A GEOMETRIC APPROACH TO THE STRING TOPOLOGY OF SPHERES EXTENDED ABSTRACT

RICARDO CABRAL BASTO RODRIGUES

1. Introduction

String topology deals with algebraic structures on the homology of the free loop space of a manifold, *i.e.* the space of continuous maps from S^1 into the manifold. This area started when the loop product was introduced in [MCDS] by M. Chas and D. Sullivan. This product is defined on the homology of the free loop space of a manifold and encodes the geometric operations of intersecting submanifolds and concatenating loops. Initially, it was hoped that the loop product might be able to distinguish between different smooth structures on a manifold. However, in [CKS], it was shown that this structure is homotopy invariant. Since then, a lot of work has been done. Recently, in [MGNH], the loop coproduct was introduced by M. Goresky and N. Hingston. Contrary to the loop product, it is not homotopy invariant in general, as was shown in [FN]. In fact, this structure distinguishes homotopy equivalent non-homeomorphic 3-dimensional lens spaces ([NRW]).

In this work we study the loop product and coproduct in the simplest non-trivial spaces - the spheres. These were the first examples studied ([CJY]). But, as far as the author knows, the loop coproduct had only been partially computed in [NHNW]. Our computations rely on geometric descriptions of the loop product and coproduct proved in [NHNW]. They have the advantage of being elementary in the sense that they don't use complicated machinery, such as spectral sequences.

2. Homology of LS^n

In this section, we study the homology of the free loop space of the sphere, $LS^n = C(S^1, S^n)$. Throughout this text we'll use $[M]$ to denote the fundamental class of an oriented manifold M. And for an oriented manifold with boundary $(N, \partial N)$, we'll use $[N, \partial N]$ or simply $[N]$ to denote its fundamental class. We start by looking at the fiber bundle $\Omega S^n \hookrightarrow LS^n \xrightarrow{ev} S^n$, where ev denotes the evaluation of the loop at $t = 0$. Let D^n_+ and D^n_- be the northern and southern hemispheres of S^n, respectively. Furthermore, let $\Omega^+ S^n$ and $\Omega^- S^n$ be the based loop spaces at the North pole, N, and South pole, S, respectively. Then, we may apply Mayer-Vietoris to trivializations $\phi_\pm : D^n_\pm \times \Omega^\pm S^n \cong ev^{-1}(D^n_\pm)$. After an algebraic simplification, we get the following long exact sequence, called the **Wang sequence**:

Theorem 2.1. *The following sequence is exact:*

$$H_{*-(n-1)}(\Omega^+ S^n) \xrightarrow{t^{Wang}} H_*(\Omega^- S^n) \xrightarrow{j_*} H_*(LS^n) \xrightarrow{\partial^{Wang}} H_{*-n}(\Omega^+ S^n)$$

Here, j denotes the inclusion and

$$\partial^{Wang} = pr^F \circ (\phi_+^{-1})_* \partial_M$$
$$t^{Wang} = proj_* \circ (\phi_-^{-1} \circ \phi_+)_*([S^{n-1}] \times -)$$

where ∂_M is the boundary map of the Mayer-Vietoris sequence associated with $\{ev^{-1}(D^n_+), ev^{-1}(D^n_-)\}$, $pr^F : H_*(S^{n-1} \times \Omega^+ S^n) \to H_{*-(n-1)}(\Omega^+ S^n)$ is the map sending $1 \times x$ to 0 and

$[S^{n-1}]\times x$ to $x\in H_{*-(n-1)}(\Omega^+S^n)$. Moreover, $proj: D^n_-\times\Omega^-S^n\to\Omega^-S^n$ is the projection on the second component. Here, $S^{n-1}\subset S^n$ is the equator.

