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I N D E X

TABLE OF CONTENTS

Chapter I . DIAGRAMMATIC CATEGORIES IN RELATION TO THE CATEGORY OF SETS

Chapter II . THE THEORY OF REGULAR CATEGORIES AND AN ABSTRACT CHARACTERIZATION OF DIAGRAMMATIC CATEGORIES

Chapter III . ISOMORPHISMS OF DIAGRAMMATIC CATEGORIES

Chapter IV . ALGEBRAIC ASPECTS OF DIAGRAMMATIC CATEGORIES

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P R E F A C E

The theory of categories was introduced by Eilenberg and Mac Lane in 1945 [4] ; it arose from the field of topology. It was soon realized that other mathematical theories as well could profit from their invention. This was initially the main reason for the increasing interest in categories. The applications brought soon attention to problems peculiar to the theory of categories, which in a few years grew enough to become another area of mathematics. Even so, the now widespread interest in category theory seems still to lie in the many virtues of its applications, such as its unifying character, elegant and concise language, fruitfulness and emphasis on results involving structure. This led to the idea that category theory might provide a more suitable foundation for mathematics than set theory. To carry out this program it was necessary to have also a theory of the (meta)category of categories. Lawvere [17] has recently provided such a theory; this seems to be the proper framework in which to develop mathematics on a categorical basis.

An important step in the program of categorizing mathematics has been accomplished by Lawvere himself [16] upon reformulating set theory in terms of categorical concepts alone, namely, those of mapping, domain, codomain and composition.

In this paper we study a class of categories closely related to the category of sets and mappings. An essential prerequisite will be an acquaintance with [16] . To study this class of categories we introduce what we call regular categories, which are weakened abelian categories , especially as axiomatized by Freyd [8] , so that [8] is also assumed as a prerequisite. A general knowledge of category theory is required as well. Among the various sources, Freyd [8] , Mac Lane [22] and Mitchell [23] seem to be the more introductory ones. Also, an acquaintance with the literature on adjoint functors, starting with Kan [13] and following with several others , e.g., Freyd [6 , 8] , Lawvere [14] , will be assumed.

The formation of functor categories is one of the basic constructions in the (meta)category of categories. Given any two categories $\mathcal{X}$ and $\mathcal{Y}$, the functor category denoted by $\mathcal{Y}^{\mathcal{X}}$ has as objects all functors with domain $\mathcal{X}$ and codomain $\mathcal{Y}$ and as maps, all natural transformations between these. We will be concerned in this paper with a special type of functor categories : those for which the codomain categofy is $\mathcal{S}$, the category of sets and mappings.

A motivation for this choice can be found in the following ; any category with small hom-sets is a full subcategory of a category of this type. Explicitly : if the category $\mathcal{X}$ has small hom-sets, there is a bifunctor HOM : $\mathcal{X}^* \times \mathcal{X} \longrightarrow \mathcal{S}$, which induces by exponential adjointness a functor H : $\mathcal{X} \longrightarrow \mathcal{S}^{\mathcal{X}^*}$. The latter is full, faithful and preserves all left roots existing in $\mathcal{X}$: it is called the regular representation of $\mathcal{X}$.

However, if $\mathcal{X}$ is not small, then $\mathcal{S}^{\mathcal{X}^*}$ will not have small hom-sets, and thus a not very manageable category. Fortunately there are many interesting categories which, though not small admit a regular representation into a category with small hom-sets. These are categories which have a small subcategory , let $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{X}$ be the inclusion functor, and such that the composite functor

$$\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathbb{A}^*}$$

is still full and faithful. The functor is called the subregular representation of $\mathcal{X}$ over $\mathbb{A}$, and $\mathbb{A}$ is said to be an adequate subcategory of $\mathcal{X}$. Therefore, if we restrict ourselves - as we will- to the study of categories of set valued functors with small domain category, the class of categories admitting a representation as full subcategories of these does not reduce to the class of small categories. The broader class of categories with adequate subcategories are investigated by Isbell [12] and it includes, e.g., every algebraic category in the sense of Lawvere [14 , 15] : in this case, the dual of the corresponding algebraic theory is canonically embedded as an adequate subcategory.

