Infinitesimal Higher Symmetries and Higher Connections

Severin Bunk
joint with L. Müller, J. Nuiten, and R. J. Szabo
[ArXiv: to appear]
• Principal bundles, parallel transport and connections
• Higher principal bundles
• Higher connections as infinitesimal symmetries
• Case study: connections on $n$-gerbes
The classical story

Principal bundles, parallel transport and connections
Fix a finite-dimensional smooth manifold $M$.

**Definition (Principal $G$-bundle; global)**

Let $G$ be a Lie group. A principal $G$-bundle over $M$ is a fibre bundle $\pi: P \to M$, together with a smooth $G$-action on $P$ which preserves the fibres and satisfies that following map is a diffeomorphism:

$$P \times G \to P \times_M P, \quad (p, g) \mapsto (p, pg).$$

**Definition (Principal $G$-bundle; local)**

Let $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ be a good open covering of $M$. A principal $G$-bundle is a family of smooth maps $g_{ab}: U_{ab} \to G$ satisfying the Čech 1-cocycle condition,

$$g_{ab} g_{bc} = g_{ac} \quad \text{(restricted to } U_{abc}) \quad \forall a, b, c \in \Lambda, \quad g_{aa} = 1.$$
Morphisms of principal bundles

**Definition (Morphisms of principal $G$-bundles; global)**

Let $P, Q$ be principal $G$-bundles over $M$. A morphism $P \to Q$ is a smooth map $f: P \to Q$ which preserves fibres and the $G$-action.

**Definition (Morphisms of principal $G$-bundles; local)**

Let $(g_{ab})$ and $(g'_{ab})$ define two principal $G$-bundles (over the same cover, for simplicity). A morphism $(g_{ab}) \to (g'_{ab})$ is a collection of smooth maps $h_a: U_a \to G$ which are a Čech coboundary:

$$h_b g_{ab} = g'_{ab} h_a, \quad \forall a, b \in \Lambda.$$

**Remark:** Any morphism of principal $G$-bundles is an isomorphism; we obtain a groupoid $\mathcal{B}un(M; G)$.  ```
### Definition (Parallel transport)

Let $P \to M$ be a principal $G$-bundle. A parallel transport on $P$ is assignment as follows: to each smooth path $\gamma: [0, 1] \to M$, we assign a diffeomorphism $\text{PT}_\gamma: P_{\gamma(0)} \to P_{\gamma(1)}$, such that

1. $\text{PT}_\gamma$ preserves the $G$-action,
2. $\text{PT}_{\gamma'} \circ \text{PT}_\gamma = \text{PT}_{\gamma' \circ \gamma}$,
3. $\text{PT}_\gamma = \text{id}$ whenever $\gamma$ is a constant path,
4. $\text{PT}_\gamma$ depends smoothly on $\gamma$,
5. $\text{PT}_\gamma = \text{PT}_{\gamma'}$ whenever $\gamma$ and $\gamma'$ are thin-homotopic.

---

**Proposition** A parallel transport is flat if and only if it is invariant under all homotopies $h: \gamma_0 \to \gamma_1$.

**Relevance:** Gauge theory; general relativity (geodesics); Aharonov-Bohm effect; machine learning; ...
Parallel transport

Definition (Parallel transport)

Let $P \to M$ be a principal $G$-bundle. A parallel transport on $P$ is assignment as follows: to each smooth path $\gamma : [0, 1] \to M$, we assign a diffeomorphism $PT_\gamma : P|_{\gamma(0)} \to P|_{\gamma(1)}$, such that

1. $PT_\gamma$ preserves the $G$-action,
2. $PT_{\gamma'} \circ PT_\gamma = PT_{\gamma' \circ \gamma}$,
3. $PT_\gamma = id$ whenever $\gamma$ is a constant path,
4. $PT_\gamma$ depends smoothly on $\gamma$,
5. $PT_\gamma = PT_{\gamma'}$ whenever $\gamma$ and $\gamma'$ are thin-homotopic.

Proposition

A parallel transport is flat if and only if it is invariant under all homotopies $h: \gamma_0 \to \gamma_1$.

Relevance: Gauge theory; general relativity (geodesics); Aharonov-Bohm effect; machine learning; ...
A parallel transport is a rule for comparing fibres of $P \to M$ over different points.

**Goal now:** **Infinitesimal** parallel transport.

- Locally, we can trivialise $P$ to $P|_U \cong U \times G \xrightarrow{\text{Pr}} U$.
- PT within $U$ then assigns to a path $\gamma$ an element $g \in G$ (fibre translation).
- Passing to infinitesimals, this is a pair $(X, \xi) \in T_xM \times g$. 

