Notes on Arakelov Theory

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1 Introduction

These notes provide a more or less detailed account on the intersection theory for divisors on an arithmetic surface, which was first introduced by Suren Arakelov in 1974 [1]. Important contributions to the topic were made by Gerd Faltings in 1984 [5], developing a series of ideas that would lead to the proof of Mordell's conjecture.

These works constitute the starting point of the theory that later received the name of arithmetic intersection theory, developed after 1990 after the introduction of arithmetic Chow groups by Henri Gillet and Christophe Soulé [6] and the generalisation of the theory of Arakelov to arithmetic schemes of arbitrary dimension. A very important part of the theory is the development of a theory of arithmetic K-groups and the connections between arithmetic K-theory and arithmetic Chow theory. An important example of such a connection is the arithmetic Riemann-Roch theorem of Gillet and Soulé [7], which was first proved in the case of surfaces by Faltings in [5].

The contents of these notes cover the basics which are necessary to define the Arakelov intersection pairing and to show its invariance under linear equivalence. Very little knowledge of analytic geometry has been assumed, and this is why there is a whole section containing the necessary ingredients: integration of differential forms on and the existence of the Green function of a compact analytic manifold.

It is not discarded that these notes will be enlarged in the future to include the proof of the adjunction formula, Riemann-Roch theorem and some results on Arakelov intersection theory on models of elliptic curves.

2 Ingredients of complex geometry

We review a few notions of complex geometry that play a very important role in Arakelov theory. We refer to [12] for a more detailed account on the topics covered in this section.

Let $i = \sqrt{-1}$.

2.1 Complex structure on and complexification of a real vector space

Definition 2.1. Let V be a real vector space. A complex structure on V is an $\mathbb{R}$-linear endomorphism $J : V \to V$ such that $J^2 = -\mathrm{Id}_V$.

A real vector space V together with a complex structure J can be equipped with the structure of a complex vector space by defining

$$(x+iy)v := xv + yJ(v), \quad x, y \in \mathbb{R},\, v \in V. \tag{1}$$

We shall denote the resulting complex vector space by V_J.

Conversely, if V is a complex vector space, then it may also be considered as a real vector space by restriction of scalars. Scalar multiplication by i is an $\mathbb{R}$-linear endomorphism which induces a complex structure on V.

Example 2.2. The components of any element $(z_1, \ldots, z_n) \in \mathbb{C}^n$ may be written in the form $z_j = x_j + iy_j$ for some $x_j, y_j \in \mathbb{R}$, $1 \le j \le n$. The vector $(z_1, \ldots, z_n)$ may then be identified with the vector $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$. The complex structure on $\mathbb{R}^{2n}$ induced by scalar multiplication by i is

$$J(x_1, y_1, \ldots, x_n, y_n) = (-y_1, x_1, \ldots, -y_n, x_n). \tag{2}$$

Of course, there are other different complex structures on $\mathbb{R}^{2n}$, some of which are not $\mathbb{C}$-linearly equivalent to $\mathbb{C}^n$.

Definition 2.3. Let V be a real vector space. The complex vector space

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \tag{3}$$

obtained by base change is called the complexification of V.

A complex structure J on V extends to a $\mathbb{C}$-linear endomorphism of $V_{\mathbb{C}}$ by setting

$$v \otimes z \mapsto J(v) \otimes z \tag{4}$$

for any $v \in V$, $z \in \mathbb{C}$. We will also denote this map by J. The identity $J^2 = -\mathrm{Id}_{V_{\mathbb{C}}}$ still holds; it follows that J has two eigenvalues $\pm i$.

Definition 2.4. We will denote by $V^{1,0}$ (resp. $V^{0,1}$) the eigenspace of $V_{\mathbb{C}}$ of eigenvalue i (resp. $-i$).

Hence, we have a decomposition of complex vector spaces

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}. \tag{5}$$

Definition 2.5. Conjugation of complex numbers extends to an $\mathbb{R}$-linear automorphism of $V_{\mathbb{C}}$ by setting

$$\overline{v \otimes z} := v \otimes \overline{z}, \quad v \in V,\, z \in \mathbb{C}. \tag{6}$$

It is easy to check that $\overline{V^{1,0}} \simeq_{\mathbb{R}} V^{0,1}$ and that V_J is $\mathbb{C}$-linearly isomorphic to $V^{1,0}$.

2.2 Affine complex manifolds, complex manifolds and local coordinates

We start discussion of complex manifolds by showing the very important example of open sets in $\mathbb{C}^n$.

Definition 2.6. The polydisk of radius $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \mathbb{R}_{>0}$ centered at $a \in \mathbb{C}^n$ is the following set:

$$\Delta^n_{a,\varepsilon} := \{z \in \mathbb{C}^n;\ |z_i - a_i| < \varepsilon_i\}. \tag{7}$$

The topology on $\mathbb{C}^n$ obtained by taking the set of all polydisks as a basis is called the complex topology. It is equivalent to the euclidean topology.

Definition 2.7. Let U be a domain in $\mathbb{C}^n$ (i.e. U is open and connected). A function $f : U \to \mathbb{C}$ is holomorphic (or analytic) at $a \in U$ if and only if there exists a polydisk $\Delta^n_{a,\varepsilon} \subset U$ in which f can be represented as a convergent power series

$$f(z) = \sum_{\alpha \geq 0} c_\alpha (z-a)^\alpha, \tag{8}$$

where $c_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $(z-a)^\alpha = (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}$. We say that f is holomorphic on U if it is holomorphic at every point in U.

As being holomorphic is a local property, the rings of holomorphic functions on open sets of U define a sheaf $\mathcal{H}_U$. This sheaf is obviously a subsheaf of the sheaf of continuous functions on U, and the pair $(U, \mathcal{H}_U)$ is a ringed space.

Definition 2.8. An affine complex manifold of dimension n is a ringed space of the form $(U, \mathcal{H}_U)$, with U a domain in $\mathbb{C}^n$ and $\mathcal{H}_U$ the sheaf of holomorphic functions on it.

Definition 2.9. A complex manifold of dimension n is a ringed topological space $(X, \mathcal{H})$ which is Hausdorff and regular and such that it is locally isomorphic to an affine complex manifold. We will call the sheaf $\mathcal{H}$ the sheaf of holomorphic functions on X.

