CHAPTER I.6

DIFFERENTIAL GEOMETRY OF COSET MANIFOLDS

I.6.1 - Introduction

Coset manifolds are a natural generalization of group manifolds (see I.3), and play an important role in Kaluza-Klein (super) gravity theories, to be discussed later (Part V).

We begin by defining homogeneous spaces.

Def. A metric space is said to be homogeneous if it admits as an isometry the transitive action of a group G. A group acts transitively if any point of the space can be reached from any other by the group action.

Example. The unit sphere in $\mathbb{R}^3$ is isometric under the transitive action of SO(3): any point (x,y,z) on the sphere can be carried into any other point (x',y',z') by a three dimensional rotation R

$$R\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad \begin{array}{l} x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = 1 : \\ R \subset SO(3) \end{array} \tag{I.6.1}$$

Def. The subgroup H of G which leaves a point X fixed is called the isotropy subgroup. Because of the transitive action of G, any other point $X' = gX$ $(g \in G,\ g \notin H)$ is invariant under a subgroup gHg^{-1} of G, isomorphic to H.

In our example, the North pole (0,0,1) is invariant under that SO(2) subgroup of SO(3) which rotates the sphere around the z-axis.

It is natural to label the point X of a homogeneous space by the parameters describing the G-group element which carries a conventional X_0 (origin) into X. However these parameters are redundant: because of H-isotropy

$$HX_0 = X_0 \tag{I.6.2}$$

there are infinitely many ways to reach X from X_0. Indeed, if g carries X_0 into X, any other G-element of the form gH does the same and one is led to characterize the points of a homogeneous space by the coset gH.

A homogeneous space is therefore a coset space G/H, i.e. the set of equivalence classes of elements of G, where the equivalence is defined by right H multiplication $(g \sim g'$ if $g = g'h$, with $g,$ $g' \in G$ and $h \in H)$.

The two-sphere S^2 can be written as the coset space SO(3)/SO(2). In general, for a n-sphere

$$S^n = \frac{SO(n+1)}{SO(n)} \quad . \tag{I.6.3}$$

Taking G to be a Lie group we obtain coset manifolds (endowed with a Riemannian structure, see Chapter I.2), parametrized by D coordinates $(y^1,\ldots,y^D)$, with

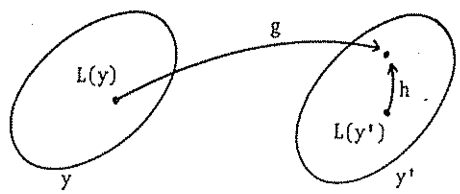
$$D = \dim G - \dim H \tag{I.6.4}$$

(Cfr. (I.6.17) below)

In each coset, corresponding to a D-plet of coordinates $y = (y^1, \ldots, y^D)$ we can choose a representative group element $L(y) \in G$. Under left multiplication by $g \in G$, $L(y)$ is in general carried into another coset, with representative element $L(y')$. Thus

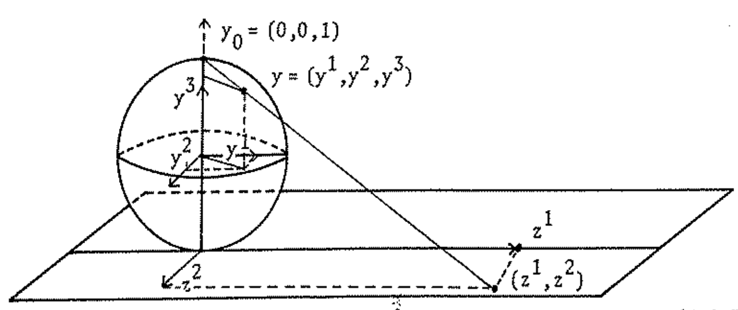
$$gL(y) = L(y')h \quad , \quad h \in H \tag{I.6.5}$$

where y' and h are functions of y and g, and depend on the way of choosing coset representatives. Pictorially:



(I.6.6)

In the case of $S^2 = SO(3)/SO(2)$, we assign to each point y, i.e. each coset, an element of SO(3) (the coset representative $L(y)$) which maps the North pole y_0, chosen as origin, into y.



(I.6.7)

Denoting by $L(y)^\Lambda{}_\Sigma$ $(\Lambda, \Sigma = 1, \ldots, 3)$ the SO(3) matrix element, we must have

$$L(y)^\Lambda{}_\Sigma \, L(y)_\Pi{}^\Sigma = \delta^\Lambda_\Pi \qquad \text{(orthogonality)}$$

$$L(y)^\Lambda{}_\Sigma \, y_0{}^\Sigma = y^\Lambda \qquad (L(y) \text{ maps the North pole into } y) \tag{I.6.8}$$

Using the stereographic coordinates z^1, z^2, it is easy to prove that(*)

$$L(z)^\Lambda{}_\Sigma = \left(\begin{array}{c|c} L(z)^\lambda{}_\sigma & L(z)^\lambda{}_3 \\ \hline L(z)^3{}_\sigma & L(z)^3{}_3 \end{array}\right) \tag{I.6.9}$$

with

$$L(z)^\lambda{}_\sigma = \delta^\lambda_\sigma - \frac{2z^\lambda z_\sigma}{z^2+4} \quad , \quad L(z)^\lambda{}_3 = \frac{4z^\lambda}{z^2+4}$$

$$L(z)^3{}_\sigma = \frac{4z_\sigma}{z^2+4} \quad , \quad L(z)^3{}_3 = \frac{z^2-4}{z^2+4} \tag{I.6.10}$$

satisfies (I.6.8).

Notice that any

$$L'(z) = L(z) \begin{pmatrix} A_{2\times 2}(z) & 0 \\ 0 & 1 \end{pmatrix} \tag{I.6.11}$$

(*) From $\dfrac{z^\lambda}{2} = \dfrac{y^\lambda}{1-y^3}$ one derives $y^3 = \dfrac{z^2-4}{z^2+4}$, and $y^\lambda = \dfrac{4z^\lambda}{z^2+4}$. $(z^2 \equiv z^\lambda z_\lambda)$.

with $A_{2\times 2} \in SO(2)$ still satisfies (I.6.8). The coset representative is chosen to have $A_{2\times 2} = \mathbb{1}_{2\times 2}$. Under left multiplication by a general SO(3) matrix S we have:

$$S\,L(z) = L(z') \begin{pmatrix} H_{2\times 2}(z,S) & 0 \\ 0 & 1 \end{pmatrix} \tag{I.6.12}$$

where $H_{2\times 2} \in SO(2)$. Eq. (I.6.12) is an illustration of the general formula (I.6.5). We leave as an exercise to compute $H_{2\times 2}(z,S)$ and z'. The general method to obtain y' and h of Eq. (I.6.5) is discussed in Sect. (I.6.4).

The Lie algebra of G can be split as

$$\mathbb{G} = \mathbb{K} \oplus \mathbb{H} \tag{I.6.13}$$

where $\mathbb{H}$ is the Lie algebra of H. $\mathbb{K}$ contains the remaining generators, henceforth referred to as "coset generators".

The structure constants of G are defined by

$$[H_i, H_j] = C^k{}_{ij} H_k \qquad H_i \in \mathbb{H} \quad ,$$

$$[H_i, K_a] = C^j{}_{ia} H_j + C^b{}_{ia} K_b \qquad K_a \in \mathbb{K} \quad ,$$

$$[K_a, K_b] = C^j{}_{ab} H_j + C^c{}_{ab} K_c \tag{I.6.14}$$

and we use the index conventions

$a, b, c \ldots$	flat coset indices	
$\alpha, \beta, \gamma \ldots$	curved coset indices	
$i, j, k \ldots$	H-indices	
$A, B, C \ldots$	G-indices .	(I.6.15)

Any g is expressible in the form (*)

$$g = e^{y^a K_a} e^{x^i H_i} \qquad H_i \in \mathbb{H}, \quad K_a \in \mathbb{K} \tag{I.6.16}$$

which is just a particular way to exponentiate $\mathbb{G}$ to obtain finite elements of G. Eq. (I.6.16) suggests a natural parametrization of coset spaces by the representative choice

$$L(y) = e^{y^a K_a} \tag{I.6.17}$$

corresponding to $x^i = 0$.

An explicit matrix representation of $L(y)$ is given in Section I.6.3.

I.6.2 - Classification of coset manifolds

The simple Lie algebras (L.A.) are classified in the A, B, C, D series, corresponding to the classical matrix groups, and in the five exceptional G_2, F_4, E_6, E_7, E_8 algebras. Any semisimple L.A. is the direct sum of simple L.A., and any L.A. is the semidirect sum of a semisimple and a solvable L.A.

In what follows we shall consider G/H spaces with $\mathbb{G}$ semisimple, or at most semisimple $\oplus$ abelian algebras (a particular case of the general decomposition semisimple $\subset\!\!\!\!+$ solvable). These cover most of the G/H spaces used in the course of this book. There are, however, some physically interesting G/H spaces with nonsemisimple $\mathbb{G}$ that do not admit the semisimple abelian decomposition, e.g. $\mathbb{G}$ = Poincaré and super Poincaré algebras.

To completely specify a coset space G/H, both topologically and metrically, two informations are necessary:

i) The particular embedding of H in G. This determines the topology.

(*) This holds true for $g \in$ compact G. For noncompact G, g must be "not too far" from the origin.

ii) The particular invariant metric on G/H. In general there are infinitely many, labelled by a finite number of rescaling parameters (see Section I.6.10).

The G isometry is realized on G/H in the most economical way, i.e. G/H has the lowest dimension, when H is a maximal subgroup of G.

There exist tables listing all the maximal subgroups of a given (simple) G (see for example ref. [7]); these provide, therefore, also a classification of $G_{SIMPLE}/H_{MAXIMAL}$ coset spaces. The same tables can be used to find maximal subgroups of semisimple G, and also, more generally, to find any subgroup of a semisimple G (since one can find the maximal subgroups of $H_{MAXIMAL}$ and so on). Thus, for a given G, or for a given $\dim G/H$, all G/H are in principle known. An example of such a classification is provided by Table V.6.1, which lists all the coset spaces of dimension 7.

Reductive G/H

We can in general perform a tensor transformation

$$T^A = S^A{}_B\, T^B \qquad \text{(I.6.18)}$$

on the generators T^B of a semisimple group so that its Cartan-Killing metric [see (I.3.95)]

$$g_{AB} = C^C{}_{AD}\, C^D{}_{BC} \qquad C^C{}_{AD} = \text{structure const. of } G \qquad \text{(I.6.19)}$$

becomes diagonal. *On that basis, G/H (for any subgroup H) is reductive, i.e. the decomposition (I.6.13) satisfies

$$[\mathbb{H}, \mathbb{K}] \subset \mathbb{K} \quad . \qquad \text{(I.6.20)}$$

* g_{AB} is a real symmetric matrix, transforming as $g' = S^T g\, S$ under (I.6.18). Then g can always be diagonalized with a particular (orthogonal) S.

Proof: $C^j{}_{ia}$ is proportional to $C_{jia} = C_{aji} = 0$ (because $[\mathbb{H}, \mathbb{H}] \subset \mathbb{H}$).

Indices are lowered with the Killing metric, and we recall that C_{ABC} is totally antisymmetric because of Jacobi identities.

It is straightforward to prove the existence of a reductive decomposition (I.6.13) also for semisimple $\oplus$ abelian L.A. In the following we shall always consider diagonal g_{AB} (unless otherwise specified) and hence reductive G/H spaces.

Symmetric G/H: when

$$[\mathbb{K}, \mathbb{K}] = \mathbb{H} \qquad \text{(I.6.21)}$$

G/H is said to be symmetric. This typically happens when G is simple and H is maximal (for a proof see Ref. [7], Chapter 9).

For a discussion on the freedom in choosing bases of generators in $\mathbb{H}$ and $\mathbb{K}$ (preserving reductivity) see P. van Nieuwenhuizen in Ref. [16].

I.6.3 - Coordinates on G/H and finite G-transformation

Compact G/H

When the decomposition $\mathbb{G} = \mathbb{H} + \mathbb{K}$ is reductive, the regular representation $(T_A)^B{}_C = C^B{}_{AC}$ has a simple block diagonal structure:

$$\mathbb{G} \xrightarrow[\text{repres.}]{\text{regular}} \underbrace{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}}_{\mathbb{H}} \oplus \underbrace{\begin{array}{cc} \overline{\dim \mathbb{H}} & \overline{\dim \mathbb{K}} \\ \left(\begin{array}{cc} 0 & B \\ -B^T & 0 \end{array}\right) & \end{array}}_{\mathbb{K}} \begin{array}{l} \}\ \dim \mathbb{H} \\ \}\ \dim \mathbb{K} \end{array} \qquad \text{(I.6.22)}$$

since only $C^a{}_{ib}$, $C^j{}_{ik}$ and $C^i{}_{ab}$, $C^b{}_{ai}$ are nonvanishing. They correspond to the real matrices A_1, A_2, B, $-B^T$ respectively(*).

If $\mathbb{G}$ is a simple classical Lie algebra, the defining matrix representation reduces in all but two cases to the block diagonal structure

$$\mathbb{G} \xrightarrow[\text{repres.}]{\text{defining}} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \tag{I.6.23}$$

similar to (I.6.22)(**). Note that here the submatrices A_1, A_2, B are in general complex.

The coset representatives are obtained by exponentiating the coset generators:

$$L(B) = \exp\begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{BB^\dagger} & B\,\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ -\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}\,B^\dagger & \cos\sqrt{B^\dagger B} \end{pmatrix} \tag{I.6.24}$$

With the substitution

$$X = B\,\frac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \tag{I.6.25}$$

(*) G is compact by assumption, implying g_{AB} negative definite. Then $C^i{}_{ab} = -C^b{}_{ai}$ and A_1, A_2 are antisymmetric.

(**) The two cosets G/H whose defining matrix representatives do not have the form (I.6.23) are $S\ell(n)/SO(n)$ and $SU^*(2n)/Usp(2n)$.

Eq. (I.6.24) becomes

$$L(X) = \begin{array}{c} \begin{array}{cc} \dim \mathbb{H} & \dim \mathbb{K} \end{array} \\ \left(\begin{array}{c|c} (I - XX^\dagger)^{\frac{1}{2}} & X \\ \hline -X^\dagger & (I - X^\dagger X)^{\frac{1}{2}} \end{array}\right) \end{array} \begin{array}{l} \\ \dim \mathbb{H} \\ \dim \mathbb{K} \end{array} \tag{I.6.26}$$

The range of the parameters describing the submatrix X is limited by the requirement

$$0 \le X^\dagger X \le I_k \ , \quad \begin{pmatrix} k \equiv \dim \mathbb{K}\ , \\ I_k = k \times k \text{ identity matrix} \end{pmatrix} \tag{I.6.27}$$

where the inequality refers to the k positive real eigenvalues of the $k \times k$ hermitian matrix $X^\dagger X$. Since $X^\dagger X$ and $XX^\dagger$ have the same non-zero set of eigenvalues,

$$0 \le X^\dagger X \le I_k \Leftrightarrow 0 \le XX^\dagger \le I_h \qquad (h \equiv \dim \mathbb{H})\ . \tag{I.6.28}$$

Conditions (I.6.28) are necessary for $L(X)$ to be a group element. The coset coordinates X are bounded by (I.6.27), and therefore describe a compact coset space.

Example: $S^n = \dfrac{SO(n+1)}{SO(n)}$, where $SO(n)$ leaves the (n+1)-direction fixed.

generators of $SO(n+1)$: T_{AB} $\quad A,B = 1,\ldots,n+1$

generators of $SO(n)$: T_{ab} $\quad a,b = 1,\ldots,n$

coset generators : $T_a \equiv T_{a\,n+1}$

In the vector (defining) representation $(T_{AB})^{CD} = \delta^{CD}_{AB}$. Hence the n coset generators $(T_a)^{CD}$ take the off-diagonal form:

$$\begin{array}{cc} T_a = \left(\begin{array}{c|c} & 0 \\ & \vdots \\ 0 & 1 \\ & \vdots \\ & 0 \\ \hline 0..-1..0 & 0 \end{array}\right) & \begin{array}{l} \\ \\ \leftarrow \text{a-th row} \\ \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ \text{a-th} \\ \text{column} \end{array} & \end{array} \tag{I.6.29}$$

The general element of the "coset algebra"

$$b^a\,T_a$$

in the vector representation is therefore

$$\mathbb{K}:\left(\begin{array}{c|c} & b_1 \\ 0 & \vdots \\ & b_n \\ \hline -b_1 \dots \quad -b_n & 0 \end{array}\right) = \begin{pmatrix} 0 & b \\ -b^T & 0 \end{pmatrix} \tag{I.6.30}$$

Coset representatives:

$$\exp\,\mathbb{K}:\exp\begin{bmatrix} 0 & b \\ -b^T & 0 \end{bmatrix} = \begin{bmatrix} (I_n - xx^T)^{\frac{1}{2}} & x \\ -x^T & (1-x^T x)^{\frac{1}{2}} \end{bmatrix} \tag{I.6.31}$$

with $x_a \equiv b_a \dfrac{\sin(\Sigma b_j^2)^{\frac{1}{2}}}{(\Sigma\, b_j^2)^{\frac{1}{2}}}$.

The range of the parameters x_a is defined by

$$\sum_{a=1}^{n} x_a^2 \leq 1 \quad . \tag{I.6.32}$$

Setting $x_{n+1} = \pm\sqrt{1-\sum\limits_a x_a^2}$, the x_a satisfy:

$$x_1^2 + x_2^2 + \dots x_n^2 + x_{n+1}^2 = 1$$

and the coset $SO(n+1)/SO(n)$ can be identified with the n-dimensional sphere embedded in R^{n+1}.