---

**Definition (Atiyah Lie-algebroid, local version)**

Let $g^ab$ be the cocycle defining the bundle $P \to M$. The **Atiyah (Lie-)algebroid** of $g$ is the $C^\infty(M)$-module $\text{At}_{P,g}$ with anchor map $\rho: \text{At}_{P,g} \to X|_M$, $\rho(X, \xi)^a \mapsto g^ab(\xi)$ and bracket $\{X, \xi\}^a = \text{Ad}_{g^ab}\xi`g^ab\mu G$. 

---
Infinitesimal deformations of bundles—the Atiyah algebroid

A parallel transport is a rule for comparing fibres of $P \to M$ over different points.

**Goal now:** **Infinitesimal** parallel transport.

- Locally, we can trivialise $P$ to $P|_U \cong U \times G \xrightarrow{P_T} U$.
- PT within $U$ then assigns to a path $\gamma$ an element $g \in G$ (fibre translation).
- Passing to infinitesimals, this is a pair $(X, \xi) \in T_x M \times g$.

**Definition (Atiyah Lie-algebroid, local version)**

Let $g = (g_{ab})$ be the cocycle defining the bundle $P \to M$. The **Atiyah (Lie-)algebroid** of $g$ is the $C^\infty(M)$-module

$$\text{At}(g) = \{(X, \xi) \mid X \in \mathfrak{X}(M), \xi = (\xi_a : U_a \to \mathfrak{g}), \xi_b = \text{Ad}_{g_{ab}} \xi_a + g_{ab}^* \mu_G(X)\}$$

with anchor map $\rho : \text{At}(g) \to \mathfrak{X}(M)$, $(X, \xi) \mapsto X$ and bracket

$$\left[(X, \xi), (Y, \eta)\right]_{\text{At}(g)} = ([X, Y]_{\mathfrak{X}(M)}, \mathcal{L}_X \eta - \mathcal{L}_Y \xi - [\xi, \eta]_\mathfrak{g}).$$
An infinitesimal parallel transport, also called a connection on $P$, assigns to an infinitesimal path an infinitesimal fibre translation.

Globalising, it is a smooth map

$$(\text{id}_{\mathfrak{X}(M)}, A): \mathfrak{X}(M) \to \text{At}(g), \quad X \mapsto (X, A(X)) .$$

This does not respect the Lie structures: the failure is called the curvature of $A$,

$$[[X, A(X)), (Y, A(Y))]_{\text{At}(g)} - (\text{id}_{\mathfrak{X}(M)}, A)([X, Y]) = (0, F_A(X, Y)) .$$

**Proposition**

A connection on a principal bundle defined by $g$ corresponds to a collection of 1-forms $A = (A_a \in \Omega^1(U_a; g))$ such that

$$A_b = \text{Ad}_{g_{ab}} A_a + g_{ab}^* \mu_G = g_{ab} A_a g_{ab}^{-1} + g_{ab}^{-1} d g_{ab} .$$

Parallel transports and connections on $P$ are in 1:1-correspondence.
The universal symmetry group $\text{Sym}(P)$ of $P \to M$ has the following descriptions:

- The group of smooth maps $\hat{f}: P \to P$ which preserve the $G$-action and cover a diffeomorphism $f: M \to M$.
- The group of pairs $(f, \alpha)$, where $f \in \text{Diff}(M)$ and $\alpha: P \to \alpha^*P$ is a morphism of $G$-bundles.

There is a canonical smooth group homomorphism $\text{Sym}(P) \to \text{Diff}(M)$. 

Previously: $\text{Sym}(P)$ for gerbes [SB, Müller, Szabo] and general smooth principal 8-bundles [SB, Shahbazi]; applications to QFT anomalies and NSNS supergravity.
The universal symmetry group $\text{Sym}(P)$ of $P \rightarrow M$ has the following descriptions:

- The group of smooth maps $\hat{f} : P \rightarrow P$ which preserve the $G$-action and cover a diffeomorphism $f : M \rightarrow M$.
- The group of pairs $(f, \alpha)$, where $f \in \text{Diff}(M)$ and $\alpha : P \rightarrow \alpha^* P$ is a morphism of $G$-bundles.

There is a canonical smooth group homomorphism $\text{Sym}(P) \rightarrow \text{Diff}(M)$.