Remark 2.10. Polynomials are a good example of holomorphic functions. We may establish relations between algebraic sets and affine complex manifolds, which lead to very deep relations between complex varieties and complex manifolds. This is the object of study of Serre's GAGA [11]. At the heart of these relations there is a functor from projective complex varieties to compact complex manifolds which is an equivalence of categories.

On a different level, the identification $\mathbb{C} \simeq \mathbb{R}^2$ lets us realise that X is locally isomorphic to an open set of $\mathbb{R}^{2n}$, and thus it has the structure of a (real) smooth manifold of dimension $2n$.

According to this, a complex variety can be viewed both as a complex manifold and as a real smooth manifold. The interplay between these different structures is one key ingredient of Arakelov theory.

Remark 2.11. In complex analysis of several variables, the adjectives *holomorphic* and *analytic* are used as synonyms, and their definition is as in 2.7. This is a stronger property than being complex-differentiable (i.e: satisfying the Cauchy-Riemann equations). More precisely, for an open set $U \subset \mathbb{C}^n$, a function $f : U \to \mathbb{C}$ is holomorphic if and only if it satisfies the Cauchy-Riemann equations and it is square-integrable.

A complex manifold $(X, \mathcal{H})$ may be covered by open sets U_i such that, for every i, U_i is isomorphic to an affine complex manifold $U \subset \mathbb{C}^n$. This means that points in $P \in U_i$ may be identified with their complex coordinates $z = (z_1, \dots, z_n) \in U$ and that holomorphic functions in $\mathcal{H}(U_i)$ may be identified with holomorphic functions on U.

If $z_1, \dots, z_n$ are complex coordinates on an open set of X, we may write $z_j = x_j + iy_j$, $1 \leq j \leq n$ with $x_1, y_1, \dots, x_n, y_n$ real coordinates on the same open set of $X_\mathbb{R}$. By setting $\overline{z_j} = x_j - iy_j$, we also have $z_1, \overline{z_1}, \dots, z_n, \overline{z_n}$ as real coordinates on the same open set.

2.3 Differential study of complex manifolds

Until the end of the section, $(X, \mathcal{H})$ will denote a complex manifold of dimension n.

The symbol $\mathcal{C}_\mathbb{R}$ will denote the sheaf of real-valued smooth functions on X, in the sense of differential geometry, and $\mathcal{C} := \mathcal{C}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ will denote the sheaf of complex-valued smooth functions on X. We have natural inclusions of sheaves $\mathcal{C}_\mathbb{R} \subset \mathcal{C}$ and $\mathcal{H} \subset \mathcal{C}$.

In order to avoid confusion, we will denote the real manifold given by $(X, \mathcal{C}_\mathbb{R})$ by $X_\mathbb{R}$.

Let TX and T^*X denote respectively the holomorphic tangent and cotangent bundles of X. These are locally free $\mathcal{H}$-modules of rank n. Similarly, denote by $TX_\mathbb{R}$ and $T^*X_\mathbb{R}$ the smooth tangent and cotangent bundles on $X_\mathbb{R}$. Let us consider the complexifications

$$TX_\mathbb{C} = TX_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}, \quad T^*X_\mathbb{C} = T^*X_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}. \tag{9}$$

The linear algebra developed in the previous section applies in this situation without much effort.

Let us focus on $TX_\mathbb{C}$; the definitions, results and notations for $T^*X_\mathbb{C}$ are completely analogous.

Proposition 2.12. *Let $x \in X$. The fibre $T_xX_\mathbb{R}$ is canonically isomorphic (as real vector spaces) to the underlying real vector space of T_xX. This induces a complex structure J_x on $T_xX_\mathbb{R}$.*

Proof. We take three steps to construct the complex structure. Further details may be found in [12], Chap. 1, Example 3.2.

Step 1. For the affine manifolds $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, the complex structure is defined by example 2.2.

Step 2. Fix an affine holomorphic neighbourhood $x \in U \subset X$. Define $J_x : T_xX_\mathbb{R} \to T_xX_\mathbb{R}$ by pull-back of the structure defined in Step 1.

Step 3. J_x is well-defined. This follows from the Cauchy-Riemann equations of the biholomorphic change of coordinates that arises when picking a different affine holomorphic neighbourhood. □

Definition 2.13. If $J : TX_\mathbb{R} \to TX_\mathbb{R}$ is a $\mathcal{C}_\mathbb{R}$-linear endomorphism such that for every $x \in X$ J_x is a complex structure on $T_xX_\mathbb{R}$, we shall say that $(X_\mathbb{R}, J)$ is an almost complex manifold.

Proposition 2.14. *The complex manifold $(X, \mathcal{H})$ induces the structure of an almost complex manifold on $X_{\mathbb{R}}$.*

Proof. By proposition 2.12, we only need to show that $J : TX_{\mathbb{R}} \to TX_{\mathbb{R}}$ is a morphism of $\mathcal{C}_{\mathbb{R}}$-modules. In other words, we need to show that the map J_x is smooth with respect to $x \in X$. We check $\mathcal{C}_{\mathbb{R}}$-linearity locally. Let $U \subset X$ be an open set such that

$$TX_{\mathbb{R}|U} \simeq \mathcal{C}_{\mathbb{R}}^{2n}{}_{|U}. \tag{10}$$

In this trivialization, for sections $(\xi_1, \eta_1, \dots \xi_n, \eta_n)$, we have

$$J(\xi_1, \eta_1, \dots, \xi_n, \eta_n) = (-\eta_1, \xi_1, \dots, -\eta_n, \xi_n), \tag{11}$$

which is easily seen to be $\mathcal{C}_{\mathbb{R}}$-linear. □

Remark 2.15. Almost complex structures on complex manifolds do not necessarily arise from the complex structure. As an example, the sphere S^6 carries an almost complex structure induced by S^7 viewed as the set of octonions of norm 1. It is unknown if S^6 carries a complex structure.

As $TX_{\mathbb{R}}$ has a complex structure J, we can extend J to a $\mathcal{C}$-linear automorphism of $TX_{\mathbb{C}}$ satisfying $J^2 = -\mathrm{Id}_{TX_{\mathbb{C}}}$. The linear algebra from the previous section applies without further effort. In particular,

- We have a $\mathcal{C}$-linear decomposition $TX_{\mathbb{C}} = TX^{1,0} \oplus TX^{0,1}$.
- Complex conjugation interchanges the eigenspaces: $\overline{TX^{1,0}} = TX^{0,1}$.
- There are canonical $\mathcal{C}$-linear isomorphisms $TX \simeq (TX_{\mathbb{R}})_J \simeq TX^{1,0}$.