<u>Example</u>: $\dfrac{SO(n+m)}{SO(n)\times SO(m)}$.

generators of $SO(n+m)$: T_{AB} $A,B = 1,\dots,n+m$

generators of $SO(n)$: T_{ij} $i,j = 1,\dots,n$

generators of $SO(m)$: $T_{\alpha\beta}$ $\alpha,\beta = 1,\dots,m$

coset generators : $T_{i\alpha} \subset \mathbb{K}$

In the vector representation

$$\mathbb{K}:(T_{i\alpha})^{CD} = \begin{array}{cc} \begin{array}{r} \text{n}+\alpha\text{-th column} \end{array} & \\ \left(\begin{array}{c|c} & 0 \quad \vdots \quad 0 \\ 0_{n\times n} & \dots\dots 1 \dots\dots \\ & 0 \quad \vdots \quad 0 \\ \hline 0 \quad \vdots \quad 0 & \\ \dots\dots -1 \dots\dots & 0_{m\times m} \\ 0 \quad \vdots \quad 0 & \end{array}\right) & \begin{array}{l} \\ \text{i-th row} \\ \\ \\ \\ \\ \end{array} \end{array} \tag{I.6.33}$$

and for a general element of $\mathbb{K}$

$$b^{i\alpha}T_{i\alpha} = \left(\begin{array}{c|c} 0 & b \\ \hline -b^T & 0 \end{array}\right) \tag{I.6.34}$$

b is now an $n\times m$ matrix, and one can use formulas (I.6.24-26) with $b = B$.

<u>Noncompact G/H</u>

Given a compact G/H, the "Weyl unitary trick"

$$\mathbb{K} \longrightarrow i\,\mathbb{K} \tag{I.6.35}$$

yields the noncompact coset space G^*/H, provided it is consistent with the commutation relations

$$[\mathbb{H},\mathbb{K}] = \mathbb{K}$$

$$[\mathbb{K},\mathbb{K}] = \mathbb{K} + \mathbb{H} \quad . \tag{I.6.36}$$

While the first trivially becomes

$$[\mathbb{H}, i\mathbb{K}] = i\,\mathbb{K} \tag{I.6.37}$$

in the second one we must have $C^a{}_{bc} = 0$:

$$[\mathbb{K},\mathbb{K}] = \mathbb{H} \to [i\,\mathbb{K}, i\,\mathbb{K}] = -\mathbb{H} \tag{I.6.38}$$

i.e. G/H must be <u>symmetric</u>.

The structure constants $C^a{}_{ib}$ are unchanged, whereas $C^i{}_{ab}$ do change sign, and the metric g_{AB} becomes:

$$g_{AB} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \xrightarrow[\mathbb{K} \to i\,\mathbb{K}]{} \begin{array}{cc} \left[\begin{array}{ccc|ccc} -1 & & & & & \\ & -1 & & & & \\ & & \ddots\,{-1} & & & \\ \hline & & & +1 & & \\ & & & & +1 & \\ & & & & & \ddots\,{+1} \end{array}\right] & \begin{array}{c} \mathbb{H} \\ \\ \\ \mathbb{K} \end{array} \\ \quad\mathbb{H} \qquad\qquad \mathbb{K} & \end{array} \tag{I.6.39}$$

The regular and defining representations have the same block diagonal structure as in (I.6.22), with $-B^T$, $-B^\dagger$ respectively replaced by $+B^T$, $+B^\dagger$ in the coset generators (since $C^i{}_{ab} = C^b{}_{ai}$)

$$\mathbb{G} \xrightarrow[\text{repr.}]{\text{regular}} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \oplus \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

$$\mathbb{G} \xrightarrow[\text{repr.}]{\text{defining}} \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{\mathbb{H}} \oplus \underbrace{\begin{bmatrix} 0 & B \\ B^\dagger & 0 \end{bmatrix}}_{\mathbb{K}} \tag{I.6.40}$$

The coset representative $\exp(i\mathbb{K})$, with hermitian generators $i\mathbb{K}$, is now "unbounded". Indeed

$$\exp(i\mathbb{K}):\ \exp\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} = \begin{bmatrix} \cosh\sqrt{BB^\dagger} & B\,\dfrac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ +\dfrac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}\,B^\dagger & \cosh\sqrt{B^\dagger B} \end{bmatrix} \tag{I.6.41}$$

and after substitution:

$$X = B\,\frac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \tag{I.6.42}$$

we find

$$L(X) = \begin{bmatrix} [I + XX^\dagger]^{1/2} & X \\ X^\dagger & [I + X^\dagger X]^{1/2} \end{bmatrix} \tag{I.6.43}$$

without bounds on $X^\dagger X$ or $XX^\dagger$.

In the previous examples, the "Weyl unitary trick" brings G/H into G^*/H as follows:

$$\frac{SO(n+1)}{SO(n)} \xrightarrow[\mathbb{K}\to i\mathbb{K}]{} \frac{SO(n,1)}{SO(n)}$$

$$\frac{SO(n+m)}{SO(n)\times SO(m)} \xrightarrow[\mathbb{K}\to i\mathbb{K}]{} \frac{SO(n,m)}{SO(n)\times SO(m)} \tag{I.6.44}$$

I.6.4 - Finite transformations on G/H

We now derive an explicit expression for y' in the transformation law

$$gL(y) = L(y')h \tag{I.6.45}$$

giving the mapping of G/H into itself under the (left) action of G:

$$g : y \to y' \quad . \tag{I.6.46}$$

An arbitrary element of G has the structure

$$g \in G = \begin{array}{cc} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & \begin{array}{c} \updownarrow m \\ \updownarrow n \end{array} \\ \begin{array}{cc} \leftarrow m \rightarrow & \leftarrow n \rightarrow \end{array} & \end{array} \tag{I.6.47}$$

If g is in the adjoint representation, $m = \dim H$, $n = \dim K$.

Depending on which classical G we consider, there are various relations between the submatrices A, B, C, D. Using the parametrization (I.6.26) of coset representatives, the abstract formula (I.6.45) becomes

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} (I_m - XX^\dagger)^{1/2} & X \\ \hline -X^\dagger & (I_n - X^\dagger X)^{1/2} \end{array}\right] = \left[\begin{array}{c|c} (I_m - X'X'^\dagger)^{1/2} & X' \\ \hline -X'^\dagger & (I_n - X'^\dagger X')^{1/2} \end{array}\right]\left[\begin{array}{c|c} H_{m\times m} & 0 \\ \hline 0 & H_{n\times n} \end{array}\right] \tag{I.6.48}$$

i.e.:

$$AX + B(I_n - X^\dagger X)^{1/2} = X' H_{n\times n} \tag{I.6.49a}$$

$$CX + D(I_n - X^\dagger X)^{1/2} = (I_n - X'^\dagger X')^{1/2} H_{n\times n} \tag{I.6.49b}$$

$$A(I_m - XX^\dagger)^{1/2} - BX^\dagger = (I_m - X'X'^\dagger)^{1/2} H_{m\times m} \tag{I.6.49c}$$

$$C(I_m - XX^\dagger)^{1/2} - DX^\dagger = -X'^\dagger H_{m\times m} \quad . \tag{I.6.49d}$$

From these equations one finds X', $H_{n\times n}$ and $H_{m\times m}$ in terms of X and A, B, C, D.

Notice that multiplying (I.6.49a) by the inverse of (I.6.49b) yields:

$$\left[AX + B(I_n - X^\dagger X)^{1/2}\right]\left[CX + D(I_n - X^\dagger X)^{1/2}\right]^{-1} = X'(I_n - X'^\dagger X')^{-1/2}$$

or

$$\left[AX(I_n - X^\dagger X)^{-1/2} + B\right]\left[CX(I_n - X^\dagger X)^{-1/2} + D\right]^{-1} = X'(I_n - X'^\dagger X')^{-1/2} \ . \tag{I.6.49e}$$

This last equation suggests the use of new coordinates Z:

$$Z = X(I_n - X^\dagger X)^{-1/2} \Rightarrow X = Z(I_n + Z^\dagger Z)^{-1/2} \tag{I.6.50}$$

Z and X are 1-1 related (projectively). For compact cosets G/H, X is bounded and Z is unbounded:

$$0 \le X^\dagger X \le 1 \;\Leftrightarrow\; 0 \le Z^\dagger Z \le \infty \quad . \tag{I.6.51}$$

The Z's are called projective coordinates on G/H, and have simple transformation properties under left action of G (cfr. (I.6.49e)):

$$Z' = (AZ + B)(CZ + D)^{-1} \quad . \tag{I.6.52}$$

Thus, the group action is realized on the projective coset representatives by a fractional linear transformation.

Exercise: prove (I.6.52).

Example - $SO(3)/SO(2) = S^2$

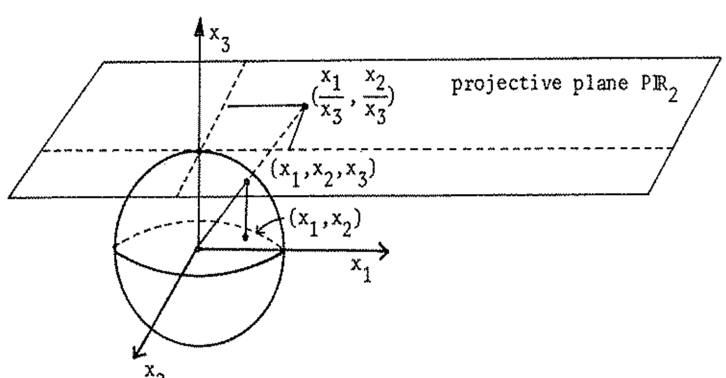
The coset representatives of the sphere $S^2 \subset \mathbb{R}^3$ are

$$\exp\left[\begin{array}{c|c} 0 & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hline -b_1, -b_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\left[I_2 - \binom{x_1}{x_2}(x_1, x_2)\right]^{\frac{1}{2}}} & \begin{matrix} x_1 \\ x_2 \end{matrix} \\ \hline -x_1 & -x_2 & x_3 \end{array}\right] \tag{I.6.53}$$

with $x_1^2 + x_2^2 + x_3^2 = 1$, $x_3 > 0$ (see (I.6.31)).

The projective coordinates of the upper hemisphere are

$$Z_i = \frac{x_i}{\sqrt{1 - x_1^2 - x_2^2}} \leq \infty \qquad i = 1,2 \tag{I.6.54}$$



(I.6.55)

The figure illustrates three different parametrizations of the upper hemisphere of $SO(3)/SO(2)$. A fourth one was given in Eq. (I.6.7) (stereographic coordinates).

Example - $SO(2,1)/SO(2) = H^2$ (hyperboloid)

H^2 is obtained from $SO(3)/SO(2)$ via $\mathbb{K} \to i\mathbb{K}$.
The coset representatives are

$$\exp\left(\begin{array}{c|c} 0 & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \hline +b_1, +b_2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{I_2 + \binom{x_1}{x_2}(x_1, x_2)} & \begin{matrix} x_1 \\ x_2 \end{matrix} \\ \hline x_1 & x_2 & x_3 \end{array}\right) \tag{I.6.56}$$

with $x_3^2 - x_2^2 - x_1^2 = 1$, $-\infty \leq x_1, x_2 \leq +\infty$.

Projective coordinates:

$$-1 < z_i = \frac{x_i}{\sqrt{1 + x_1^2 + x_2^2}} < +1 \quad . \tag{I.6.57}$$

Example - $SU(n+1)/SU(n) \times U(1) = \mathbb{C}P_n$

The generators of $SU(n+1)$ are antihermitian traceless complex matrices A^{ij}, $(i,j = 1,\ldots,n+1)$.

$$A^{ij} = -(A^{ji})^* \quad , \quad A^{ii} = 0 \quad . \tag{I.6.58}$$

There are $(n+1)^2-1$ independent matrices satisfying (I.6.58). These can be decomposed in a real and an imaginary part:

$$A^{ij} = B^{ij} + i\, C^{ij} \quad ; \quad B^{ij} = -B^{ji} \in \mathbb{R}$$
$$C^{ij} = C^{ji} \in \mathbb{R} \quad . \tag{I.6.59}$$

Note: the antisymmetric B^{ij} generate the maximal $SO(n+1)$ subgroup of $SU(n+1)$.

A convenient basis for the SU(n+1) Lie algebra is provided by the $(n+1)n/2$ antisymmetric matrices

$$[E_{ij}]^{k\ell} = \delta^{k\ell}_{ij} \Rightarrow E_{ij} = \begin{array}{c c} & \begin{array}{ccccc} & i & \cdots & j & \end{array} \\ \begin{array}{c} \\ i \\ \vdots \\ j \\ \\ \end{array} & \left[\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & \cdot & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{array}\right] \end{array} \tag{I.6.60}$$

and by the $\frac{(n+1)(n+2)}{2} - 1$ symmetric traceless matrices:

$$[F_{ij}]^{k\ell} = i\left[(\delta^k_i\delta^\ell_j + \delta^\ell_i\delta^k_j)/2 - \delta_{ij}\delta^k_{n+1}\delta^\ell_{n+1}\right] \quad \begin{array}{l} \text{i and j} \\ \text{not both} \\ \text{equal to} \\ \text{n+1} \end{array}$$

$$\Rightarrow F_{ij}\ (i \neq j) = \begin{array}{c c} & \begin{array}{ccccc} & i & \cdots & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \left[\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & \cdot & \cdots & \sqrt{-1} & \cdots \\ & \vdots & & \vdots & \\ \cdots & \sqrt{-1} & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{array}\right] \end{array}$$

$$F_{ij}\ (i = j) = \begin{array}{c c} & \begin{array}{ccccc} & j & & & \end{array} \\ \begin{array}{c} \\ j \\ \\ n+1 \\ \\ \end{array} & \left[\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & \sqrt{-1} & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \\ \cdots & \cdot & \cdots & -\sqrt{-1} & \cdots \\ & \vdots & & \vdots & \end{array}\right] \\ & \begin{array}{ccccc} & & & n+1 & \end{array} \end{array} \tag{I.6.61}$$

The $SU(n) \times U(1) = U(n)$ maximal subgroup of $SU(n+1)$ is generated by the E_{ij} and F_{ij} matrices with $i,j = 1,\ldots,n$. (Note: the extra $U(1)$ can be thought of as generated by F_{nn}. It is easy to check that F_{nn} commutes with all the E_{ij} and F_{ij}, $i,j = 1,\ldots,n$).

The generators of the coset $\mathbb{K} = \dfrac{SU(n+1)}{SU(n) \times U(1)}$ are therefore given by the $2n$ matrices

$$E_{i\,n+1}\ ,\quad F_{i\,n+1} \qquad i = 1,\ldots,n \tag{I.6.62}$$

and an arbitrary element of $\mathbb{K}$ takes the form

$$\mathbb{K} : \left[\begin{array}{ccc|c} & & & b_1 \\ & 0 & & \vdots \\ & & & b_n \\ \hline -b_1^* & \cdots & -b_n^* & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & b \\ \hline -b^\dagger & 0 \end{array}\right] \tag{I.6.63}$$

Coset representatives are obtained by exponentiating:

$$\exp \mathbb{K} = \left[\begin{array}{c|c} I_n - \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}(x_1^* \ldots x_n^*) & \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \\ \hline -x_1^* \ldots \qquad -x_n^* & r_{n+1} \end{array}\right] \tag{I.6.64}$$

with $r_{n+1} = (1 - x^\dagger x)^{\frac{1}{2}}$.

The representatives (I.6.64) are bounded by $0 \le x^\dagger x \le 1$ whereas the projective coordinates

$$z_i = \frac{x_i}{\sqrt{1 - x^\dagger x}} \tag{I.6.65}$$

are unbounded.

The projective coset representatives are points in the <u>complex projective space</u> $\mathbb{C}P_n$.

I.6.5 - Infinitesimal transformations and Killing vectors

We consider the transformation law

$$gL(y) = L(y')h \qquad (I.6.66)$$

for infinitesimal g:

$$g = 1 + \varepsilon^A T_A \quad , \quad T_A \in \mathbb{G} \qquad (I.6.67)$$

$$h = 1 - \varepsilon^A W^i{}_A(y) T_i \quad , \quad T_i \in \mathbb{H} \qquad (I.6.68)$$

$$y'^a = y^a + \varepsilon^A K_A{}^a(y) \quad . \qquad (I.6.69)$$

The induced h transformation depends in general on the infinitesimal G-parameters ε^A and on y, as shown in Eq. (I.6.68). The y-dependent matrix $W^i_A(y)$, defined by (I.6.68), is sometimes called the <u>H-compensator</u>. The shift in the coordinates $\{y\}$ is also proportional to ε^A, and the y-dependent differential operator

$$K_A(y) \equiv K_A{}^a(y) \frac{\partial}{\partial y^a} \qquad (I.6.70)$$

is the <u>Killing vector</u> on G/H associated to the G-generator T_A.