**Proposition**

1. Let $H$ be a Lie group which acts smoothly on $M$ by a map $\Phi : H \rightarrow \text{Diff}(M)$. Then, $H$-equivariant structures on $P$ are in bijections with lifts of $\Phi$ to $\hat{\Phi} : H \rightarrow \text{Sym}(P)$.

2. The Lie algebra of $\text{Sym}(P)$ is

$$\mathfrak{sym}(P) = \mathfrak{at}(P).$$

**Previously:** $\text{Sym}(P)$ for gerbes [SB, Müller, Szabo] and general smooth principal $\infty$-bundles [SB, Shahbazi]; applications to QFT anomalies and NSNS supergravity.
Higher structure

∞-groups and ∞-bundles
Higher groups

2-groups:

- Let $G$ be a Lie group. Consider its fundamental groupoid $\pi_{\leq 1} G$:
  - objects = points $g \in G$, morphisms = {paths $\gamma$ in $G$}/homotopies fixing endpoints.
  This inherits a monoidal structure from the group structure of $G$.
- A 2-group is a monoidal groupoid in which every object has an inverse [Baez? Older?].
- Example: $\text{BU}(1)$ is the groupoid with objects = {$*$} and morphisms = $\text{U}(1)$.
  We set $* \otimes * := *$, $z \circ z' := zz'$ and $z \otimes z' := zz'$. 
Higher groups

2-groups:

- Let $G$ be a Lie group. Consider its fundamental groupoid $\pi_{\leq 1}G$:
  - objects = points $g \in G$, morphisms = \{paths $\gamma$ in $G$\}/homotopies fixing endpoints.
  - This inherits a monoidal structure from the group structure of $G$.

- A 2-group is a monoidal groupoid in which every object has an inverse [Baez? Older?].

- Example: $\text{BU}(1)$ is the groupoid with objects = $\{\ast\}$ and morphisms = $\text{U}(1)$.
  - We set $\ast \otimes \ast := \ast$, $z \circ z' := z z'$ and $z \otimes z' := z z'$.

Even higher groups:

A group is (1) a set with a multiplication and (...), or (2) a groupoid with a single object.

An $\infty$-group is, equivalently, [Stasheff; Lurie]

- a coherently monoidal $\infty$-groupoids where each object has an inverse (multiplication encoded as monoidal structure), or

- an $\infty$-groupoid with a single object (multiplication encoded as composition).

Example: The based loop group $\Omega_x X$ of a topological space $X$ [Stasheff].
Smooth spaces and smooth groups

Let $\text{Cart}$ denote the category with objects $\{\mathbb{R}^n \mid n \in \mathbb{N}_0\}$ and morphisms all smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. This has a Grothendieck coverage of good open covers [Fiorenza, Schreiber, Stasheff].

Let $S$ denote the $\infty$-category of spaces (‘spaces’ = ‘$\infty$-groupoids’).

**Definition (Smooth space) [Schreiber]**

A smooth space is an $\infty$-sheaf $X \in \mathcal{S}h_\infty(\text{Cart})$. We write $\mathbb{H}$ for the $\infty$-category of smooth spaces. A smooth $\infty$-group is a group object in $\mathcal{S}h_\infty(\text{Cart})$.

- Interpretation: $X$ assigns to each $\mathbb{R}^n$, $n \in \mathbb{N}_0$, the space of smooth maps $\mathbb{R}^n \to X$. 

Let \( \mathbf{Cart} \) denote the category with objects \( \{ \mathbb{R}^n \mid n \in \mathbb{N}_0 \} \) and morphisms all smooth maps \( \mathbb{R}^n \to \mathbb{R}^m \). This has a Grothendieck coverage of good open covers [Fiorenza, Schreiber, Stasheff]. Let \( \mathcal{S} \) denote the \( \infty \)-category of spaces (‘spaces’ = ‘\( \infty \)-groupoids’).

**Definition (Smooth space)** [Schreiber]

A smooth space is an \( \infty \)-sheaf \( X \in \mathbf{Sh}_\infty(\mathbf{Cart}) \). We write \( \mathbb{H} \) for the \( \infty \)-category of smooth spaces. A smooth \( \infty \)-group is a group object in \( \mathbf{Sh}_\infty(\mathbf{Cart}) \).

- **Interpretation:** \( X \) assigns to each \( \mathbb{R}^n, n \in \mathbb{N}_0 \), the space of smooth maps \( \mathbb{R}^n \to X \).