Remark 2.16. We shall identify TX and $TX^{1,0}$ from now on. We shall also use the notations $T^*X^{1,0}$, $T^*X^{0,1}$, etc. for the objects which are analogous to those we have constructed when TX is replaced by T^*X.

We briefly indicate how complex manifolds are oriented real manifolds.

Definition 2.17. Let $(X, \mathcal{C}_{\mathbb{R}})$ be a smooth real manifold, and let $\mathcal{F}$ be a rank k locally free $\mathcal{C}_{\mathbb{R}}$-sheaf on X. We say that $\mathcal{F}$ is orientable if there exists an affine open cover $X = \cup_\alpha U_\alpha$ and a non-zero section $\omega_\alpha \in \wedge^k \mathcal{F}_{|U_\alpha}$ for every α such that at every point $x \in U_\alpha \cap U_\beta$ there is a positive real number λ such that $(\omega_\alpha)_x = \lambda(\omega_\beta)_x$. If $\mathcal{F}$ is orientable, then the collection $\{(U_\alpha, \omega_\alpha)\}$ is called an *orientation* of $\mathcal{F}$. Finally, we say that X is orientable if its tangent bundle admits an orientation.

We remark that having chosen an orientation on a complex real manifold, the coordinate change maps

$$\rho_{\alpha,\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta \tag{12}$$

have positive jacobian.

Proposition 2.18. *A complex manifold is an oriented real smooth manifold.*

Proof. This is a consequence of the Cauchy-Riemann equations. □

It is important to stress the fact that if $(X, \mathcal{H})$ is a complex manifold, then $X_{\mathbb{R}}$ has a natural orientation. From now on, and especially when defining integration, we will assume that this is the integration given on $X_{\mathbb{R}}$.

2.4 Sheaves of complex differential forms

We shall work directly with the sheaf $\mathcal{C}$ of complex valued smooth functions on X, as we are primarily interested in the relation between this sheaf and $\mathcal{H}$.

Definition 2.19. We denote by $\mathcal{E}^k$ the sheaf of complex differential k-forms on $(X, \mathcal{C})$. Hence,

- $\mathcal{E}^0 := \mathcal{C}$,
- $\mathcal{E}^1 := T^*X_{\mathbb{C}}$,
- $\mathcal{E}^k := \bigwedge^k \mathcal{E}^1$, $k \geq 0$.

Lemma 2.20 (Poincaré)**.** *Exterior differentiation extends to the complex $\mathcal{E}^\bullet$ to give an exact sequence*

$$0 \to \mathbb{C} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to \cdots . \tag{13}$$

Definition 2.21. We have $\mathcal{E}^1_X = T^*X_{\mathbb{C}}$. Let $\mathcal{E}^{1,0} := T^*X^{1,0}$ and $\mathcal{E}^{0,1} := T^*X^{0,1}$. By taking exterior powers, the decomposition $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ carries on to a decomposition

$$\mathcal{E}^k = \bigoplus_{p+q=k} \mathcal{E}^{p,q}, \tag{14}$$

where

$$\mathcal{E}^{p,q} := \bigwedge^p \mathcal{E}^{1,0} \wedge \bigwedge^q \mathcal{E}^{0,1}. \tag{15}$$

We will refer to $\mathcal{E}^{p,q}$ as the sheaves of complex differential forms on X of type (p, q).

Exterior differentiation maps forms of type (p, q) to $\mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}$, and hence can be written as the sum of two morphisms $d = \partial + \overline{\partial}$, with

$$\partial : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}, \quad \overline{\partial} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}. \tag{16}$$

Definition 2.22. We will say that ∂ is the holomorphic exterior differential operator, and that $\overline{\partial}$ is the antiholomorphic exterior differential.

Proposition 2.23. *We have $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$.*

Proof. We have $0 = d^2 = (\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$. By comparison of the different types of forms, we reach the conclusion. □

Definition 2.24. The module Ω^p of holomorphic differential p-forms on X consists of exactly those forms in $\mathcal{E}^{p,0}$ that are mapped to zero by the antiholomorphic exterior differential. In other words,

$$\Omega^p = \operatorname{Ker}\left(\overline{\partial} : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1}\right). \tag{17}$$

It is not difficult to see that the sequence

$$0 \to \mathbb{C} \to \Omega^0 \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \Omega^2 \to \cdots \tag{18}$$

is a resolution of the constant sheaf $\mathbb{C}$.

What has been explained in this section constitutes a starting point for the study of Hodge theory. This theory is concerned with the study of the cohomology groups $H^{p,q}(X) := H^p(X, \Omega^q)$ and their properties.

2.5 Integration of forms on a complex manifold

In this section we review the basic definitions of integrals of differential forms on a complex manifold $(X, \mathcal{H})$ of complex dimension n. In fact, it is only necessary to take the underlying differential structure of $X_{\mathbb{R}}$ into account. A simple introduction to the topic may be found in [4]. We do not proof the existence of partitions of unity. Integration of differential forms on a smooth manifold depends on the choice of an orientation; we assume $X_{\mathbb{R}}$ is endowed with the orientation coming from the complex structure.

For simplicity in the exposition, let us assume that X is compact: an analogous theory of integration may be obtained if we replace differential forms on a compact manifold by compactly supported differential forms on a not necessarily compact manifold. Also for simplicity, we will only define integration of volume forms, that is, sections of $\mathcal{E}^{2n}$; integration of forms of lower degree may also be defined by using the corresponding definitions of integrals of forms in euclidean space.

Let $\omega \in \mathcal{E}^{2n}(U)$ for some open set $U \subset X$. Let K be the support of ω, that is the closure of the set of points $x \in X$ such that $\omega_x \neq 0$. The set K is compact in X. We define the integral of ω on U in three steps.

Step 1. Assume there is an affine neighbourhood U_α such that $K \subset U_\alpha$. Then, in local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ we have

$$\omega = f_\alpha \cdot dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \tag{19}$$

for some $f_\alpha \in \mathcal{C}(U_\alpha)$. We define

$$\int_X \omega = \int_K \omega = \int_{U_\alpha} f_\alpha \cdot dx_1 dy_1 \cdots dx_n dy_n, \tag{20}$$

where the right-hand side is to be computed as an integral in $\mathbb{R}^{2n}$.