Every category whose objects are all set valued functors with a given small domain category is seen to be equivalent to a category of diagrams in $\mathcal{S}$ with a given diagram scheme (Grothendieck [10], Mac Lane [21] , Mitchell [23]). This suggests the name "diagrammatic" or "$\mathcal{S}$ - diagrammatic" for these categories. We adopt throughout this paper the name "diagrammatic" for any category of the form $\mathcal{S}^{\mathbb{C}}$, with $\mathbb{C}$ any small category.

In chapter I we study diagrammatic categories in general, simultaneously comparing them with $\mathcal{S}$, which is the basic diagrammatic category.

The aim of chapter II is to characterize abstractly the class of diagrammatic categories. We first introduce the theory of regular categoris, the name being suggested by a consequence of the axioms according to which every map factors uniquely into an epi followed by a mono, and which is usually called a regularity condition. It is strong enough to exclude most algebraic categories, and those which satisfy a regularity condition are called regular. All diagrammatic categories are regular, and they are by no means the only regular categories : all abelian categories are regular as well, and none is diagrammatic. Therefore, if we hope to characterize diagrammatic categories from regular categories, the strenghtening of the axioms has to be done in a different way than abelianess.

At this point we notice a striking analogy between the regular representation theorem for any category with a small adequate subcategory, and the representation theorem for Boolean algebras which says that every Boolean algebra is isomorphic to a field of sets. Thus, if we let regular categories with small adequate subcategories correspond to Boolean algebras, then regular categories of set-valued functors with a small domain category (not necessarily all such functors) must correspond to fields of sets if the analogy between the two theorems is to be mantained. Also, fields of all subsets of a set must correspond to diagrammatic categories. It is now that the analogy gives some fruits : since the fields of all subsets of some set are precisely the complete atomic Boolean algebras, we might try an analogous characterization of diagrammatic categories. With the analogy in mind, we first stipulate which objects in a regular category should be called "atoms" , and with this, when should a regular category be called "atomic" . It turns out that complete atomic regular categories have the atoms as an adequate subcategory, so that the existence of a small adequate subcategory need not be postulated before. And what is more important, complete atomic regular categories are precisely the diagrammatic categories. That is, just as any complete atomic Boolean algebra is isomorphic to the field of all subsets of the set of its atoms, so any complete (right-complete is enough) atomic regular category is isomorphic to the diagrammatic category with domain category the dual of the full subcategory determined by its atoms.

In chapter III we aim at the question of when are isomorphic any two given diagrammatic categories, which is the same question that Morita [24] asked for categories of modules (see also Bass [2]). For this purpose we first study functors between diagrammatic categories which have adjoint or coadjoint. Our results can also be found in André [1] , though the methods of proof are different , as a result of dispensing with generality from our side. Next, we use these results to establish, as Freyd noticed in [7 , 8] , that it is not the small domain category which determines completely the functor category (in his case these were categories of additive group-valued functors) but its amenable closure. The main theorem of the chapter is called "Morita isomorphism theorem for diagrammatic categories" and states that any two given diagrammatic categories are isomorphic iff the idempotent-splitting closures of the corresponding small domain categories are isomorphic. This is used to investigate the question of the uniqueness of the representation of a category as a diagrammatic category.

Chapter IV is a study of the algebraic side of every algebraic category. For this we need the theory of triples and triplable categories as introduced and developped by Huber, Beck, and Eilenberg and Moore. To avoid further requirements, we review briefly the ideas employed in the chapter. We next discuss some relations between triples and cotriples which form an adjoint pair as well, and use this information to find out which are all coadjoint triples in $\mathcal{S}$. They are given by all sets, so that Coadj Triples($\mathcal{S}$) $\cong \mathcal{S}^*$, since the correspondence is contravariantly functorial. On the other hand, adjoint triples on $\mathcal{S}$ are given by monoids. Similar questions arise for categories of the form $\mathcal{S}^I$, with I a set , regarded as a discrete category. Adjoint triples on a category $\mathcal{S}^I$, are given by all small categories whose set of objects are isomorphic to I . And the diagrammatic categories with these small domain categories come close to being the algebras of the triple. Actually, to see better which are the algebras, we introduced the notions of relative category and relative functor. These ideas have further potentialities which are beyond the scope of this paper.