The variation of $L(y)$ is then expressed as

$$L(y') - L(y) = \varepsilon^A K_A \; L(y) \qquad (I.6.71)$$

and Eq. (I.6.66), after insertion of (I.6.67-69), becomes

$$T_A L(y) = K_A(y) L(y) - L(y) T_i W_A{}^i(y) \quad . \qquad (I.6.72)$$

Consider now the commutator $g_2^{-1} g_1^{-1} g_2 g_1$ acting on $L(y)$. If $g_1 = 1 + \varepsilon_1{}^A T_A$, $g_2 = 1 + \varepsilon_2{}^B T_B$

$$g_2^{-1} g_1^{-1} g_2 g_1 L(y) \sim (1 - \varepsilon_1{}^A \varepsilon_2{}^B [T_A, T_B]) L(y) \quad . \qquad (I.6.73)$$

Let us compute $[T_A, T_B] L(y)$:

$$\begin{aligned}
[T_A, T_B] L(y) &= T_A[T_B L(y)] - T_B[T_A L(y)] = \\
&= T_A[K_B L - L T_i W_B{}^i] - (A \leftrightarrow B) = \\
&= K_B(T_A L) - (T_A L) T_j W_B{}^j - (A \leftrightarrow B) = \\
&= K_B(K_A L - L T_i W_A{}^i) - (K_A L - L T_i W_A{}^i) T_j W_B{}^j - (A \leftrightarrow B) = \\
&= [K_B, K_A] L - L T_i [(K_B W_A{}^i) - (K_A W_B{}^i) + 2C^i{}_{jk} W_A{}^j W_B{}^k] \; .
\end{aligned} \qquad (I.6.74)$$

On the other hand:

$$[T_A, T_B] L = C^C{}_{AB} T_C L = C^C{}_{AB} [K_C L - L T_i W_C{}^i] \quad . \qquad (I.6.75)$$

Equating the r.h.s. of (I.6.74) and (I.6.75) yields

$$[K_A, K_B] = - C^C{}_{AB} K_C \qquad (I.6.76)$$

$$K_B W_A{}^i - K_A W_B{}^i + 2C^i{}_{jk} W_A{}^i W_B{}^k = - C^C{}_{AB} W_C{}^i \qquad (I.6.77)$$

where we have separately compared terms with and terms without W's, since the decomposition of a group element into $L(y)h$ is unique, i.e.

$$L(y)h = L'(y)h' \qquad (I.6.78)$$

implies $L(y) = L'(y)$, $h = h'$.

Eq. (I.6.76) shows that the Killing vectors $-K_A$ satisfy the G-Lie algebra. Eq. (I.6.77) is the integrability condition for the H-covariant Lie derivatives (see later).

I.6.6 - Vielbeins and metric on G/H

Consider the 1-form

$$V(y) = L^{-1}(y)\, dL(y) \tag{I.6.79}$$

generalizing the left-invariant 1-form $g^{-1}dg$ defined on group manifolds (see Chapter I.3). $V(y)$ is Lie algebra-valued and may be expanded on the $\mathbb{G}$ generators:

$$V(y) = V^a(y)T_a + \Omega^i(y)T_i \tag{I.6.80}$$

$V^a(y) = V_\alpha{}^a(y)dy^\alpha$ is a covariant frame (vielbein) on G/H and $\Omega^i(y) = \Omega_\alpha{}^i(y)dy^\alpha$ is called the H-connection.

Under left multiplication by a constant $g \in G$, $L^{-1}dL$ is not invariant, but transforms as

$$V(y') = hL(y)^{-1}g^{-1}\, d(gL(y)h^{-1}) =$$

$$= hV(y)h^{-1} + hdh^{-1} \quad . \tag{I.6.81}$$

Projecting on the coset generators:

$$V^a(y') = (hV(y)h^{-1})^a = V^b(y)D_b{}^a(h^{-1}) \tag{I.6.82}$$

where $D_A{}^B(g)$ is the adjoint representation defined by

$$g^{-1}T_A g = D_A{}^B(g)T_B \quad . \tag{I.6.83}$$

The infinitesimal form of (I.6.82) reads

$$V^a(y+\delta y) - V^a(y) = -\,\varepsilon^A W_A{}^i(y)C^a{}_{ib}V^b(y)$$

$$\delta y^a = \varepsilon^A K_A{}^a(y) \tag{I.6.84}$$

easily derived by observing that C^B_{iA} are the generators of the adjoint representation of H, and $C^a{}_{ij} = 0$. Eq. (I.6.84) implies that the left action of G on $V^a(y)$ is equivalent to an SO(N) rotation on $V^a(y)$ ($N = \dim G/H$), since $C^a{}_{ib}$ for semisimple G is antisymmetric in a,b.

Projecting (I.6.81) on the $\mathbb{H}$ generators yields:

$$\Omega^i(y') = (hV(y)h^{-1})^i + (hdh^{-1})^i =$$

$$= \Omega^j(y)D_j{}^i(h^{-1}) + (hdh^{-1})^i \qquad (D_a{}^i(h^{-1}) = 0) \tag{I.6.85}$$

whose infinitesimal version is

$$\Omega^i(y+\delta y) - \Omega^i(y) = -\,C^i{}_{kj}W_A{}^k\varepsilon^A\Omega^j - \varepsilon^A dW_A{}^i \quad . \tag{I.6.86}$$

From Eq. (I.6.72), and using the definition in (I.6.79), we derive an explicit expression for the Killing vector $K_A{}^\alpha(y)$ and the H compensator $W_A{}^i(y)$. Multiplying (I.6.72) by $L^{-1}(y)$ from the left yields:

$$D_A{}^B(L(y))T_B = L^{-1}(y)\,\frac{\partial L(y)}{\partial y^\alpha}\,K_A{}^\alpha(y) - T_i W_A{}^i(y) =$$

$$= V_\alpha{}^a K_A{}^\alpha T_a + \Omega_\alpha{}^i K_A{}^\alpha T_i - W_A{}^i T_i \quad . \tag{I.6.87}$$

Projecting on the $\mathbb{K}$ and $\mathbb{H}$ generators gives respectively

$$K_A{}^\alpha(y) = D_A{}^a(L(y))V_a{}^\alpha(y) \tag{I.6.88}$$

$$W_A{}^i(y) = V_\alpha{}^i(y)K_A{}^\alpha(y) - D_A{}^i(L(y)) \quad . \tag{I.6.89}$$

A G-left invariant metric on G/H is given by

$$g_{\alpha\beta}(y) = \gamma_{ab}V_\alpha{}^a(y)V_\beta{}^b(y) \tag{I.6.90}$$

where γ_{ab} is the Cartan-Killing group metric (I.6.19) restricted to G/H. The invariance of $g_{\alpha\beta}(y)dy^\alpha dy^\beta$ under the infinitesimal transformations (I.6.84) is easy to prove:

$$\delta g = \gamma_{ab}C^a{}_{ic}V^cV^b + \gamma_{ab}V^aC^b{}_{ic}V^c =$$

$$= C_{cib}V^bV^c + C_{bic}V^bV^c = 0$$

$$\text{since} \quad C_{bic} = -C_{cib} \quad . \tag{I.6.91}$$

To show that $g_{\alpha\beta}$ is invariant under the finite transformations (I.6.82), it is sufficient to prove the following identity:

$$\gamma_{ab} = D_a{}^c(h)D_b{}^d(h)\gamma_{cd} \qquad h \in \mathbb{H} \tag{I.6.92}$$

which can be obtained by squaring the definition

$$h^{-1}T_ah = D_a{}^B(h)T_B = D_a{}^b(h)T_b \quad . \tag{I.6.93}$$

The last equality is due to $D_a{}^i(h) = 0$.

Indeed

$$h^{-1}T_ah = e^{-x^iT_i}\,T_a\,e^{x^iT_i} = T_a + x^i[T_a,T_i] + $$

$$+ \frac{x^ix^j}{2!}\,[[T_a,T_i],T_j] + \ldots \tag{I.6.94}$$

produces only $\mathbb{K}$ generators ($C^j{}_{ai} = 0$).

Squaring (I.6.93) gives

$$h^{-1}T_ah\,h^{-1}T_bh = D_a{}^c(h)D_b{}^d(h)T_cT_d \quad . \tag{I.6.95}$$

The trace of (I.6.95), with the $\mathbb{K}$ generators in the adjoint representation, yields the identity in (I.6.92). G-invariance of $g_{\alpha\beta}$ easily follows:

$$g_{\alpha\beta}(y')dy'^\alpha dy'^\beta = g_{\alpha\beta}(y)dy^\alpha dy^\beta \quad . \tag{I.6.96}$$

Exercise: show that $g_{\alpha\beta}(y)$ is insensitive to the particular choice of coset representative $L(y)$.

Exercise: prove that

$$e^{y^aK_a}\,e^{\delta y^aK_a} = e^{(y^a+\delta y^a)K_a}\,e^{-\delta y^aV_a{}^\alpha\omega_\alpha{}^iK_i} \tag{I.6.97}$$

i.e.

$$L(y)L(\delta y) = L(y+\delta y)h$$

with

$$h = e^{-\delta y^aV_a{}^\alpha(y)\omega_\alpha{}^i(y)K_i} \quad . \tag{I.6.98}$$

$V_a{}^\alpha(y)$ is defined as the inverse of $V_\alpha{}^a(y)$:

$$V_a{}^\alpha(y)V_\alpha{}^b(y) = \delta_a^b \quad , \quad V_\alpha{}^a V_a{}^\beta = \delta_\alpha^\beta \tag{I.6.99}$$

(I.6.97) is a particular case of (I.6.66), and expresses the transformation of an infinitesimal displacement dx (at the origin) under left multiplication by a coset representative.

Curved and flat coset indices are connected by $V_\alpha{}^a(y)$:

$$(\text{vector})^a = V_\alpha{}^a\,(\text{vector})^\alpha \tag{I.6.100}$$

Example: vielbein, H-connection and metric on S^n

We use the stereographic coordinates (I.6.10).

$$L^{-1}(z) = L^T(z) = \left(\begin{array}{c|c} \delta^\lambda_\sigma - \dfrac{2z^\lambda z_\sigma}{z^2+4} & \dfrac{4z^\lambda}{z^2+4} \\ \hline \dfrac{4z_\sigma}{z^2+4} & \dfrac{z^2-4}{z^2+4} \end{array}\right)$$

$$dL(z) = \left(\begin{array}{c|c} \dfrac{-2z^\sigma dz_\rho - 2z^\sigma z_\rho}{z^2+4} + \dfrac{4z^\sigma z_\rho z\cdot dz}{(z^2+4)^2} & \dfrac{4dz^\sigma}{z^2+4} - \dfrac{8z^\sigma z\cdot dz}{(z^2+4)^2} \\ \hline \dfrac{4dz_\rho}{z^2+4} - \dfrac{8z_\rho z\cdot dz}{(z^2+4)^2} & \dfrac{16z\cdot dz}{(z^2+4)^2} \end{array}\right)$$

$$L^{-1}(z)dL(z) = \left(\begin{array}{c|c} \dfrac{2(z^\lambda dz_\rho - z_\rho dz^\lambda)}{z^2+4} & \dfrac{4dz^\lambda}{z^2+4} \\ \hline \dfrac{-4dz_\rho}{z^2+4} & 0 \end{array}\right) \tag{I.6.101}$$

$$= V_\alpha{}^a(z)T_a dz^\alpha + \Omega_\alpha{}^i(y)T_i dz^\alpha \quad .$$

Vielbein:

$$V_\alpha{}^a(z) = \frac{4\delta^a_\alpha}{z^2+4} \quad . \tag{I.6.102}$$

H-connection:

$$\Omega_\alpha{}^i(z) = \Omega_\alpha{}^{ab}(z) = \frac{2(\delta^a_\alpha z^b - \delta^b_\alpha z^a)}{(z^2+4)} \quad . \tag{I.6.103}$$

since

$$(T_a)^{AB} = (T_{a\ n+1})^{AB} = \delta^A_a\delta^B_{n+1} - \delta^B_a\delta^A_{n+1}$$

$$(T_i)^{AB} = (T_{ab})^{AB} = \delta^{[A}_a\delta^{B]}_b \quad . \tag{I.6.104}$$

Metric:

$$g_{\alpha\beta}(z) = \gamma_{ab}V_\alpha{}^a(z)V_\beta{}^b(z) = -\frac{16\,\delta_{\alpha\beta}}{(z^2+4)^2} \tag{I.6.105}$$

$$(\gamma_{ab} = \mathrm{Tr}[C_aC_b] = -\delta_{ab}) \quad .$$

As a check, we compute the length of half a meridian on S^n:

$$\int_{\text{equator}}^{\text{South pole}} ds = \int\sqrt{-g_{\alpha\beta}dz^\alpha dz^\beta} = \int_0^2 \frac{4}{z_1^2+4}\,dz_1 \underset{(z=2t)}{=} \int_0^1 \frac{1}{t^2+1}\,2dt$$

$$= 2\,\mathrm{arc\,tg}\Big|_0^1 = \frac{\pi}{2} \qquad (0 \le z_1 \le 2,\ z_2 = 0)\ . \tag{I.6.106}$$

The metric tensor at the origin of G/H is just the Cartan-Killing metric of G restricted to G/H:

$$g_{\alpha\beta}(0) = \gamma_{ab}V_\alpha{}^a(0)V_\beta{}^b(0) = \gamma_{\alpha\beta} \tag{I.6.107}$$

since

$$V_\alpha{}^a(0) = \delta^a_\alpha \qquad (L^{-1}(y)dL(y)\Big|_{y=0} = dy^a T_a) \quad . \tag{I.6.108}$$

Invariance of the metric means

$$ds^2 = g_{\alpha\beta}(Y)dy^\alpha(Y)dy^\beta(Y) = g_{\alpha\beta}(0)dy^\alpha(0)dy^\beta(0) \tag{I.6.109}$$

or

$$g_{\gamma\delta}(Y) = \gamma_{\alpha\beta}\,\frac{\partial y^\alpha(0)}{\partial y^\gamma(Y)}\,\frac{\partial y^\beta(0)}{\partial y^\delta(Y)} \quad . \tag{I.6.110}$$

Comparing (I.6.110) with the definition:

$$g_{\gamma\delta}(Y) = \gamma_{ab}V_\gamma{}^a(Y)V_\delta{}^b(Y) \tag{I.6.111}$$

we arrive at the following expression for the vielbein $V_\alpha{}^a(Y)$:

$$V_\alpha{}^a(Y) = \frac{\partial y^a(0)}{\partial y^\alpha(Y)} \tag{I.6.112}$$

and we can interpret the vielbein at a point Y as the matrix connecting the two infinitesimal displacements $dy^a(0)$ and $dy^\alpha(Y)$:

$$dy^a(0) = V_\alpha{}^a(Y)dy^\alpha(Y) \quad . \tag{I.6.113}$$

I.6.7 - Covariant Lie derivative

For an arbitrary tensor $T_{\alpha\beta\ldots\gamma}$, the (ordinary) Lie derivative along a vector $\vec{v}$ is defined by (cfr. Section I.1.7)

$$\ell_{\vec{v}}\, T_{\alpha\ldots\beta}(x) = v^\alpha \frac{\partial}{\partial x^\gamma} T_{\alpha\ldots\beta} + (\partial_\alpha v^\gamma)T_{\gamma\ldots\beta} + \ldots + (\partial_\beta v^\gamma)T_{\alpha\ldots\gamma} \tag{I.6.114}$$

so that $\ell_{\vec{v}}$ generates a general coordinate transformation with (infinitesimal) parameter v^γ. For example, the Lie derivatives along the Killing vectors of the vielbein and the H-connection are:

$$\ell_{K_A} V_\alpha{}^a = K_A V_\alpha{}^a + (\partial_\alpha K_A{}^\beta)V_\beta{}^a \tag{I.6.115}$$

$$\ell_{K_A} \Omega_\alpha{}^i = K_A \Omega_\alpha{}^i + (\partial_\alpha K_A{}^\beta)\Omega_\beta{}^i \quad . \tag{I.6.116}$$

Note that $\ell_{\vec{v}}$ only acts on curved indices.

The Lie derivative on p-forms is defined by

$$\ell_{\vec{v}}\omega = \vec{v}\rfloor d\omega + d\,\vec{v}\rfloor\omega \tag{I.6.117}$$

(cfr. Section I.1.7).

Writing (I.6.117) in components one retrieves the definition (I.6.114) in the case of antisymmetric tensors.

<u>Exercise</u>: prove this.

Since $\ell_{\vec{v}}$ generates coordinate transformations $\vec{y} \to \vec{y} + \varepsilon\vec{v}$, we have

$$\ell_{\vec{v}}\omega(y) = \lim_{\varepsilon\to 0} \frac{\omega(\vec{y}+\varepsilon\vec{v}) - \omega(\vec{y})}{\varepsilon} \tag{I.6.118}$$

and the transformation laws of the vielbein and H-connection (Eqs. (I.6.84) and (I.6.86)) can be written as

$$\ell_{K_A} V^a(y) = W_A{}^i(y) C^a{}_{ib} V^b(y) \qquad (I.6.119)$$

$$\ell_{K_A} \Omega^i(y) = C^i{}_{jk} W_A{}^k(y) \Omega^j - dW_A{}^i(y) \quad . \qquad (I.6.120)$$

The H-connection transforms as a gauge-field, but note that the functions $W_A{}^i(y)$ are not arbitrary, but are fixed by (I.6.89). Eqs. (I.6.119) and (I.6.120) can be combined into a single formula for the infinitesimal variation of $V(y) = L^{-1}(y)\, dL(y)$:

$$\ell_{K_A} V(y) = dW_A(y) - [W_A(y), V(y)] \quad . \qquad (I.6.121)$$

We recall some properties of Lie derivatives:

i) $[\ell_{\vec{v}}, d] = 0$ i.e. the Lie derivative commutes with the exterior derivative.

ii) $[\ell_{\vec{v}}, \ell_{\vec{u}}] = \ell_{[\vec{v},\vec{u}]}$. Hence

$$[\ell_{K_A}, \ell_{K_B}] = - C^C{}_{AB}\, \ell_{K_C}$$

with K_A, K_B, K_C : Killing vectors (I.6.122)

since $[K_A, K_B] = - C^C{}_{AB} K_C$ and const. $\ell_{\vec{v}} = \ell_{\text{const. } \vec{v}}$.