Step 2. It is necessary to check that this definition is consistent with changes of coordinates. Assume $K \subset U_\beta$ for some other affine open set of X. Let $(\xi_1, \eta_1, \ldots, \xi_n, \eta_n)$ be coordinates on U_β. The isomorphism

$$\rho_{\alpha,\beta} : U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta \tag{21}$$

that changes coordinates from U_β to U_α is a smooth function between two open sets in $\mathbb{R}^{2n}$ which preserves orientation, i.e: its jacobian $J(\rho_{\alpha,\beta})$ is positive. We have

$$\omega_\beta = J(\rho_{\alpha,\beta}) f_\beta \cdot d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n, \tag{22}$$

where $f_\beta = f_\alpha \circ \rho_{\alpha,\beta}$. The jacobian transformation formula tells us

$$\int_{U_\alpha} f_\alpha \cdot dx_1 dy_1 \ldots dx_n dy_n = \int_{U_\beta} J(\rho_{\alpha,\beta}) f_\beta \cdot d\xi_1 d\eta_1 \ldots d\xi_n d\eta_n, \tag{23}$$

giving the invariance of the definition of the integral with respect to the affine neighbourhood in which the support of ω is contained. Notice that this definition is not correct if an orientation of X is not chosen beforehand.

Step 3. We need to extend the definition to forms whose support is not necessarily contained in an affine neighbourhood. Let $\{U_i\}_{1\leq i\leq r}$ be a finite affine covering of X, and let $\{\varphi_i\}_i$ be a partition of unity subordinated to this covering. Let us recall what this means: for every i, φ_i is a smooth real-valued function on X, the support of φ_i is contained in U_i, and $0 \leq \varphi_i \leq 1$. Finally, we have $\sum_{i=1}^r \varphi_i = 1$.

The support of the form $\varphi_i\omega$ is contained in U_i. By using the previous steps, we may define

$$\int_X \omega = \sum_{i=1}^r \int_X \varphi_i\omega. \tag{24}$$

Step 4. This last definition does not depend on the choice of finite affine covering and partition of unity subordinated to the covering. Suppose $\{V_j\}_{1\leq j\leq s}$ is another finite affine covering of X and that $\{\psi_j\}_j$ is a partition of unity subordinated to this covering.

We have that $\{U_i \cap V_j\}_{i,j}$ is a new finite affine covering of X, and that $\{\varphi_i\psi_j\}_{i,j}$ is a partition of unity subordinated to this covering. We have

$$\sum_{i=1}^r \int_X \varphi_i\omega = \sum_{i=1}^s \int_X \varphi_i \left(\sum_{j=1}^s \psi_j\right) \omega = \sum_{i,j} \int_X \varphi_i\psi_j\omega. \tag{25}$$

On the other hand, we also have

$$\sum_{j=1}^s \int_X \psi_j\omega = \sum_{j=1}^s \int_X \psi_j \left(\sum_{i=1}^r \varphi_i\right) \omega = \sum_{i,j} \int_X \varphi_i\psi_j\omega, \tag{26}$$

and hence the two definitions of the integral are equivalent.

2.6 Hermitian metrics on a locally free sheaf

Let $\mathcal{F}$ be a locally free coherent $\mathcal{H}$-module.

Definition 2.25. An hermitian metric on $\mathcal{F}$ is a pairing

$$\langle\cdot,\cdot\rangle : \mathcal{F} \times \mathcal{F} \to \mathcal{C} \tag{27}$$

which defines an hermitian inner product on the fibres $\mathcal{F}_{(P)} := \mathcal{F}_P/\mathfrak{m}_{X,P}\mathcal{F}_P$ of $\mathcal{F}$ above any point $P \in X$. We recall that the fibre $\mathcal{F}_{(P)}$ is a finite dimensional $\mathbb{C}$-vector space. We shall say that $(\mathcal{F}, \langle\cdot,\cdot\rangle)$ is an hermitian sheaf. The complex manifold $(X, \mathcal{H})$ is said to be an hermitian manifold if TX is equipped with an hermitian metric.

Remark 2.26. The following are consequences of the previous definition, for an hermitian sheaf $(\mathcal{F}, \langle\cdot,\cdot\rangle)$ on X:

1. The pairing $\langle\cdot,\cdot\rangle$ is $\mathcal{H}$-linear in the first component.
2. For any two sections ξ, η of $\mathcal{F}$, we have $\langle\xi,\eta\rangle = \overline{\langle\eta,\xi\rangle}$.
3. The pairing is positive definite, which means that for any section ξ of $\mathcal{F}$, the smooth function $\langle\xi,\xi\rangle$ takes positive real values.

Theorem 2.27 ([12], Chap. 3, Theorem 1.2)**.** *Every locally free coherent $\mathcal{H}$-module admits an hermitian metric.* □

There is a natural way to define metrics on all the usual $\mathcal{H}$-module operations such as direct sums, tensor products, homomorphism spaces, duals, etc. provided that the modules we start with are equipped with an hermitian metric.

2.7 Riemann surfaces. Curvature and volume forms

Let us specialise the previous results to the case of Riemann surfaces, that is, to complex manifolds $(X, \mathcal{H})$ of complex dimension 1. For the rest of this section, X will denote a compact, connected Riemann surface of genus $g > 0$.

Definition 2.28. Let $\mathcal{L}$ be an hermitian sheaf on X. The curvature of $\mathcal{L}$ is a $(1,1)$-form on the surface defined as follows. Let $s \in \mathcal{L}(X)$, $s \neq 0$. Then,

$$\mathrm{curv}_{\mathcal{L}} := \partial\overline{\partial} \log \|s\|^2 \in \mathcal{E}^{1,1}. \tag{28}$$

The definition of the curvature does not depend on the choice of global section s, since the quotient of two different non-zero global sections of $\mathcal{L}$ is a constant function on X.

Proposition 2.29. *The curvature form of an hermitian line bundle satisfies*

$$\int_X \mathrm{curv}_{\mathcal{L}} = 2\pi i \deg \mathcal{L}. \tag{29}$$

Proof. The result follows easily from Stokes' formula. See [8], Proposition 4.1. We remark that this book refers to the curvature of the line bundle as the *first Chern form of the metric*, and denotes it by $c_1(\rho)$. □

It is important to be aware that in the case of Riemann surfaces, $\dim_{\mathbb{C}} \mathcal{E}^{1,0} = \dim_{\mathbb{C}} \mathcal{E}^{0,1} = 1$. From here, we deduce that $\mathcal{E}^{2,0} = \mathcal{E}^{0,2} = 0$.

