TABLE III.4.I continued

G) Supersymmetry Transformation Laws

$$\delta_\varepsilon V^a = i\, \bar{\varepsilon}_A \gamma^a \psi_A$$

$$\delta_\varepsilon \psi_A = \mathscr{D}\varepsilon_A + \bar{e}\, \varepsilon_{AB} A \wedge \varepsilon_B + i\, \varepsilon_{AB} F_{ab} V^b \gamma^a \varepsilon_B + \frac{1}{2} \varepsilon_{AB} F_{cd} \gamma_5 \gamma_a \varepsilon_B V_b \varepsilon_{abcd}$$

$$\delta_\varepsilon A = 2\, \varepsilon_{AB} \bar{\psi}_A \varepsilon_B$$

$$\delta_\varepsilon \omega^{ab} = 2\, \bar{e}\, \bar{\psi}_A \wedge \gamma^{ab} \varepsilon_A + \bar{\Theta}^{ab}_{A|c} \varepsilon_A V^c + 2\, \varepsilon_{AB} F_{ab} \bar{\psi}_B \varepsilon_A + i\, \varepsilon^{abcd} \varepsilon_{AB} F_{cd} \bar{\psi}_B \gamma_5 \varepsilon_A \,.$$

If $\Theta^{ab}_{A|c} \equiv \overset{(I)}{\Theta}{}^{ab}_{A|c}$ they are a symmetry ($\equiv$ closed algebra) of the inner field equations; if $\Theta^{ab}_{A|c} = \overset{(II)}{\Theta}{}^{ab}_{A|c}$ they are an invariance (on-shell closed, off-shell open algebra) of the action.

CHAPTER III.5

THE D=5, N=2 SUPERGRAVITY THEORY

III.5.1 - Introduction

In this chapter we discuss the simplest example of a supergravity theory in more than four dimensions, namely the D=5, N=2 supergravity model.

Using the results of Chapter II.7 on the spinor properties in every dimension we easily conclude that D=5, N=1 supergravity does not exist. Indeed on a 5-dimensional spinor we cannot impose the Majorana nor the Weyl condition so that it is intrinsically complex. Using an O(2) notation we can say that we have actually 2 gravitino 1-forms ψ_A labelled by an O(2) index $A = 1,2$, the same notation we used for the D=4, N=2 case.

As it will be evident from the explicit construction, this theory fits pretty well into the general scheme of building rules for supergravity theories given in Chapter III.3. It is, however, the first and last case where a $D > 4$ theory can be interpreted as a theory of 1-forms associated to a given (super)-Lie algebra. From D=6 to D=11 it will prove necessary to enlarge the concept of (super)-Lie algebra to a more general structure called free differential (super)-Algebra, allowing the formulation of the $D>5$ supergravity theories in the same geometrical setting used for the D=4 and D=5 cases. This will be in fact the topic of the next chapter; for the moment we stick to D=5 and to super-Lie algebras. From a purely aesthetic point of view the D=5 theory is a beautiful theory since it satisfies the "strong" geometricity requirement even if it contains a spin 1 field in the supergravity multiplet. We have seen in the N=2, D=4 case that the presence of a spin 1 field generally forces the introduction of new fields which are 0-forms. They are needed for the construction of the spin 1 kinetic terms, thus avoiding the use of the Hodge duality operator. There are however a few exceptions to this state of affairs and the D=5 theory is one of them. The existence in this theory of a non zero torsion allows a dynamical reconstruction of the spin 1 kinetic term in the second order formulation, without using the 0-forms in the first order formulation. The theory is therefore strongly geometrical.

To understand how this can happen we recall the mechanism by which the kinetic term of the spin 2 vielbein field is reconstructed in pure D=4 gravity and supergravity from the torsion equation $R^a{}_{mn}=0$; one eliminates the spin connection ω^{ab}_m in terms of the vielbein field (and of the gravitino field in supergravity):

$$\omega^{ab}{}_\mu = \omega^{ab}{}_\mu (V, \psi) . \qquad \text{(III.5.1)}$$

Then, in second order one finds that the Einstein term, which does not contain the Hodge * operator, generates the vielbein kinetic term on space time:

$$R^{ab}(\omega(V,\psi)) \wedge V^c \wedge V^d \varepsilon_{abcd} \to V^{\mu|a} \Box V_{a|\mu} + \text{more terms} . \qquad \text{(III.5.2)}$$

Let us now assume that in our D=5 theory there exists a 3-index antisymmetric field F_{abc} such that the torsion equation is modified as follows:

$$R^a{}_{bc} = \text{const} \times F_{mbc}\, \eta^{ma} . \qquad \text{(III.5.3)}$$

In D=5 supergravity F_{abc} can be actually identified with the dual of F_{ab}, $F_{abc} = \varepsilon_{abcpq} F^{pq}$ where F_{ab} are the inner components of the curvature $R^{\otimes}$ of the spin 1 gauge field B.

Solving (III.5.3) for ω^{ab} one finds that ω^{ab} depends also on F_{ab}:

$$\omega^{ab}{}_\mu = \omega^{ab}{}_\mu (V^a{}_\mu, \psi_\mu, F_{abc}) . \qquad \text{(III.5.4)}$$

It is then not surprising that after substitution of (III.5.4) into the terms containing the curvature R^{ab}, present in the D=5 Lagrangian, one can generate the kinetic term for F, namely $F \wedge {}^*F$. In the sequel we will describe exactly how this happens in the D=5 theory.

We conclude this section by explicitly writing down the correct generalization of the 4-D gravitational term when $D>4$; all the higher dimensional theories in fact must contain this term. The correct generalization of D-dimensional gravity in Einstein-Cartan first order formalism is the following:

$$\mathcal{L}^{(D)}_{(grav)} = \text{const} \times R^{ab} \wedge V^{c_1} \wedge \dots \wedge V^{c_{D-2}} \varepsilon_{abc_1\dots c_{D-2}} \qquad \text{(III.5.5)}$$

where R^{ab} is the Lorentz curvature of the soft SO(1,D-1) group, V^c is the D-dimensional vielbein, and $\varepsilon_{abc_1\dots c_{D-2}}$ is the D-dimensional completely antisymmetric Ricci tensor.