The integrability condition for (I.6.119) yields

$$\ell_{K_A} W_B - \ell_{K_B} W_A - [W_A, W_B] = - C^C{}_{AB} W_C \qquad (I.6.123)$$

with $(W_A)^{ab} \equiv W_A{}^i C_i{}^{ab}$. This formula was already derived in (I.6.77).

The transformation laws (I.6.119), (I.6.120) suggest the definition of an H-covariant Lie derivative L_{K_A}:

$$L_{K_A} \equiv \ell_{K_A} - W_A{}^i(y) T_i \qquad (I.6.124)$$

where T_i acts as $(C_i)_a{}^b \equiv C^b{}_{ia}$ on $\mathbb{K}$ and as $(C_i)_j{}^k \equiv C^k{}_{ij}$ on $\mathbb{H}$. Then

$$L_{K_A} V^a(y) = 0 \qquad (I.6.125)$$

$$L_{K_A} \Omega^i(y) = dW_A{}^i(y) \quad . \qquad (I.6.126)$$

For later use we define the action of L_{K_A} on the coset representative $L(y)$ as

$$L_{K_A} L(y) = K_A(y) L(y) - L(y) T_i W_A{}^i(y) \qquad (I.6.127)$$

with T_i in the same G-representation as $L(y)$. Eq. (I.6.72) implies

$$L_{K_A} L(y) = T_A\, L(y) \qquad (I.6.128)$$

so that

$$[L_{K_A}, L_{K_B}] L(y) = - C^C{}_{AB} L_{K_C} L(y) = L_{-C^C{}_{AB} K_C} L(y) =$$

$$= L_{[K_A, K_B]} L(y) \quad . \qquad (I.6.129)$$

Exercise: prove that

$$[L_{K_A}, L_{K_B}] = L_{[K_A,K_B]} \qquad (I.6.130)$$

also on the vielbein V^a and on the H-connection Ω^i.

I.6.8 - Geodesics

Geodesics through the origin of G/H are obtained by exponentiating straight lines through the origin of $\mathbb{K}$ (the "coset algebra"):

$$\exp(tA) \qquad A \in \mathbb{K} \quad . \qquad (I.6.131)$$

In the off-diagonal representation of $\mathbb{K}$ (cfr. (I.6.23))

$$\exp(tA) = \exp t \begin{bmatrix} 0 & B \\ -B^\dagger & 0 \end{bmatrix} = \begin{bmatrix} \cos t\sqrt{B^\dagger B} & B\,\dfrac{\sin t\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ -\dfrac{\sin t\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}\,B^\dagger & \cos t\sqrt{B^\dagger B} \end{bmatrix} \qquad (I.6.132)$$

for compact G/H.

The geodesic coordinates are then given by:

$$X(t) = B\,\frac{\sin t\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \quad . \qquad (I.6.133)$$

Theorem:

The length of the geodesic connecting the origin $t=0$ and the point $t=1$ (i.e. the element e^A) is equal to the length of the vector $A \in \mathbb{K}$, i.e.

$$d(\mathbb{1}, e^A) = \|A\| = \sqrt{-\gamma_{ab}A^aA^b} \qquad (I.6.134)$$

with $A = A^aX_a$, $\{X_a\}$ = basis for $\mathbb{K}$ and γ_{ab} = Cartan Killing metric restricted on G/H.

Proof: $d(\mathbb{1}, e^A) =$

$$\int_{t=0}^{t=1} \sqrt{-g_{\alpha\beta}(t)dX^\alpha(t)dX^\beta(t)} =$$

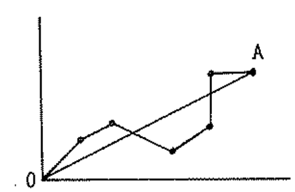
$$= \int_{t=0}^{t=1} \sqrt{-\gamma_{ab}V^a_\alpha(t)V^b_\beta(t)dX^\alpha(t)dX^\beta(t)}$$

$$= \int_{t=0}^{t=1} \sqrt{-\gamma_{ab}[L^{-1}(t)dL(t)]^a[L^{-1}(t)dL(t)]^b}$$

$$= \int_{t=0}^{t=1} \sqrt{-\gamma_{ab}(e^{-tA}Adte^{tA})^a\ (e^{-tA}Adte^{tA})^b}$$

$$= \int_{t=0}^{t=1} \sqrt{-\gamma_{ab}A^aA^b}\ dt = \sqrt{-\gamma_{ab}A^aA^b} = \|A\| \quad . \qquad (I.6.135)$$

Any other line connecting the origin and e^A has greater length, since in the $\mathbb{K}$ linear vector space it would correspond to a curved line connecting 0 with A



$\mathbb{K}$-vector space

(I.6.136)

Hence e^{tA} $(0 \le t \le 1)$ is really the coset representative of a geodesic between $\mathbb{1}$ and the point represented by e^A on G/H.

Example: S^n

$$A = b^a T_{a,n+1} = \left[\begin{array}{c|c} 0 & \begin{matrix} b^1 \\ \vdots \\ b^n \end{matrix} \\ \hline -b^1 \ldots \quad -b^n & 0 \end{array}\right]$$

$$\exp(tA) = \left[\begin{array}{c|c} \cos t\sqrt{bb^T} & \begin{matrix} x^1 = b^1 \dfrac{\sin tb}{b} \\ \vdots \\ x^n = b^n \dfrac{\sin tb}{b} \end{matrix} \\ \hline -x^1 \ldots \quad -x^n & x^{n+1} = \cos tb \end{array}\right] \qquad \begin{matrix} b \equiv \sqrt{b^T b} \\ \sum_{i=1}^{n+1} (x^i)^2 = 1 \end{matrix} \tag{I.6.137}$$

A vector $A \in \mathbb{K}$ such that e^A represents a point on the equator must have(*)

$$\|A\| = \sqrt{-\gamma_{ab} b^a b^b} = b = \pi/2 \tag{I.6.138}$$

so that $x^1 = x^2 = \ldots x^n = 0$, $x^{n+1} = \cos b = 1$ (coordinates of the equator). Thus: $\pi/2$ = length of a geodesic connecting the pole and equator of an n-sphere of unit radius.

Def: the distance of a point $y \in G/H$ and the origin 0 is the length of the shortest geodesic connecting 0 and y:

$$d(0,y) \qquad (= d(\mathbb{1}, L(y)) \quad . \tag{I.6.139}$$

As entries of the distance function d, we use indifferently coordinates y of points in G/H or their coset representatives $L(y)$.

(*) We take here the "natural" metric $\gamma_{ab} = -\delta_{ab}$ rather than the Cartan-Killing metric restricted to S^n (proportional to $-\delta_{ab}$, see the example at the end of Section I.6.10).

Theorem:

The length of a curve on G/H is group invariant. This is intuitively obvious since the infinitesimal lengths are G-invariant

$$ds(y') = ds(y) \quad . \tag{I.6.140}$$

A corollary of (I.6.140) is that geodesics through the origin are mapped into geodesics through y via (left) multiplication by $L(y)$:

$$\begin{matrix} e^{tA} & \longrightarrow & L(y)\, e^{tA} \\ \text{geodesics through } 0 & & \text{geodesics through } y \end{matrix} \tag{I.6.141}$$

We can therefore compute the length of a geodesic connecting any two points x, y on G/H:

$$d(L(x), L(y)) = d(\mathbb{1}, L^{-1}(x)L(y)) \tag{I.6.142}$$

and define the distance between x and y as the length $d(L(x), L(y))$ of the shortest geodesic $x \to y$.

I.6.9 - Invariant measure

An element of volume at the origin

$$dV(0) = dx^1(0) \wedge dx^2(0) \wedge \ldots \wedge dx^n(0) \tag{I.6.143}$$

can be moved from 0 to y by the group operation $L(y)$, and becomes:

$$dV(y) = dx^1(y) \wedge dx^2(y) \wedge \ldots \wedge dx^n(y) = \\ = \det|V_\alpha{}^a(y)|^{-1}\, dx^1(0) \wedge \ldots \wedge dx^n(0) \quad . \tag{I.6.144}$$

An invariant measure can therefore be defined:

$$d\mu(y) = \det|V_\alpha{}^a(y)|\,dV(y) \qquad d\mu(y) = d\mu(0) \quad . \tag{I.6.145}$$

Theorem:

$$d\mu(g) = d\mu(y)d\mu(h) \quad \text{if} \quad \begin{array}{l} g \in G \\ y \in G/H \\ h \in H \end{array} \quad . \tag{I.6.146}$$

Proof: $\det|V_A{}^B(g)| = \det|V_\alpha{}^a(y)|\ \det|V_i{}^j(h)|$

since $h^{-1}dh$ has no components along the $\mathbb{K}$ generators.

Integration of (I.6.146) yields

$$\text{vol } G = \int_{g\in G} d\mu(g) = \int_{y\in G/H} d\mu(y)\int_{h\in H} d\mu(h) =$$

$$= \text{vol }(G/H)\cdot\text{vol }(H) \quad . \tag{I.6.147}$$

so that the volume of coset spaces G/H is just vol G/vol H.

Exercise: find the volume of S^n.

I.6.10 - Connection and curvature

The differential properties of $V = L^{-1}dL$ are expressed by the Maurer-Cartan equation:

$$dV + V \wedge V = 0 \quad , \tag{I.6.150}$$

an immediate consequence of the definition of V:

$$dV = dL^{-1} \wedge dL = -L^{-1}dL\,L^{-1} \wedge dL = -V \wedge V \quad . \tag{I.6.151}$$

In components (see (I.6.80)) we have:

$$dV^a + \frac{1}{2}C^a{}_{bc}V^b \wedge V^c + C^a{}_{bi}V^b \wedge \Omega^i = 0 \tag{I.6.152}$$

$$d\Omega^i + \frac{1}{2}C^i{}_{ab}V^a \wedge V^b + \frac{1}{2}C^i{}_{jk}\Omega^j \wedge \Omega^k = 0 \tag{I.6.153}$$

The torsion 2-form is defined by

$$T^a = dV^a - B^a{}_b \wedge V^b \tag{I.6.154}$$

where the 1-form $B^a{}_b$ is the spin connection (see Chapter I.2).

The spin connection defines parallel transport on the manifold. The simplest choice for $B^a{}_b$ corresponds to vanishing torsion on G/H

$$dV^a - B^a{}_b \wedge V^b = 0 \tag{I.6.155}$$

and $B^a{}_b$ is then called a Riemannian connection.

Combining Eqs. (I.6.152) and (I.6.155) yields

$$B^a{}_b = +\frac{1}{2}C^a{}_{bc}V^c + C^a{}_{bi}\Omega^i \quad . \tag{I.6.156}$$

For symmetric G/H spaces, $B^a{}_b$ takes the simple form

$$B^a{}_b = C^a{}_{bi}\Omega^i \quad . \tag{I.6.157}$$

The curvature 2-form is defined in terms of $B^a{}_b$ as

$$R^a_b = dB^a{}_b - B^a{}_e \wedge B^e{}_b = R^a{}_{bcd}V^c \wedge V^d \quad . \tag{I.6.158}$$

Substituting (I.6.156) into (I.6.158), using the Maurer-Cartan Eqs. (I.6.152-153) for dV^a and $d\Omega^i$, and using Jacobi identities for products of structure constants, one derives the following formula for the curvature tensor:

$$R^a{}_{bcd} = -\frac{1}{4} C^a{}_{be} C^e{}_{cd} - \frac{1}{2} C^a{}_{bi} C^i{}_{cd} - \frac{1}{8} C^a{}_{ec} C^e{}_{bd} + \frac{1}{8} C^a{}_{ed} C^e{}_{bc} \ . \qquad (I.6.159)$$

Exercise: derive (I.6.159).

Exercise: prove the symmetry of (I.6.159) under $(ab) \leftrightarrow (cd)$ interchange.

The Ricci tensor

$$R_{bd} \equiv R^a{}_{bad} \qquad (I.6.160)$$

is easily obtained from (I.6.159) by contracting a and c.

Notice that for symmetric algebras

$$R_{ab} = -\frac{1}{4} \gamma_{ab} \qquad (I.6.161)$$

because

$$\gamma_{ab} = -C^A{}_{aD} C^D{}_{bA} = -C^i{}_{ad} C^d{}_{bi} - C^d{}_{ai} C^i{}_{bd} = 2C^d{}_{ai} C^i{}_{db} \ . \qquad (I.6.162)$$

Exercise: prove that for symmetric G/H

$$D_e R^a{}_{bcd} = 0 \qquad (I.6.163)$$

where the covariant derivative D_e is constructed via the Riemannian connection $B^a{}_b$ in (I.6.156):

$$D_e = \partial_e + \Omega_e{}^i(y) C_i \ . \qquad (I.6.164)$$

Example: the round S^n:

SO(n+1) algebra:

$$[T_{AB}, T_{CD}] = \frac{1}{2} \delta_{AD} T_{BC} + \frac{1}{2} \delta_{BC} T_{AD} - \frac{1}{2} \delta_{AC} T_{BD} - \frac{1}{2} \delta_{BD} T_{AC}$$

$$A, B \ldots = 1, \ldots, n+1 \qquad (I.6.165)$$

Structure constants

$$C^{[AB]}{}_{[CD][EF]} = \frac{1}{2} \delta_{CF} \delta_D^{[A} \delta_E^{B]} + \frac{1}{2} \delta_{DE} \delta_C^{[A} \delta_F^{B]} - \frac{1}{2} \delta_{CE} \delta_D^{[A} \delta_F^{B]} - \frac{1}{2} \delta_{DF} \delta_C^{[A} \delta_F^{B]} \ . \qquad (I.6.166)$$

SO(n) generators: T_{ab}, $a,b = 1,\ldots,n$

coset generators: $T_{a,\, n+1} \equiv T_a$

SO(n+1)/SO(n) is a symmetric coset, so that structure constants with all three indices in $\mathbb{K}$ directions vanish. This greatly simplifies the expressions for the connection and the curvature.

metric:

$$\gamma_{ab} \equiv \gamma_{[a,\, n+1][b,\, n+1]} = C^{[EF]}{}_{[a,\, n+1][CD]} C^{[CD]}{}_{[b,\, n+1][EF]} =$$

$$= -4C^{[e,\, n+1]}{}_{[a,\, n+1][cd]} C^{[cd]}{}_{[e,\, n+1][b,\, n+1]} = \qquad (I.6.167)$$

$$= -\frac{1}{2}\delta^e_c\delta_{da}\left[\delta^c_e\delta^d_b - \delta^d_e\delta^c_b\right] = -\frac{1}{2}\left[n\delta_{ab} - \delta_{ab}\right] =$$

$$= -\frac{1}{2}(n-1)\delta_{ab} \quad .$$

Riemann connection:

a, b run on $K = S^n$

i runs on $H = SO(n)$

$$B_\alpha{}^a{}_b = C^a{}_{bi}\Omega^i_\alpha = C^a{}_{b[cd]}\Omega^{[cd]}_\alpha = -\frac{1}{2}\delta^a_{[c}\,\delta_{d]b}\Omega^{[cd]}_\alpha(z) =$$

$$= \frac{-\delta^a_\alpha z_b + \delta_{\alpha b}z^a}{z^2 + 4} \quad . \qquad (I.6.168)$$

We have used the stereographic coordinates z^a (cfr. (I.6.7)) and the explicit expression (I.6.103) for $\Omega_\alpha{}^{cd}$. Notice that for S^n the H-connection coincides with the Riemann connection.

Curvature:

$$R^a{}_{bcd} = -\frac{1}{2}C^a{}_{bi}C^i{}_{cd} = -\frac{1}{4}\delta^a_{[e}\,\delta_{f]b}\,\delta^{[e}_c\,\delta^{f]}_d =$$

$$= -\frac{1}{8}\left[\delta^a_{|c}\,\delta_{d|b} - \delta_{b|c}\,\delta^a_{d|}\right] \qquad (I.6.169)$$

and

$$R^{ab}{}_{cd} = \gamma^{bb'}R^a{}_{b'cd} = -\frac{2}{(n-1)}\delta^{bb'}R^a{}_{b'cd} = \frac{1}{4(n-1)}\delta^{[ab]}_{[cd]} \qquad (I.6.170)$$

I.6.11 - Rescalings

In general the metric:

$$g_{\alpha\beta}(y) = \gamma_{ab}V^a_\alpha(y)V^b_\beta(y) \qquad (I.6.172)$$

is not the only G-invariant metric on G/H. Let us study the extent of this non-uniqueness.

First, consider the tensor γ_{ab}, i.e. minus the Killing metric restricted to G/H. By an appropriate choice of basis in $\mathbb{G}$, γ_{ab} can always be brought to the form:

$$\gamma_{ab} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & -1 & \\ & 0 & & & & \ddots \\ & & & & & & -1 \end{bmatrix} \begin{matrix} \Big\} p \\ \\ \Big\} n \end{matrix} \qquad (I.6.173)$$

The tangent group, i.e. the group of local rotations on the vielbeins leaving $g_{\alpha\beta}$ unchanged, is SO(p,n). If either n or p vanish, G/H is compact.

