From String structures to Spin structures on loop spaces

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Some notations

- We will always use bold type notations for smooth stacks.
- For a given open cover $\mathcal{U} = \{U_\alpha\}$ of a smooth manifold M and every positive integer n, we will write $U_{\alpha_{i_1}\ldots\alpha_{i_n}}$ for the n-fold intersection $U_{\alpha_{i_1}} \cap U_{\alpha_{i_2}} \cap \ldots \cap U_{\alpha_{i_n}}$.
- We will denote the standard n-simplex with Δ^n.

We want to prove that a String structure on a n-dimensional smooth manifold X ($n \geq 3$) induces a Spin structure on its loop space.
It is well known in the theoretical physics folklore that a String structure on X is essentially the same thing as a Spin structure on LX. Konrad Waldorf, in [13], gave a proof of this, showing also the validity of the inverse implication, in the case of a compact and simply connected manifold X.
We show how Waldorf's result can easily be obtained in the more general setting of smooth stacks. The crucial point in our proof is the existence of a natural morphism of smooth stacks

$$\mathbf{B}Spin \to \mathbf{B}^2(\mathbf{B}U(1)_{conn})$$

refining the first fractional Pontryagin class.

Classic results about oriented manifolds

Let X be a n-dimensional smooth manifold: to its tangent bundle TX corresponds the classifying map

$$f_{TX} : X \to BO(n)$$

- While $\pi_0(O(n)) = \mathbb{Z}_2$, $SO(n)$ is connected: endowing X with an *orientation* structure means giving a lift

$$f_{TX} : X \to BSO(n).$$

Doing so, we "kill" the first nonzero homotopy group of $BO(n)$, that is $\pi_1(BO(n)) = \mathbb{Z}_2$.

- A smooth manifold X can be endowed with an orientation structure if and only if

$$(X \xrightarrow{f_{TX}} BO(n) \xrightarrow{w_1} K(\mathbb{Z}_2, 1)) \tag{1}$$

  is homotopic to zero.

In (1), $K(\mathbb{Z}_2, 1)$ denotes the first Eilenberg MacLane space of $\mathbb{Z}_2$ and w_1 corresponds to the first universal Stiefel-Whitney class in the bijection $H^1(BO(n), \mathbb{Z}_2) \to [BO(n), K(\mathbb{Z}_2, 1)]$.

Classic results about Spin manifolds

Let X be a n-dimensional oriented manifold: to its tangent bundle TX corresponds the classifying map

$$f_{_{TX}} : X \to BSO(n)$$

- While $\pi_1(SO(n)) = \mathbb{Z}_2$, $Spin(n)$ is simply connected: endowing X with a $Spin$ structure means giving a lift

$$f_{_{TX}} : X \to BSpin(n).$$

Doing so, we "kill" the first nonzero homotopy group of $BSO(n)$, that is $\pi_2(BSO(n)) = \mathbb{Z}_2$.

- An oriented smooth manifold X can be endowed with a Spin structure if and only if

$$(X \overset{f_{_{TX}}}{\to} BSO(n) \overset{w_2}{\to} K(\mathbb{Z}_2, 2)) \qquad (2)$$

  is homotopic to zero.

In (2), $K(\mathbb{Z}_2, 2)$ denotes the second Eilenberg MacLane space of $\mathbb{Z}_2$ and w_2 corresponds to the second universal Stiefel-Whitney class in the bijection $H^2(BSO(n), \mathbb{Z}_2) \to [BSO(n), K(\mathbb{Z}_2, 2)]$.

The first fractional Pontryagin map

The first nonzero homotopy group of *Spin*(n) is the third, $\pi_3(Spin(n)) = \mathbb{Z}$. We get:

$$\begin{aligned} &\pi_4(BSpin(n)) = \mathbb{Z} && \\ &H_4(BSpin(n)) = \mathbb{Z} && \text{(Hurewicz theorem)} \\ &H^4(BSpin(n), \mathbb{Z}) = \mathbb{Z} && \text{(universal coefficient theorem)} \end{aligned}$$

The generator of $H^4(BSpin(n), \mathbb{Z}) = \mathbb{Z}$ is represented by a map (up to homotopy)

$$BSpin(n) \to K(\mathbb{Z}, 4) \tag{3}$$

called *first fractional Pontryagin class* and is denoted with the symbol $\frac{1}{2}p_1$.