- **Example:** If \( M \) is a manifold, then \( \mathbb{R}^n \mapsto Mfd(\mathbb{R}^n, M) \) defines an object \( M \in \mathbb{H} \).
  This furnishes an embedding \( Mfd \hookrightarrow \mathbb{H} \).

- \( \mathbb{B}U(1) \) is a smooth \( \infty \)-group with \( \mathbb{B}U(1)(\mathbb{R}^n) = N(\text{Mfd}(\mathbb{R}^n, \mathbb{U}(1)) \rightrightarrows *) \cong \mathbf{Bun}(\mathbb{R}^n; \mathbb{U}(1)) \).

- There are smooth \( \infty \)-groups \( \mathbf{Bun}_\nabla(\dashv; \mathbb{U}(1)) \) without an ‘underlying space/Lie groupoid’.
With a notion of (smooth) higher groups at hand, we can build higher principal bundles.

- Works in a particular type of $\infty$-categories, the $\infty$-topoi [Giraud; Rezk; Lurie].

- **Example:** Both $S$ and $\mathcal{H}$ are $\infty$-topoi. The EEpis in $S$ are those maps which are surjective on $\pi_0$.

**Definition (Principal $\infty$-bundle)** [Nikolaus, Schreiber, Stevenson; SB]

Let $\mathcal{X}$ be an $\infty$-topos and $G$ a group object in $\mathcal{X}$. A **$G$-principal $\infty$-bundle** consists of an effective epimorphism $P \to X$ in $\mathcal{X}$ and a fibre-preserving $G$-action on $P$ such that the canonical morphism $P \times G \to P \times_X P$ is an equivalence.

**Examples:**

- For $G$ a Lie group, the canonical map $\ast \to BG$ is a $G$-principal $\infty$-bundle ($G$ acts trivially).

- A **(bundle) gerbe** is equivalently a $\text{BU}(1)$-principal $\infty$-bundle.
Parallel transport for a strict type of 2-bundles was introduced by [Baez, Schreiber '04], later linked to connections [Schreiber, Waldorf '07; Faria Martins, Picken '10; Waldorf '17; Saemann, Schmidt, Kim '19;...].

For higher bundles whose structure ∞-group arises as an integration of an $L_\infty$-algebra, a general formalism was provided by [Sati, Schreiber, Stasheff '08].

For ∞-groups with another strictness condition, connections and parallel transport on trivial bundles was given by [Kapranov '07, '15] using the free Lie algebroid on the tangent bundle $TM$.

A theory of holonomy for flat connections and its relation to ∞-local systems was developed by [Abad, Schätz '14].

An approach using rational homotopy for bundles controlled/classified by a discrete space by [Fiorenza, Sati, Schreiber '20].

...
Higher connections—(some of) what happened so far

- Parallel transport for a **strict** type of **2-bundles** was introduced by [Baez, Schreiber '04], later linked to connections [Schreiber, Waldorf '07; Faria Martins, Picken '10; Waldorf '17; Saemann, Schmidt, Kim '19;...].

- For higher bundles whose structure $\infty$-group arises as an integration of an $L_\infty$-algebra, a general formalism was provided by [Sati, Schreiber, Stasheff '08].

- For $\infty$-groups with another **strictness condition**, connections and parallel transport on trivial bundles was given by [Kapranov '07, '15] using the free Lie algebroid on the tangent bundle $T M$.

- A theory of holonomy for **flat** connections and its relation to $\infty$-local systems was developed by [Abad, Schätz '14].

- An approach using rational homotopy for bundles controlled/classified by a **discrete** space by [Fiorenza, Sati, Schreiber '20].

- ...
Higher connections as infinitesimal symmetries

Derived geometry and deformation theory
For $G \in \text{Grp}(\mathcal{X})$, let $BG \in \mathcal{X}$ denote the quotient of the trivial action of $G$ on the point $\ast$.

**Definition (Classifying object)** [Nikolaus, Schreiber, Stevenson]

The object $BG$ is called the **classifying object of $G$**.

**Theorem** [Nikolaus, Schreiber, Stevenson]

Let $X \in \mathcal{X}$. There is an equivalence of $\infty$-groupoids

$$\mathcal{Bun}(X; G) \simeq \mathcal{X}(X, BG).$$
Classifying stacks

For $G \in \mathcal{G}p(X)$, let $BG \in X$ denote the quotient of the trivial action of $G$ on the point $\ast$.

**Definition (Classifying object)** [Nikolaus, Schreiber, Stevenson]

The object $BG$ is called the **classifying object of $G$**.