We can choose the basis of $\mathbb{G}$ generators once for all: as far as the metric (I.6.172) is concerned, tensor transformations on γ_{ab} are equivalent to the same transformations on the vielbeins. We therefore assume γ_{ab} as in (I.6.173), and consider the invertible linear mappings

$$V^a = M^{ab}V'^b \qquad \det M \neq 0, \quad M^{ab} \in \mathbb{R} \quad . \qquad (I.6.174)$$

Now we ask ourselves under which conditions the new metric

$$g'_{\alpha\beta} = \gamma_{ab}M^{ac}V'^c_\alpha\, M^{bd}V'^d_\beta \qquad (I.6.175)$$

is still a G-invariant one.

A real nonsingular matrix M always admits the decomposition(*)

$$M = OD \qquad \begin{array}{l} O : \text{(pseudo) orthogonal matrix} \\ D : \text{real diagonal matrix} \end{array} \tag{I.6.176}$$

The (pseudo) orthogonal part of M has a trivial action on $g_{\alpha\beta}$: by construction the metric is insensitive to SO(p,n) rotations of the vielbeins. The interesting part of (I.6.176) is D, and essentially different metrics are obtained by <u>rescaling</u> the vielbeins with D:

$$D = \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_{n+p} \end{pmatrix}, \quad V = DV' \Rightarrow V^a = r^a V'^a \quad \text{(no sum on a)} \tag{I.6.177}$$

An arbitrary rescaling (I.6.177) will in general destroy the G-isometry of G/H. For example, on S^n only the uniform dilatation of all directions maintains the SO(n+1) isometry. If some directions expand differently from others, the resulting "squashed" S^n has a lower symmetry (see next section).

The rescaled vielbeins $V^{a'}$ transform under left multiplication by G:

$$V'^a(y+\delta y) - V'^a(y) = -\varepsilon^A W_A{}^i(y)\, \frac{r_b}{r_a}\, C^a{}_{ib} V'^b(y) \quad . \tag{I.6.178}$$

The new metric

$$g'_{\alpha\beta}(y) = \gamma_{ab} V'^a_\alpha(y) V'^b_\beta(y) \tag{I.6.179}$$

(*) A particular case of the Iwasawa decomposition

$$M = ODN$$

with N = nilpotent matrix. When $\det M \neq 0$, N is absent.

is invariant under (I.6.178) only if $r_b/r_a = 1$, for a,b such that $C^b{}_{ia} \neq 0$.

If $C^b{}_{ia}$ is block-diagonal in some subspaces $S_1, S_2 \ldots$ of $\mathbb{K}$, we can satisfy $r_b/r_a = 1$ by choosing a common rescaling r_I for all the vielbeins within the same block S_I. Then $\delta g' = 0$ under the variation (I.6.178), and we have a <u>G-symmetric rescaling</u>.

We summarize this result in the following

<u>Theorem</u>: a rescaling

$$\begin{array}{lll} V^{a_1} \to r_{a_1} V^{a_1} & & a_1 \text{ labels the subspace } S_1 \\ V^{a_2} \to r_{a_2} V^{a_2} & & a_2 \text{ labels the subspace } S_2 \\ \vdots \quad \vdots \quad \vdots & & \end{array} \tag{I.6.180}$$

is a G-symmetric rescaling if and only if

$$(C_i)_a{}^b \qquad (\equiv C^b{}_{ia})$$

is block-diagonal in the spaces spanned by $V^{a_1}, V^{a_2}, \ldots$.

The number N of rescaling parameters, i.e. the number of parameters necessary to specify the particular G-invariant metric, is equal to the number of irreducible blocks of $(C_i)_a{}^b$. This matrix describes how H acts on the subspace $\mathbb{K}$. If H acts irreducibly, the coset is called <u>isotropy irreducible</u>, and only the trivial rescaling $V^a \to rV^a$ (same r for all a) is G-symmetric. If G/H is isotropy reducible, we have an independent rescaling parameter for each irreducible subspace.

The rescalings must be non singular ($r = 0$, $r = \infty$ are excluded), but are otherwise unconstrained. We now derive the rescaled expressions for the connection and the curvature.

The Cartan Maurer eqs. become(*)

(*) dropping primes on vielbeins.

$$dV^a + \frac{1}{2}\frac{r_b r_c}{r_a} C^a{}_{bc} V^b \wedge V^c + \frac{r_b}{r_a} C^a{}_{bi} V^b \wedge \Omega^i = 0 \qquad \text{(I.6.181a)}$$

$$d\Omega^i + \frac{1}{2} r_a r_b C^i{}_{ab} V^a \wedge V^b + \frac{1}{2} C^i{}_{jk} \Omega^j \wedge \Omega^k = 0 \quad . \qquad \text{(I.6.181b)}$$

The zero-torsion condition (I.6.155) determines $B^a{}_b$ up to a tensor $K^a{}_{bc}$, symmetric in b,c:

$$B^a{}_b = \frac{1}{2}\frac{r_b r_c}{r_a} C^a{}_{bc} V^c + \frac{r_b}{r_a} C^a{}_{bi} \Omega^i + K^a{}_{bc} V^c \quad . \qquad \text{(I.6.182)}$$

$K^a{}_{bc}$ is determined by the requirement $B^a{}_b + B_b{}^a = 0$ (i.e. $B^a{}_b$ is a Riemann connection. Indices are raised and lowered with γ_{ab}). Then:

$$K^a{}_{bc} = \frac{r_a}{2} C^a{}_{bc} \left[\frac{r_c}{r_b} - \frac{r_b}{r_c}\right] \quad . \qquad \text{(I.6.183)}$$

and

$$B^a{}_b = + \frac{1}{2} C^a{}_{bc} V^c \binom{ab}{c} + C^a{}_{bi} \frac{r_b}{r_a} \Omega^i \qquad \text{(I.6.184)}$$

with

$$\binom{ab}{c} \equiv \frac{r_a r_c}{r_b} + \frac{r_b r_c}{r_a} - \frac{r_a r_b}{r_c} \quad . \qquad \text{(I.6.185)}$$

The Riemann curvature, defined by (I.6.158), is now:

$$R^a{}_{bcd} = -\frac{1}{4} C^a{}_{be} C^e{}_{cd} \binom{ab}{e} \frac{r_c r_d}{r_e} - \frac{1}{2} C^a{}_{bi} C^i{}_{cd}\, r_c r_d - $$
$$- \frac{1}{8} C^a{}_{ec} C^e{}_{bd} \binom{ae}{c}\binom{be}{d} + \frac{1}{8} C^a{}_{ed} C^e{}_{bc} \binom{ae}{d}\binom{be}{c} \quad . \qquad \text{(I.6.186)}$$

The Ricci tensor for symmetric G/H is

$$R_{ab} = -\frac{1}{2} C^c{}_{ai} C^i{}_{cb} r_c r_b = -\frac{1}{4}\gamma_{ab}(r_a)^2 = -\frac{1}{4}\gamma_{ab}(r_b)^2 \quad . \qquad \text{(I.6.187)}$$

(cfr. (I.6.161). Note that $C^a{}_{ib}$ is block diagonal).

I.6.12 - A note on the isometries of G/H

The "natural" isometry group of the coset space G/H is G, and we have seen in Section I.6.6 how to construct a G-invariant metric.

Our analysis, however, has been restricted to the left action of G on G/H. For example, the metric (I.6.172) is a left invariant metric, and the transformation law considered in (I.6.5) expresses how $L(y)$ changes under left multiplication by $g \in G$.

One can also examine what happens to $L(y)$ under right action of G. The left action of G induces $SO(N)$ rotations on the vielbein, thus leaving $g_{\alpha\beta}(y)$ invariant. What will be the vielbein transformation law under right action of G? Is there a subgroup of G such that its right action on G/H only rotates the vielbein? This subgroup would be an additional isometry of G/H.

We start by studying the behaviour of the coset representative $L(y)$:

$$L(y)g = L(y')h : \quad \text{right action of } g \in G \text{ on } L(y) \ . \qquad \text{(I.6.188)}$$

For the expression $L(y)g$ to make sense, it should not depend on the choice of coset representatives. This happens if and only if g belongs to the normalizer of H in G, denoted $N(H)$, and defined by

$$g H g^{-1} = H \Leftrightarrow g \in N(H) \quad . \qquad \text{(I.6.189)}$$

Indeed one can easily verify that $L(y) \sim L(y)'$ (i.e. $L(y)$ and $L(y)'$ belong to the same coset) implies $L(y)g \sim L(y)'g$ if and only if $g \in N(H)$.

This discussion was not necessary for left multiplication since $gL(y)$ is well defined for every $g \in G$.

It is clear that if $g \in H$, its right action on $L(y)$ is trivial: it does not move the point y on the coset space. Thus we need to consider only elements of $N(H)/H$, which has a natural group structure.

We proceed to prove that $N(H)/H$ is the <u>right isometry group</u> of G/H.

Consider the transformation law of the 1-form $V(y)$ under right multiplication by $g \in N(H)/H$:

$$V(y') = L(y')dL(y') = hg^{-1}(L^{-1}(y)dL(y))gh^{-1} + hdh^{-1} \quad . \tag{I.6.190}$$

Projecting on the coset generators T_a we find:

$$\begin{aligned} V^a(y') &= (hg^{-1}V(y)gh^{-1})^a = V^b(y)D_b{}^a(gh^{-1}) = \\ &= V^b(y)D_b{}^E(g)D_E{}^a(h^{-1}) = V^b(y)D_b{}^e(g)D_e{}^a(h^{-1}) \quad . \end{aligned} \tag{I.6.191}$$

where we have used $D_i{}^a(h^{-1}) = 0$.

Infinitesimally, taking g and h as in (I.6.67-68):

$$\begin{aligned} V^a(y+\delta y) - V^a(y) &= -\varepsilon^A V^B(y)C_{AB}{}^a + \varepsilon^A W_A{}^i(y)V^b(y)C^a{}_{ib} = \\ &= \varepsilon^A(-C^a{}_{Ab} + W_A{}^i(y)C^a{}_{ib})V^b(y) - \varepsilon^b C^a{}_{bi}\omega^i(y) \quad . \end{aligned} \tag{I.6.192}$$

Thus, the right action of g on the vielbein induces an $SO(N)$ rotation of $V^b(y)$ if and only if $\varepsilon^b C^a{}_{bi} = 0$ for every a,i. This happens if the generators $K_b \subset N(H)/H$ commute with H, since this implies

$$C^a{}_{bi} = 0 \quad \text{if} \quad \begin{array}{l} b \to N(H)/H \\ i \to H \\ a \to G/H \end{array} \quad . \tag{I.6.193}$$

Now, the generators of $N(H)/H$ are defined to act on $\mathbb{H}$ as

$$[K_b, \mathbb{H}] \subset \mathbb{H} \qquad K_b \in N(H)/H \tag{I.6.194}$$

which is just the infinitesimal form of $gHg^{-1} = H$. Reductivity of G, however, requires

$$[K_b, \mathbb{H}] \subset \mathbb{K} \quad . \tag{I.6.195}$$

Eqs. (I.6.194) and (I.6.195) together imply

$$[K_b, \mathbb{H}] = 0 \quad . \tag{I.6.196}$$

Therefore the generators K_b of $N(H)/H$ commute with H, and consequently $N(H)/H$ is an isometry of G/H.

The Killing vectors $K_A(y)$ of $N(H)/H$ and the corresponding H-compensators $W_A{}^i(y)$ can be derived as in Section I.6.6, Eqs. (I.6.88-89). Consider first the infinitesimal form of $L(y)g = L(y')h$, i.e.:

$$L(y)T_A = K_A(y)L(y) - L(y)W_A{}^i(y)T_i \quad . \tag{I.6.197}$$

Multiplying by $L^{-1}(y)$ on the left:

$$\begin{aligned} T_A &= L^{-1}(y)\frac{\partial L}{\partial y^\alpha}(y)K_A{}^\alpha(y) - T_i W_A{}^i(y) = \\ &= V_\alpha{}^a K_A{}^\alpha T_a + \Omega_\alpha{}^i K_A{}^\alpha T_i - W_A{}^i T_i \quad . \end{aligned} \tag{I.6.198}$$

Projecting on the $\mathbb{K}$ generators T_a yields:

$$K_a{}^\alpha(y) = V_a{}^\alpha(y) \tag{I.6.199}$$

$$K_i^{\alpha}(y) = 0 \quad . \qquad (I.6.200)$$

Eq. (I.6.200) is consistent with the fact that the right action of H is trivial on G/H. Eq. (I.6.199) gives the Killing vectors corresponding to the right action of N(H)/H: they are just the inverse vielbeins $V_a^{\alpha}(y)$.

Projecting Eq. (I.6.198) on the $\mathbb{H}$ generators T_i yields

$$W_A^{\ i}(y) = \delta_A^i - \Omega_{\alpha}^{\ i}(y)K_A^{\ \alpha}(y) \quad . \qquad (I.6.201)$$

It is evident from Eqs. (I.6.66) and (I.6.188) that left- and right-isometries on G/H commute.

Exercise: check that left and right Killing vectors commute.

From the preceding discussion, the reader could infer that the isometries of a coset space G/H are at least

$$G \times \frac{N(H)}{H} \quad . \qquad (I.6.202)$$

In most cases this is indeed correct. However, there are two instances in which (I.6.202) fails to give the actual isometry group:

1) Some of the right Killing vectors coincide with left Killing vectors. As each right isometry commutes with each left isometry, these common Killing vectors can only correspond to explicit U(1) factors occurring in G and N(H)/H. The isometry of G/H is therefore reduced to

$$G' \times N(H)/H \qquad (I.6.203)$$

where $G = G' \times$ (common U(1) factors). This happens whenever G contains explicit U(1) factors: their right and left actions clearly coincide, as they commute with all of G. An example if provided by the coset spaces N^{pqr} discussed in Chapter V.6.

2) The symmetry may be larger than (I.6.202). This happens when the coset manifold can be described by more than one quotient G/H. If $G/H \approx \tilde{G}/\tilde{H}$, with $\tilde{G} \supset G$ the maximal group for which this is possible, the true isometry group of the coset manifold will be

$$\tilde{G} \times N(\tilde{H})/\tilde{H} \qquad (I.6.204)$$

modulo the considerations in 1). A classic example is given by the 7-sphere S^7: as a coset space, S^7 can be written in many ways:

$$\frac{SO(5)}{SO(3)} \approx \frac{SU(4)}{SU(3)} \approx \frac{SO(7)}{G_2} \approx \frac{SO(8)}{SO(7)} \quad . \qquad (I.6.205)$$

In the first two cases, the isometry group is in general $G \times N(H)/H$, but is increased to SO(8) by a particular rescaling of the vielbeins (see later the example of SO(5)/SO(3)).

On $SO(7)/G_2$, the unique SO(7)-invariant metric is also SO(8) invariant, so that $SO(7)/G_2$ is the round S^7.

Symmetric rescalings

We now discuss rescalings preserving the full $G \times N(H)/H$ isometry. Recall the transformation laws of the coset vielbeins:

left action of G:

$$V^a(y+\delta y) - V^a(y) =$$

$$- \varepsilon^A [D_A^{\ i}(L(y)) - K_A^{\ \beta}(y) V_{\beta}^{\ i}(y)]\, C^a_{\ ib} V^b(y) \quad . \qquad (I.6.206)$$

(A runs on $\mathbb{G}$)

right action of $N(H)/H$:

$$V^a(y+\delta y) - V^a(y) =$$

$$-\varepsilon^b\left[C^a{}_{bc} + V_b{}^{\beta}(y)\,\omega_\beta{}^i(y)\,C^a{}_{ic}\right]V^c(y) \qquad \text{(I.6.207)}$$

(b runs on $N(H)/H$)

By the same argument used in Section I.6.11, it is clear that if $(C_b)_c{}^a$ and $(C_i)_c{}^a$ are block diagonal in the same subspaces $S_1, S_2 \ldots$ of K, the vielbeins of these subspaces can be independently rescaled without loss of $G \times N(H)/H$ symmetry.

We have therefore the following extension of Theorem (I.6.180):

<u>Theorem:</u> A rescaling

$$\begin{aligned} V^{a_1} &\to r_{a_1} V^{a_1} \\ V^{a_2} &\to r_{a_2} V^{a_2} \\ &\vdots \end{aligned} \qquad \text{(I.6.208)}$$

is a $G \times N(H)/H$ symmetric rescaling if and only if

$(C_D)_b{}^a$ D runs on $N(H)$

is block-diagonal in the spaces spanned by $V^{a_1}, V^{a_2}, \ldots$

I.6.13 Some examples

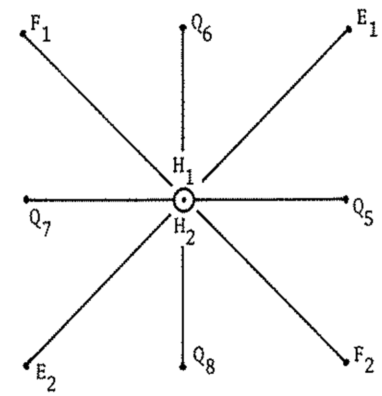
$$\frac{SO(5)}{SO(3)^I}$$

The root diagram(*) of $SO(5)$ is:

(*) We recall that in the Cartan basis (H_a, E_λ), the commutators are

$[H_a, E_\lambda] = r^\lambda_a E_\lambda$ r^λ_a = a-component of the root $\vec{r}^\lambda$ corresponding to E_λ

$[E_\lambda, E_{-\lambda}] = \sum_a r^\lambda_a H_a$



(short roots)$^2 = \frac{1}{6}$

(long roots)$^2 = \frac{1}{3}$

(I.6.209)

The $SO(3)^I \times SO(3)^J$ subalgebra is associated to the generators E_1, E_2, F_1, F_2, H_1, H_2. More precisely, it is generated by the combinations

$$SO(3)^I:\quad \mathbb{E}_1 = \frac{i\sqrt{6}}{2}(E_1+E_2),\quad \mathbb{E}_2 = \frac{\sqrt{6}}{2}(E_1-E_2),$$

$$\mathbb{E}_3 = -\frac{i\sqrt{6}}{2}(H_1+H_2) \qquad \text{(I.6.210)}$$

$$SO(3)^J:\quad \mathbb{F}_1 = \frac{i\sqrt{6}}{2}(F_1+F_2),\quad \mathbb{F}_2 = \frac{\sqrt{6}}{2}(F_1-F_2),$$

$$\mathbb{F}_3 = \frac{i\sqrt{6}}{2}(H_1-H_2) \qquad \text{(I.6.211)}$$

Note that we are forced to introduce complex combinations of E, F and H generators in order to have a compact $SO(3)\times SO(3)$. If the i factors were omitted in $\mathbb{E}_1, \mathbb{E}_3, \mathbb{F}_1, \mathbb{F}_3$, the resulting algebra would be

footnote cont'd...