Some notations and conventions are the following : (1) small categories will be denoted by $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$; (2) arbitrary categories will be denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$; (3) $\mathcal{S}$ will always denote the category of sets ; (4) the small categories which are preorders will be denoted by $\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{3}, \dots$; (5) small categories which are discrete will be denoted the same way as sets are , by I , J , K , etc. ; (6) $\mathbb{E}$ is the category pictured thus :  ; (7) the set of objects of a small category $\mathbb{C}$, will be denoted $|\mathbb{C}|$; (8) the dual of any category $\mathcal{A}$ will be denoted $\mathcal{A}^*$; (9) composition of maps is denoted in the diagrammatic order , and evaluation is on the left ; (10) the identity map of the object A is either 1_A or A .

Chapter I

DIAGRAMMATIC CATEGORIES IN RELATION TO THE CATEGORY OF SETS

Let $\mathbb{C}$ be a fixed but arbitrary small category. We denote by $\mathcal{S}$ the category of sets and mappings, and by $\mathcal{S}^{\mathbb{C}}$ the category whose objects are all covariant functors $\mathbb{C} \longrightarrow \mathcal{S}$ and whose maps are all natural transformations between these. For reasons given in the Preface, any such category will be said to be <u>diagrammatic</u>. Our aims in this chapter are: (1) to describe properties which are common to all diagrammatic categories; (2) to determine the extent to which these properties rely on properties of $\mathcal{S}$; (3) to investigate the range of validity in the class of diagrammatic categories of the axioms of Lawvere's elementary theory of $\mathcal{S}$.

§ 1 - FINITE ROOTS

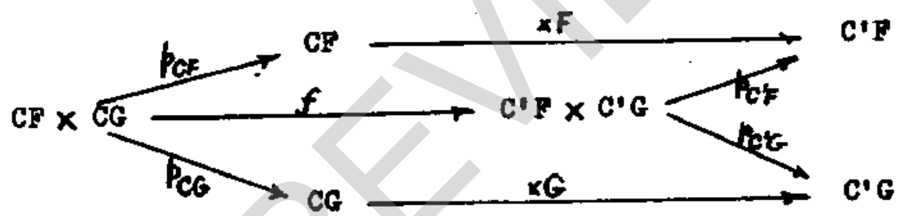
A category $\mathcal{X}$ is said to have <u>finite roots</u> iff for every small category such that its set of objects is finite, and letting $\mathbb{A}$ be one such, the functor $\mathcal{X} \longrightarrow \mathcal{X}^{\mathbb{A}}$ induced by the functor $\mathbb{A} \longrightarrow \mathbb{1}$, has both a coadjoint (insuring the existence of left roots) and an adjoint (right finite roots). It has been shown ([8] , [14]) that it is enough that the category has terminal and coterminal objects ($\mathbb{A} \cong \mathbb{0}$) , binary products and coproducts ($\mathbb{A} \cong |2|$) and equalizers and coequalizers ($\mathbb{A} \cong \mathbb{E}$) for it to have all finite roots. Among the finite roots are finite products and coproducts, pull-backs and push-outs, images and inverse images, unions and intersections. We now show that any diagrammatic category has finite roots.

<u>Proposition 1.1</u> For any small category $\mathbb{C}$, $\mathcal{S}^{\mathbb{C}}$ has finite roots.

Proof:

A terminal object for $\mathcal{S}^{\mathbb{C}}$ is given by the functor which is constantly 1 , where 1 is the name for the terminal object in $\mathcal{S}$. A coterminal object is given dually and denoted 0 .

Given any two functors F and G we define $(F \times G\ ,\ p_F\ ,\ p_G)$ as follows: let $C(F \times G) = CF \times CG$; $(p_F)_C = p_{CF}$ and $(p_G)_C = p_{CG}$, for any $C \in |\mathbb{C}|$. If $C \xrightarrow{x} C'$ is a map in $\mathbb{C}$, let $x(F \times G) = f$ where f is the unique map which renders commutative the following diagram:



By the way $x(F \times G)$ is defined, this says not only that $F \times G$ is a functor, but also that $p_F : F \times G \longrightarrow F$ and $p_G : F \times G \longrightarrow G$ are natural transformations. Dually one can define the coproduct $F + G$ together with the canonical injections i_F and i_G .