Definition 2.30. A volume form on X is a global section of $\mathcal{E}^2 = \mathcal{E}^{1,1}$ that is nowhere zero.

Remark 2.31. A volume form ω on X defines a Borel measure μ_ω on X by setting

$$\mu_\omega(U) = \int_U \omega \tag{30}$$

for any Borel set U of X.

We have a natural hermitian product on global sections of Ω^1 defined by integration

$$\begin{array}{ccc} \Omega^1(X) \times \Omega^1(X) & \longrightarrow & \mathbb{C} \\ (\omega, \eta) & \longmapsto & \frac{i}{2} \int_X \omega \wedge \overline{\eta}. \end{array} \tag{31}$$

Let $\omega_1, \ldots, \omega_g$ be an orthonormal basis for this hermitian product and let

$$\mu := \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k} \in \mathcal{E}^{1,1}. \tag{32}$$

It is very easy to check that μ is a volume form on X that does not depend on the choice of orthonormal basis. Moreover, we have

$$\int_X \mu = 1. \tag{33}$$

Definition 2.32. We say that μ is the canonical volume form on X.

2.8 The Arakelov-Green function of a Riemann surface

Theorem 2.33. *There is a unique function $G : X \times X \to \mathbb{R}_{\geq 0}$ such that*

1. *$G \in \mathcal{C}_{\mathbb{R}}(X \times X)$ and vanishes only at the diagonal $\Delta_X \subset X \times X$.*

For a point $P \in X$, let $G_P = G(P, \cdot) : X \to \mathbb{R}_{\geq 0}$.

2. *For any point $P \in X$, any affine open neighbourhood U of P with local coordinate z, we have*

$$\log G_P(z) = \log |z - P| + f(z) \tag{34}$$

for $z \in U \setminus \{P\}$ and some $f \in \mathcal{C}_{\mathbb{R}}(U)$.

3. *For any $P \in X$, $\partial\overline{\partial} \log G_P = 2\pi i \mu$ at points $Q \neq P$.*

4. *For any $P \in X$, $\int_X \log G_P \mu = 0$.*

Remark 2.34. Condition 2 in the statement of the theorem says that one may think of G_P as a uniformizer for the local ring $\mathcal{C}_{\mathbb{R},P}$.

Condition 3 says that G_P is the solution to a partial differential equation while condition 4 expresses a contour condition. Hence, we may think that the existence of G is provided by condition 3, while uniqueness is provided by condition 4.

Proof. We refer to [1], Section 2 for proof of existence.

In order to proof uniqueness of G, suppose that G, H are two functions satisfying all the conditions in the statement. At any point $P \in X$, we have

$$\partial\overline{\partial} \log \frac{G_P}{H_P} = 0 \tag{35}$$

by condition 3. This last equation implies that $\log G_P/H_P$ is a harmonic function on X, while condition 1 implies that it is smooth. Hence, $\log G_P/H_P$ is a constant function on X. Finally, condition 4 ensures that this constant is zero. Therefore, for every point $P \in X$, we have $G_P = H_P$. □

Definition 2.35. We say that G is the Arakelov-Green function of X.

Notation. Many conditions regarding the Green function of a point $P \in X$ need to be stated in terms of the logarithm of G_P. So let us denote, from now on, $g_P := \log G_P$. We have $g_p : X \setminus \{P\} \to \mathbb{R}$.

Given a divisor $D = \sum n_i P_i$ on X, we denote

$$g_D := \sum n_i g_{P_i}. \tag{36}$$

The function g_D is defined for points in $X \setminus \mathrm{Supp}(D)$.

In particular, we shall write $g(D, P) = g_D(P)$ for any point. Finally, using this convention the notation $g(D, E)$ for two effective divisors D, E on X makes sense.

Notation. Let us recall that $d = \partial + \overline{\partial}$, and let us denote

$$d^c = \frac{1}{4\pi i}(\partial - \overline{\partial}), \tag{37}$$

so that we have

$$dd^c = \frac{i}{2\pi}\partial\overline{\partial}. \tag{38}$$

Theorem 2.36 (Symmetry of the Green function)**.** *Let $P \neq Q$ be two points on X. Then $g_P(Q) = g_Q(P)$.*

Proof. The symmetry of the Green function is a direct consequence of Stokes' formula. Let us compute the integral

$$\frac{i}{2\pi}\int_X \left(g_P\partial\overline{\partial}g_Q - g_Q\partial\overline{\partial}g_P\right) \tag{39}$$

in two different ways. First, the defining properties of g_P and g_Q imply:

$$\frac{i}{2\pi}\int_X \left(g_P\partial\overline{\partial}g_Q - g_Q\partial\overline{\partial}g_P\right) = \frac{i}{2\pi}\int_X (2\pi i g_P \cdot \mu - 2\pi i g_Q \cdot \mu) = 0. \tag{40}$$

Now let z be a coordinate about P or Q, $\varepsilon > 0$ a real positive number and $U_P(\varepsilon), U_Q(\varepsilon)$ be open neighbourhoods of P and Q, respectively on X, corresponding to discs around P and Q defined by $|z| < \varepsilon$. Let $Y(\varepsilon) = X \setminus (U_P(\varepsilon) \cup U_Q(\varepsilon))$. Finally, let $\omega = g_P d^c g_Q - g_Q d^c g_P$. We have

$$\frac{i}{2\pi}\int_X \left(g_P\partial\overline{\partial}g_Q - g_Q\partial\overline{\partial}g_P\right) = \int_X (g_P dd^c g_Q - g_Q dd^c g_P) = \lim_{\varepsilon\to 0}\int_{Y(\varepsilon)} (g_P dd^c g_Q - g_Q dd^c g_P)\,. \tag{41}$$

A very quick computation yields

$$g_P dd^c g_Q - g_Q dd^c g_P = d\left(g_P d^c g_Q - g_Q d^c g_P\right) = d\omega, \tag{42}$$

and by Stokes' formula we deduce

$$\int_{Y(\varepsilon)} d\omega = -\int_{\partial U_P(\varepsilon)} \omega - \int_{\partial U_Q(\varepsilon)} \omega. \tag{43}$$