If we know $H_*(\Omega^\pm S^n)$ and t^{Wang}, we can use this sequence to compute $H_*(LS^n)$. With that in mind, let

$$\begin{aligned}\sigma_+^m:(S^{n-1})^m&\to\Omega^+S^n\\ x&\mapsto\overline{\gamma}_1\cdot\gamma_{x_1}\cdot\ldots\cdot\overline{\gamma}_1\cdot\gamma_{x_m}\end{aligned}$$

and

$$\begin{aligned}\sigma_-^m:(S^{n-1})^m&\to\Omega^-S^n\\ x&\mapsto\gamma_1\cdot\overline{\gamma}_{x_1}\cdot\ldots\cdot\gamma_1\cdot\overline{\gamma}_{x_m}\end{aligned}$$

Here, γ_y with $y\in S^{n-1}$ is the unique geodesic from S to N that passes through y; $\left(\gamma\mapsto\overline{\gamma}\right)$ is the path-reversing map; and $\left((\alpha,\beta)\mapsto\alpha\cdot\beta\right)$ denotes the concatenation of loops. Then, we may use techniques from Morse-Bott theory (see [AO]) to conclude that $\{(\sigma_\pm^m)_*[(S^{n-1})^m]\mid m\geq 0\}$ together with any constant map is a basis of $H_*(\Omega^\pm S^n)$ as a free abelian group.

Proposition 2.2. *Let $\alpha_\pm^k$ be $(\sigma_\pm^k)_*[(S^{n-1})^k]$ and let α_+^0 (α_-^0) be the constant loop based at N (S). Then, $\{\alpha_\pm^k:k\geq 0\}$ is a basis of $H_*(\Omega^\pm S^n)$.*

Consequently, t^{Wang} is completely described by the homological effect of the maps $proj\circ(\phi_-^{-1}\circ\phi_+)\circ(id\times\sigma_+^k)$. And it can be computed by relating these maps with σ_-^{k+1}. We then get the following results:

Proposition 2.3. *For n odd,*

$$t_*^{Wang}(\alpha_+^k)=0$$

For n even,

$$t_*^{Wang}(\alpha_+^k)=\begin{cases}0 & \text{, if } k \text{ is even}\\ \pm 2\alpha_-^{k+1} & \text{, if } k \text{ is odd}\end{cases}$$

Finally, we conclude that

Theorem 2.4. *Let $n\geq 3$. If n is odd, then $H_*(LS^n)$ is non-trivial only for degrees $k(n-1)$ and $k(n-1)+n$, for $k\geq 0$. The following sequences are exact for $k\geq 0$:*

$$0\longrightarrow H_{k(n-1)}(\Omega^-S^n))\xrightarrow{j_*}H_{k(n-1)}(LS^n))\longrightarrow 0$$

$$0\longrightarrow H_{k(n-1)+n}(LS^n)\xrightarrow{\partial^{Wang}}H_{k(n-1)}(\Omega^+S^n)\longrightarrow 0$$

If n is even, then $H_*(LS^n)$ is non-trivial only for degrees $k(n-1)$ and $2k(n-1)+n$, for $k\geq 0$. The following sequences are exact for $k\geq 0$:

$$0\longrightarrow H_{(2k+1)(n-1)}(\Omega^-S^n))\xrightarrow{j_*}H_{(2k+1)(n-1)}(LS^n))\longrightarrow 0$$

$$0\longrightarrow \frac{H_{(2k)(n-1)}(\Omega^-S^n))}{t^{Wang}(H_{(2k-1)(n-1)}(\Omega^+S^n))}\xrightarrow{j_*}H_{(2k)(n-1)}(LS^n))\longrightarrow 0$$

$$0\longrightarrow H_{(2k)(n-1)+n}(LS^n)\xrightarrow{\partial^{Wang}}H_{(2k)(n-1)}(\Omega^+S^n)\longrightarrow 0$$

Here,

$$\frac{H_{(2k)(n-1)}(\Omega^- S^n)}{t^{Wang}(H_{(2k-1)(n-1)}(\Omega^+ S^n))} = \frac{\mathbb{Z}\alpha_-^{2k(n-1)}}{2\mathbb{Z}\alpha_-^{2k(n-1)}}$$

for $k \geq 1$.