Given any two natural transformations η and ξ , we want to define their equalizer. For this, we look again in each coordinate , and let $e_C = \mathrm{Eq}\,(\eta_C, \xi_C)$ for each $C \in |\mathbb{C}|$. We show next that the family so obtained can be made into a natural transformation e which moreover is the equalizer of η and ξ . For this we first define a functor , the domain of e as follows : let $CE = E_C$ where $E_C \xrightarrow{e_C} CF \overset{\eta_C \cdot \xi_C}{\rightrightarrows} CG$

is an equalizer diagram. If $C \xrightarrow{x} C'$ is a map in $\mathbb{C}$, let xE be defined as the unique map $f : CE \longrightarrow C'E$ such that $fe_{C'} = e_C(xF)$. That this map f exists and is unique follows from the universal property of equalizers together with the following identity:

$$(e_C(xF))\eta_{C'} = e_C((xF)\eta_{C'}) = e_C(\eta_C(xG)) = (e_C\eta_C)(xG) = (e_C\xi_C)xG =$$
$$= e_C(\xi_C(xG)) = e_C((xF)\xi_{C'}) = (e_C(xF))\xi_{C'} .$$

With this we have that E is a functor and $e : E \longrightarrow F$ a natural transformation and it is immediate to see that it is the equalizer of η and ξ. Coequalizers are dually defined. QED .

§ 2 - THE EXISTENCE OF A GENERATING FAMILY

In $\mathcal{S}$, the terminal object 1 is a generator. Arbitrary diagrammatic categories need not have a generator, but they always have a generating family of objects. We will show that the generating property of a particular generating family in each diagrammatic category is a consequence of the generating property of 1 in $\mathcal{S}$.

As usual, a functor is said to be <u>representable</u> and denoted by H^C if it is $C \in |\mathbb{C}|$ which represents it, iff it is naturally equivalent to the functor HOM(C,). The family of representable functors in any diagrammatic category has the size of the domain category for the functors. We want to show that it is generating, for which purpose we need to state and prove (for reference) a lemma due to Yoneda.

<u>Lemma 2.1</u> (Yoneda) For any small $\mathbb{C}$, any F in $\mathcal{S}^{\mathbb{C}}$, and any

$$C \in |\mathbb{C}|, \ (H^C, F)\text{nat} \cong CF \cong \mathrm{HOM}_{\mathcal{S}}(1, CF)$$

Proof:

Let $\phi:(H^C, F)\longrightarrow CF$ be defined for $\eta\in(H^C, F)$ by $\eta\phi = 1_C\eta_C \in CF$

Let $\psi: CF\longrightarrow(H^C, F)$ be defined for $z\in CF$ as the natural transformation $z\psi: H^C\longrightarrow F$ defined for $x\in C'H^C = HOM(C,C')$ by $x(z\psi)_{C'} = z(xF)$ and naturality follows since for any $C'\xrightarrow{y}C''$ the following diagram commutes:

$$\begin{array}{ccc} C'H^C & \xrightarrow{(z\psi)_{C'}} & C'F \\ {\scriptstyle yH^C}\downarrow & & \downarrow{\scriptstyle yF} \\ C''H^C & \xrightarrow{(z\psi)_{C''}} & C''F \end{array}$$

That it is so can be seen as follows: let $x\in HOM(C,C')$, arbitrary. Then we have that $x(z\psi)_{C'}(yF) = (x(z\psi)_{C'})(yF) = (z(xF)(yF)) =$ $= z((xy)F) = (xy)(z\psi)_{C''} = (x(yH^C))(z\psi)_{C''} = x((yH^C)(z\psi)_{C''})$.

It is now easy to verify that both $\phi\psi$ and $\psi\phi$ are identities. QED .