The third lift of f_{TX}, that is the String structure

We say, *by definition*, that a Spin manifold X is endowed with a String structure if the map

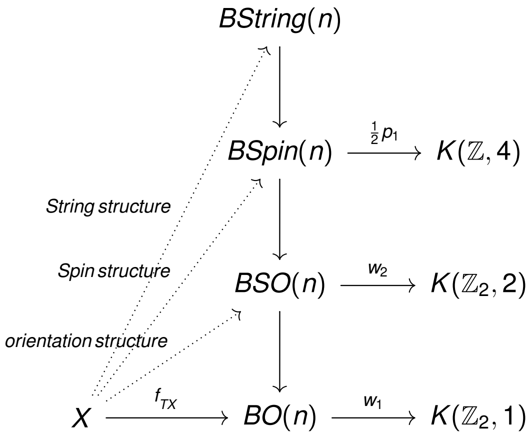
$$X \stackrel{f_{TX}}{\to} BSpin(n) \stackrel{\frac{1}{2}p_1}{\to} K(\mathbb{Z}, 4) \tag{4}$$

is homotopic to zero. In this case, the map f_{TX} can be lifted to

$$f_{TX} : X \to BString(n),$$

where $BString(n)$ is the homotopy fiber of $\frac{1}{2}p_1$.

A useful visualization of what described until now is showed by the following diagram



A Spin structure on the loop space

Let X be a Spin manifold, LX its (free) loop space and $LSpin(n)$ the loop group of the Spin group.
Since $Spin(n)$ is connected, the tangent bundle $T(LX)$ is naturally an $LSpin(n)$ bundle over LX (see [7]).
We now consider the universal central extension ([6])

$$1 \to U(1) \to \widetilde{LSpin}(n) \to LSpin(n) \to 1 \tag{5}$$

induced by the canonical 2-cocycle valued in $U(1)$ on $LSpin(n)$ and give the following natural

Definition

A Spin structure on LX is a lift of the structure group of the tangent bundle $T(LX)$ from $LSpin(n)$ to $\widetilde{LSpin}(n)$.

Waldorf, in [13], shows the existence of a natural transgression map

$$\{\text{ String structures on } X\} \ \to \ \{\text{Spin structures on } LX\}$$

which induces a bijection

$$\{\text{String structures on } X\}/\sim \to \{\text{Spin structures on } LX\}/\sim$$

if X is compact and simply connected.

What is a smooth stack?

A smooth stack **S** is the formalization of the naive idea of a sheaf of groupoids defined over the site of smooth manifolds.
That is, for every smooth manifold M, we have a groupoid $\mathbf{S}(M)$ and this association is such that:

1. a morphism $f\colon M \to N$ induces a pullback morphism $f^* : \mathbf{S}(N) \to \mathbf{S}(M)$;
2. if $\mathcal{U} = \{U_\alpha\}$ is an open cover of M, the groupoid $\mathbf{S}(M)$ is completely reconstructed from the groupoids $\mathbf{S}(U_\alpha)$, from the restriction maps $\mathbf{S}(U_\alpha) \to \mathbf{S}(U_{\alpha\beta})$ and from the compatibility conditions between these on $U_{\alpha\beta\gamma}$ (one says that the descent data are effective).

Three examples:

1. For every smooth manifold N we have the smooth stack $\mathbf{N} : M \mapsto C^\infty(M, N)$ (this is a set, and so a groupoid with only identities as morphisms).
2. For every Lie group G we have the smooth stack $\mathbf{B}G : M \mapsto \{\text{principal } G\text{-bundles over } M\}$.
3. For every Lie group G we have the smooth stack $\mathbf{B}G_{conn} : M \mapsto \{\text{principal } G\text{-bundles with connection over } M\}$.

More generally one can consider smooth higher stacks. The definition is the same but now $\mathbf{S}(M)$ is a higher groupoid (a simplicial set with some good properties). Since simplices of higher dimensions are involved in a higher stack, the descent data involve not only double (as for sheaves) and triple (as for ordinary stacks) intersections of the open sets in the given open cover, but also higher intersections.

For instance the 2-stack $\mathbf{B}^2U(1)$ of $U(1)$-bundle gerbes involves the quadruple intersections $U_{\alpha\beta\gamma\delta}$ in its definition in terms of local data.