The proof is quite simple: expanding R^{ab} in the vielbein basis one finds:

$$R^{ab} \wedge V^{c_1} \wedge \dots \wedge V^{c_{D-2}} \varepsilon_{abc_1\dots c_{D-2}} =$$

$$= R^{ab}{}_{mn} V^m \wedge V^n \wedge V^{c_1} \wedge \dots \wedge V^{c_{D-2}} \varepsilon_{abc_1\dots c_{D-2}} =$$

$$= R^{ab}{}_{mn}\, \varepsilon^{mnc_1\dots c_{D-2}}\, \varepsilon_{abc_1\dots c_{D-2}} \det V\, d^Dx =$$

$$= (-1)^{D+1}\, 2!(D-2)!\, R^{mn}{}_{mn} \det V\, d^Dx\ . \qquad \text{(III.5.6)}$$

which proves our assertion.

III.5.2 - Identification of the supergroup and construction of its curvatures

In order to construct D=5 supergravity we ought to find the proper supergroup G on which it is based. In Chapter II.2 (Sect. II.2.2) we have anticipated that 5-dimensional supergravity is based on the SU(2,2|N) superalgebra, or, since we are going to discuss the minimal case N=1, on the SU(2,2|1) algebra. Let us recall the argument.

G must of course contain the Poincaré ISO(1,4) or the de Sitter SO(2,4) group in D=5. Since ISO(1,4) may be considered as the Inonü-Wigner contraction of SO(2,4) we shall treat first the supergroup extension G in the SO(2,4) case; the supergroup extension in the ISO(1,4) case will then be retrieved as a suitable contraction of G.

The local isomorphism

$$SO(2,4) \simeq SU(2,2) \qquad \text{(III.5.7)}$$

implies that the minimal grading of SO(2,4) is the group

$$G = SU(2,2|1) \qquad \text{(III.5.8)}$$

which contains as a maximal bosonic subgroup

$$SO(2,4) \otimes U(1)\ . \qquad \text{(III.5.9)}$$

The grading is performed by adding to the 15 + 1 bosonic generators

$$T_{\hat a\hat b},\ Z_{\otimes} \quad (\hat a, \hat b = 0, 1, \dots, 5) \qquad \text{(III.5.10)}$$

the eight fermionic generators corresponding to two complex anti-commuting charges(*)

$$Q_\alpha\ ; \qquad \bar Q^\alpha = (Q^\dagger \Gamma^0)^\alpha \qquad \text{(III.5.11)}$$

in the four-dimensional representation of SU(2,2). Here and in the following the "hat" indices are SO(2,4) indices with metric

$$\eta_{\hat a\hat b} = (1,-1,-1,-1,-1,+1)\ .$$

From a physical point of view we may guess what the structure of G will be by examining the field content of the theory. Here we quote a general formula, valid in every dimension D, yielding the number of on-shell degrees of freedom for fields of spin running from two to zero, including antisymmetric tensor fields in p-indices (these latter will appear in $D>5$ supergravities):

Field	Spin	On-shell degrees of freedom
$V^a{}_\mu$	2	$D(D-3)/2$
$\psi^\alpha{}_\mu$	3/2	$\frac{1}{2}(D-3)\cdot 2^{[D/2]}$
$A_{\mu_1\dots\mu_p}$	1	$\binom{D-2}{p}$
χ^α	1/2	$\frac{1}{2}\, 2^{[D/2]}$
ϕ	0	1

(III.5.12)

with the proviso that an extra factor 2 is to be included in the fermionic case $(\psi^\alpha{}_\mu;\ \chi^\alpha)$ if the spinor representation of SO(1,D-1) is

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(*) In $D>4$ the γ-matrices are indicated with capital letters according to the conventions established in part two.

intrinsically complex and an extra factor 1/2 if the spinors obey a chirality constraint (Weyl spinor).

The justification of the preceding formula is easy if one remembers that:

i) only transverse components contribute to the on-shell degrees of freedom: $D \to D-2$.

ii) The dimension of the spinor representation in D dimensions is $2^{[D/2]}$ and the Dirac equation reduces the degrees of freedom by a factor 1/2. ($[x]$ denotes the integer part of x).

iii) In the spin 2 case Lorentz gauge invariance reduces the degrees of freedom to those of a symmetric tensor: $[V^a_\mu] \sim [g_{\mu\nu}]$. Taking into account transversality $(D \to D-2)$ and subtracting the spinless trace we get:

$$[V^a{}_\mu] = \frac{(D-2)(D-1)}{2} - 1 = \frac{D(D-3)}{2} \qquad \text{(III.5.12a)}$$

iv) In the case of spin 3/2 we have, besides transversality, the gauge condition $\gamma^\mu \psi^\alpha_\mu = 0$ which eliminates the spin 1/2 part out of ψ^α_μ. Hence the coordinate index μ adds a factor $[(D-2)-1]$ to the spinorial degrees of freedom and we find

$$[\psi^\alpha_\mu] = (D-3)\,\frac{1}{2}\,2^{[D/2]} \qquad \text{(III.5.12b)}$$

Using (III.5.12) we find that in D=5 the spin 2 graviton has 5, and the spin 3/2 complex gravitino field has 8 (on-shell) degrees of freedom. The mismatch $8-5=3$ between the fermionic and bosonic degrees of freedom may be compensated by the presence of a spin 1 vector field (Maxwell field). Thus we expect the minimal supergroup G to contain, besides the nonpropagating Lorentz connection ω^{ab}_μ, a fünfbein V^a_μ, a Maxwell field B_μ and complex gravitino field ξ^α_μ. This is exactly the field content of the potentials in the Lie algebra of SU(2,2|1) (or in one of its contractions).

Indeed let

$$\mu = \mu^\Lambda T_\Lambda = \frac{i}{2}\,\Omega^{\hat{a}\hat{b}} T_{\hat{a}\hat{b}} + B\,Z_\otimes + \bar{\xi}Q + \bar{Q}\xi \qquad \text{(III.5.13)}$$

be the $G \equiv SU(2,2|1)$ Lie algebra valued soft 1-form defined on the G soft manifold; Ω^{ab} and B are SO(2,4) (anti de Sitter) and U(1) real bosonic 1-forms and ξ, $\bar{\xi} \equiv \xi^\dagger \Gamma_0$, a fermionic 1-form in the SU(2,2) spinor representation.