Consequently, for $n \geq 3$ odd and $m \geq 4$ even,

$$H_i(LS^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = k(n-1), k \geq 0 \\ \mathbb{Z} & \text{if } i = k(n-1) + n, k \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad H_i(LS^m) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = (2k+1)(m-1), k \geq 0 \\ \frac{\mathbb{Z}}{2\mathbb{Z}} & \text{if } i = (2k)(m-1), k \geq 1 \\ \mathbb{Z} & \text{if } i = (2k)(m-1) + n, k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

However, this is not enough to compute the loop product and coproduct geometrically. We need to represent the generators in homology by closed manifolds. We have already determined part of the manifold generators of $H_*(LS^n)$: those coming from the inclusion of the fiber, *i.e.*, $j_*\big(H_*(\Omega^- S^n)\big)$. For the sake of consistency, we'll consider the generators coming from $H_*(\Omega^+ S^n)$ instead of $H_*(\Omega^- S^n)$. Note that this choice makes no difference, since $\Omega^+ S^n$ and $\Omega^- S^n$ are isotopic submanifolds of LS^n.

The other generators we are missing are the ones that yield a generator of $H_*(\Omega^+ S^n)$, when we apply ∂^{Wang}. Accordingly, we introduce the manifolds Y_j and $s^* Y_j$ together with maps $\varphi_j : Y_j \to \Lambda S^n$ and $s^*\varphi_j : s^* Y_j \to \Lambda S^n$. It should be mentioned that the manifolds Y_j and the maps φ_j were first introduced in [AO] as completing manifolds of the Energy functional in the free loop space.

Definition 2.5. *Let* $Y \coloneqq \{(x, v, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : x \in S^n,\ v \in STS^n_x,\ y \in v^\perp \cap S^n\}$ *and*

$$p : Y \to S^n$$
$$(x, v, y) \mapsto x$$

Consider as well the map $\varphi : Y \to \Lambda S^n$ *that associates to each* (x, v, y) *the unique constant speed loop with initial velocity in the direction of* v *that parametrizes the circle* $(x + span_{\mathbb{R}}(v, y - x)) \cap S^n$*, if* $y \neq x$*. If* $y = x$*, then* $\varphi(x, v, x)$ *is the constant loop at* x*.*

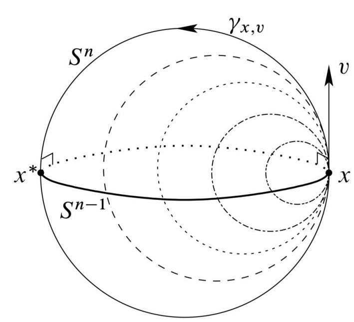

FIGURE 1. Manifold Y and map φ. Here $\gamma_{x,v}$ denotes the unique geodesic starting from x with initial velocity v and $x^* = -x$. Figure from [AO].

We know that $p:Y\to S^n$ is a $S^{n-1}\times S^{n-1}$-bundle and $\pi:Y\to STS^n$ is a S^{n-1}-bundle, where $\pi(x,v,y)=(x,v)$. We may then define its fibered product. Moreover, one easily extends the map φ to the fibered product by concatenating the loops resulting from each fiber.

Definition 2.6.

$$Y_k := Y\times_{S^n}\dots\times_{S^n} Y$$

k times. Moreover, $p_k := p\times\dots\times p$ $\quad k$ times. And,

$$\begin{aligned}\varphi_k : Y_k &\to LS^n\\ (x,u_1,y_1,\dots,u_n,y_n) &\mapsto \varphi(x,u_1,y_1)\cdot\dots\cdot\varphi(x,u_k,y_k)\end{aligned}$$

When n is odd, one can choose a section $s:S^n\to STS^n$. Hence we may pullback the previous fiber bundles and make the following definition:

Definition 2.7. *Let $s^*Y := \{(x,y) : x\in S^n,\ y\in s(x)^\perp\cap S^n\}$ and*

$$\begin{aligned}s^*p : s^*Y &\to S^n\\ (x,y) &\mapsto x\end{aligned}$$

Furthermore, let $s^*\varphi(x,y) := \varphi(x,s(x),y)$.