Refining $\frac{1}{2}p_1$ to a morphism of smooth stacks

- $BSpin$ is the classifying space for principal Spin-bundles. We can refine it and consider $\mathbf{B}Spin$ which is the stack of principal Spin bundles. Note that $BSpin = |\mathbf{B}Spin|$.
- $K(\mathbb{Z}, 4)$ can be refined: this leads to $\mathbf{B}^3(U(1))$, the stack of 3-$U(1)$ bundles.
- $\frac{1}{2}p_1 : BSpin \to K(\mathbb{Z}, 4)$ can be refined to a morphism of stacks $\frac{1}{2}\mathbf{p_1} : \mathbf{B}Spin \to \mathbf{B}^3(U(1))$ (see [14]).

A rereading of the pullback defining *BString*

We previously defined $BString(n)$ as the homotopic fiber of $BSpin(n) \to K(\mathbb{Z}, 4)$. The above smooth refinement permits us to define the stack $\mathbf{B}String$ by means of

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}Spin & \xrightarrow[\frac{1}{2}\mathbf{p}_1]{} & \mathbf{B}^3(U(1)) \end{array} \tag{6}$$

A rereading of the central extension (5)

- The central extension of the loop Spin group considerd above induces the evident commutative diagram

$$\begin{array}{ccc} \mathbf{B}U(1) & \longrightarrow & \mathbf{B}\widetilde{LSpin} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}LSpin \end{array}$$

- Refining the canonical 2-cocycle valued in $U(1)$ on $LSpin$ gives a morphism of stacks

$$\mathbf{B}LSpin \to \mathbf{B}^2U(1)$$

A rereading of the central extension (5)

- The existence of the central extension (5) and its relation with this cocycle is encoded in the commutative diagram

$$\begin{array}{ccccc} \mathbf{B}U(1) & \longrightarrow & \mathbf{B}L\widetilde{Spin} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}LSpin & \longrightarrow & \mathbf{B}^2U(1) \end{array} \tag{7}$$

The lift of $\frac{1}{2}p_1$: structure of the proof

A very crucial point for the remainder of our work is the following

Theorem

The first fractional Pontryagin class can be lifted to a map $\mathbf{B}Spin \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^2(\mathbf{B}U(1)_{conn})$.

This lift is realized in two steps:

1. the differential refinement of $\frac{1}{2}\mathbf{p}_1$, inducing a morphism $\mathbf{B}Spin_{conn} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1)_{conn}$ (see [14])
2. the passage of the previous map to the quotient by the action of $\Omega^1(-;\mathfrak{so})$.

Given a smooth manifold X with a good open cover $\mathcal{U} = \{U_\alpha\}$, the generic object of $\mathbf{B}Spin_{\mathrm{conn}}(X)$, that is a *Spin*-principal bundle with connection on X, consists in the following collection of local data:

1. smooth functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to Spin$
2. local 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{so})$

satisfying the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$$

on the triple intersections $U_{\alpha\beta\gamma}$ and the compatibility condition

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta}$$

on the double intersections $U_{\alpha\beta}$.

How locally $\mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ looks like

We produce the extension of this data given by:

- a smooth family of based paths in Spin

$$\hat{g}_{\alpha\beta} : U_{\alpha\beta} \times \Delta^1 \to Spin$$

with $\hat{g}_{\alpha\beta}(0) = e$ and $\hat{g}_{\alpha\beta}(1) = g_{\alpha\beta}$ together with a 1-form

$$\hat{A}_{\alpha\beta} = \hat{g}_{\alpha\beta}^{-1} A_\alpha \hat{g}_{\alpha\beta} + \hat{g}_{\alpha\beta}^{-1} d\hat{g}_{\alpha\beta} \in \Omega^1(U_{\alpha\beta} \times \Delta^1, \mathfrak{so})$$

with $\hat{A}_{\alpha\beta}(0) = A_\alpha$ and $\hat{A}_{\alpha\beta}(1) = A_\beta$ on double intersections;