<u>Theorem 2.2</u> For any small $\mathbb{C}$, the family $\{H^C\}_{C\in|\mathbb{C}|}$ is generating for $\mathcal{S}^{\mathbb{C}}$.

Proof:

Given any two natural transformations $F\xrightarrow{\eta,\xi}G$ such that they are different, there must exist at least a $C\in|\mathbb{C}|$ for which $\eta_C\neq\xi_C$. This implies that there exists a map $1\xrightarrow{z}CF$ in $\mathcal{S}$, such that $z\eta_C\neq z\xi_C$. By Yoneda, let $z\psi: H^C\longrightarrow F$ be the corresponding natural transformation. We want to show that $(z\psi)\eta\neq(z\psi)\xi$. This will be so iff $\exists\, C'\in|\mathbb{C}|$ such that $(z\psi)_{C'}\eta_{C'}\neq(z\psi)_{C'}\xi_{C'}$. Take $C' = C$. For $(z\psi)_C\eta_C$ to be different from $(z\psi)_C\xi_C$ it is enough that there exists $x\in HOM(C,C)$ for which $x(z\psi)_C\eta_C$ be different from $x(z\psi)_C \xi_C$. Let $x = 1_C$, then we have that

$$(1_C (z\psi)_C)\eta_C = (z(1_C F))\eta_C = z\eta_C \neq z\xi_C = (z(1_C F))\xi_C = (1_C(z\psi)_C)\xi_C$$

which implies the desired result. QED .

§ 3 - EXPONENTIATION

A category with products is said to have <u>exponentiation</u> iff for any object A the functor $A \times (\ )$ has a coadjoint, denoted $(\ )^A$. The category of sets has exponentiation and for every set A, we have that $(\ )^A = \mathrm{HOM}(A,\ )$. However, $\mathcal{S}$ is the only category in which exponentiation is given by HOM , precisely because $(\ )^A$ has to be an endofunctor while the only category for which $\mathrm{HOM}(A,\ )$ is an endofunctor for every object A, is $\mathcal{S}$. All diagrammatic categories have exponentiation. However, the proof that it is so is not straightforward as the proof of the existence of finite roots was, and this is so because exponentiation is not defined coordinatewisely.

<u>Theorem 3.1</u> For any small $\mathbb{C}$, and any object F in $\mathcal{S}^{\mathbb{C}}$, the endofunctor $F \times (\ ) : \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ has a coadjoint.

Proof:

Define a functor $(\ )^F : \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ as follows:

if G is any object of $\mathcal{S}^{\mathbb{C}}$, let the value at $C \in |\mathbb{C}|$ of G^F be given by

$$C\,G^F = (H^C \times F,\ G)_{nat}$$

and extend it to the maps $C \longrightarrow C'$ in the obvious fashion so that it becomes a functor. We can now define a natural transformation

$$F \times G^F \xrightarrow{\ ev\ } G$$

called <u>evaluation</u>, as follows: given $C \in |\mathbb{C}|$ one has to say what is $ev_C : CF \times C(G^F) \longrightarrow CG$, that is, $ev_C : CF \times (H^C \times F, G) \longrightarrow CG$

If $z \in CF$ and $\eta \in (H^C \times F, G)$, define $(z, \eta)ev_C = (1_C, z)\eta_C$.

If $C \xrightarrow{x} C'$, there are induced maps $CF \xrightarrow{xF} C'F$ and $((H^x \times F), G) : (H^C \times F, G) \longrightarrow (H^{C'} \times F, G)$ and these two induce $xF \times ((H^x \times F), G) : CF \times (H^C \times F, G) \longrightarrow C'F \times (H^{C'} \times F, G)$,

and the following diagram is commutative:

$$\begin{array}{ccc} CF \times (H^C \times F, G) & \xrightarrow{\;ev_C\;} & CG \\ {\scriptstyle xF \times ((H^x \times F), G)}\big\downarrow & & \big\downarrow{\scriptstyle xG} \\ C'F \times (H^{C'} \times F, G) & \xrightarrow{\;ev_{C'}\;} & C'G \end{array} \qquad (*)$$

To see that the diagram is commutative we take any $z \in CF$ and any $\eta \in (H^C \times F, G)$, and travel in the two orientations. We have

$(z, \eta)\, ev_C\, (xG) = (1_C, z)\, \eta_C(xG)$ and

$(z, \eta)(xF \times ((H^x \times F), G))ev_{C'} = (z(xF), (H^x \times F)\eta)ev_{C'} =$

$= (1_{C'}, z(xF)((H^x \times F)\eta)_{C'} = (x, z(xF))\eta_{C'}$.