In order to perform the contraction giving the minimal grading of ISO(1,4) (i.e. the 5-dimensional Poincaré group), it is convenient to split the anti de Sitter connection and generators according to their SO(1,4) Lorentz content:

$$T_{a5} \equiv P_a\,; \qquad \Omega^{a5} \equiv V^a \qquad \text{(III.5.14a)}$$

$$T_{ab} \equiv J_{ab}\,; \qquad \Omega^{ab} \equiv \omega^{ab} \qquad \text{(III.5.14b)}$$

where $a,b = (0,1,\dots,4)$, and $\eta_{ab} = (1,-1,-1,-1,-1)$.

Then (III.5.13) can be rewritten as follows:

$$\mu = \frac{i}{2}\,\omega^{ab} J_{ab} + \frac{i}{2}\,V^a P_a + B\,Z_\otimes + \bar{\xi}Q + \bar{Q}\xi\,. \qquad \text{(III.5.15)}$$

To compute the curvatures (or, equivalently, the Maurer-Cartan equations) related to μ it is convenient to give an explicit matrix representation of μ. We introduce a $(4+1)\times(4+1)$ super-matrix

$$Q = \begin{array}{c} \begin{matrix} \;\;4 & \;\;\;1 \end{matrix} \\ \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{array} \begin{matrix} 4 \\ 1 \end{matrix} \qquad \text{(III.5.16)}$$

satisfying Eqs. (II.2.58) and (II.2.60) with

$$H_4 = -\Gamma_0 \qquad \text{(III.5.17a)}$$

$$H_1 = \mathbb{1} \qquad \text{(III.5.17b)}$$

where Γ_0 is the 4×4 0-component of the Dirac Γ-matrices in 5 dimensions. (For conventions and properties of spinor algebra in D=5 see Chapter II.7).

The SU(2,2|1) (pseudo)connection μ is a 1-form supermatrix satisfying Eqs. (II.2.58) and (II.2.60). Its most general structure is the following

$$\mu = \left(\begin{array}{c|c} \frac{1}{4} B\,\mathbb{1} + \frac{i}{4}\omega^{ab}\Gamma_{ab} + \frac{1}{2} V^a\Gamma_a & \xi \\ \hline -\bar{\xi} & B \end{array}\right) \qquad \text{(III.5.18)}$$

where Γ_a, Γ_{ab} are a complete set of Dirac matrices in D=5. From the definition

$$R = R^A T_A = d\mu + \mu \wedge \mu \qquad \text{(III.5.19)}$$

we find

$$R = \left(\begin{array}{c|c} \frac{1}{4} R^{\otimes} - \frac{1}{4} R^{ab}\Gamma_{ab} + \frac{1}{2} R^a\Gamma_a & \rho \\ \hline -\bar{\rho} & R^{\otimes} \end{array}\right) \qquad \text{(III.5.20)}$$

where

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b + V^a \wedge V^b + \frac{i}{2}\bar{\xi} \wedge \Gamma^{ab}\xi \qquad \text{(III.5.21a)}$$

$$R^a = \mathcal{D}V^a - \frac{i}{2}\bar{\xi} \wedge \Gamma^a\xi \qquad \text{(III.5.21b)}$$

$$R^{\otimes} = dB - i\,\bar{\xi} \wedge \xi \qquad \text{(III.5.21c)}$$

$$\rho = \mathcal{D}\xi + \frac{i}{2} V^a \wedge \Gamma_a\xi - i\,\frac{3}{4} B \wedge \xi \qquad \text{(III.5.21d)}$$

and where as usual

$$\mathcal{D}V^a = dV^a - \omega^{ab} \wedge V_b \qquad \text{(III.5.22a)}$$

$$\mathcal{D}\xi = d\xi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\xi \qquad \text{(III.5.22b)}$$

are the Lorentz (SO(1,4)) covariant derivatives.

For many purposes it is convenient to use an SO(2) notation to describe the complex gravitino field ξ: we set

$$\xi = \psi_1 + i\,\psi_2 \qquad \text{(III.5.23)}$$

where ψ_1 and ψ_2 satisfy the <u>pseudo-Majorana condition</u>

$$\psi_1{}^c \equiv C(\bar{\psi}_1)^t = \psi_2 \qquad \text{(III.5.24a)}$$

$$\psi_2{}^c \equiv C(\bar{\psi}_2)^t = -\psi_1 \,. \qquad \text{(III.5.24b)}$$

Equations (III.5.24) imply:

$$\xi^c = \psi_1 + i\,\psi_1 \qquad \text{(III.5.25a)}$$

$$(\xi^c)^c = -\psi_1 - i\,\psi_2 \equiv -\xi \,. \qquad \text{(III.5.25b)}$$

(The last equation confirms the impossibility of imposing the Majorana condition on ξ).

Introducing SO(2) indices $(A,B \equiv \{1,2\})$ Eqs. (III.5.24) may be rewritten as follows:

$$C(\bar{\psi}_A)^t = \varepsilon_{AB}\,\psi_B \qquad \text{(III.5.26)}$$

which implies the following definition of the adjoint of a pseudo-Majorana spinor

$$\bar{\psi}_A = \varepsilon_{AB}\,\psi_B^t\, C \,. \qquad \text{(III.5.27)}$$

Decomposing the 5-dimensional spinors ρ and Q as in (III.5.23) we find analogous properties for the pseudo-Majorana curvatures ρ_A and the supersymmetry charges Q_A. From the previous formulae one easily obtains:

$$\bar{\xi} \wedge \Gamma_{ab}\xi = i\, \varepsilon_{AB}\, \bar{\psi}_A \wedge \Gamma_{ab}\, \psi_B \tag{III.5.28a}$$

$$\bar{\xi} \wedge \Gamma_a \xi = \bar{\psi}_A\, \Gamma^a\, \psi_A \tag{III.5.28b}$$

$$\bar{\xi} \wedge \xi = \bar{\psi}_A \wedge \psi_A\,. \tag{III.5.28c}$$

Moreover, taking into account that $C\Gamma_{ab}$ is symmetric while C and $C\Gamma_a$ are antisymmetric in spinor space one also finds:

$$\varepsilon_{AB}\, \bar{\psi}_A \wedge \Gamma_a\, \psi_B = 0 \tag{III.5.29a}$$

$$\varepsilon_{AB}\, \bar{\psi}_A \wedge \psi_B = 0 \tag{III.5.29b}$$

$$\bar{\psi}_A \wedge \Gamma_{ab}\, \psi_A = 0\,. \tag{III.5.29c}$$

Finally, in order to be able to perform the Inonü-Wigner contraction, we rescale the generators and the 1-forms as follows:

$$J_{ab} \to J_{ab}\,; \quad \omega^{ab} \to \omega^{ab}\,; \quad R^{ab} \to R^{ab} \tag{III.5.30a}$$

$$P_a \to \frac{1}{2\bar{e}}\, P_a\,; \quad V^a \to 2\bar{e}V^a\,; \quad R^a \to 2\bar{e}R^a \tag{III.5.30b}$$

$$Z_\otimes \to \frac{1}{2\bar{e}}\, Z_\otimes\,; \quad B \to 2\bar{e}B\,; \quad R^\otimes \to 2\bar{e}R^\otimes \tag{III.5.30c}$$

$$Q_A \to \frac{1}{\sqrt{2\bar{e}}}\, Q_A\,; \quad \psi_A \to \sqrt{2\bar{e}}\,\psi_A\,; \quad \rho_A \to \sqrt{2\bar{e}}\,\rho_A\,. \tag{III.5.30d}$$

Conventions have been chosen in such a way that formulae (III.5.30) be analogous to Eqs. (II.2.141) and (II.3.25-26) in the case of the Osp(4/N) groups.

Let us now rewrite the curvatures (III.5.21) using the rescaled quantities (III.5.30) and the pseudo-Majorana notation for the spinor-fields. Equations (III.5.21) assume the following final form:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} + 4\bar{e}^2\, V^a \wedge V^b + {}$$
$$+ i\, \bar{e}\, \varepsilon_{AB}\, \bar{\psi}_A \wedge \Gamma^{ab}\, \psi_B \tag{III.5.31a}$$

$$R^a = \mathscr{D} V^a - \frac{i}{2}\, \bar{\psi}_A \wedge \Gamma^a\, \psi_A \tag{III.5.31b}$$

$$R^\otimes = dB - i\, \bar{\psi}_A \wedge \psi_A \tag{III.5.31c}$$

$$\rho_A = \mathscr{D}\psi_A - \bar{e}\, V^a\, \Gamma_a\, \psi_B\, \varepsilon_{AB} + \bar{e}\, \frac{3}{2}\, \varepsilon_{AB}\, B \wedge \psi_B \tag{III.5.31d}$$

$$(\bar{\rho}_A = \mathscr{D}\bar{\psi}_A + \bar{e}\, \bar{\psi}_B\, \varepsilon_{AB}\, \Gamma_a\, V^a + \bar{e}\, \frac{3}{2}\, B\, \bar{\psi}_B\, \varepsilon_{AB})\,. \tag{III.5.31e}$$

Applying d to both sides of Eqs. (III.5.31) we find the Bianchi identities:

$$(\nabla R)^{ab} \equiv \mathscr{D} R^{ab} + 8\bar{e}^2\, V^{[a} \wedge R^{b]} + 2\, i\, \bar{e}\, \varepsilon_{AB}\, \bar{\psi}_A \wedge \Gamma^{ab}\, \rho_B = 0 \tag{III.5.32a}$$

$$(\nabla R)^a \equiv \mathscr{D} R^a + R^{ab} \wedge V_b + i\, \bar{\rho}_A \wedge \Gamma^a\, \psi_A = 0 \tag{III.5.32b}$$

$$(\nabla R)^\otimes \equiv \mathscr{D} R^\otimes - 2\, i\, \bar{\psi}_A \wedge \rho_A = 0 \tag{III.5.32c}$$

$$\nabla\rho_A \equiv \mathscr{D}\rho_A - \bar{e}\, V^a\, \Gamma_a\, \rho_B\, \varepsilon_{AB} + \frac{3}{2}\, \bar{e}\, B \wedge \rho_B\, \varepsilon_{AB} + {}$$
$$+ \frac{1}{4}\, \Gamma_{ab}\, \psi_A \wedge R^{ab} - \bar{e}\, \varepsilon_{AB}\, \Gamma_a\, \psi_B \wedge R^a + {}$$
$$+ \frac{3}{2}\, \bar{e}\, \varepsilon_{AB}\, \psi_B \wedge R^\otimes = 0 \tag{III.5.32d}$$

$$(\nabla\bar{\rho}_A \equiv \mathscr{D}\bar{\rho}_A - \bar{e}\, V^a\, \varepsilon_{AB}\, \bar{\rho}_B\, \Gamma_a + \frac{3}{2}\, \bar{e}\, B \wedge \varepsilon_{AB}\, \bar{\rho}_B - {}$$
$$- \frac{1}{4}\, \bar{\psi}_A \wedge \Gamma_{ab}\, R^{ab} - \bar{e}\, \varepsilon_{AB}\, \bar{\psi}_B \wedge \Gamma_a\, R^a + {}$$
$$+ \frac{3}{2}\, \bar{e}\, \varepsilon_{AB}\, \bar{\psi}_B \wedge R^\otimes = 0)\,. \tag{III.5.32e}$$

Let us now perform the Inonü-Wigner contraction $\bar{e} \to 0$. Equations (III.5.31) and (III.5.32) become:

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \equiv \mathcal{R}^{ab} \qquad \text{(III.5.33a)}$$

$$R^a = \mathcal{D} V^a - \frac{i}{2} \bar{\psi}_A \wedge \Gamma^a \psi_A \qquad \text{(III.5.33b)}$$

$$R^{\otimes} = dB - i \bar{\psi}_A \wedge \psi_A \qquad \text{(III.5.33c)}$$

$$\rho_A = \mathcal{D} \psi_A \qquad \text{(III.5.33d)}$$

and

$$\mathcal{D} R^{ab} = 0 \qquad \text{(III.5.34a)}$$

$$\mathcal{D} R^a + R^{ab} \wedge V_b + i \bar{\rho}_A \wedge \Gamma^a \psi_A = 0 \qquad \text{(III.5.34b)}$$

$$\mathcal{D} R^{\otimes} - 2 i \bar{\psi}_A \wedge \rho_A = 0 \qquad \text{(III.5.34c)}$$

$$\mathcal{D} \rho_A + \frac{1}{4} R^{ab} \Gamma_{ab} \psi_A = 0 \qquad \text{(III.5.34d)}$$

respectively.