$[E_\lambda, E_\mu] = N_{\lambda\mu} E_{\lambda+\mu}$ with $N^2_{\lambda\mu} = n_\lambda(1+m_\lambda)\,\frac{\vec{r}^\lambda \cdot \vec{r}^\lambda}{2}$

where n_λ, m_λ are defined by the following conditions:

$r^\mu + n r^\lambda$ is a root and $r^\mu + (n+1) r^\lambda$ is <u>not</u> a root

$r^\mu - n r^\lambda$ is a root and $r^\mu - (m+1) r^\lambda$ is <u>not</u> a root

Furthermore note that $\vec{r}^{-\lambda} = -\vec{r}^\lambda$.

$SO(2,1)\times SO(2,1)$. In a compact coset space, the coset generators must be antihermitian (see Section (I.6.3)). Since $Q_5^\dagger = Q_7$, $Q_6^\dagger = Q_8$, we consider the four antihermitian combinations:

$$\begin{aligned} \mathbb{Q}_0 &= i(Q_5 + Q_7) \\ \mathbb{Q}_1 &= Q_6 - Q_8 \\ \mathbb{Q}_2 &= -i(Q_6 + Q_8) \\ \mathbb{Q}_3 &= Q_5 - Q_7 \quad . \end{aligned} \tag{I.6.212}$$

The SO(5) structure constants in the $\mathbb{E}$, $\mathbb{F}$, $\mathbb{Q}$ basis are

$$\begin{aligned} &C^i_{jk} = \varepsilon_{ijk} \quad , \quad C^{\hat i}_{\hat j\hat k} = \varepsilon_{\hat i\hat j\hat k} \\ &C^0_{ia} = \frac{1}{2}\delta_{ai} \quad , \quad C^a_{i0} = -\frac{1}{2}\delta_{ai} \quad , \quad C^i_{a0} = \frac{1}{2}\delta_{ai} \\ &C^0_{\hat i a} = -\frac{1}{2}\delta_{a\hat i} \quad , \quad C^a_{\hat i 0} = \frac{1}{2}\delta_{a\hat i} \quad , \quad C^{\hat i}_{a0} = -\frac{1}{3}\delta_{a\hat i} \\ &C^b_{ia} = \frac{1}{2}\varepsilon_{iab} \quad , \quad C^i_{ab} = \frac{1}{3}\varepsilon_{iab} \\ &C^b_{\hat i a} = \frac{1}{2}\varepsilon_{\hat i ab} \quad , \quad C^{\hat i}_{ab} = \frac{1}{3}\varepsilon_{\hat i ab} \quad . \end{aligned} \tag{I.6.213}$$

with the index conventions:

$\hat i, \hat j, \hat k$ run on $SO(3)^I$

i, j, k run on $SO(3)^J$

$$\left.\begin{array}{ll} a, b, c & \text{correspond to } \mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3 \\ 0 & \text{corresponds to } \mathbb{Q}_0 \end{array}\right\} \text{coset indices}$$

The Killing metric

$$\gamma_{AB} = C^D{}_{AC}C^C{}_{BD} \qquad A, B.. \text{ run on } SO(5)$$

is given by

$$\gamma_{ij} = -3\delta_{ij}, \quad \gamma_{\hat i\hat j} = -3\delta_{\hat i\hat j}, \quad \gamma_{ab} = -2\delta_{ab}, \quad \gamma_{00} = -2$$

with vanishing off-diagonal parts.

We examine next the possible SO(5)-symmetric rescalings of the coset vielbeins V^i, V^a, V^0.

From the root diagram, it appears that the 7-dimensional coset space $SO(5)/SO(3)^I$ splits into 5 irreducible subspaces under the action of $SO(3)^I$, namely the three singlets $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$ and the two doublets (Q_6,Q_7) and (Q_8,Q_5). However, because of the change of basis (I.6.212) necessary to obtain a compact coset, the matrix (C_i) is not reducible any more in the $\mathbb{Q}$ subspace via a real tensor transformation on the Q's, and mixes the 0,a directions (C^a_{i0}, C^a_{ib} are nonvanishing). According to Theorem (I.6.180), the rescalings that preserve the SO(5) isometry involve four independent parameters r^i $(i=1,2,3)$, r

$$V^{i\prime} = r^i V^i \qquad V^{a\prime} = rV^a \qquad V^{0\prime} = rV^0 \tag{I.6.214}$$

Before proceeding to compute the rescaled curvature, we observe that the symmetry of $SO(5)/SO(3)^I$ is actually greater than SO(5).

Indeed the normalizer N of $SO(3)^I$ in SO(5) is

$$SO(3)^I \times SO(3)^J \tag{I.6.215}$$

and N/H is therefore $SO(3)^J$. According to Theorem (I.6.208), the full isometry of $SO(5)/SO(3)^I$ is

$$G \times N/H = SO(5) \times SO(3)^J \tag{I.6.216}$$

and the rescaling preserving this isometry involves only <u>two</u> parameters, since

$$C^{\alpha}{}_{D\beta} \qquad \begin{array}{l} \alpha,\beta : a,\ i,\ 0 \to G/H \\ D : \hat{i},\ i \to N(H) \end{array} \tag{I.6.217}$$

is block diagonal in the spaces $\{i\}$ and $\{a,0\}$.

We therefore consider the rescalings

$$V^{i'} \to aV^i \qquad V^{a'} \to bV^a$$
$$V^{0'} \to bV^0 \quad . \tag{I.6.218}$$

Applying formula (I.6.186), the rescaled Riemann curvature reads:

$$R^a{}_{bcd} = -\frac{1}{48}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(8 - \frac{3b^2}{a^2})b^2$$

$$R^a{}_{0b0} = -\frac{1}{48}\delta_{ab}(8 - \frac{3b^2}{a^2})b^2$$

$$R^i{}_{0j0} = -\frac{1}{48}\delta_{ij}\frac{b^4}{a^2}$$

$$R^a{}_{ibj} = -\frac{1}{32}\delta_{ij}\delta_{ab}\frac{b^4}{a^2} + \frac{1}{32}(\delta_{aj}\delta_{ib} - \delta_{ai}\delta_{bj})(2 - \frac{b^2}{a^2})b^2$$

$$R^i{}_{jk\ell} = -\frac{1}{4}\delta^{ij}_{k\ell}\,a^2$$

$$R^i{}_{jab} = -\frac{1}{12}(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{aj})(\frac{b^4}{a^2} - 2b^2)$$

$$R^i{}_{j0a} = +\frac{1}{24}\varepsilon_{ija}(2 - \frac{b^2}{a^2})b^2 \quad . \tag{I.6.219}$$

The corresponding Ricci tensor is

$$R_{ab} = -\delta_{ab}(\frac{1}{2} - \frac{1}{8}\frac{b^2}{a^2})b^2$$

$$R_{ij} = -\delta_{ij}(\frac{a^2}{4} + \frac{1}{8}\frac{b^4}{a^2})$$

$$R_{00} = -(\frac{1}{2} - \frac{1}{8}\frac{b^2}{a^2})b^2 \quad . \tag{I.6.220}$$

Let us now look for rescalings a,b such that the resulting space becomes an <u>Einstein space</u>, i.e. a space for which the Ricci tensor $R_{\alpha\beta}$ is proportional to the metric $\gamma_{\alpha\beta}$. As discussed in Part V, Einstein spaces are of special relevance in Kaluza-Klein supergravity. It is an easy exercises to check that if

$$\frac{b^2}{a^2} = 2 \tag{I.6.221}$$

the Riemann curvature becomes that of the round 7-sphere(*)

$$R^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{24}\delta^{\alpha\beta}_{\gamma\delta} \quad . \tag{I.6.222}$$

This is an interesting illustration of how the symmetry $G \times N/H$ of a coset G/H can be increased by a rescaling that brings G/H to be equivalent to $\tilde{G}/\tilde{H}$, with $G \subset \tilde{G}$. Here the $SO(5) \times SO(3)$ symmetry of $SO(5)/SO(3)^I$ becomes the full $SO(8)$ of the round S^7.

Another Einstein space can be reached continuously from S^7 at the value

$$\frac{b^2}{a^2} = \frac{2}{5} \tag{I.6.223}$$

of the rescalings.

(*) Cfr. the example of S^n at the end of Section I.6.10, with $n = 7$.

This space has only the "canonical" symmetry of $SO(5)/SO(3)^I$, i.e.

$$SO(5) \times SO(3)^J \qquad (I.6.224)$$

and is called the "squashed" seven-sphere; it is the only other Einstein space with the topology of S^7. Its use in 11-dimensional supergravity is discussed in Chapter V.6.

The M^{pqr} spaces

Topology and symmetries

Consider the 7-dimensional coset manifolds

$$\frac{G}{H} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \qquad (I.6.225)$$

where $SU(2)$ is embedded as an "isospin" subgroup of $SU(3)$, i.e. the triplet $\underline{3}$ of SU(3) decomposes as $\underline{2}+\underline{1}$ under $SU(2)$. $SU(3) \times SU(2) \times U(1)$ has three commuting U(1) generators:

λ_8 : hypercharge of SU(3), commutes with the "isospin" subgroup $SU(2) \subset SU(3)$

τ_3 : the isospin of SU(2)

Y : the U(1) charge .

The surviving U(1) generator Z in the "coset algebra" is in general a linear combination of λ_8, τ_3 and Y:

$$Z = \frac{i}{2} p\sqrt{3}\lambda_8 + \frac{i}{2} q\tau_3 + irY \quad . \qquad (I.6.226)$$

The embedding of the two U(1) factors of H is thus characterized by the three integers p,q,r, and the corresponding spaces are denoted by M^{pqr}.

For Z to be a compact generator, p,q and r must be rational numbers. Since an overall rescaling is inessential, we can always choose p,q,r to be coprime integers.

The topology of the M^{pqr} spaces can be understood by considering the quotient in separate pieces: $SU(3)/SU(2)$ is topologically S^5, $SU(2)$ is S^3, and thus

$$M^{pqr} = \frac{S^5 \times S^3 \times U(1)}{U(1) \times U(1)} \quad . \qquad (I.6.227)$$

For simplicity, consider first M^{pq0}. In this case the generator of one of the U(1) factors of H is mapped into Y. Then

$$M^{pq0} = \frac{S^5 \times S^3}{U(1)} \quad . \qquad (I.6.228)$$

Next observe that S^{2n+1} may be considered as a U(1) bundle over $\mathbb{C}P^n$, since

$$S^{2n+1} = \frac{SU(n+1) \times U(1)}{SU(n) \times U(1)} \qquad (I.6.229)$$

(remember that $\mathbb{C}P^n = \frac{SU(n+1)}{SU(n) \times U(1)}$, see Section I.6.3).

Then S^5 is a U(1) bundle over $\mathbb{C}P^2$, and S^3 is a U(1) bundle over $\mathbb{C}P^1 \simeq S^2$. The factoring by U(1) in (I.6.228) causes the identification of the two fibers, and therefore M^{pq0} can be considered as a U(1) bundle over $\mathbb{C}P^2 \times S^2$. The identification of the two fibers is done in such a way that going q times around the U(1) fiber of S^5 is equivalent to going $-3/2p$ times around the fiber in S^3 (see Eq. (I.6.231)); this implies that the topology of M^{pq0} only depends on the ratio p/q.

The isometry group of M^{pq0} is at least $SU(3)\times SU(2)\times U(1)$. Indeed M^{pq0} may also be written as

$$\frac{SU(3)\times SU(2)}{SU(2)\times U(1)} \tag{I.6.230}$$

with the $SU(3)\times SU(2)$ symmetry generated by the left action of the group on the coset space. However, the metric is also invariant under right multiplication by the U(1) generated by Z, i.e. the extra generator (outside $SU(2)\times U(1)$ itself) in the normalizer of $SU(2)\times U(1)$ in $SU(3)\times SU(2)$.

In certain cases the symmetry of M^{pq0} may be even larger. For example, in $M^{010}\simeq \mathbb{C}P^2\times S^3$ or $M^{100}\simeq S^5\times S^2$ the symmetries are $SU(3)\times SU(2)\times SU(2)$ and $SO(6)\times SO(3)$ respectively.

Returning to the general case of M^{pqr}, we may choose one of the U(1) factors in H to be generated by

$$Z'' = -\frac{i}{2}q\sqrt{3}\,\lambda_8 + \frac{3i}{2}p\,\tau_3 \tag{I.6.231}$$

and consequently

$$M^{pqr} \simeq \frac{\dfrac{S^5\times S^3}{U(1)}\times U(1)}{U(1)} \simeq \frac{M^{pq0}\times U(1)}{U(1)} \quad . \tag{I.6.232}$$

The last quotient by U(1) is almost trivial: it has the effect of cancelling the U(1) in the numerator, and factoring the space M^{pq0} by a finite cyclic group. If p and q are not both zero, the embedding of $U(1)\to M^{pq0}\times U(1)$ may be defined by

$$e^{i\Phi} \to e^{\Phi Z'/k} \tag{I.6.233}$$

where

$$Z' = 2r\Big[\frac{i}{2}p\sqrt{3}\,\lambda_8 + \frac{i}{2}q\,\tau_3\Big] - i(3p^2+p^2)Y \tag{I.6.234}$$

and k is the highest common factor of $2rp$, rq and $(3p^2+q^2)$. Dividing by k ensures that (I.6.233) is a one-to-one embedding. If Φ changes by $2\pi k/(3p^2+q^2)$, then one has returned to the identity in the U(1) generated by Y: it follows that points in M^{pq0} which differ by an integer power of

$$\exp\left[\frac{4i\pi r}{(3p^2+q^2)}\left(\frac{1}{2}p\sqrt{3}\,\lambda_8 + \frac{1}{2}q\,\tau_3\right)\right] \tag{I.6.235}$$

must be identified with each other. Thus

$$M^{pqr} = \frac{M^{pq0}}{\mathbb{Z}_\ell} \tag{I.6.236}$$

with $\ell = (3p^2+q^2)/k$ and $\mathbb{Z}_1 \equiv \{1\}$, and the spaces M^{pqr} have fundamental group (see Section I.6.14):

$$\Pi_1(M^{pqr}) = \mathbb{Z}^\ell \quad . \tag{I.6.237}$$

Their universal covering space is M^{pq0}.

<u>Rescaled curvatures</u>

We now turn to study the geometry of M^{pqr} spaces. As for any G/H space, this involves three steps:

i) to determine the G structure constants in a suitable generator basis for $\mathbb{G} = \mathbb{H}\oplus\mathbb{K}$

ii) to determine how the coset vector space $\mathbb{K}$ decomposes under ad (H), the adjoint representation of H whose matrix elements are $(C_H)_K{}^K$. A rescaling parameter is assigned to each irreducible subspace of $\mathbb{K}$.

iii) to compute the rescaled curvature of G/H.

Of particular relevance for d=11 supergravity are the Einstein metrics, for which

$$R_{\alpha\beta} = \text{const.}\ g_{\alpha\beta} \quad . \tag{I.6.238}$$

Condition (I.6.238) translates into algebraic equations for the rescaling parameters. If these can be solved, G/H can be rescaled to an Einstein space.

A suitable basis for $G = SU_3 \times SU_2 \times U_1$ generators is given by*

$$\frac{i}{2}\,[\lambda_1\ \lambda_2\ \lambda_3 \quad ; \quad \lambda_4\ \lambda_5\ \lambda_6\ \lambda_7]$$

$$\frac{i}{2}\,[\tau_1,\tau_2]$$

$$Z,\ Z',\ Z'' \quad . \tag{I.6.239}$$

where Z, Z' and Z'' are the generators of the three commuting $U(1)$'s (see Eqs. (I.6.226-231-234)) in G. The subgroup $H = SU(2) \times U(1) \times U(1)$ is generated by

$$\frac{i}{2}\,[\lambda_1,\ \lambda_2,\ \lambda_3] \ : SU(2)$$

$$Z',\ Z'' \ : U(1) \times U(1) \tag{I.6.240}$$

so that the coset directions correspond to

$$\frac{i}{2}\,[\lambda_4,\ \lambda_5,\ \lambda_6,\ \lambda_7 \quad ; \quad \tau_1,\ \tau_2] \quad ; \quad Z \quad . \tag{I.6.241}$$

---

* In the following $\lambda_1,\dots,\lambda_8$ are the standard Gell-Mann matrices satisfying $[\lambda_i,\lambda_j] = 2i\, f_{ijk}\,\lambda_k$ where the non vanishing components of the SU(3) structure constants f_{ijk} are $f_{123} = 1$; $f_{147} = f_{246} = f_{257} = f_{345} = f_{367} = -f_{156} = 1/2$; $f_{458} = f_{678} = \sqrt{3}/2$. τ_1 and τ_2 are the first two Pauli matrices.