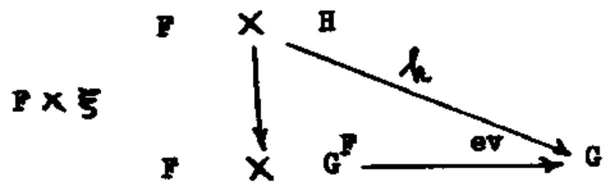
We now use the fact that η is a natural transformation, so that the following diagram commutes:

$$\begin{array}{ccc} CH^C \times CF & \xrightarrow{\;\eta_C\;} & CG \\ {\scriptstyle H^x \times xF}\big\downarrow & & \big\downarrow{\scriptstyle xG} \\ C'H^C \times C'F & \xrightarrow{\;\eta_{C'}\;} & C'G \end{array}$$

and so, for $1_C \in CH^C$ and $z \in xF$, this says precisely that $(1_C, z)\eta_C(xG) = (1_C, z)(H^x \times xF)\eta_{C'} = (x, z(xF))\eta_{C'}$ so that (*) above is commutative, and so evaluation is indeed a natural transformation.

We still have to show that $(\;)^F$ is coadjoint to $F \times (\;\;)$, and it is for this purpose that we will use the evaluation map just defined.

Suppose given any functor H and a natural transformation $h: F \times H \longrightarrow G$ to show that there exists a unique natural transformation $\xi : H \longrightarrow G^F$ such that $(F \times \xi)ev = h$, i.e., such that the following diagram is commutative:



Let be given for each $C \in |\mathbb{C}|$ as follows: if $y \in CH$,
let $y(\xi_C) \in (H^C \times F, G)$ be given by , for $x' \in C'H^C$ and $z' \in C'F$
let $(x', z')(y\,\xi_C)_{C'} = (z'(x'F), y)h.$

We verify now that $(F \times \xi)ev = h$: given $C \in |\mathbb{C}|$, $z \in CF$ and $y \in CH$ then $((z, y)(F \times \xi))ev_C = (z, y\xi_C)ev_C = (1_{C'}, z)(y\xi_C)_C = (z(1_C F), y)h =$ $= (z, y)h$. The definition of ξ was forced to make the diagram commute and it is easy to see that it is the only possible choice. QED .

A functor which has a coadjoint preserves all right roots that exist, so that the existence of exponentiation for any diagrammatic category implies that products distribute over coproducts and that products preserve coequalizers.

It is known that if $\mathbb{C}$ is any small category, the regular representation functor $H : \mathbb{C} \longrightarrow \mathcal{S}^{\mathbb{C}^*}$ defined by $CH = HOM(\ , C)$, is full and faithful and preserves all left roots which might exist in $\mathbb{C}$. In fact, if $\mathcal{X}$ is not small, but has a small adequate subcategory (Isbell [12]) $\mathbb{A}$, the subregular representation functor of $\mathcal{X}$ over $\mathbb{A}$, which is just the composition $\mathcal{X} \xrightarrow{H} \mathcal{S}^{\mathcal{X}^*} \xrightarrow{\mathcal{S}^{j^*}} \mathcal{S}^{\mathbb{A}^*}$ is by definition, full and faithful and it preserves left roots since each of the composite functors does.

What is not known is that if exponentiation exists , then the regular representation functor or the subregular representation functor preserve it. We prove two separate theorems to that effect:

<u>Theorem 3.2</u> Let $\mathbb{C}$ be small and with exponentiation. Then, the regular representation functor $H: \mathbb{C} \longrightarrow \mathcal{S}^{\mathbb{C}^*}$ preserves exponentiation.