- a smooth family of based 2-simplices in Spin

$$\hat{g}_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \times \Delta^2 \to Spin$$

together with a 1-form

$$\hat{A}_{\alpha\beta\gamma} = \hat{g}_{\alpha\beta\gamma}^{-1} A_\alpha \hat{g}_{\alpha\beta\gamma} + \hat{g}_{\alpha\beta\gamma}^{-1} d\hat{g}_{\alpha\beta\gamma} \in \Omega^1(U_{\alpha\beta\gamma} \times \Delta^2, \mathfrak{so})$$

satisfying suitable conditions on $U_{\alpha\beta\gamma} \times \partial\Delta^2$ on triple intersections ;

- a smooth family of based 3-simplices in Spin

$$\hat{g}_{\alpha\beta\gamma\delta} : U_{\alpha\beta\gamma\delta} \times \Delta^3 \to Spin$$

together with a 1-form

$$\hat{A}_{\alpha\beta\gamma\delta} = \hat{g}^{-1}_{\alpha\beta\gamma\delta} A_\alpha \hat{g}_{\alpha\beta\gamma\delta} + \hat{g}^{-1}_{\alpha\beta\gamma\delta} d\hat{g}_{\alpha\beta\gamma\delta} \in \Omega^1(U_{\alpha\beta\gamma\delta} \times \Delta^3, \mathfrak{so})$$

satisfying suitable conditions on $U_{\alpha\beta\gamma\delta} \times \partial\Delta^3$ on quadruple intersections.

Finally, we define an object of $\mathbf{B}^3U(1)_{conn}(X)$ by means of the following collection:

$$(\mathrm{cs}(A_\alpha), \int_{\Delta^1} \mathrm{cs}(\hat{A}_{\alpha\beta}), \int_{\Delta^2} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta}) \bmod \mathbb{Z})$$

where $\mathrm{cs}(A)$ is the Chern-Simons 3-form obtained by evaluating a $\mathfrak{so}$-valued 1-form A in the Chern-Simons element cs.

The passage to the quotient by the action of $\Omega^1(-;\mathfrak{so})$:

By composing the differential refinement of $\frac{1}{2}\mathbf{p}_1$ with the forgetful morphism $\mathbf{B}^3 U(1)_{\mathrm{conn}} \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$ we get the morphism

$$\mathbf{B}Spin_{\mathrm{conn}} \to \mathbf{B}^2(\mathbf{B}U(1)_{\mathrm{conn}})$$

locally defined by:

$$(A_\alpha, g_{\alpha\beta}) \mapsto (0,0, \int_{\Delta^2} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma}), \int_{\Delta^3} \mathrm{cs}(\hat{A}_{\alpha\beta\gamma\delta}) \bmod \mathbb{Z}).$$

To conclude the proof, we only need to verify this map is independent, up to homotopy, on the connection data $\{A_\alpha\}$.

The main result

We now derive the sequence of commutative diagrams at the end of which, we get our result. The starting point is

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}Spin & \xrightarrow[\frac{1}{2}\mathbf{p_1}]{} & \mathbf{B}^3(U(1)) \end{array} \tag{8}$$

The main result

Thanks to the previous theorem, the bottom map of the commutative diagram (8) factors and we have

$$\begin{array}{ccccc} \mathbf{B}String & & \longrightarrow & & * \\ \big\downarrow & & & & \big\downarrow \\ \mathbf{B}Spin & \longrightarrow & \mathbf{B}^2(\mathbf{B}U(1)_{conn}) & \longrightarrow & \mathbf{B}^3U(1) \end{array} \qquad (9)$$

The main result

which induces

$$\begin{array}{ccccc} \mathbf{B}String & \longrightarrow & \mathbf{B}^2\Omega^1 & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}Spin & \longrightarrow & \mathbf{B}^2(\mathbf{B}U(1)_{conn}) & \longrightarrow & \mathbf{B}^3U(1) \end{array} \qquad (10)$$

The main result

Now forget the rightmost part of the diagram

$$\begin{array}{ccc} \mathbf{B}String & \longrightarrow & \mathbf{B}^2\Omega^1 \\ \downarrow & & \downarrow \\ \mathbf{B}Spin & \longrightarrow & \mathbf{B}^2(\mathbf{B}U(1)_{conn}) \end{array} \qquad (11)$$

The main result

and apply the free loop space functor L, i.e., the internal hom $[S^1, -]$

$$\begin{array}{ccc} [S^1, \mathbf{B}String] & \longrightarrow & [S^1, \mathbf{B^2}\Omega^1] \\ \downarrow & & \downarrow \\ [S^1, \mathbf{B}Spin] & \longrightarrow & [S^1, \mathbf{B^2}(\mathbf{B}U(1)_{conn})] \end{array} \tag{12}$$