**Theorem** [Nikolaus, Schreiber, Stevenson]

Let $X \in \mathcal{X}$. There is an equivalence of $\infty$-groupoids

\[ \mathcal{B}un(X; G) \simeq \mathcal{X}(X, BG) . \]

If $p : X \to BG$ classifies a $G$-principal $\infty$-bundle $P \to X$, the symmetries of $P$ are the ‘deformations’
For infinitesimal symmetries of $P$, study infinitesimal deformations of its classifying map
$$p: X \to BG.$$ 

Incorporate infinitesimals into the formalism of smooth spaces: derived differential geometry (DDG) [Lawvere; Dubuc; Moerdijk, Reyes; Kock; Spivak; Carchedi, Steffens; Nuiten; ...].

- This works by incorporating algebra: the functions $C^\infty(M; \mathbb{R})$ on each manifold form a $C^\infty$-ring.
- Roughly speaking, DDG is algebraic geometry over dg- or simplicial $C^\infty$-rings; it behaves differently form (derived) algebraic geometry, e.g. due to existence of partitions of unity.
- Strongly related to dg/higher Lie geometry [Xu, Zhu, Behrend, Weinstein, Gualtieri, Ševera, ...].
We replace Cart by Cart\(_{th}\), whose function algebras are of the form \(C^\infty(\mathbb{R}^n; \mathbb{R}) \otimes W\), where \(W\) is a local algebra with nilpotent ideal. These are the infinitesimal thickenings of the \(\mathbb{R}^n\)s.

**Definition (Formal smooth space)**

A **formal smooth space** is an \(\infty\)-sheaf \(X \in Sh_\infty(\text{Cart}_{th})\). We denote this \(\infty\)-topos by \(\mathbb{H}_{th}\).
We replace \( \text{Cart} \) by \( \text{Cart}_{th} \), whose function algebras are of the form \( C^\infty(\mathbb{R}^n; \mathbb{R}) \otimes W \), where \( W \) is a local algebra with nilpotent ideal. These are the infinitesimal thickenings of the \( \mathbb{R}^n \)s.

**Definition (Formal smooth space)**

A **formal smooth space** is an \( \infty \)-sheaf \( X \in \text{Sh}_\infty(\text{Cart}_{th}) \). We denote this \( \infty \)-topos by \( \mathcal{H}_{th} \).

**Example:** We now indeed capture infinitesimal deformations of smooth geometric data intrinsically: Consider the space \( \mathbb{R}_\epsilon \) with function \( C^\infty \)-ring \( \mathbb{R} \otimes \mathbb{R}[\epsilon]/\epsilon^2 \). Then,

\[
\mathcal{H}_{th}(\mathbb{R}_\epsilon, M) \cong TM \quad \text{(as a set)}.
\]

**Remark:** There is a fully faithful embedding \( \mathcal{H} \hookrightarrow \mathcal{H}_{th} \).
Let $k$ be a field of characteristic zero and $A$ a connective commutative dg algebra over $k$.

### Definition ($L_\infty$-algebroid)

A $L_\infty$-algebroid over $A$ is a dg module $E$ over $A$ together with an anchor map $\rho: E \to T_A$ and a family of brackets $[-]_{n,E}: E^\otimes n \to E$ of degree $2 - n$ such that

1. the brackets turn $E$ into an $L_\infty$-algebra (antisymmetry, coherent Jacobi),
2. $\rho$ is a morphism of $L_\infty$-algebras, and
3. $[-]_{n,E}$ satisfies the Leibniz rule

$$[[\xi, f \cdot \eta], E] = (-1)^{|f|} f \cdot [\xi, \eta]_E + \rho(\xi)(f) \cdot \eta,$$

for $n = 2$ and is graded $A$-linear for $n > 2$.

If $[-]_{n,E} = 0$ for all $n > 2$, then $(E, [-, -]_E, \rho)$ is called a dg Lie algebroid over $A$. 

---

**Morphisms of $L_\infty$-algebroids:** tower of $\phi_1: g \to h$ and $\phi_n: g^b_n \to h^r_n$ with coherences.

Conveniently encoded using Chevalley-Eilenberg CDGCs. Our situation: $A = C_8^p M_q$; we then speak of $L_\infty$-algebroids on $M$.
Let $k$ be a field of characteristic zero and $A$ a connective commutative dg algebra over $k$.