On the basis (I.6.239), the non vanishing structure constants read:(*)

$$C^m{}_{nZ} = \frac{q}{2}\,\varepsilon_{mn} \qquad C^A{}_{iB} = f_{iAB}$$

$$C^Z{}_{mn} = \frac{q}{2}\,\varepsilon_{mn} \qquad C^A{}_{Z'B} = 2\sqrt{3}\, rp\, f_{8AB}$$

$$C^Z{}_{AB} = \frac{\sqrt{3}}{2}\, p\, f_{8AB} \qquad C^A{}_{Z''B} = -\sqrt{3}\, q\, f_{8AB}$$

$$C^A{}_{BZ} = \frac{\sqrt{3}}{2}\, p\, f_{8AB} \qquad C^m{}_{Z'n} = 2rq\,\varepsilon_{mn}$$

$$C^m{}_{Z''n} = 3p\,\varepsilon_{mn} \quad . \tag{I.6.242}$$

By inspection of the $C^K{}_{HK'}$ structure constants, we see that the coset linear space $\mathbb{K}$ splits into three irreducible subspaces spanned respectively by $[\lambda_4,\ \lambda_5,\ \lambda_6,\ \lambda_7]$, $[\tau_1,\ \tau_2]$ and Z. We can therefore introduce 3 rescaling parameters a, b, c:

$$V^A = a\, V^{A'}$$

$$V^m = b\, V^{m'}$$

$$V^Z = c\, V^{Z'} \tag{I.6.243}$$

and the $SU_3 \times SU_2 \times U_1$ - invariant metrics on M^{pqr} spaces are characterized by the values of a, b, c.

The rescaled M^{pqr} curvatures are easily derived from the general formula (I.6.186):

$$R^{mn}{}_{rs} = b^2\left(1 - \frac{3}{4}\frac{b^2}{c^2}\, q^2\right)\varepsilon_{rs}$$

$$R^{mn}{}_{AB} = -\sqrt{3}\, pq\, \frac{a^2 b^2}{4c^2}\,\varepsilon^{mn}\, f_{8AB}$$

$$R^{Zm}{}_{Zn} = q^2\, \frac{b^2}{4c^2}$$

(*) The indices have the following ranges: $A,B = 4,5,6,7$; $m,n = 1,2$.

$$R^{ZA}{}_{ZB} = \frac{9}{16} p^2 \frac{a^4}{c^2} \delta^A_B$$

$$R^{mA}{}_{nB} = -\sqrt{3}\, pq \frac{a^2 b^2}{4c^2} f_{8AB}\, \varepsilon^{mn}$$

$$R^{AB}{}_{CD} = \frac{1}{2} a^2 (f_{iAB} f_{iCD} + (1 - \frac{3p^2a^2}{2c^2}) f_{8AB} f_{8CD}) - $$

$$- \frac{3}{4} p^2 \frac{a^2}{c^2} f_{8AC} f_{8BD}$$

$$R^{AB}{}_{mn} = -\sqrt{3}\, pq \frac{a^2 b^2}{4c^2} \varepsilon^{mn} f_{8AB} \qquad (I.6.244)$$

and the Ricci tensor is block diagonal in the A,m,Z indices:

$$R^m{}_n = \frac{1}{2} \delta^m_n \left[b^2 - \frac{1}{2}\frac{b^4}{c^4} q^2\right] \quad , \quad R^A{}_B = \frac{3}{8} a^2 \delta^A_B \left(2 - \frac{9}{2}\frac{a^2}{c^2} p^2\right)$$

$$R^Z{}_Z = \frac{1}{4}\left[\frac{b^4}{c^4} q^2 + \frac{9}{2} p^2 \frac{a^4}{c^2}\right] \quad . \qquad (I.6.245)$$

Einstein metrics on M^{pqr}

In view of the later applications to Kaluza-Klein supergravity, we investigate here the possibility of having Einstein metrics on M^{pqr}.

The question is whether there exists a triplet a,b,c such that the components of the diagonal Ricci tensor (I.6.245) are all equal.

We distinguish 4 cases:

i) $q = p = 0$. The topology is that of $\mathbb{C}P_2 \times \mathbb{C}P_1 \times S^1$, which obviously cannot be rescaled to an Einstein space, since S^1 has vanishing curvature.

ii) $p = 0$, $q \neq 0$. In this case the topology is that of $\mathbb{C}P_2 \times S^3$, and the rescalings

$$a = 4e \quad , \quad b = 4\sqrt{3}\, e \quad , \quad c = 2\, qe \qquad (I.6.246)$$

bring the Ricci tensor $R^\alpha{}_\beta$ to be

$$R^\alpha{}_\beta = 12\, e^2\, \delta^\alpha_\beta \quad . \qquad (I.6.247)$$

iii) $p \neq 0$, $q = 0$. The topology is $S^5 \times S^2$ and

$$a = 2\sqrt{6}\, e \quad , \quad b = 2\sqrt{6}\, e \quad , \quad c = 3\sqrt{6}\, pe \qquad (I.6.248)$$

are the rescalings for condition (I.6.247) to hold.

iv) $p \neq 0$, $q \neq 0$. The topology is no longer that of a direct product of spaces, and is different for different ratios q/p. For each q/p there exists an Einstein metric, corresponding to the rescalings

$$a = \frac{q}{p} \gamma \sqrt{\frac{2\alpha}{3}} \quad , \quad b = \gamma\sqrt{2\beta} \quad , \quad c = q\gamma \qquad (I.6.249)$$

where β is a real positive root of the following cubic equation:

$$4\beta^3 - 6\beta^2 + \left(\frac{9}{4} + \frac{q^2}{p^2}\right)\beta - \frac{1}{2}\frac{q^2}{p^2} = 0 \qquad (I.6.250)$$

α and γ are linked to β by the relations:

$$\alpha = \frac{p^2}{q^2}(3\beta - 4\beta^2) \quad , \quad \gamma = \sqrt{\frac{12e^2}{\beta(1-\beta)}} \quad . \qquad (I.6.251)$$

Equation (I.6.250) has always (for all values of q/p) one and only one positive real root β whose range is

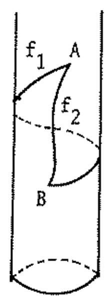
$$0 < \beta < \frac{1}{2} \quad . \qquad (I.6.252)$$

We can conclude that for all values q,p (except $p=q=0$) there is always one (and only one) $SU(3)\times SU(2)\times U(1)$ Einstein metric on M^{pqr}.

I.6.14 - Elements of algebraic topology

This Section contains a micro-review of homotopy and homology, and is quite non-rigorous. Its purpose is to recall some definitions and theorems that will be useful in Section I.6.15, and in later parts of this book.

A path in a topological space X is a continuous map of some closed interval I into X. Two paths, with the same end points, are said to be equivalent if they can be continuously deformed into one another.



The two paths f_1, f_2 on the two-dimensional cylinder are inequivalent: $f_1 \not\sim f_2$.

(I.6.253)

The product of two paths AB and BC, defined when the terminal point of the first path coincides with the initial point of the second path, is just the path ABC.

If two paths f_1, f_2 are respectively equivalent to g_1, g_2, their product $f_1 \cdot f_2$ is equivalent to $g_1 \cdot g_2$ (the simple proof is left to the reader), and we can consider the multiplication of equivalence classes. This multiplication is associative.

A path, or path class, is a loop if the initial and terminal points are the same. The loop is said to be based at the common end point.

The set of all loop classes based at any point x of X is a group, with the above-defined multiplication. The identity is the trivial loop, and the inverse of a loop is just the same loop traversed in the opposite direction.

This group is called the fundamental group of X at the base point x, denoted by $\pi(X,x)$. If x and y are two points of X connected by a path γ, we can define an isomorphism $u: \pi(X,x) \to \pi(X,y)$ induced by $\Phi \to \gamma^{-1}\Phi\gamma$ (we use Greek letters to denote path classes).

Exercise: prove that u really is an isomorphism.

The group structure of $\pi(X,x)$ is therefore independent of the particular point $x \in X$.

Example: the fundamental group of a circle is $\mathbb{Z}$ (infinite cyclic). Class representatives are paths wrapping around the circle $0,1,2\ldots\infty$ times both in clockwise and anticlockwise directions.

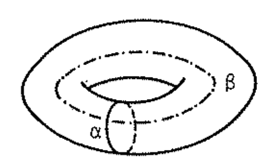
$0,1,2\ldots$ are called the winding numbers of the loop classes. Opposite winding numbers conventionally refer to the same loop traversed in opposite directions.

Example: the fundamental group of a n-torus is $\mathbb{Z}\times\mathbb{Z}\times\ldots\times\mathbb{Z}$ (n-times).

This is immediately proved by using the

Theorem: the fundamental group of a product space is isomorphic to the product of the fundamental groups, i.e.

$$\pi(X\times Y,(x,y)) \approx \pi(X,x) \times \pi(Y,y) \tag{I.6.254}$$



The fundamental group of a 2-torus is generated by the paths α and β.

(I.6.255)

The isomorphism between $\pi(X\times Y,(x,y))$ and $\pi(X,x)\times\pi(Y,y)$ is defined by assigning to each element $\alpha\in\pi(X\times Y,(x,y))$ the ordered pair $(p\alpha,q\alpha)$, where $p: X\times Y\to X$ and $q: X\times Y\to Y$ denote the projections of the product space into its factors.

Higher homotopy groups can be introduced via a generalization of the mapping defining a loop, and are essentially sets of equivalence classes of closed hypersurfaces. Consider the continuous map

$$f : I^n \to X \tag{I.6.256}$$

where I^n is the n-dimensional unit hypercube, satisfying:

$$f(\partial I^n = \text{boundary of } I^n) = x_0 \in X \quad . \tag{I.6.257}$$

If the images of two maps f_1, f_2 can be continuously deformed into each other, f_1 is equivalent to f_2. The set of equivalence classes of mappings (I.6.256) is easily seen to form a group, the n-th homotopy group of X about the point x_0:

$$\pi_n(X,\ x_0) \quad .$$

As for the fundamental group, the particular point x_0 is inessential up to isomorphisms. Examples of higher homotopy groups are provided in the next section.

From their definition, it is clear that homotopy groups are topological invariants: if two spaces are homeomorphic, their homotopy groups are isomorphic.

The converse is not true: see the example of $\mathbb{C}P^2\times S^3$ and $S^5\times S^2$ of next section. These spaces have isomorphic π_n for all n, but are topologically inequivalent.

For a more complete information on topological differences between two spaces, it is necessary to study their homology or cohomology groups.

These also provide a powerful link between the topological aspects of manifolds and their differentiable structure.

Let M be a smooth connected manifold. A p-chain a_p is defined by the formal sum:

$$a_p = \sum_i c_i\, N_i \tag{I.6.258}$$

where the N_i are smooth p-dimensional oriented submanifolds of M. The coefficients c_i can be taken complex, real, integers, $\mathbb{Z}_2 \ldots$; for present purposes $c_i\in\mathbb{R}$.

Let us denote by ∂ the operation of taking the oriented boundary. Then

$$\partial a_p \equiv \sum_i c_i\, \partial N_i \tag{I.6.259}$$

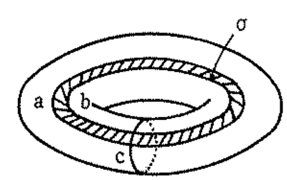
is a (p-1)-chain. Cycles are defined to be p-chains without boundary. Boundaries are those chains which can be written as $a_p = \partial a_{p+1}$ for some a_{p+1}. Since the boundary of a boundary is always empty $(\partial\partial a_p = 0)$, the set of boundaries is contained in the set of cycles.

The set of equivalence classes of cycles of M differing only by boundaries is called the simplicial homology of M:

$$H_p = \frac{\text{set of p-cycles}}{\text{set of p-boundaries}} \tag{I.6.260}$$

and two p-cycles z_p, z'_p are equivalent if

$$z'_p = z_p + \partial a_{p+1} \qquad \text{for some } a_{p+1} \, . \tag{I.6.261}$$



The curves a and b belong to the same homology class, since they bound the two-dimensional strip σ $(a+b=\partial\sigma)$. The curves a and c are not in the same homology class.

(I.6.262)

Example: the homology groups of the 2-torus are

$$H_0 = \mathbb{R}$$

$$H_1 = \mathbb{R} \oplus \mathbb{R}$$

$$H_2 = \mathbb{R} \tag{I.6.263}$$

De Rahm cohomology

We define the De Rahm cohomology groups as the set of equivalence classes of closed forms which differ only by exact forms

$$H^p_{DR}(M) = \frac{\text{set of closed p-forms}}{\text{set of exact p-forms}} \tag{I.6.264}$$

Two forms ω_p and ω'_p are equivalent if

$$\omega'_p = \omega_p + d\alpha_{p-1} \tag{I.6.265}$$

for some α_{p-1}.

Note: since the exterior derivative of a constant is zero

$$H^0_{DR}(M) = \{\text{space of constant functions}\} \tag{I.6.266}$$

and

$$\dim H^0_{DR}(M) = \text{number of connected pieces of } M. \tag{I.6.267}$$

Poincaré's lemma: The De Rahm cohomology of $\mathbb{R}^n$ is trivial, since any closed p-form can be expressed as the exterior derivative of a p-1 form in $\mathbb{R}^n$ (for $p > 0$).

Hence for any manifold M, the De Rahm cohomology is locally trivial, in the sense that it is trivial in any local $\mathbb{R}^n$ coordinate patch. Only when local coordinate neighbourhoods are patched together in a globally non-trivial way, the resulting manifold has non-trivial De Rahm cohomology.

Inner product: the inner product of a cycle c_p and a closed form ω_p is defined as

$$(c_p, \omega_p) \equiv \int_{c_p} \omega_p \in \mathbb{R} \tag{I.6.268}$$

By Stokes' theorem:

$$\int_{c_p} (\omega_p + d\alpha_{p-1}) = \int_{c_p} \omega_p + \int_{\partial c_p} \alpha_{p-1} = \int_{c_p} \omega_p \quad (\partial c_p = 0) \tag{I.6.269}$$

$$\int_{c_p + \partial a_{p+1}} \omega_p = \int_{c_p} \omega_p + \int_{a_{p+1}} d\omega_p = \int_{c_p} \omega_p \quad (d\omega_p = 0) \tag{I.6.270}$$

The inner product is therefore independent of the choice of representatives in the equivalence classes.

When M is a compact manifold without boundary, the following theorem (De Rahm) holds:

Let $\{c_i\}$ be a set of independent p-cycles forming a basis for $H_p(M)$; let $\{\omega_j\}$ be a basis for $H_{DR}(M)$.

Then the matrix (c_i, ω_j) is invertible:

$$\det (c_i, \omega_j) \neq 0 \quad . \tag{I.6.271}$$

Hence $H^p_{DR}(M)$ is dual to $H_p(M)$ with respect to the inner product: simplicial homology and De Rahm cohomology are naturally isomorphic.

The <u>p-th Betti number</u> (b_p) of M is defined to be the dimension of the p-th homology (or cohomology) group:

$$b_p \equiv \dim H_p(M) = \dim H^p_{DR}(M) \quad . \tag{I.6.272}$$

The alternating sum of the Betti numbers is the <u>Euler characteristic</u>:

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \, b_p \quad . \tag{I.6.273}$$

<u>Poincaré duality</u>: $H^p(M)$ is dual to $H^{n-p}(M)$ $(n = \dim M)$ with respect to the inner product

$$(\omega_p, \omega_{n-p}) \equiv \int_M \omega_p \wedge \omega_{n-p} \quad . \tag{I.6.274}$$

$H^p(M)$ and $H^{n-p}(M)$ are therefore isomorphic vector spaces, and

$$\dim H^p(M) = \dim H^{n-p}(M) \quad . \tag{I.6.275}$$

As a consequence the Betti numbers are related by

$$b_p = b_{n-p} \quad . \tag{I.6.276}$$

<u>Product formula</u> (Kunneth)

$$H^k(M_1 \times M_2) = \bigoplus_{p+q=k} H^p(M_1) \times H^q(M_2) \quad . \tag{I.6.277}$$

For Betti numbers:

$$b_k(M_1 \times M_2) = \sum_{p+q=k} b_p(M_1) \, b_q(M_2) \quad . \tag{I.6.278}$$

Hence the Euler characteristic satisfies:

$$\chi(M_1 \times M_2) = \chi(M_1) \, \chi(M_2) \quad . \tag{I.6.279}$$

Using Hodge's decomposition theorem on compact manifolds without boundary:

$$\omega_p = d\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p \qquad (\gamma_p \text{ harmonic}) \tag{I.6.280}$$

de Rahm cohomology classes are seen to be isomorphic to the set of harmonic forms. Indeed $d\omega = 0$ implies $d\delta\beta = 0$ so that $\delta\beta = 0$, and $\omega = d\alpha + \gamma$ is in the same cohomology class of the harmonic form γ. Conversely, if ω is harmonic, $\omega = \gamma$, and ω is closed but not exact (also: ω is co-closed but not coexact).

$$\text{Thus} \quad \{\text{ set of harmonic p-forms on } M \} \approx H^p(M) \quad . \tag{I.6.281}$$

<u>Examples</u>: $\mathbb{R}^n$:

All closed forms are exact except 0-forms εH^o. Hence $\dim H^o(\mathbb{R}^n) = 1$ (space of constant functions) and $\dim H^o(\mathbb{R}^n) = 0$ for $k \neq 0$.