When $R^{\Lambda} = 0$ Eqs. (III.5.33) define a new group denoted by $\overline{SU(2,2|1)}$, which contains $ISO(1,4) \otimes U(1)$ as a maximal bosonic subgroup. Note that the definition of Eq. (III.5.15) is unchanged. In the contracted case J_{ab} and P_a are Lorentz and translation generators, respectively; correspondingly we shall refer to ω^{ab} and V^a as Lorentz connection and fünfbein 1-forms and to R^{ab} and R^a as the Lorentz curvature and torsion 2-forms.

Finally we notice that Eqs. (III.5.31) and (III.5.32) are invariant under the following rigid scalings:

$$\omega^{ab} \to \omega^{ab} \; ; \quad R^{ab} \to R^{ab} \qquad \text{(III.5.35a)}$$

$$V^a \to w V^a \; ; \quad R^a \to w R^a \qquad \text{(III.5.35b)}$$

$$B \to wB \; ; \quad R^{\otimes} \to w R^{\otimes} \qquad \text{(III.5.35c)}$$

$$\psi_A \to w^{1/2} \psi_A \; ; \quad \rho_A \to w^{1/2} \rho_A \qquad \text{(III.5.35d)}$$

$$\bar{e} \to w^{-1} \bar{e} \, . \qquad \text{(III.5.35e)}$$

The same scaling invariance obviously holds for the contracted case, Eqs. (III.5.33-34), (Eq. (III.5.35e) being absent in this case).

III.5.3 - Construction of the Lagrangian

In this section we derive the action and the equations of motion of D=5 supergravity in the non contracted case, namely when $G \equiv SU(2,2|1)$. As usual the relevant formulae for $G = SU(2,2|1)$ can be straightforwardly obtained by setting the contraction parameter $\bar{e} = 0$.

First we notice that the $SU(2,2|1)$ superalgebra has the required structure discussed in Sect. III.3.9 of Chapter III.3 (see Eqs. (III.3.136-141)).

Indeed we can write (using the same notation for the (super)-Lie algebra as for the (super)-group):

$$\begin{array}{ccccc} SU(2,2|1) & = & \underbrace{SO(1,4) \oplus U(1)} & \oplus & (P_a, Q_A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G} & = & \mathbb{H} & \oplus & \mathbb{K} \end{array} \qquad \text{(III.5.36)}$$

where $SO(1,4)$ and $U(1)$ are spanned by the generators $\tilde{D}_{ab}$ and $\tilde{D}_{\otimes}$ which are dual to ω^{ab} and B respectively:

$$\omega^{ab}(\tilde{D}_{cd}) = \delta^{ab}_{cd} \; ; \quad B(\tilde{D}_{\otimes}) = 1 \, . \qquad \text{(III.5.37)}$$

On the other hand the subspace $\mathbb{K}$ is spanned by the translation $(\tilde{D}_a)$ and supersymmetry $(\tilde{D}_A)$ generators which are, respectively, dual to V^a and ψ_A. We can write:

$$\mathbb{K} = \mathbb{I} \oplus \mathbb{O} \quad \text{where} \quad \mathbb{I} \equiv \{\tilde{D}_a\} \; ; \quad \mathbb{O} \equiv \{\tilde{D}_A\} \qquad \text{(III.5.38)}$$

and using the definitions:

$$V^a(\tilde{D}_b) = \delta^a_b \; ; \quad \psi^\alpha_A(\tilde{D}_{\beta|B}) = \delta^\alpha_\beta\, \delta_{AB} \tag{III.5.39}$$

together with the equations (III.5.21) we can verify that the structure (III.3.136-141) is indeed respected. Hence we can write the following associations:

$(\omega^{ab}, B) \to \mathbb{H}$-subalgebra

$(V^a) \to$ inner directions ($\mathbb{I}$)

$(\psi_A) \to$ outer directions ($\mathbb{O}$) . (III.5.40)

At this point we can proceed to the construction of the Lagrangian using the building rules A-E of Sect. III.3.9. For simplicity we shall impose from the beginning the horizontality of the curvatures R^Λ with respect to the gauge group $H \equiv SO(1,4)\times SO(2)$; namely,

$$\underline{D_{ab}}\rfloor R^\Lambda = \underline{D_\otimes}\rfloor R^\Lambda = 0 \ . \tag{III.5.41}$$

Consequently

$$\tilde{G} = \tilde{G}(\tilde{G}/H, H) \tag{III.5.42}$$

and Eqs. (III.5.31) will be considered from now on as the structure equations of the superspace

$$\tilde{G}/H \equiv \widetilde{SU(2,2|1)}\Big/SO(1,4) \otimes SO(2) \ . \tag{III.5.43}$$

Accordingly $(V^a(P), \psi_A(P))$ will be interpreted as a (super)-coframe on $T_P^{\,*}(\tilde{G}/H)$.

Following the building rule A, (i.e. Eq. (III.3.148)), the action of D=5 supergravity has the general form:

$$\mathscr{A} = \int_{M^5 \subset \tilde{G}/H} (\Lambda + R^\Lambda \wedge \nu_\Lambda + R^\Lambda \wedge R^\Sigma \wedge \nu_{\Lambda\Sigma}) \tag{III.5.44}$$

where R^Λ are the Lie algebra valued 2-forms (III.5.31) and

$$\Lambda = C_{\Lambda_1\ldots\Lambda_5}\, \mu^{\Lambda_1} \wedge \ldots \wedge \mu^{\Lambda_5} \tag{III.5.45a}$$

$$\nu_\Lambda = C_{\Lambda\Lambda_1\Lambda_2\Lambda_3}\, \mu^{\Lambda_1} \wedge \mu^{\Lambda_2} \wedge \mu^{\Lambda_3} \tag{III.5.45b}$$

$$\nu_{\Lambda\Sigma} = C_{\Lambda\Sigma|\Pi}\, \mu^\Pi \tag{III.5.45c}$$

are polynomials of order 5, 3 and 1 respectively, in the μ^Λ potentials with constant coefficients $C_{\Lambda_1\ldots\Lambda}$, $C_{\Lambda|\Lambda_1\ldots\Lambda_3}$, $C_{\Lambda\Sigma|\Pi}$. Equations (III.5.45a), (III.5.45b), (III.5.45c) have indices in the scalar, coadjoint, coadjoint ⊗ coadjoint representations, respectively.

