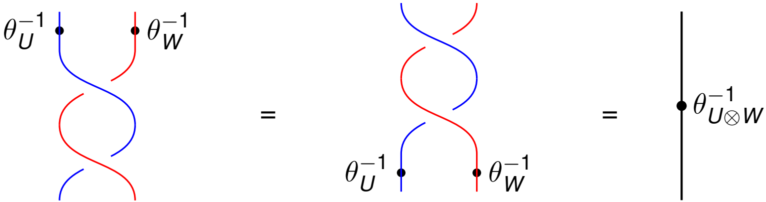
Naturality means if $f : V \to W$ then $\theta_W f = f\theta_V$:



# Ribbon axioms, continued

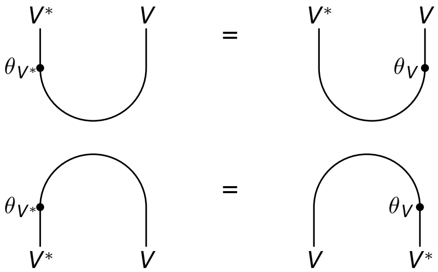
We must have (using naturality of $c_{U,W}$ and $c_{W,U}$:

$$\theta_{U\otimes W}^{-1} = c_{W,U}\circ c_{U,W}\circ \theta_U^{-1}\otimes\theta_W^{-1} = \theta_U^{-1}\otimes\theta_W^{-1}\circ c_{W,U}\circ c_{U,W}$$



## Ribbon axioms, concluded

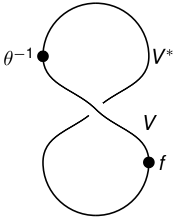
Finally it is necessary to assume that if $I$ is the unit object in the category then $\theta_I = 1_I$, and that $\theta_{V^*} = \theta_V^*$. This last axiom means a compatibility with evaluation and coevalution:




Example: The category of framed tangles is a ribbon category.

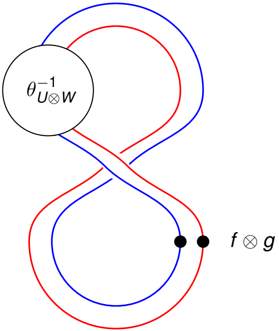
# The trace in a ribbon category

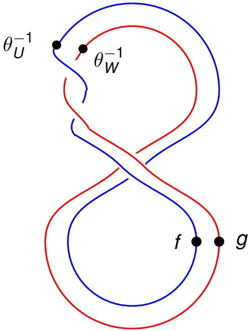
The definition of a ribbon category contains all we need to define a multiplicative trace. It is an endomorphism of $I$, which in a category of vector spaces means a scalar.



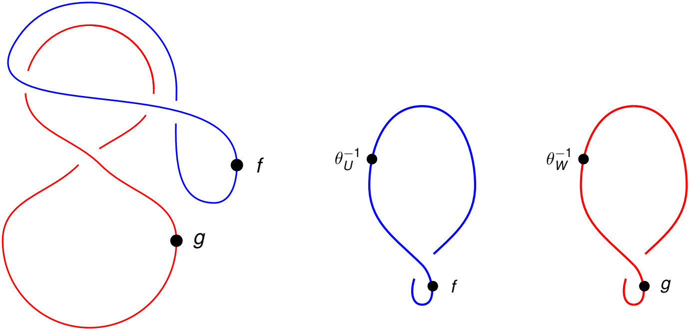
# Multiplicativity of the ribbon trace (formal)

Let $f : U \to U$ and $g : W \to W$.



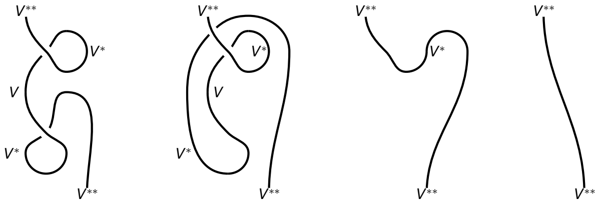
# Multiplicativity of the ribbon trace (continued)




This proves $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$.

## Exercises

Exercise 1. In the slides called Checking the inverse we proved that $u_V$ and $u_V^{-1}$ as we defined it were inverses by the following manipulations.




Explain carefully the justification of each step.

Exercise 2. What is the inverse of $v_V$?

Exercise 3. Prove that $\theta_{V^*} = \theta_V^*$ implies the $\mathrm{ev}_V$ and $\mathrm{coev}_V$ compatibilities under Ribbon axioms concluded.