Differential cohomology in geometry and analysis

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19.9.2008/ DMV-Tagung 2008

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1. Differentiable cohomology

2. e-invariant

3. Application of differentiable K-theory

Notation

- $X, Y, \ldots$ - smooth compact manifolds
- h - generalized cohomology theory, $h_* := h(*)$
- $\Omega(X, h_*) := \Omega(X) \otimes_{\mathbb{Z}} h_*$ smooth differential forms with coefficients in h_*
- $\Omega_{d=0}(X, h_*) \subseteq \Omega(X, h_*)$ - closed forms
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Why?

- Chern-Simons invariants
- Characteristic classes for flat vector bundles
- Invariants of elements in stable homotopy groups

- topological terms in σ-models
- Configuration spaces of field theories with differential form field strength

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- Existence of $\hat{h}$?
- Uniqueness of $\hat{h}$?
- What is $\hat{h}^{flat}(X)$?
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Natural questions

- Cup product?
  - Assume that h is multiplicative.
  - Require hat I and R are homomorphisms of rings.
  - Then we call $\hat{h}$ a multiplicative extension.
  - $\widehat{\mathbf{H}\mathbb{Z}}$ is multiplicative - Cheeger-Simons
  - $\hat{\mathbf{K}}$ and bordism theories like $\hat{\mathbf{S}}$, $\mathbf{M}\hat{\mathbf{U}}$ have multiplicative extensions (B.-Schick)
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  - Extensions constructed using Landweber exactness from $\hat{\mathbf{MU}}$ are multiplicative.
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- Orientation and integration?
  - $f : X \to Y$ proper submersion, $\hat{h}$-multiplicative.
    - No additional structures needed for $\widehat{\mathbf{H}\mathbb{Z}}$ (Brylinski, Dupont-Ljungman, Gomi)
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- Existence of $\hat{h}$?
- Uniqueness of $\hat{h}$?
- What is $\hat{h}^{flat}(X)$?
- Cup product?
- Orientation and integration?
- Riemann-Roch?

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There are various methods to construct smooth extensions:

- Sheaf theory (Deligne cohomology) for $\widehat{\mathbf{H}\mathbb{Z}}$
- Differential characters $\widehat{\mathbf{H}\mathbb{Z}}$ (Cheeger-Simons), $\hat{\mathbf{K}}$ (Maghfoul, 2008).
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Stably framed manifolds

- M manifold
- TM stably framed
- consider $M \sim M'$ if M and M' are framed bordant
- $[M] \in \Omega^{fr}$ - class of M in the group of framed bordism classes
- Question: Is $[M]$ trivial?
- Pontrjagin-Thom: $\Omega^{fr} \cong \pi^S$. This group is complicated and only partially known.

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  There exists a manifold W such that TW is stably framed and $\partial W \cong M \sqcup -M'$
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Primary Invariants

Construct homomorphisms $\epsilon : \Omega^{fr} \to A$ to known groups A and study image $\epsilon([M]) \in A$.

Ω^{fr} are coefficients of a generalized homology theory represented by spectrum **S**.

$[M]$ corresponds to homotopy class $f : \Sigma^{\dim(M)}\mathbf{S} \to \mathbf{S}$

Idea: map to simpler homology theories.

Choices: $\mathbf{H}\mathbb{Z}$, **K**, **MU**

use unit $\epsilon : \mathbf{S} \to \mathbf{K}$

Problem: The primary invariant vanishes for $\dim(M) > 0$ since $\Omega^{fr}_{>0}$ is torsion (Serre) and $\mathbf{K}_*$ is free.

$*$	-2	-1	0	1	2	3	4	5	6	7
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Problem: Lift $\bar{f} \in \bar{\mathbf{K}}_{\dim(M)+1}$ is not unique!

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$$\begin{tikzcd} & \Sigma^{-1}\bar{\mathbf{K}} \arrow[d] \\ \Sigma^{\dim(M)}\mathbf{S} \arrow[ur, dotted, "\bar{f}"] \arrow[r, "f"] \arrow[dr, dotted, "0"'] & \mathbf{S} \arrow[d, "\epsilon"] \\ & \mathbf{K} \end{tikzcd}$$

Problem: Lift $\bar{f} \in \bar{\mathbf{K}}_{\dim(M)+1}$ is not unique!
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Secondary invariants

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$$\begin{tikzcd} & & \Sigma^{-1}\bar{\mathbf{K}} \arrow[d] \\ \Sigma^{\dim(M)}\mathbf{S} \arrow[urr, dotted, "\bar{f}"] \arrow[rr, "f"] \arrow[drr, dotted, "0"'] & & \mathbf{S} \arrow[d, "\epsilon"] \\ & & \mathbf{K} \end{tikzcd}$$

Problem: Lift $\bar{f} \in \bar{\mathbf{K}}_{\dim(M)+1}$ is not unique!
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Need simpler targets!

$\mathbb{R}/\mathbb{Z}$-invariants

$\mathbf{K}_{\mathbb{R}/\mathbb{Z},*}$ is known

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$$\mathbf{K}_{\mathbb{R}/\mathbb{Z},ev} \cong \mathbb{R}/\mathbb{Z}\ , \quad \mathbf{K}_{\mathbb{R}/\mathbb{Z},odd} \cong 0$$

$\mathbb{R}/\mathbb{Z}$-invariants

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$\mathbf{K}_{\mathbb{R}/\mathbb{Z},ev} \cong \mathbb{R}/\mathbb{Z}\ , \quad \mathbf{K}_{\mathbb{R}/\mathbb{Z},odd} \cong 0$

Use $\mathbf{K}_{\mathbb{R}/\mathbb{Z}}$ as target!