**Definition ($L_\infty$-algebroid)**

A $L_\infty$-algebroid over $A$ is a dg module $E$ over $A$ together with an anchor map $\rho: E \to TA$ and a family of brackets $[-]_{n,E}: E^\otimes n \to E$ of degree $2 - n$ such that

1. the brackets turn $E$ into an $L_\infty$-algebra (antisymmetry, coherent Jacobi),
2. $\rho$ is a morphism of $L_\infty$-algebras, and
3. $[-]_{n,E}$ satisfies the Leibniz rule

$$[\xi, f \cdot \eta]_E = (-1)^{|\xi||f|} f \cdot [\xi, \eta]_E + \rho(\xi)(f) \cdot \eta.$$ 

for $n = 2$ and is graded $A$-linear for $n > 2$.

If $[-]_{n,E} = 0$ for all $n > 2$, then $(E, [-, -]_E, \rho)$ is called a **dg Lie algebroid** over $A$.

**Morphisms** of $L_\infty$-algebroids: tower of $\phi_1: \mathfrak{g} \to \mathfrak{h}$ and $\phi_n: \mathfrak{g}^\otimes n \to \mathfrak{h}[n]$ with coherences. Conveniently encoded using Chevalley-Eilenberg CDGCs.

**Our situation:** $A = C^\infty(M)$; we then speak of $L_\infty$-algebroids on $M$. 

**$L_\infty$-algebroids and dg Lie algebroids**
**Definition (Formal moduli problem over $A$) [Nuiten '17]**

Let $k$ be a field of characteristic zero and $A$ be a connective commutative $k$-algebra. A **formal moduli problem (FMP) over $A$** is a functor $F: (\text{CAlg}_k^{\text{Art}})_A \to \mathcal{S}$ such that

1. $F(A) \simeq \ast$, and
2. $F$ maps square-zero extensions to pullbacks.

---

**Theorem [Nuiten '17]**

There is an equivalence of $\infty$-categories

$$\text{MC}: L_{\infty} \text{Agd}_A \xrightarrow{\sim} \text{FMP}(A).$$
We are interested in the FMP describing deformations of the classifying map \( p: M \to BG \).

**Definition (Atiyah \( L_\infty \)-algebroid) [SB, Müller, Nuiten, Szabo]**

Let \( G \) be a smooth \( \infty \)-group and \( P \to M \) a \( G \)-principal \( \infty \)-bundle classified by a morphism \( p: M \to BG \) in \( \mathbb{H}_{(th)} \). The **Atiyah \( L_\infty \)-algebroid** \( \text{At}(P) \) **of** \( P \) is the \( L_\infty \)-algebroid corresponding to the above FMP under Nuiten’s theorem.
Defining higher connections, circumventing flatness

We are interested in the FMP describing deformations of the classifying map $p: M \to BG$.

**Definition (Atiyah $L_\infty$-algebroid) [SB, Müller, Nuiten, Szabo]**

Let $G$ be a smooth $\infty$-group and $P \to M$ a $G$-principal $\infty$-bundle classified by a morphism $p: M \to BG$ in $\mathbb{H}_{(th)}$. The **Atiyah $L_\infty$-algebroid** $\text{At}(P)$ of $P$ is the $L_\infty$-algebroid corresponding to the above FMP under Nuiten’s theorem.

**Goal:** Define (not necessarily flat) connections on generic $\infty$-bundles $P$.

For $l \in \mathbb{N}$, there is an $\infty$-functor $Q^{(l)}: L_\infty \text{Agd}_A \to L_\infty \text{Agd}_A$ which truncates away all terms in $CE_\ast(g)$ containing more than $l$ tensor factors, i.e. $CE_\ast(Q^{(l)}g) = \text{Sym}^{1 \leq \bullet \leq l}_{C^\infty(M)}(g)$ [Nuiten].

**Definition (Space of $l$-connections) [SB, Müller, Nuiten, Szabo]**

The $\infty$-groupoid of $l$-connections on $P$ is the mapping space

$$\text{Con}_l(P) := L_\infty \text{Agd}_{C^\infty(M)}(Q^{(l)}\mathcal{X}(M), \text{At}(P)) \in S.$$
Case studies

Testing the new model
First check: If $P \to M$ is an ordinary principal bundle ($G$ a Lie group), then

$$\text{Con}_1(P) = \{\text{classical connections on } P\}, \quad \text{Con}_l(P) = \{\text{flat conns. on } P\}, \quad \forall l > 1. \quad \checkmark$$
Higher $U(1)$-bundles/$n$-gerbes

**First check:** If $P \to M$ is an ordinary principal bundle ($G$ a Lie group), then

$$\text{Con}_1(P) = \{\text{classical connections on } P\}, \quad \text{Con}_l(P) = \{\text{flat conn. on } P\}, \quad \forall l > 1. \quad \checkmark$$

**Particularly well-known higher case:** connections on $n$-gerbes/higher $U(1)$-bundles.

### Definition ($n$-gerbe with $l$-connection) [Deligne; Gajer; SB, Shahbazi]

Let $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ be a good open covering of $M$.