The integration of (III.5.44) is made on any five dimensional surface M_5 embedded in $\tilde{G}/H$, to be identified later with the physical space time.

The action must be stationary with respect to arbitrary $\delta\mu^\Lambda$ variations and to any arbitrarily chosen surface M^5.

We apply next the requirements of H-gauge invariance and scale invariance under the transformations (III.5.35) (building rules B and C). Decomposing the adjoint and coadjoint indices of Eq. (III.5.44) in an H-invariant way we can write:

$$\mathscr{L} = \Lambda + R^\Lambda \wedge \nu_\Lambda + R^\Lambda \wedge R^\Sigma \wedge \nu_{\Lambda\Sigma} = \Lambda + R^{ab} \wedge \nu_{ab} + R^a \wedge \nu_a + R^\otimes \wedge \nu_\otimes + \bar{\rho}_A \wedge \nu_A + R^\Lambda \wedge R^\Sigma \wedge \nu_{\Lambda\Sigma} \tag{III.5.46}$$

where

$$\Lambda = a_1 \bar{e}^2\, \varepsilon_{abijk}\, V^a \wedge V^b \wedge V^i \wedge V^j \wedge V^k + a_2 \bar{e}\, \varepsilon_{abijk}\, \bar{\psi}_A \wedge \Gamma^{ab} \psi_B\, \varepsilon_{AB} \wedge V^i \wedge V^j \wedge V^k +$$

$$+ a_3 \bar\psi_A \wedge \psi_A \wedge \bar\psi_B \wedge \Gamma_a \psi_B \wedge V^a +$$

$$+ (a_4 \bar\psi_A \wedge \psi_A \wedge \bar\psi_B \wedge \psi_B +$$

$$+ a_5 \bar{e} \bar\psi_A \wedge \Gamma^{ab} \psi_B \varepsilon_{AB} \wedge V_a \wedge V_b) \wedge B \qquad \text{(III.5.47a)}$$

$$\nu_{ab} = \frac{1}{3} \varepsilon_{abijk} V^i \wedge V^j \wedge V^k + b_1 V^a \wedge V^b \wedge B \qquad \text{(III.5.47b)}$$

$$\nu_a = i\, c_1 \bar\psi_A \wedge \psi_A \wedge V_a + i\, c_2 \bar\psi_A \wedge \Gamma_{ab} \psi_B \varepsilon_{AB} \wedge V^b +$$

$$+ i\, c_3 \bar\psi_A \wedge \Gamma_a \psi_A \wedge B \qquad \text{(III.5.47c)}$$

$$\nu_\otimes = i\, d_1 \bar\psi_A \wedge \Gamma_a \psi_A \wedge V^a + i\, d_2 \bar\psi_A \wedge \psi_A \wedge B \qquad \text{(III.5.47d)}$$

$$\nu_A = i\, k_1 \Gamma_{ab} \psi_A \wedge V^a \wedge V^b + i\, h_2 \Gamma_a \psi_A \wedge V^a \wedge B \qquad \text{(III.5.47e)}$$

and

$$R^\Lambda \wedge R^\Sigma \wedge \nu_{\Lambda\Sigma} = \eta R^a \wedge R^\otimes \wedge V_a + (\beta R^a \wedge R_a + \gamma R^\otimes \wedge R^\otimes) \wedge B \,. \qquad \text{(III.5.48)}$$

A more detailed explanation of the previous "ansatz", is obtained if we take into account the following points:

a) First of all Λ, $\{\nu_{ab}, \nu_a, \nu_\otimes, \nu_A\}$ and ν_{AB} must be forms of degree 5, 3 and 1 respectively, since $\mathscr{L}$ is a 5-form.

b) All the terms we have written correspond to terms in the Lagrangian with the scaling behaviour $[w^3]$ under the transformations (III.5.30). This is in fact the scale power of the Einstein term $R^{ab} \wedge V^c \wedge V^d \wedge V^f \varepsilon_{abcdf}$ in D=5.

c) The polynomials $(\Lambda, \nu_{ab}, \nu_a, \nu_\otimes, \nu_A)$ are good $H(\equiv SO(1,4) \otimes SO(2))$-tensors, so that the corresponding Lagrangian is an H-scalar, i.e. invariant under H-transformations. Also the terms (III.5.48) are, of course, H-scalars. Apparently there are some missing structures in Λ (III.5.47a), namely:

$$\bar\psi_A \wedge \Gamma_{ab} \psi_A \wedge \bar\psi_B \wedge \Gamma^a \psi_B \wedge V^b \;;$$

$$\bar\psi_A \wedge \Gamma^a \psi_A \wedge \bar\psi_B \wedge \Gamma_a \psi_B \wedge B \;;$$

$$\bar\psi_A \wedge \Gamma_{ab} \psi_A \wedge \bar\psi_B \wedge \Gamma^{ab} \psi_B \wedge B \,. \qquad \text{(III.5.49)}$$

The point is that they are related by Fierz identities to the terms with coefficients a_3, a_4, a_5 respectively. Indeed from Table II.8.IX one easily finds:

$$\bar\psi_A \wedge \psi_A \wedge \bar\psi_B \wedge \psi_B = \bar\psi_A \wedge \Gamma_a \psi_A \wedge \bar\psi_B \wedge \Gamma^a \psi_B =$$

$$= -\frac{1}{4} \bar\psi_A \wedge \Gamma^{ab} \psi_B \wedge \bar\psi_A \wedge \Gamma_{ab} \psi_B \qquad \text{(III.5.50a)}$$

$$\bar\psi_A \wedge \Gamma_a \psi_A \wedge \bar\psi_B \wedge \Gamma^{ab} \psi_B = 0 \,. \qquad \text{(III.5.50b)}$$

d) To obtain H-gauge invariance we still have to impose that the coefficient in the Lagrangian of the gauge field B be a closed 4-form. (See the discussion of H-gauge invariance given in Sect. II.3.9 and observe that B is in fact the gauge field of the $U(1) \equiv SO(2)$ subgroup of $H = SO(1,4) \times SO(2)$). Therefore we can have non trivial solutions of Eq. (III.3.154).