Let us fix our attention at P in order to compute the first integral on the right-hand side of this last equation. The computation of the second integral is completely analogous. We claim:

$$\lim_{\varepsilon\to 0}\int_{\partial U_P(\varepsilon)} g_P d^c g_Q = 0, \tag{44}$$

$$\lim_{\varepsilon\to 0}\int_{\partial U_P(\varepsilon)} g_Q d^c g_P = g_Q(P). \tag{45}$$

Checking this claim is a computation that we leave to the reader: $g_P = \log r^2 + v$ in polar coordinates (r, θ), and for h smooth on the circle of radius r,

$$d^c h = \frac{r}{4\pi}\frac{\partial h}{\partial r} \cdot d\theta. \tag{46}$$

Finally, we may compare the two computations of (39) to obtain:

$$0 = \frac{i}{2\pi}\int_X \left(g_P\partial\overline{\partial}g_Q - g_Q\partial\overline{\partial}g_P\right) = g_Q(P) - g_P(Q). \tag{47}$$

□

The Arakelov-Green function of X gives rise to certain hermitian metrics on invertible holomorphic sheaves that we will distinguish. To show how these metrics may be obtained, it is enough to consider the case of an hermitian invertible sheaf $\mathcal{H}(P)$ that is associated to the divisor of a point $P \in X$, as we may obtain every invertible sheaf as a tensor product of sheaves associated to a divisors of points on X.

In this situation, the constant function 1 is a distinguished section of $\mathcal{H}(P)$ and we may set, for any $Q \in X$,

$$\|1\|_Q := G(P, Q). \tag{48}$$

This defines an hermitian metric on $\mathcal{H}(P)$, and it is very easy to check, using property 3 of the Arakelov-Green function, that the curvature of $\mathcal{H}(P)$ as an hermitian invertible sheaf with this metric is equal to μ. In general, if D is any divisor on X, the metric that we have constructed on $\mathcal{H}(D)$ satisfies

$$\mathrm{curv}_{\mathcal{H}(D)} = \mathrm{deg} D \cdot \mu \tag{49}$$

Definition 2.37. We will say that an hermitian line bundle is admissible if its curvature form is a multiple of μ. In such a situation, we will also call the metric $\|\cdot\|$ admissible.

Proposition 2.38. *Two admissible metrics on a holomorphic invertible sheaf on X are equal up to multiplication by a constant.*

Proof. Let $\|\cdot\|$ and $\|\cdot\|'$ be two admissible metrics on a holomorphic invertible sheaf $\mathcal{L}$. We have

$$\partial\overline{\partial} \log \frac{\|\cdot\|}{\|\cdot\|'} = \partial\overline{\partial} \log \|\cdot\| - \partial\overline{\partial} \log \|\cdot\|' = \frac{\deg \mathcal{L}}{2} \cdot \mu - \frac{\deg \mathcal{L}}{2} \cdot \mu = 0. \tag{50}$$

Hence, the function $\|\cdot\|/\|\cdot\|'$ is smooth and harmonic on X. Therefore, it is constant. □

Let us briefly sketch the steps which are necessary in order to prove the existence of the Green function using admissible metrics on invertible sheaves.

Proposition 2.39 ([1], Proposition 2.1)**.** *Let $\mathcal{L}$ be an invertible sheaf on X. There exists an admissible hermitian metric on $\mathcal{L}$.*

The proof of this theorem is complicated, but the metric on $\mathcal{L}$ is not defined using the Green function of X.

Example 2.40. We define a metric on $\mathbb{P}^1_{\mathbb{C}}$ by stating that its volume form is

$$\mu = \frac{i}{2\pi} \frac{dz \wedge d\overline{z}}{(1 + |z|^2)^2}. \tag{51}$$

We stress the fact that this is a genus zero surface, and as such it does not have a canonical metric defined via a scalar product on $\Omega^1(X)$; this space is zero.

The green function of $\mathbb{P}^1_{\mathbb{C}}$ with respect to μ is given at a point P by

$$G_P(z) = \frac{e^{\frac{1}{2}} |P - z|}{(1 + |P|^2)^{\frac{1}{2}} (1 + |z|^2)^{\frac{1}{2}}}. \tag{52}$$

3 Arakelov intersection theory on arithmetic surfaces

The main references for this part of the notes is [3]. The original work of Arakelov [1] may also provide a good insight of the topic. Other available references are [2], Chapter XII and [8].

Let us fix notations. Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $S := \operatorname{Spec} \mathcal{O}_K$ and denote its generic point by η.

Let $p : X \to S$ be an arithmetic surface. That is: X is a regular 2-dimensional scheme, p is a proper and flat morphism of schemes and the generic fibre X_η is a geometrically connected curve over K.

Let Σ denote the set of archimedean places of K. An archimedean place $\sigma \in \Sigma$ is an embedding of K into $\mathbb{R}$ or $\mathbb{C}$, with the convention that if we are dealing with the latter case we shall assume that $\overline{\sigma} \notin \Sigma$. For a complex place of K, $|\cdot|_\sigma$ will denote the square of the (usual) complex module, so that the product formula for K may be written as

$$\prod_{\mathfrak{p} \in S} |a|_\mathfrak{p} \times \prod_{\sigma \in \Sigma} |a|_\sigma = 1, \quad a \in K^\times. \tag{53}$$

Every archimedean place of K gives rise to a Riemann surface related to X in the following way. For $\sigma \in \Sigma$, let K_σ denote the completion of K at σ. Let $X_{K_\sigma} = X \times_S K_\sigma = X \times_K K_\sigma$. Then

$$X_\sigma := X_{K_\sigma}(\overline{K_\sigma}) \tag{54}$$

is the set of complex points of a complex algebraic curve, and may be viewed as a Riemann surface in a natural way.

Definition 3.1. We shall call X_σ the fibre of X above σ.

We will also need a formal symbol F_σ for every $\sigma \in \Sigma$. We will also call F_σ the fibre of X over σ; no confusion shall arise as we may identify F_σ and X_σ from a formal point of view.

For every archimedean place $\sigma \in \Sigma$, let μ_σ denote the canonical volume form on X_σ previously introduced, and let $G_\sigma : X_\sigma \times X_\sigma \to \mathbb{R}_{\geq 0}$ denote the corresponding Arakelov-Green function. Correspondingly, denote $g_\sigma = \log G_\sigma$.