Similarly, we can extend these definitions to the fibered products:

Definition 2.8.

$$s^*Y_k := s^*Y\times_{S^n}\dots\times_{S^n} s^*Y$$

k times. Moreover, $s^*p_k := s^*p\times_{S^n}\dots\times_{S^n} s^*p$ $\quad k$ times. And,

$$\begin{aligned}s^*\varphi_k : s^*Y_k &\to LS^n\\ (x,y_1,\dots,y_k) &\mapsto s^*\varphi(x,y_1)\cdot\dots\cdot s^*\varphi(x,y_n)\end{aligned}$$

Recalling the formula for ∂^{Wang}, we can use the naturality of the Mayer-Vietoris sequence to get $\partial_M\circ(\varphi_j)_* = (\varphi_j)_*\circ\partial_M$. And when we apply ∂_M to Y_j we get the fundamental class of $p_j^{-1}(S^{n-1})\cong S^{n-1}\times(S^{n-1}\times S^{n-1})^j$, when given the boundary orientation. Finally, when we apply pr^F we essentially remove the first S^{n-1} resulting in $(\varphi_j)_*[p_j^{-1}(N)]$, *i.e.*, the image via $(\varphi_j)_*$ of the fundamental class of the fiber of Y_j at the North pole. Then, the following holds:

Proposition 2.9. *For n even,*

$$\partial^{Wang}((\varphi_j)_*[Y_j]) = (-1)^j\alpha_+^{2j}$$

And, for n odd,

$$\partial^{Wang}((s^*\varphi_j)_*[s^*Y_j]) = -\alpha_+^j$$

Finally, we have a full description of the generators of $H_*(LS^n)$ as an abelian group:

Definition 2.10. *Let $a_m := \sigma_+^m[(S^{n-1})^m]$ for $m \geq 0$. Let $b_0 = c_*[S^n]$, where $c : S^n \to LS^n$ is the constant loop section. For n odd, let $b_m := (s^*\varphi_m)_*[s^*Y_m]$ for $m \geq 1$.*
For n even, let $b_{2m} := (\varphi_m)_*[Y_m]$ for $m \geq 1$.

Theorem 2.11. *For n odd, the homology of $H_*(LS^n)$ is generated (as an abelian group) by a_m and b_m for $m \geq 0$. Moreover, the relative homology $H_*(LS^n, S^n)$ is generated by a_m and b_m for $m \geq 1$.*
For n even, the homology of $H_*(LS^n)$ is generated by a_m and b_{2m} for $m \geq 0$. Furthermore, the relative homology $H_*(LS^n, S^n)$ is generated by a_m and b_{2m} for $m \geq 1$.

3. Loop Product

The loop product is a binary operation on $H_*(LM)$ that is derived from the intersection product of the base space and the Pontryagin product (concatenation of loops) on the fibers. Indeed, the original construction in [MCDS] tries to capture the following geometric idea: consider two families of loops parametrized by a manifolds A and B, respectively, such that their basepoints intersect transversely; then the loop product yields the a family of loops parametrized by the intersection of the basepoints, where at each intersection point we concatenate the original loops.

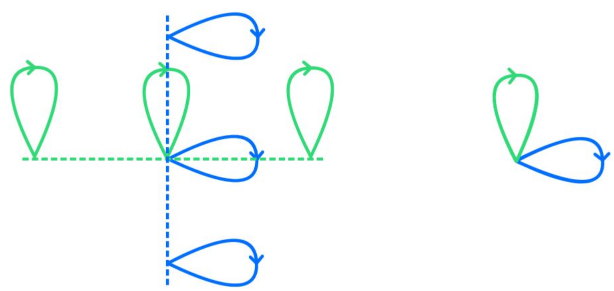

Figure 2. Loop product of two families of loops whose basepoints are parametrized by the dashed lines.