1. An $(n-1)$-gerbe/$B^nU(1)$-bundle on $M$ is a collection $g = (g_{a_0\ldots a_n} : U_{a_0\ldots a_n} \to U(1))$ satisfying the Čech cocycle condition, $\delta g := \prod_{i=0}^{n} (-1)^i g_{a_0\ldots \hat{a}_i\ldots a_n} = 1$. 


Higher $U(1)$-bundles/$n$-gerbes

**First check:** If $P \rightarrow M$ is an ordinary principal bundle ($G$ a Lie group), then

$$\text{Con}_1(P) = \{\text{classical connections on } P\}, \quad \text{Con}_l(P) = \{\text{flat conn. on } P\}, \quad \forall l > 1. \quad \checkmark$$

**Particularly well-known higher case:** connections on $n$-gerbes/higher $U(1)$-bundles.

---

**Definition ($n$-gerbe with $l$-connection)** [Deligne; Gajer; SB, Shahbazi]

Let $\mathcal{U} = \{U_a\}_{a \in \Lambda}$ be a good open covering of $M$.

1. An $(n-1)$-gerbe/B$^nU(1)$-bundle on $M$ is a collection $g = (g_{a_0\ldots a_n} : U_{a_0\ldots a_n} \rightarrow U(1))$ satisfying the Čech cocycle condition, $\delta g := \prod_{i=0}^{n} (-1)^i g_{a_0\ldots \hat{a}_i\ldots a_n} = 1$.

2. An $l$-connection on an $n$-gerbe $g$ is a tuple $(A^{(1)}, \ldots, A^{(l)})$, where

$$A^{(p)} = (A^{(p)}_{a_0\ldots a_{n-p}} \in \Omega^p(U_{a_0\ldots a_{n-p}}))$$

and such that

$$d \log(g) = \delta A^{(1)}, \quad dA^{(p)} = (-1)^p \delta A^{(p+1)} \quad \forall \; p = 1, \ldots, l - 1.$$

For each $0 \leq l \leq n + 1$, there is an $\infty$-groupoid $\text{Grb}^n_{\nabla| l}(M)$ of $n$-gerbes with $l$-connections.
Example: 0-gerbes are the same as $U(1)$-bundles.
1-gerbes with connections model the B-field in string theory/SuGra [Kapustin; Witten]
n-gerbes with $(n+1)$-connection model differential cohomology [Deligne; Brylinski; Gajer; Schreiber]
*connections on \( n \)-gerbes

**Example:** 0-gerbes are the same as \( U(1) \)-bundles.

1-gerbes with connections model the B-field in string theory/SuGra \([\text{Kapustin; Witten}]\)

\( n \)-gerbes with \((n+1)\)-connection model differential cohomology \([\text{Deligne; Brylinski; Gajer; Schreiber}]\)

(Space of connections) \([\text{SB, Shahbazi}]\)

The **space of \( l \)-connections** on an \( n \)-gerbe \( g \) is the (homotopy) fibre \([\text{SB, Shahbazi}]\)

\[
\begin{align*}
\text{Con}_{\text{geo},l}(g) & \longrightarrow \text{Grb}_{\nabla|l}^n(M) \\
\downarrow & \\
\ast & \longrightarrow \text{Grb}^n(M)
\end{align*}
\]

**Question:** Is this space equivalent to the one obtained from our \( L_\infty \)-algebroid picture?
The Atiyah $L_\infty$-algebroid of an $n$-gerbe

**Theorem** [Nuiten; SB, Müller, Nuiten, Szabo]

Let $g$ describe an $n$-gerbe on $M$. Its Atiyah $L_\infty$-algebroid is the dg Lie algebroid

$$
C^\infty(U^{[0]}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^\infty(U^{[n-1]}) \xrightarrow{(0, \delta)} E_n(g),
$$

where $E_n(g) = \{ (X, f) \in \mathcal{X}(M) \times C^\infty(U^{[n]}) \mid \delta f = (-1)^{n+1} d \log(g)(X) \}$. 