Proof:

Let A and B be objects in $\mathbb{C}$, we have to show that

$$H(B^A) = (B^A)H \cong BH^{AH} = H_B{}^{H_A}$$

By definition, given $C \in |\mathbb{C}|$ $C\,H(B^A) = HOM\,(C, B^A)$ and

$C\,(H_B{}^{H_A}) = (H_C \times H_A\,,\, H_B) \cong (H_{C \times A}\,,\, H_B) \cong HOM\,(\,C \times A\,,\, B)$

And since $\mathbb{C}$ is assumed to have exponentiation , we have that

$HOM\,(C \times A,\, B\,) \cong HOM\,(\,C\,,\, B^A\,)$ which finishes the proof. QED .

<u>Theorem 3.3</u> Let $\mathfrak{X}$ be any category and let $\mathbb{A}$ be an adequate subcategory of $\mathfrak{X}$. Then, if $\mathfrak{X}$ has exponentiation, the subregular representation of $\mathfrak{X}$ over $\mathbb{A}$, that is, the functor

$$\mathfrak{X} \xrightarrow{\quad H \quad} \mathcal{S}^{\mathfrak{X}^*} \xrightarrow{\quad \mathcal{S}^{j^*} \quad} \mathcal{S}^{\mathbb{A}^*}$$

preserves exponentiation.

Proof:

Let X and Y be any two objects in $\mathfrak{X}$. We have to show that

$Y^X\,(H\mathcal{S}^{j^*}) \cong (Y(H\mathcal{S}^{j^*}))^{(X(H\mathcal{S}^{j^*}))}$ Let $A \in |\mathbb{A}|$, arbitrary.

Onthe one hand, $A\,(Y^X(H\mathcal{S}^{j^*})) = A(H_{(Y^X)}\mathcal{S}^{j^*}) = A\,j^*\,H_{(Y^X)} = HOM\,(Aj^*,\, Y^X\,)$.

On the other hand we have:

$A(YH\mathcal{S}^{j^*})^{(XH\mathcal{S}^{j^*})} = (H_A \times j^*\,H_Y\,,\, j^*\,H_X) \cong (j^*\,H_{Aj^*} \times j^*\,H_Y\,,\, j^*\,H_X) \cong$

$\cong j^*(H_{Aj^*} \times H_Y, H_X) \cong j^*(H_{Aj^* \times Y}, H_X) \cong \mathrm{HOM}(Aj^* \times Y, X) =$

$\cong \mathrm{HOM}(Aj^*, Y^X)$. QED .

§ 4 - AUTONOMY

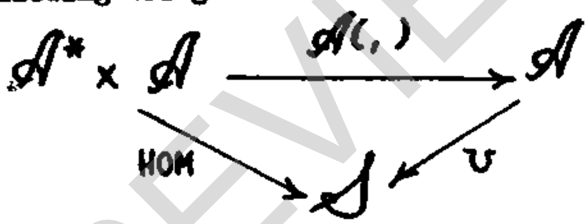
An <u>autonomous</u> category (Linton [18]) is a category $\mathcal{A}$ together with a bifunctor

$$\mathcal{A}(\ ,\ ) : \mathcal{A}^* \times \mathcal{A} \longrightarrow \mathcal{A}$$

and a forgetful functor

$$U : \mathcal{A} \longrightarrow \mathcal{S}$$

such that the following tringle is commutative:



Moreover, there is a law of composition for $\mathcal{A}(\ ,\ )$, which is given by a collection of maps , one for each triple (A,B,C) of objects in $\mathcal{A}$, and which is natural in each of the three variables, it is associative and behaves well with respect to a ground object if there is any. The domain and range of the maps are

$$L^A_{B,C} : \mathcal{A}(B, C) \longrightarrow \mathcal{A}(\mathcal{A}(A, B), \mathcal{A}(A, C))$$

With the above one can introduce "tensor products" as follows: let $L^A : \mathcal{A} \longrightarrow \mathcal{A}$ be defined by $B L^A = \mathcal{A}(A, B)$, for any A and B in $\mathcal{A}$. Given A and B , consider L^A and L^B . If we assume that the composition $L^A L^B$ is representable, and denoting the objects which represents it by $A \otimes B$, we have that