S^n :

$\dim H^o(S^n) = 1$: space of constant functions

$\dim H^n(S^n) = 1$: constant multiples of volume element

all other $H^k(S^n)$ vanish.

I.6.15 - Homotopy and (co)homology of coset spaces

Homotopy

It is surprisingly easy to obtain detailed information concerning the homotopy group of a coset space G/H. While knowledge of the fundamental group of a coset space is particularly valuable, one finds that the higher homotopy groups yield only very basic details of the topology. More precisely, the higher homotopy groups tell one about how the topology is changed under various embeddings of the non-abelian factors of H, such as $SU(2)$ and $SU(3)$. Since there are only very few ways in which such factors may be embedded, and because such embeddings can usually be understood quite directly, the information gleaned from higher homotopy is not great. The more subtle variations in topology occur through the embeddings of the $U(1)$ factors, and essentially since $\pi_n[U(1)] = 0$ for $n \geq 2$, the higher homotopy groups do not measure these differences in topology.

For these reasons we will concentrate principally on the fundamental group, and return to the question of the other homotopy groups later.

There are two reasons why one can easily calculate the homotopy of the coset space G/H. First, the homotopy groups of Lie groups are known, at least up to π_{15} (See for ex. Encyclopedic dictionary of mathematics, ref. [10]). Furthermore, for products of spaces one has $\pi_n(A \times B) = \pi_n(A) \times \pi_n(B)$. The second reason is the homotopy exact sequence for fiber spaces. This says that if (E,p,B) is a fiber space with fiber F, then there is an exact sequence

$$\ldots \to \pi_{n+1}(B) \xrightarrow{\Delta} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \ldots \qquad (I.6.282)$$

where the maps i_* and p_* are induced by the inclusion $i: F \to E$ and the projection $p: E \to B$. The definition of the map Δ is somewhat complicated, and will not concern us here. Knowledge that such a map exists is usually sufficient. It should be recalled that the notation

(E,p,B) means that E is the total space, B is the base manifold and p is the projection map of E onto B. The fiber F is isomorphic to $p^{-1}(x)$ where x is any point in the base space B. Furthermore an exact sequence

$$\to A \xrightarrow{f} B \xrightarrow{g} C \to \qquad (I.6.283)$$

by definition means that Image f = Kernel g at each point in the sequence.

The relevance to coset spaces is that $(G,p,G/H)$ is a fiber space with fiber H, where p is the projection which takes $g \in G$ to the coset gH. Thus we have an exact sequence:

$$\to \pi_{n+1}(G/H) \xrightarrow{\Delta} \pi_n(H) \xrightarrow{i_*} \pi_n(G) \xrightarrow{p_*} \pi_n(G/H) \to \ldots \qquad (I.6.284)$$

where i_* is induced by the embedding $i: H \to G$. It is important to stress that the map i must be one to one for the sequence (I.6.284) to be exact. When considering $U(1)$ factors, i may be written in many ways, but for (I.6.284) to be applicable, i must be written in its unique one to one form.

In order to calculate the fundamental group of G/H, consider the last part of the sequence. For any Lie group, one has $\pi_2(G) = 0$, and so one obtains

$$0 \to \pi_2(G/H) \xrightarrow{\Delta} \pi_1(H) \xrightarrow{i_*} \pi_1(G) \xrightarrow{p_*} \pi_1(G/H) \xrightarrow{\Delta} \pi_0(H) \xrightarrow{i_*} \pi_0(G) \to \qquad (I.6.285)$$

For any connected space M, $\pi_0(M) = \mathbb{Z}$ (the dimension of π_0 over $\mathbb{Z}$ measures the number of components), and the fact that the only component of H maps into the only component of G, means that $i_*: \pi_0(H) \to \pi_0(G)$ is an isomorphism. (We are assuming that G and H are both connected). Consequently $\operatorname{Im} \Delta|_{\pi_0(H)} = \operatorname{Ker} i_*|_{\pi_0(H)} = 0$. Hence we may replace the sequence with the exact sequence

$$0 \to \pi_2(G/H) \xrightarrow{\Delta} \pi_1(H) \xrightarrow{i_*} \pi_1(G) \xrightarrow{p_*} \pi_1(G/H) \xrightarrow{\Delta} 0 \quad . \qquad \text{(I.6.286)}$$

Since $\operatorname{Ker} \Delta|_{\pi_1(G/H)} = \pi_1(G/H)$ one has

$$\pi_1(G/H) = \operatorname{Im} p_* \simeq \frac{\operatorname{Dom} p_*}{\operatorname{Ker} p_*} = \frac{\pi_1(G)}{\operatorname{Im} i_*} \quad . \qquad \text{(I.6.287)}$$

Furthermore $\operatorname{Ker} \Delta|_{\pi_2(G/H)} = 0$, and hence

$$\pi_2(G/H) \simeq \operatorname{Im} \Delta = \operatorname{Ker} i_* \ \text{in} \ \pi_1(H) \quad . \qquad \text{(I.6.288)}$$

Therefore the properties of the map i_* completely determine $\pi_1(G/H)$ and $\pi_2(G/H)$. The map i_* is readily understood: it simply takes a noncontractible loop with a particular set of winding numbers in H, and gives its winding numbers as a loop in G.

Consider the example of M^{pqr} (see the previous section). The fundamental groups of G and H are given by

$$\pi_1(H) = \mathbb{Z} \oplus \mathbb{Z} \quad , \quad \pi_1(G) = \mathbb{Z} \quad . \qquad \text{(I.6.289)}$$

These groups measure the winding numbers of a closed loop around the $U(1)$ factors of the group. Since one of the $U(1)$ factors of H is mapped directly into $SU(3) \times SU(2)$, and because $\pi_1(SU(3) \times SU(2)) = 0$, one sees that i_* maps one of the $\mathbb{Z}$'s in $\pi_1(H)$ to zero. If $p = q = 0$, then both $U(1)$ factors of H are mapped into $SU(3) \times SU(2)$ and consequently

$$\operatorname{Ker} i_* = \pi_1(H) = \mathbb{Z} \oplus \mathbb{Z} \quad , \quad \operatorname{Im} i_* = 0 \quad . \qquad \text{(I.6.290)}$$

Hence, from (I.6.287) and (I.6.288):

$$\pi_1(G/H) = \pi_1(G) = \mathbb{Z} \qquad \text{(I.6.291)}$$

$$\pi_2(G/H) = \operatorname{Ker} i_* = \mathbb{Z} \oplus \mathbb{Z} \quad . \qquad \text{(I.6.292)}$$

This is consistent with the earlier observation that $M^{00r} \simeq \mathbb{C}P^2 \times S^2 \times S^1$, when one recalls that $\pi_1(\mathbb{C}P^2) = \pi_1(S^2) = 0$ and $\pi_2(\mathbb{C}P^2) = \pi_2(S^2) = \mathbb{Z}$.

If $p \neq 0$ or $q \neq 0$ then the generator Z' in (I.6.234) defines the embedding, i. In particular this shows that a loop which goes once around the second $U(1)$ factor of H, goes around the $U(1)$ factor of G precisely $\ell = (3p^2 + q^2)/k$ times. Therefore, on the fundamental groups, the map

$$i_* : \pi_1(H) = \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(G) = \mathbb{Z} \qquad \text{(I.6.293)}$$

takes (a,b) into $b\ell$ and so

$$\pi_2(G/H) = \operatorname{Ker} i_* = \mathbb{Z}$$

$$\pi_1(G/H) = \frac{\pi_1(G)}{\operatorname{Im} i_*} = \frac{\mathbb{Z}}{\ell \mathbb{Z}} = \mathbb{Z}_\ell \quad . \qquad \text{(I.6.294)}$$

One now sees the importance of using the one to one map, i, which defines the embedding of H into G. If one had used an "m to one" embedding, one would have obtained $\pi_1(G/H) = \mathbb{Z}_{m\ell}$.

Having classified π_1 and π_2 of M^{pqr} let us now consider the higher homotopy groups. To do this, we have to refine our techniques a little. Recall that M^{pqr} may be considered as the quotient space

$$\frac{S^5 \times S^3 \times S^1}{U(1) \times U(1)} \quad . \qquad \text{(I.6.295)}$$

Clearly this is also a fiber space. That is, $S^5 \times S^3 \times S^1$ is an $S^1 \times S^1$ fibration over M^{pqr}. Applying the homotopy exact sequence one obtains

$$\to \pi_{n+1}(M^{pqr}) \to \pi_n(S^1 \times S^1) \to \pi_n(S^5 \times S^3 \times S^1) \to$$

$$\to \pi_n(M^{pqr}) \to \pi_{n-1}(S^1 \times S^1) \to \quad . \tag{I.6.296}$$

However for $n \geq 2$, $\pi_n(S^1 \times S^1) = 0$ and so one has

$$0 \to \pi_n(S^5 \times S^3 \times S^1) \to \pi_n(M^{pqr}) \to 0 \quad . \tag{I.6.297}$$

Hence

$$\pi_n(M^{pqr}) \simeq \pi_n(S^5 \times S^3 \times S^1) \quad , \quad n \geq 3 \quad . \tag{I.6.298}$$

In other words, the higher homotopy groups give no further information, in particular they do not tell anything about the role of the ratio p/q in determining the topology of M^{pqr}.

A direct consequence of the foregoing is that if p and q are not both zero, then $\pi_n(M^{pq0})$ is the same for all p and q. That is, homotopy cannot tell the difference between any of the spaces M^{pq0}. In particular

$$\pi_n(\mathbb{C}P^2 \times S^3) = \pi_n(S^5 \times S^2) \quad . \tag{I.6.299}$$

Thus one sees both the power, and the limitations of the homotopy groups.

The calculation of $\pi_1(M^{pqr})$ and $\pi_2(M^{pqr})$ can be generalized to arbitrary G/H.

Suppose that $G = G' \times U(1)$ and $H = H' \times U(1)$ where G' is simply connected, and H' is mapped by the embedding, i, into G'. Then

i) If i maps all of H into G',

$$\pi_1(G/H) = \pi_1(G) = \mathbf{Z}$$

$$\pi_2(G/H) = \pi_1(H) = \mathbf{Z} \oplus \pi_1(H') \quad . \tag{I.6.300}$$

ii) If i maps the explicit U(1) factor of $H' \times U(1)$ so that a simple loop in this U(1) winds ℓ times around the U(1) of $G' \times U(1)$, then

$$\pi_1(G/H) = \mathbf{Z}_\ell$$

$$\pi_2(G/H) = \pi_1(H') \quad . \tag{I.6.301}$$

This theorem enables one to calculate the $\pi_1(G/H)$ and $\pi_2(G/H)$ for every coset space in the TABLE V.6.1 of Chapter V.6.

A word of caution is advisable concerning cosets with nontrivial fundamental groups. Let M be a manifold with $\pi_1(M) \neq 0$, and let $\tilde{M}$ be its universal covering space (for example $M = M^{pqr}$ and $\tilde{M} = M^{pq0}$). If we solve Killing's equation in M, or solve the Killing spinor equation to determine the surviving supersymmetry (see Section V.4.3), then these calculations are performed locally, that is, in some coordinate patch. If one is not careful one might conclude that solutions to these equations exist in M if and only if they exist in $\tilde{M}$, since these equations are only expressed in local terms. However, for the existence of a supersymmetry, or a global Lie group symmetry, these Killing spinors and Killing vectors must exist globally. When the manifold is simply connected, the solutions which exist locally may be consistently patched together to produce a global solution, and so there is no problem on $\tilde{M}$. However, the manifold M is isomorphic to $\tilde{M}/X$, where X is some discrete group, and the condition that Killing vectors or Killing spinors exist on M is that they exist on $\tilde{M}$, and are consistent with the factoring by the discrete group X. As a simple example of this, consider the harmonics(*) cos $(m\Phi)$ and sin $(m\Phi)$ on the unit circle. If one divides the circle by the finite group $\mathbb{Z}_\ell$, then the only well defined harmonics are those where $m = k\ell$ for some integer k.

The conclusion is that a coset space G/H will always have a Lie group symmetry G, however it may appear locally to have more Killing vectors than those obtained from G. These extra Killing vectors may not be globally well defined on G/H, and thus one may not have a larger Lie group symmetry. If the space G/H is simply connected, then the extra local Killing symmetries can be made into global ones; but if the space is not simply connected, the possibility of extending to global symmetries depends on the details of the particular situation, and it will usually be impossible.

Homology

It is interesting to note at this juncture that there are also exact sequences on the homology and cohomology of fiber spaces, and that the complete homology and cohomology of Lie groups is known (cfr. Encyclopedic Dictionary of Mathematics, Ref. [10]). Thus one can, in principle, determine the homology and cohomology modules of G/H. It turns out that these modules are considerably more informative about the structure of G/H. The difficulty is that it is no longer quite so straightforward to determine the behavior of the induced maps i_* and p_* on the homology. Furthermore, the homology of a product space is not the product of the homologies. Instead one has to apply the Künneth formula (I.6.277) to find the homology of a product, and to trace the action of i_* and p_* through these formulae is somewhat complicated.

However, for a large class of cosets G/H there exists a straightforward way to obtain the Betti numbers, based on the Poincaré polynomials. These are defined as follows:

(*) harmonic analysis on G/H is discussed in Chapter V.3.

$$P_M(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n$$

$$M = \text{n-dim. manifold}$$

$$b_i = \text{Betti numbers} \quad . \qquad (I.6.302)$$

For the classical groups, the Poincaré polynomials are given by (see for example, Ref. [8]):

$$P_{U(1)} = 1 + t$$

$$P_{A(\ell)} = (1+t^3)(1+t^5)\ldots(1+t^{2\ell+1}) \qquad \leftarrow SU(\ell+1)$$

$$P_{B(\ell)} = (1+t^3)(1+t^7)\ldots(1+t^{4\ell-1}) \qquad \leftarrow SO(2\ell+1)$$

$$P_{C(\ell)} = (1+t^3)(1+t^7)\ldots(1+t^{4\ell-1}) \qquad Sp(2\ell)$$

$$P_{D(\ell)} = (1+t^3)(1+t^7)\ldots(1+t^{4\ell-5})(1+t^{2\ell-1}) \qquad \leftarrow SO(2\ell) \quad .$$

(I.6.303)

Examples:

$P_{SU(2)} = P_{SO(3)} = 1 + t^3$. The Betti numbers of $SO(3) \approx S^3$ are therefore $b_0 = 1$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$ and the Euler characteristic $\chi = \sum_{k=1}^{n} (-)^k b_k$ is zero. This generalizes to all odd spheres, while all even spheres have $\chi = 2$; also, for all Lie group manifolds $\chi = 0$.

$$P_{SU(3)} = (1+t^3)(1+t^5) = 1 + t^3 + t^5 + t^8 \quad , \quad \chi = 0$$

$$P_{SO(5)} = P_{Sp(4)} = (1+t^3)(1+t^7) = 1 + t^3 + t^7 + t^{10} \quad , \quad \chi = 0$$

$$P_{SO(4)} = P_{SU(2)\times SU(2)} = (1+t^3)(1+t^3) = 1 + 2t^3 + t^6 \quad , \quad \chi = 0 \quad .$$

Some useful theorems are:

i) $P_{G1 \times G2} = P_{G1} \times P_{G2}$ (Cfr. the example of $P_{SU(2) \times SU(2)}$)

ii) $\chi_G = 0$ for G = Lie group manifold.

This can be easily inferred from the structure

(even power of t + odd power) (even power + odd power) ...

of the G-Poincaré polynomials in (I.6.303).

The beauty of (I.6.303) is that it can also be applied to coset manifolds G/H when G and H are of equal rank. The theorem is:

$$P_{G/H} = P'_G / P'_H \tag{I.6.304}$$

where P' is the polynomial obtained from P in (I.6.303) by changing all + signs into - signs and raising all powers of t by 1 unit.

Examples:

i) $$P_{S^2} = P_{\frac{SO(3)}{SO(2)}} = \frac{P'_{SO(3)}}{P'_{SO(2)}} = \frac{1-t^4}{1-t^2} = 1 + t^2$$

so that the Betti numbers of S^2 are $b_0 = 1$, $b_1 = 0$, $b_2 = 1$, yielding $\chi = 2$

ii) $$P_{S^{2n}} = \frac{P'_{SO(2n+1)}}{P'_{SO(2n)}} = \frac{(1-t^4)(1-t^8)\dots(1-t^{4n})}{(1-t^4)(1-t^8)\dots(1-t^{4n-4})(1-t^{2n})} =$$
$$= 1 + t^{2n}$$

iii) in general $P_{S^n} = 1 + t^n$ for any n.

iv) $$P_{\mathbb{C}P^2 \times S^3} = P_{\mathbb{C}P^2} \times P_{S^3} = P_{\frac{SU(3)}{SU(2) \times U(1)}} \times P_{S^3} =$$
$$= \frac{P'_{SU(3)}}{P'_{SU(2) \times U(1)}} \times P_{S^3} =$$
$$= \frac{(1-t^4)(1-t^6)}{(1-t^4)(1-t^2)} \cdot (1+t^3) =$$
$$= 1 + t^2 + t^3 + t^4 + t^5 + t^7 \quad , \quad \chi = 0$$

v) $$P_{S^2 \times S^5} = P_{S^2} \times P_{S^5} = (1+t^2)(1+t^5) = 1 + t^2 + t^5 + t^7 \quad .$$

The last two examples show that homology indeed distinguishes between $\mathbb{C}P^2 \times S^3$ and $S^2 \times S^5$ (whereas homotopy does not, cfr. (I.6.299)).

Tables of Poincaré polynomials for more general G/H coset spaces can be found in Ref. [8], Vol. III, pp. 492-497.