The anchor map is the projection onto $\mathcal{X}(M)$, and the bracket is the Lie derivative of functions and vector fields.
The Atiyah $L_\infty$-algebroid of an $n$-gerbe

**Theorem** [Nuiten; SB, Müller, Nuiten, Szabo]

Let $g$ describe an $n$-gerbe on $M$. Its Atiyah $L_\infty$-algebroid is the dg Lie algebroid

$$C^\infty(U[0]) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^\infty(U[n-1]) \xrightarrow{(0, \delta)} E_n(g),$$

where

$$E_n(g) = \{(X, f) \in \mathfrak{X}(M) \times C^\infty(U[n]) \mid \delta f = (-1)^{n+1} d \log(g)(X)\}.$$

The anchor map is the projection onto $\mathfrak{X}(M)$, and the bracket is the Lie derivative of functions and vector fields.

**Theorem** [SB, Müller, Nuiten, Szabo]

For any $n$-gerbe $g$ on $M$, there is an equivalence of $(l-1)$-groupoids

$$\text{Con}_{\text{geo}, l}(g) \simeq \text{Con}_{l}(g).$$

This is an algebraic description of differential cohomology.
Proof (sketch)

- Goal: compute explicitly the mapping space $\text{Map}_{L_{\infty}\mathcal{A}dg_{C^{\infty}(M)}}(Q^{(l)}\mathcal{X}(M), \text{At}(g))$.

- Use model structure on $L_{\infty}\mathcal{A}dg_{C^{\infty}(M)}$: $Q^{(l)}\mathcal{X}(M)$ is $C^{\infty}(M)$-cofibrant.
Proof (sketch)

- Goal: compute explicitly the mapping space \( \text{Map}_{L^\infty \mathcal{A} \text{gd}^{dg}_{C^\infty(M)}} \left( Q^{(l)} \mathcal{X}(M), \text{At}(g) \right) \).
- Use model structure on \( L^\infty \mathcal{A} \text{gd}^{dg}_{C^\infty(M)} \): \( Q^{(l)} \mathcal{X}(M) \) is \( C^\infty(M) \)-cofibrant.
- Find manageable simplicial resolution of \( \text{At}(g) \): we give a general, explicit construction for ‘semi-abelian extensions’ of a dg Lie algebroid structure on
  \[
  [n] \mapsto \text{Hom}_k \left( C_\bullet(\Delta^n; k), \text{ch}_k(g) \right),
  \]
  which in this case allows us to simplify formal constructions of [Getzler; Robert-Nicoud, Vallette].
- Lemma: if \( g \) is fibrant (surjective anchor map) this produces a simplicial resolution \( \hat{g} \) of \( g \).
Proof (sketch)

- **Goal:** Compute explicitly the mapping space \( \text{Map}_{L_\infty \mathcal{A} \text{gd}^{\text{dg}}_{C^{\infty}(M)}} (Q^{(l)} \mathcal{X}(M), \text{At}(g)) \).

- **Use model structure on** \( L_\infty \mathcal{A} \text{gd}^{\text{dg}}_{C^{\infty}(M)}: Q^{(l)} \mathcal{X}(M) \) is \( C^{\infty}(M) \)-cofibrant.

- **Find manageable simplicial resolution of** \( \text{At}(g) \): we give a general, explicit construction for ‘semi-abelian extensions’ of a dg Lie algebroid structure on

\[
[n] \mapsto \text{Hom}_k(C_\ast(\Delta^n; k), \text{ch}_k(\mathfrak{g})),
\]

which in this case allows us to simplify formal constructions of [Getzler; Robert-Nicoud, Vallette].

- **Lemma:** If \( \mathfrak{g} \) is fibrant (surjective anchor map) this produces a simplicial resolution \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \).

- **The mapping space is thus modelled by the simplicial set**

\[
[n] \mapsto L_\infty \mathcal{A} \text{gd}^{\text{dg}}_{C^{\infty}(M)} (Q^{(l)} \mathcal{X}(M), \hat{\mathfrak{g}}_n).
\]

- **Explicit computation:** We have an isomorphism of simplicial sets

\[
L_\infty \mathcal{A} \text{gd}^{\text{dg}}_{C^{\infty}(M)} (Q^{(l)} \mathcal{X}(M), \hat{\mathfrak{g}}_n) \cong \text{Con}_{\text{geo,l}}(g).
\]
Thank you for your attention!