Notation. Let $D = \overline{\{y\}}$ be an irreducible horizontal divisor on X, with $y \in X_\eta$. Let $\sigma \in \Sigma$. There are at most $n = [k(y) : K]$ points $y_1, \ldots, y_n \in X_\sigma$ which lie over y. These are the points of D on the archimedean fibre X_σ. For a shorthand notation, we shall denote

$$D^\sigma := \sum_{i=1}^{n} y_i \tag{55}$$

as a divisor on X.

3.1 Arakelov divisors. Linear equivalence

Definition 3.2. The group of Arakelov divisors on X is the group

$$\widehat{\operatorname{Div}}(X) := \operatorname{Div}(X) \oplus \bigoplus_{\sigma \in \Sigma} \mathbb{R} \cdot F_\sigma, \tag{56}$$

where $\mathrm{Div}(X)$ denotes the group of Weil divisors on X. Thus, an Arakelov divisor on X is an expression of the type

$$D = D_{\mathrm{fin}} + D_{\mathrm{inf}}, \tag{57}$$

where D_{fin} is a Weil divisor on X and

$$D_{\mathrm{inf}} = \sum_{\sigma \in \Sigma} \alpha_\sigma F_\sigma, \quad \alpha_\sigma \in \mathbb{R}. \tag{58}$$

Definition 3.3. Let $f \in K(X)^\times$. We associate an Arakelov divisor to f in the following way:

$$(f) = (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}}, \tag{59}$$

where $(f)_{\mathrm{fin}}$ is the principal Weil divisor associated to f and

$$(f)_{\mathrm{inf}} = \sum_{\sigma \in \Sigma} v_\sigma(f) \cdot F_\sigma, \tag{60}$$

with

$$v_\sigma(f) := -\int_{X_\sigma} \log |f|_\sigma \cdot \mu_\sigma. \tag{61}$$

Arakelov divisors of the form (f) for some $f \in K(X)^\times$ shall be called principal Arakelov divisors in the sequel.

It is very easy to check that principal Arakelov divisors form a subgroup of $\widehat{\mathrm{Div}}(X)$.

Definition 3.4. Two Arakelov divisors are said to be linearly equivalent if their difference is a principal Arakelov divisor. We shall denote the group of Arakelov divisors modulo linear equivalence by $\widehat{\mathrm{Cl}}(X)$. We shall refer to this group as the class group of Arakelov divisors on X.

The analogous definitions of Arakelov divisors, principal Arakelov divisors and class group of Arakelov divisors may be given for S, we recover the definitions in [10], Chapter III. In particular,

$$\widehat{\mathrm{Div}}(S) = \mathrm{Div}(S) \oplus \bigoplus_{\sigma \in \Sigma} \mathbb{R} \cdot F_\sigma, \tag{62}$$

the group $\mathrm{Div}(S)$ being isomorphic to the group of fractional ideals of K. The group $\widehat{\mathrm{Cl}}(S)$ may be defined in a similar manner. We have surjective group homomorphisms

$$\widehat{\mathrm{Div}}(X) \twoheadrightarrow \mathrm{Div}(X) \tag{63}$$

$$\widehat{\mathrm{Div}}(S) \twoheadrightarrow \mathrm{Div}(S) \tag{64}$$

which factor through linear equivalence and give two epimorphisms

$$\widehat{\mathrm{Cl}}(X) \twoheadrightarrow \mathrm{Cl}(X) \tag{65}$$

$$\widehat{\mathrm{Cl}}(S) \twoheadrightarrow \mathrm{Cl}(S). \tag{66}$$

Finally, we have a pull-back group homomorphism $p^* : \widehat{\mathrm{Cl}}(S) \to \widehat{\mathrm{Cl}}(X)$ which is given by the pull-back of Weil divisors along the morphism $p : X \to S$.

3.2 Arakelov intersection pairing

We define the Arakelov intersection pairing by its properties in the following proposition.

Proposition 3.5. *The Arakelov intersection pairing is the unique symmetric bilinear pairing*

$$(\cdot,\cdot) : \widehat{\mathrm{Div}}(X) \times \widehat{\mathrm{Div}}(X) \to \mathbb{R} \tag{67}$$

satisfying the following conditions:

1. *If $D_1 = (D_1)_{\mathrm{fin}}$ is a Weil divisor and D_2 is a vertical Weil divisor that has no common components with D_1,*

$$(D_1, D_2) = \sum_{s \in S_0} i_s(D_1, D_2) \log \#k(s), \tag{68}$$

   where i_s is the intersection multiplicity of D_1 and D_2 above a closed point $s \in S$ (cf. [9], Theorem 9.1.12; see remark 3.6).

2. *If D_1 is a horizontal divisor, $(D_1, F_\sigma) = e_\sigma \deg D_1$ for every $\sigma \in \Sigma$, where $e_\sigma = [K_\sigma : \mathbb{R}]$.*

3. *If D_1 and D_2 are horizontal irreducible divisors, define*

$$(D_1, D_2) = (D_1, D_2)_{\mathrm{fin}} + (D_1, D_2)_{\mathrm{inf}} \tag{69}$$

   with

$$(D_1, D_2)_{\mathrm{fin}} = \sum_{s \in S_0} i_s(D_1, D_2) \log \#k(s) \tag{70}$$

   and

$$(D_1, D_2)_{\mathrm{inf}} = -\sum_{\sigma \in \Sigma} e_\sigma g_\sigma(D_1^\sigma, D_2^\sigma). \tag{71}$$

4. *If D is a vertical Weil divisor, $(D, F_\sigma) = 0$ for every $\sigma \in \Sigma$.*

5. *For every $\sigma, \tau \in \Sigma$, $(F_\sigma, F_\tau) = 0$.*

Proof. The proof of the proposition amounts to showing that the conditions exposed in the statement determine a unique symmetric bilinear pairing. □

Remark 3.6. For greater convenience, we review the definition of the usual intersection pairing for Weil divisors on a regular fibred surface, given in [9], Sections 9.1.1 and 9.1.2.

First, for two effective divisors D_1, D_2 on X with no common component and a closed point $x \in X$, define the local intersection index of D_1 and D_2 at x as

$$i_x(D_1, D_2) := \mathrm{length}_{\mathcal{O}_{X,x}} \left(\frac{\mathcal{O}_{X,x}}{\mathcal{O}_X(-D_1)_x + \mathcal{O}_X(-D_2)_x} \right). \tag{72}$$

Second, if D_1 and D_2 are two divisors on X with no common component, write each of them as a difference of effective divisors and use bilinearity and (72) to define $i_x(D_1, D_2)$.