Recall that the intersection product on an oriented manifold M, usually defined using Poincaré duality, may be expressed via the Thom isomorphism. Indeed, let A and B be two closed oriented submanifolds that intersect transversally. Then the intersection product of their fundamental classes, $[A]\cdot[B]$, is given by the cap product of $[A]$ with the Thom class of the normal bundle of B, $\tau_B \cap [A]$ (see section 11 of chapter VI of [GB]). Accordingly, in this section we give a definition of loop product (definition 3.2) using a Thom-Pontryagin construction and present the geometric result from [NHNW] (proposition 3.3). Then, we apply the latter to the generators a_i and b_j of the previous section to compute the loop product in the case of spheres. But before introducing the loop product, we need the following definition.

Definition 3.1. *Let (M, g) be a closed Riemannian manifold. Let U_M be the tubular neighbourhood of $\Delta M \subset M \times M$ defined as follows:*

$$U_M := \{(x,y) \in M \times M |\ |x-y| < \epsilon\}$$

Moreover, let $\tau_M \in H^n(N(\Delta M), N(\Delta M) \setminus \Delta M) \cong H^n(U_M, U_M \setminus \Delta M)$ *be the Thom class of* $N(\Delta M)$*, when this vector bundle has the orientation making the following isomorphism orientation-preserving:*

$$T(M \times M)|_{\Delta M} \cong N(\Delta M) \bigoplus T(\Delta M)$$

Here, $\epsilon > 0$ *is a constant smaller than the injectivity radius.*

Now Let

$$\Lambda M \times_M \Lambda M := \{(\gamma, \eta) \in \Lambda M \times \Lambda M |\ \gamma(0) = \eta(0)\}$$

and let $U_{CS} := (ev \times ev)^{-1}(U_M) = \{(\gamma,\eta)|\ |\gamma(0) - \eta(0)| < \epsilon\}$. Then there is a retraction $R_{CS} : U_{CS} \to \Lambda M \times_M \Lambda M$. Given $(c,l) \in U_{CS}$, $R_{CS}(c,l)$ can be described as follows: it follows c from 0 to 1; then it follows the minimal geodesic connecting $c(0)$ and $l(0)$; then it follows l from 0 to 1; and, finally, it follows the minimal geodesic connecting $l(0)$ and $c(0)$. Now we can give the definition of the loop product.

Definition 3.2. *Let* $\tau_{CS} = (ev \times ev)^*(\tau_M)$*. Then, the loop product of two homology classes* $x, y \in H_*(LM)$*, denoted by* $x \wedge_{Th} y$ *, is the defined as follows:*

$$concat_* \circ (R_{CS})_*(\tau_{CS} \cap exc(x \times y))$$

Here exc *denotes the excision isomorphism between* $(\Lambda M^2, \Lambda M^2 \setminus \Lambda M \times_M \Lambda M)$ *and* $(U_{CS}, U_{CS} \setminus \Lambda M \times_M \Lambda M)$*, and* $concat$ *denotes the concatenation of loops.*

The geometric computation result of [NHNW] is the following:

Proposition 3.3. *Assume an* i*-cycle* $a \in H_i(\Lambda M)$ *and a* j*-cycle* $b \in H_j(\Lambda M)$ *are represented by oriented closed manifolds* $f_1 : Z_1 \to \Lambda M$ *and* $f_2 : Z_2 \to \Lambda M$*. Suppose the maps* $ev \circ f_1 : Z_1 \to M$ *and* $ev \circ f_2 : Z_2 \to M$ *are transverse. Then,*

$$(f_1)_*[Z_1] \wedge_{Th} (f_2)_*[Z_2] = (f_1 * f_2)_*[Z_1 \times_{ev} Z_2]$$

Here,

$$\begin{aligned}(f_1 * f_2) : Z_1 \times_{ev} Z_2 &\to \Lambda M \\ (x,y) &\mapsto concat(f_1(x), f_2(y))\end{aligned}$$

and $Z_1 \times_{ev} Z_2 := (ev \circ f_1 \times ev \circ f_2)^{-1}(\Delta(M)) \subset Z_1 \times Z_2$ *has the orientation making the following isomorphism orientation-preserving:*

$$T(Z_1 \times Z_2)_{Z_1 \times_{ev} Z_2} \cong (ev \circ f_1 \times ev \circ f_2)^* N(\Delta M) \bigoplus T(Z_1 \times_{ev} Z_2)$$

Then, we can apply the previous result to express the loop product in terms of the generators a_i and b_j of the previous section. The transversality condition is always satisfied, except in $a_i \wedge_{Th} a_j = (\sigma_+^i)_*[(S^{n-1})^i] \wedge_{Th} (\sigma_+^j)_*[(S^{n-1})^j]$. In this case, we can change the basepoint of the map σ_+^j. For instance, let R be a small planar rotation such that $R(N) \neq N$. Then $ev \circ R \circ \sigma_+^j \neq N = ev \circ \sigma_+^i$ and $R \circ \sigma_+^j \simeq \sigma_+^j$. Consequently, $ev \circ \sigma_+^i$ and $ev \circ R \circ \sigma_+^j$ are transverse because they don't intersect and, hence, $a_i \wedge_{Th} a_j = 0$. For the other cases, we observe that $(S^{n-1})^i \times_{ev} Y_j = (S^{n-1})^i \times p_j^{-1}(N) \cong (S^{n-1})^{i+2j}$; $Y_i \times_{ev} Y_j = Y_{i+j}$; $(S^{n-1})^i \times_{ev} s^*Y_j = (S^{n-1})^i \times s^*p_j^{-1}(N) \cong (S^{n-1})^{i+j}$ and $s^*Y_i \times_{ev} s^*Y_j = s^*Y_{i+j}$.

Theorem 3.4.
For n even,

- $a_i \wedge_{Th} a_j = 0$
- $a_i \wedge_{Th} b_{2j} = (-1)^j a_{i+2j}$
- $b_{2i} \wedge_{Th} b_{2j} = b_{2(i+j)}$

For n odd,

- $a_i \wedge_{Th} a_j = 0$
- $a_i \wedge_{Th} b_j = -a_{i+j}$
- $b_i \wedge_{Th} b_j = b_{i+j}$

4. Loop Coproduct

The loop coproduct is a coproduct on the relative homology of the free loop space, $H_*(\Lambda M, M)$. Geometrically this amounts to an operation taking $H_*(\Lambda M, M)$ to $H_*(\Lambda M \times \Lambda M, M \times \Lambda M \cup \Lambda M \times M)$, obtained by cutting loops along their self-intersections with their base points. The relative homology appears since we want to exclude the trivial self-intersections on constant loops. It is a more recent construction that was introduced in [MGNH].

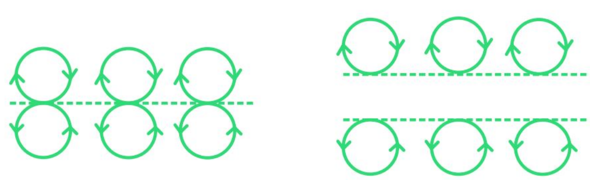

Figure 3. Loop coproduct of a family of loops whose basepoints are parametrized by the dashed line.