The main result

Since $L = [S^1, -]$ commutes with **B** (this is the stacky refinement of the classical result that forming the classifying space B commutes with forming the loop space L, see [15]), all the nodes of (12) can be rewritten to give the following diagram:

$$\begin{array}{ccc} \mathbf{B}LString & \longrightarrow & \mathbf{B^2}[S^1, \Omega^1] \\ \downarrow & & \downarrow \\ \mathbf{B}LSpin & \longrightarrow & \mathbf{B^2}[S^1, (\mathbf{B}U(1)_{conn})] \end{array} \qquad (13)$$

The main result

The holonomy morphism $[S^1, \mathbf{B}U(1)_{conn}] \stackrel{hol}{\to} U(1)$ induces the evident morphism $\mathbf{B^2}[S^1, \mathbf{B}U(1)_{conn}] \to \mathbf{B^2}U(1)$, which permits us to write

$$\begin{array}{ccccc} \mathbf{B}LString & \longrightarrow & \mathbf{B^2}[S^1, \Omega^1] & \xrightarrow{\int_{S^1}} & \mathbf{B^2}\mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{B}LSpin & \longrightarrow & \mathbf{B^2}[S^1, \mathbf{B}U(1)_{conn}] & \xrightarrow{hol} & \mathbf{B^2}U(1) \end{array} \tag{14}$$

The main result

Let us keep only the outer diagram of (14)

$$\begin{array}{ccc} \mathbf{B}LString & \longrightarrow & \mathbf{B^2}\mathbb{R} \\ \downarrow & & \downarrow \\ \mathbf{B}LSpin & \longrightarrow & \mathbf{B^2}U(1) \end{array} \tag{15}$$

and denote with $\mathbf{B}\widetilde{LSpin}_{\mathbb{Z}}$ the pullback of

$$\mathbf{B}LSpin \to \mathbf{B^2}U(1) \leftarrow \mathbf{B^2}\mathbb{R}.$$

For the universal property of homotopy pullbacks, exists a canonical morphism $\mathbf{B}LString \to \mathbf{B}\widetilde{LSpin}_{\mathbb{Z}}$:

The main result

$$
\begin{array}{ccccc}
\mathbf{B}LString & & & & \\
& \searrow & & & \\
& & \mathbf{B}\widetilde{LSpin}_{\mathbb{Z}} & \longrightarrow & \mathbf{B}^2\mathbb{R} \\
& & \downarrow & & \downarrow \\
& & \mathbf{B}LSpin & \longrightarrow & \mathbf{B}^2U(1)
\end{array}
\qquad (16)
$$

Since the bottom horizontal map here is the stacky refinement of the canonical 2-cocycle valued in $U(1)$ on $LSpin$...

…we can extend (16) to

$$
\begin{array}{ccccc}
\mathbf{B}LString & & \mathbf{B}\widetilde{LSpin} & \longrightarrow & * \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mathbf{B}\widetilde{LSpin}_{\mathbb{Z}} & \longrightarrow & \mathbf{B}^2\mathbb{R} \\
 & & \downarrow & & \downarrow \\
 & & \mathbf{B}LSpin & \longrightarrow & \mathbf{B}^2U(1)
\end{array}
\tag{17}
$$

The main result

The crucial point of (17) is the cospan

$$\mathbf{B}LString \to \mathbf{B}\widetilde{LSpin}_{\mathbb{Z}} \leftarrow \mathbf{B}\widetilde{LSpin}.$$

Its right arrow becomes an isomorphism

$$|\mathbf{B}\widetilde{LSpin}| \xrightarrow{\sim} |\mathbf{B}\widetilde{LSpin}_{\mathbb{Z}}|$$

at the level of topological realizations (since the Lie group $\mathbb{R}$ is contractible). Therefore there is a canonical map

$$BLString \xrightarrow{f} B\widetilde{LSpin}.$$

The main result

To read it at the level of String structures on a given manifold X, notice that, by definition, for a String manifold X, there is a distinguished map

$$X \to BString.$$

Taking the free loop spaces gives a map

$$LX \to LBString \simeq BLString.$$

Finally, after a composition with f, we get the map

$$LX \to B\widetilde{LSpin}.$$

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