Third, if $s \in S$ is a closed point, let $\mathrm{Div}_s(X)$ denote the subgroup of $\mathrm{Div}(X)$ generated by the irreducible components of X_s. There exists a unique bilinear map

$$i_s : \mathrm{Div}(X) \times \mathrm{Div}_s(X) \to \mathbb{Z} \tag{73}$$

which, among other characterising properties, is invariant under linear equivalence on the first component, is symmetric when restricted to $\mathrm{Div}_s(X) \times \mathrm{Div}_s(X)$ and satisfies

$$i_s(D_1, D_2) = \sum_{x \in (X_s)_0} i_x(D_1, D_2)\,[k(x) : k(s)] \tag{74}$$

whenever D_1 and D_2 have no common component.

The pairing i_s may be correctly defined a posteriori if we use formula (74), invariance under linear equivalence and the *moving lemma*.

The strength of this new intersection index for Arakelov divisors is a consequence of the following theorem, first proved in [1].

Theorem 3.7. *The pairing defined above is invariant under linear equivalence. That is, the map (67) factors to a bilinear symmetric pairing*

$$\widehat{\mathrm{Cl}}(X) \times \widehat{\mathrm{Cl}}(X) \to \mathbb{R}. \tag{75}$$

Before proving this result, we will compute the Green function attached to the divisor of a meromorphic function on a Riemann surface. We do this in the lemma below.

Lemma 3.8. *Let $\sigma \in \Sigma$. For a meromorphic function f on X_σ and a point $P \in X_\sigma$, we have*

$$G_\sigma\left((f), P\right) = e^{v_\sigma(f)} \cdot |f|(P) \tag{76}$$

Proof. Let $P \in X_\sigma \setminus \mathrm{Supp}(f)$. Since f is holomorphic away from $\mathrm{Supp}(f)$, at these points we have $\partial\overline{\partial} \log |f| = 0$. We also have that $\partial\overline{\partial} g_\sigma\left((f), P\right) = 0$, because $\deg(f) = 0$. Combining these two facts, we deduce that

$$G_\sigma\left((f), P\right) = e^{\alpha} \cdot |f|(P) \quad \text{ for some } \alpha \in \mathbb{R}. \tag{77}$$

Finally, we may take logarithms and integrate to find

$$0 = \int_{X_\sigma} \log G_\sigma\left((f), P\right) \cdot \mu_\sigma = \alpha + \int_{X_\sigma} \log |f|_\sigma \cdot \mu_\sigma. \tag{78}$$

From here we deduce that $\alpha = v_\sigma(f)$. □

Proof of Theorem 3.7. It is enough to show that, given $f \in K(X)^\times$ and $D = \overline{\{y\}}$ an irreducible horizontal curve on X for some $y \in X_\eta$, we have $((f), D) = 0$. Let $n = [k(y) : k]$ and denote the n points of X_σ over y by $y_1^\sigma, \dots, y_n^\sigma$, for every $\sigma \in \Sigma$. Let us write

$$((f), D) = ((f)_{\mathrm{fin}}, D)_{\mathrm{fin}} + ((f)_{\mathrm{fin}}, D)_{\mathrm{inf}} + ((f)_{\mathrm{inf}}, D) \tag{79}$$

and consider the three summands in the right-hand side separately.

The third summand is easy to compute:

$$((f)_{\mathrm{inf}}, D) = \sum_{\sigma \in \Sigma} v_\sigma(f) \cdot (F_\sigma, D) = n \sum_{\sigma \in \Sigma} e_\sigma v_\sigma(f). \tag{80}$$

The other two summands are a bit harder to tackle. We have

$$((f)_{\text{fin}}, D)_{\text{fin}} = \sum_{s \in S_0} i_s\left(D, (f)_{\text{fin}}\right) \log \#k(s). \tag{81}$$

We may express i_s as a sum over closed points $x \in X_s$ in the following way:

$$i_s\left((f)_{\text{fin}}, D\right) = \sum_{x \in X_s} i_x\left((f)_{\text{fin}}, D\right)\left[k(x) : k(s)\right]. \tag{82}$$

Since $D \geq 0$, we may also restrict to D and compute the intersection index i_x in the following way (cf. [9], Lemma 9.1.4):

$$i_x\left((f)_{\text{inf}}, D\right) = \text{mult}_x\left((f)_{\text{fin}}|_D\right) = v_x\left(f|_D\right), \tag{83}$$

with $x \in D$, and v_x is the corresponding valuation on $k(y)$.

We get

$$((f)_{\text{fin},D})_{\text{fin}} = \sum_{s \in S_0} i_s\left((f)_{\text{fin}}, D\right) \log \#k(s) = \tag{84}$$

$$\sum_{s \in S_0} \sum_{x \in D \cap X_s} v_x\left(f|_D\right)\left[k(x) : k(s)\right] \log \#k(s) = \tag{85}$$

$$\sum_{x \in D_0} v_x\left(f|_D\right) \log \#k(x). \tag{86}$$

Regarding the last summand, we use Lemma 3.8:

$$((f)_{\text{fin}}, D)_{\text{inf}} = -\sum_{\sigma \in \Sigma} e_\sigma g_\sigma\left((f)_{\text{fin}}, D\right) = \tag{87}$$

$$-\sum_{\sigma \in \Sigma} e_\sigma \sum_{i=1}^{n} \log\left(e^{v_\sigma(f)} \cdot |f|_\sigma(y_i^\sigma)\right) = \tag{88}$$

$$-\sum_{\sigma \in \Sigma} e_\sigma n v_\sigma(f) - \sum_{\sigma \in \Sigma} \sum_{x \in D^\sigma} e_\sigma \log |f|_\sigma(x). \tag{89}$$

Finally, we can add up all three expressions to get

$$((f), D) = \sum_{x \in D_0} v_x(f|_D) \log \#k(x) - \sum_{\sigma \in \Sigma} \sum_{x \in D^\sigma} e_\sigma \log |f|_\sigma(x) = 0 \tag{90}$$

by the product formula for $k(y)$. We note that the points on D^σ correspond to the places of $k(y)$ which extend σ. □

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