$\mathbb{R}/\mathbb{Z}$-invariants

Rationalization

$$\mathbf{S} \longrightarrow \mathbf{S}_{\mathbb{R}}$$

take homotopy cofibre

$\mathbb{R}/\mathbb{Z}$-invariants

$$\Sigma^{-1}\mathbf{S}_{\mathbb{R}/\mathbb{Z}} \xrightarrow{\delta} \mathbf{S} \longrightarrow \mathbf{S}_{\mathbb{R}} \quad .$$

Relate with K-theory.

$\mathbb{R}/\mathbb{Z}$-invariants

$$\begin{tikzcd} & \Sigma^{-1}\bar{\mathbf{K}} \arrow[d] \arrow[dr, dotted, "0"] & \\ \Sigma^{-1}\mathbf{S}_{\mathbb{R}/\mathbb{Z}} \arrow[r, "\delta"] & \mathbf{S} \arrow[r] \arrow[d, "\epsilon"] & \mathbf{S}_{\mathbb{R}} \\ & \mathbf{K} & \end{tikzcd} \quad .$$

$\mathbb{R}/\mathbb{Z}$-invariants

$$\begin{tikzcd} & \Sigma^{-1}\bar{\mathbf{K}} \arrow[dl, dashed, "u"'] \arrow[d] \arrow[dr, dotted, "0"] & \\ \Sigma^{-1}\mathbf{S}_{\mathbb{R}/\mathbb{Z}} \arrow[r, "\delta"] & \mathbf{S} \arrow[r] \arrow[d, "\epsilon"] & \mathbf{S}_{\mathbb{R}} \\ & \mathbf{K} & \end{tikzcd} .$$

observe existence and uniqueness of u!

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$$\begin{tikzcd}
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$$\begin{tikzcd}
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Observe that $e \in \mathbf{K}_{\mathbb{R}/\mathbb{Z},\dim(M)+1}$ is well-defined.

A family version

Consider space B and stable cohomotopy class

$$f : \Sigma^k B_+ \to \mathbf{S}$$

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$$e \in \frac{\mathbf{K}^{-k-1}_{\mathbb{R}/\mathbb{Z}}(B)}{\mathbf{S}^{-k-1}_{\mathbb{R}}(B)}$$ is well-defined.

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Note that for $k \geq 0$ we have

$$e \in K_{\mathbb{R}/\mathbb{Z}}^{-k-1}(B) \ .$$

Special Examples

Of particular interest is the following special case.

- $\pi : W \to B$ - locally trivial fibre bundle
- framing of vertical bundle $T^v\pi := \ker(d\pi)$
- $\pi : W \to B$ with framing represents class

$$\Sigma^k B_+ \xrightarrow{f} \mathbf{S}\,, \quad k = \dim(B) - \dim(W)$$

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- more special case: $\pi : W \to B$ a G-principal bundle
  basis of $Lie(G)$ induces trivialization of $T^v\pi$ by fundamental vector fields

A secondary index theorem

$\pi : W \to B$, $T^v\pi$ framed, $f \in [\Sigma^k B_+, \mathbf{S}]$, $e \in K_{\mathbb{R}/\mathbb{Z}}^{-k-1}(B)$

Theorem (B.-Schick)

π has canonical $\hat{\mathbf{K}}$-orientation.

Define

$$\hat{\pi}_!(1) \in \hat{\mathbf{K}}(B) .$$

Note that $\hat{\pi}_!(1)$ is flat.
Define

$$e^{an} := \hat{\pi}_!(1) \in \hat{\mathbf{K}}^{flat,-k}(B) \cong \mathbf{K}_{\mathbb{R}/\mathbb{Z}}^{-k-1}(B)$$

Theorem (Secondary index theorem)

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Bordism formula

Assume that $q : V \to B$ is a zero bordism of $\pi : W \to B$ as K-oriented bundle.

Can extend the smooth K-orientation of π over to q.

Theorem (bordism formula)

$$\hat{\pi}_!(1) = a\left(\int_{V/B} \mathbf{Td}\right)$$

This integral can often be evaluated!

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Principal bundles

$\pi : W \to B$ a G-principal bundle
$T \subseteq G$ maximal torus
choose $U(1) \subseteq T$
let $U(1)$ act on D^2 in the standard way, $\partial D^2 \cong U(1)$.
$q : V := W \times_{U(1)} D^2 \to B$ is K-oriented zero bordism of $\pi : W \to B$

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use Chern character

$$\mathbf{ch} : K_{\mathbb{R}} \to H\mathbb{R}^{per}$$

in order to identify

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Theorem

$$\int_{V/B} \mathbf{Td} = \frac{1}{1-\mathbf{e}^{-z}} - \frac{1}{z}\ .$$

this power series starts with

$$\frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \frac{1}{30240}z^5 - \frac{1}{1209600}z^7 + \frac{1}{47900160}z^9 - \dots\ .$$

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pairing e with higher dimensional primitive classes in $\mathbf{K}_*(BU(1))$ reproduces results by Miller and Knapp (late 70ties) on the transfer for $U(1)$-bundles

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pairing e with higher dimensional primitive classes in $\mathbf{K}_*(BU(1))$ reproduces results by Miller and Knapp (late 70ties) on the transfer for $U(1)$-bundles

for higher rank, e.g. $U(1)^n$ must replace $\mathbf{K}$ by more complicated cohomology theory, e.g. elliptic cohomology

this is work in progress with N. Naumann.

Remarks

$$\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\frac{1}{30240}z^5-\frac{1}{1209600}z^7+\frac{1}{47900160}z^9-\dots .$$

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