Before implementing this equation, however, let us observe that the terms we have written in (III.5.47-48) are actually redundant. Indeed some of them can be related to each other by adding to the Lagrangian (III.5.46) a total derivative, namely an exact 5-form $\omega^{(5)}$ of the type

$$\omega^{(5)} = d\omega^{(4)} \qquad \text{(III.5.51)}$$

where $\omega^{(4)}$ must be an H-scalar 4-form with scaling power $[w^3]$ in order not to violate the previously imposed conditions. There are only three 4-forms satisfying these requirements, namely

$$\omega_1^{(4)} = \mathscr{D} V^a \wedge V_a \wedge B \qquad \text{(III.5.52a)}$$

$$\omega_2^{(4)} = \bar\psi_A \wedge \Gamma_a \psi_A \wedge V^a \wedge B \qquad \text{(III.5.52b)}$$

$$\omega_3^{(4)} = \bar\psi_A \wedge \Gamma_{ab} \psi_B \,\varepsilon_{AB} \wedge V^a \wedge V^b \,. \qquad \text{(III.5.52c)}$$

Let us differentiate these forms; we obtain:

$$\begin{aligned} d\omega_1^{(4)} \equiv \mathscr{D}\omega_1^{(4)} = {} & R^{ab} \wedge V_a \wedge V_b \wedge B + R^a \wedge R_a \wedge B + \\ & + i\, R^a \wedge \bar\psi_A \wedge \Gamma_a \psi_A \wedge B - \frac{1}{4} \bar\psi \wedge \Gamma^a \psi_A \wedge \bar\psi_B \wedge \Gamma_a \psi_B \wedge B \\ & + R^a \wedge R^{\otimes} \wedge V_a - i\, R^a \wedge \bar\psi_A \wedge \psi_A \wedge V_a - \\ & - \frac{i}{2} R^{\otimes} \wedge \bar\psi_A \wedge \Gamma_a \psi_A \wedge V^a + \\ & + \frac{i}{2} \bar\psi_A \wedge \Gamma_a \psi_A \wedge \bar\psi_B \wedge \psi_B \wedge V^a \end{aligned} \qquad \text{(III.5.53a)}$$

$$\begin{aligned} d\omega_2^{(4)} = {} & 2\, \bar\psi_A \wedge \Gamma_a \rho_A \wedge V^a \wedge B + R^a \wedge \bar\psi_A \wedge \Gamma_a \psi_A \wedge B + \\ & + \frac{i}{2} \bar\psi_A \wedge \Gamma^a \wedge \psi_A \wedge \bar\psi_B \wedge \Gamma_a \psi_B \wedge B - \\ & - R^{\otimes} \wedge \bar\psi_A \wedge \Gamma_a \psi_A \wedge V^a - \\ & - i\, \bar\psi_A \wedge \Gamma_a \psi_A \wedge \bar\psi_B \wedge \psi_B \wedge V^a \end{aligned} \qquad \text{(III.5.53b)}$$

$$\begin{aligned} d\omega_3^{(4)} = {} & 2\, \bar\psi_A \wedge \Gamma_{ab} \rho_A \wedge V^a \wedge V^b + \\ & + 2\, \varepsilon_{AB} \bar\psi_A \wedge \Gamma_{ab} \psi_B \wedge R^a \wedge V^b \end{aligned} \qquad \text{(III.5.53c)}$$

where we have made repeated use of the definitions (III.5.31).

The previous expressions contain most of the terms appearing in the Lagrangian. By adding them to $\mathscr{L}$ with suitable coefficients we can annihilate three arbitrarily chosen terms. We chose to annihilate the terms whose coefficients are c_2, c_3 and h_2 which indeed appear in (III.5.53a), (III.5.53b) and (III.5.53c) respectively. Hence without loss of generality we can set

$$c_2 = c_3 = h_2 = 0 \,. \qquad \text{(III.5.54)}$$

Let us now impose SO(2)-gauge invariance. Calling $\mathscr{L}_B$ the part of $\mathscr{L}$ which is linear in B, we set (according to Eq. (III.3.150)),

$$\mathscr{L}_B = \Gamma \wedge B \,. \qquad \text{(III.5.55)}$$

Taking into account Eqs. (III.5.46) and (III.5.47-48) we have

$$\begin{aligned} \Gamma = {} & a_4\, \bar\psi_A \wedge \psi_A \wedge \bar\psi_B \wedge \psi_B + a_5\, \bar e\, \varepsilon_{AB}\, \bar\psi_A \wedge \Gamma_{ab} \psi_B \wedge V^a \wedge V^b + \\ & + i\, c_3\, R^a \wedge \bar\psi_A \wedge \Gamma_a \psi_A + i\, d_2\, R^{\otimes} \wedge \bar\psi_A \wedge \psi_A + \\ & + \beta\, R^a \wedge R_a + \gamma\, R^{\otimes} \wedge R^{\otimes} \,. \end{aligned} \qquad \text{(III.5.56)}$$

Under SO(2) gauge variation, $\delta B = d\lambda$, we find:

$$\delta \mathscr{L}_B = \Gamma \wedge \delta B = \Gamma \wedge d\lambda = d\Gamma \wedge \lambda + d(\Gamma \wedge \lambda) \,. \qquad \text{(III.5.57)}$$

Therefore we must set

$$d\Gamma = 0 \qquad \text{(III.5.58)}$$

according to the general discussion.