Definition 4.1. *Let $\mathcal{F} := \{(\gamma, s) \in \Lambda M \times I | \, \gamma(0) = \gamma(s)\}$. Then we define the map*

$$cut : \mathcal{F} \to \Lambda M \times \Lambda M$$
$$(\gamma, s) \mapsto (\gamma_{[0,s]}, \gamma_{[s,1]})$$

Now let e_I be the following map:

$$\begin{aligned} e_I : \Lambda M \times I &\to M \times M \\ (\gamma, t) &\mapsto (\gamma(0), \gamma(t)) \end{aligned}$$

and let $U_{GH} := e_I^{-1}(U_M) = \{(\gamma, s) | \, |\gamma(0) - \gamma(s)| < \epsilon\}$. Then there is a retraction $R_{GH} : U_{GH} \to \mathcal{F}$. Given $(c, s) \in U_{GH}$, $R_{GH}(c, s)$ can be described as follows: it follows c until time s; then goes from $c(s)$ to $c(0)$ via the minimal geodesic connecting them; then goes from $c(0)$ to $c(s)$ via the minimal geodesic connecting them; and finally it follows c from s until 1. Now we can give the definition of loop coproduct.

Definition 4.2. *Let $\tau_{GH} = e_I^*(\tau_M) \in H^n(U_{GH}, \mathcal{F}^c)$ be its Thom class. Then, the loop coproduct, denoted by $\vee_{Th}$, is the degree $1-n$ map on $H_*(LM, M)$ that is defined as follows for a class $x \in H_*(LM, M)$:*

$$cut_* \circ (R_{GH})_* (\tau_{GH} \cap exc(x \times I))$$

Here $I \in H_1([0,1], \{0,1\})$ denotes the (relative) fundamental class of $[0,1]$ and exc denotes the excision isomorphism between $(\Lambda M \times I, \mathcal{F}^c \cup M \times I \cup \Lambda M \times \partial I)$ and $(U_{GH}, \mathcal{F}^c \cup M \times I \cup \Lambda M \times \partial I)$.

The geometric computation result of [NHNW] is the following:

Proposition 4.3. *Assume a relative k-cycle $a \in H_k(\Lambda M, M)$ is represented by an oriented manifold pair $f : (Y, L) \to (\Lambda M, M)$. Let $Y_{\mathcal{B}} := L \times I \cup Y \times \partial I$ and*

$$\begin{aligned} E(f) : (Y \times I) \setminus Y_{\mathcal{B}} &\to M \times M \\ (x, s) &\mapsto (f(x)(0), f(x)(s)) \end{aligned}$$

Suppose that $E(f)$ is smooth and transverse to the diagonal map $\Delta : M \to M \times M$.
Now let $Y_\Delta := E(f)^{-1}(\Delta M)$ and $\overline{Y_\Delta}$ be its closure inside $Y \times I$. There is a natural orientation on $\overline{Y_\Delta}$ induced by the isomorphism $T_{(x,s)}(Y \times I) \cong N_{E(f)(x,s)} \Delta M \bigoplus T_{(x,s)} Y_\Delta$. Then,

$$\vee_{Th} a = \big(cut \circ (f \times id)\big)_* [\overline{Y_\Delta}]$$

I.e., $\vee_{Th} a$ is represented by $\overline{Y_\Delta}$ via $cut \circ (f \times id)$.

Contrary to the previous section, our manifold representatives don't satisfy the transversality condition. To overcome this challenge, we must suitably deform the maps. After this, we can apply the previous geometric result to get the following:

Theorem 4.4.

$$\vee_{Th} a_i = \sum_{k=2}^{i-1} (-1)^{(i+k+1)(n-1)} a_{k-1} \times a_{i-k}$$

For n odd,

$$\vee_{Th} b_j = -\sum_{k=1}^{j-1} \big(b_{k-1} \times a_{j-k} + a_{k-1} \times b_{j-k}\big)$$

For n even,

$$\vee_{Th} b_{2j} = \sum_{k=1}^{j-1} \big((-1)^{j-k-1} b_{2k} \times a_{2(j-k)-1} + (-1)^k a_{2k-1} \times b_{2(j-k)}\big)$$

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