Differentiating (III.5.56) one obtains:

$$\begin{aligned} & (-4a_4 - 2d_2)\bar\psi_A \wedge \rho_A \wedge \bar\psi_B \wedge \psi_B + 2a_5\, \bar e\, \varepsilon_{AB}\, \bar\psi_A \wedge \Gamma_{ab} \rho_B \wedge V^a \wedge V^b + \\ & + 4a_5\, \bar e\, \varepsilon_{AB}\, \bar\psi_A \wedge \Gamma_{ab} \psi_B \wedge R^a \wedge V^b + (2b_1 - 2\beta) R^{ab} \wedge R_a \wedge V_b + \\ & + i b_1\, R^{ab} \wedge \bar\psi_A \wedge \Gamma_a \psi_A \wedge V_b + 2i\beta\, \bar\rho_A \wedge \Gamma_a \psi_A \wedge R^a + \\ & + (4i\gamma - 2id_2) R^{\otimes} \wedge \bar\psi_A \wedge \rho_A = 0 \,. \end{aligned} \qquad \text{(III.5.59)}$$

Equation (III.5.59) implies:

$$b_1 = \beta = 0 \,; \quad \gamma = \frac{d_2}{2} = -\, a_4 \,. \qquad \text{(III.5.60)}$$

Inserting the results (III.5.54) and (III.5.60) into the general ansatz we get the simpler form:

$$\Lambda = a_1 \bar{e}^2 \varepsilon_{abijk} V^a \wedge V^b \wedge V^i \wedge V^j \wedge V^k + \\ + a_2 \bar{e} \varepsilon_{AB} \bar{\psi}_A \wedge \Gamma^{ab} \psi_B \wedge V^i \wedge V^j \wedge V^k \varepsilon_{abijk} + \\ + a_3 \bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \Gamma_a \psi_B \wedge V^a - \\ - \gamma \bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \psi_B \wedge B . \quad \text{(III.5.61a)}$$

$$\nu_{ab} = \frac{1}{3} \varepsilon_{abijk} V^i \wedge V^j \wedge V^k \quad \text{(III.5.61b)}$$

$$\nu_a = i c_1 \bar{\psi}_A \wedge \psi_A \wedge V_a \quad \text{(III.5.61c)}$$

$$\nu_\otimes = i d_1 \bar{\psi}_A \wedge \Gamma_a \psi_A \wedge V^a + 2i\gamma \bar{\psi}_A \wedge \psi_A \wedge B \quad \text{(III.5.61d)}$$

$$\nu_A = i k_1 \Gamma_{ab} \psi_A \wedge V^a \wedge V^b \quad \text{(III.5.61e)}$$

$$R^\Lambda \wedge R^\Sigma \wedge \nu_{\Lambda\Sigma} = \eta R^a \wedge R^\otimes \wedge V_a + \gamma R^\otimes \wedge R^\otimes \wedge B . \quad \text{(III.5.62)}$$

All the coefficients entering the coadjoint multiplet $(\Lambda, \nu_{ab}, \nu_a, \nu_\otimes, \nu_A)$ will now be obtained from the requirement that $R^\Lambda = 0$ be a solution of the equations of motion derived from (III.5.46) (vacuum existence, i.e. building rule D).

As it has been already remarked in the D=4, N=2 case, it is sufficient to vary Λ and R^Λ in the general Lagrangian (III.5.46) and to use the Maurer-Cartan equations of SU(2,2|1) (that is, Eqs. (III.5.31)) with $R^\Lambda = 0$ so to express the derivatives of μ^Λ in terms of the μ^Λ themselves. Alternatively (and equivalently) one may use directly the equations (III.3.58). In our case they read

$$\nabla \nu_{ab} = 0 \qquad (\text{at } R^\Lambda = 0) \quad \text{(III.5.63a)}$$

$$\frac{\delta\Lambda}{\delta\nu_a} + \nabla\nu_a = 0 \qquad (\text{at } R^\Lambda = 0) \quad \text{(III.5.63b)}$$

$$\frac{\delta\Lambda}{\delta\nu_\otimes} + \nabla\nu_\otimes = 0 \qquad (\text{at } R^\Lambda = 0) \quad \text{(III.5.63c)}$$

$$\frac{\delta\Lambda}{\delta\nu_A} + \nabla\nu_A = 0 \qquad (\text{at } R^\Lambda = 0) . \quad \text{(III.5.63d)}$$

We choose this last method as an illustration of the general formula. In this case one must compute the explicit form of the G-covariant derivative ∇ of the coadjoint multiplet ν_A. We use the following trick: from

$$\nabla(R^\Lambda \wedge \nu_\Lambda) \equiv D(R^\Lambda \wedge \nu_\Lambda) \quad \text{(III.5.64)}$$

where D denotes the SO(1,4) ⊗ SO(2) covariant derivative, using Bianchi identities $\nabla R^\Lambda = 0$, we find:

$$R^\Lambda \wedge (\nabla - D)\nu_\Lambda = D R^\Lambda \wedge \nu_\Lambda = \\ = (-8\bar{e}^2 V^a \wedge R^b - 2\bar{e} \varepsilon_{AB} \bar{\psi}_A \wedge \Gamma^{ab} \rho_B) \wedge \nu_{ab} + \\ + (-R^{ab} \wedge V_b - i \bar{\rho}_A \wedge \Gamma^a \psi_A) \wedge \nu_a + 2i \bar{\psi}_A \wedge \rho_A \wedge \nu_\otimes + \\ + (\bar{e} V^a \varepsilon_{AB} \wedge \bar{\rho}_B \Gamma_a + \frac{1}{4} \bar{\psi}_A \wedge \Gamma_{ab} R^{ab} + \bar{e} \varepsilon_{AB} \bar{\psi}_B \wedge \Gamma^a R_a - \\ - \frac{3}{2} \bar{e} \varepsilon_{AB} \bar{\psi}_B \wedge R^\otimes) \wedge \nu_A \quad \text{(III.5.65)}$$

where we used again the Bianchi identities (III.5.32) in order to express DR^Λ in terms of the curvatures themselves. Equating the coefficients of the curvatures on both sides of Eq. (III.5.65) one obtains:

$$\nabla\nu_{ab} = V_{[a} \wedge \nu_{b]} + \frac{1}{4} \bar{\psi}_A \wedge \Gamma_{ab} \nu_A + \mathscr{D}\nu_{ab} \quad \text{(III.5.66a)}$$

$$\nabla\nu_a = 8 \bar{e}^2 V^b \wedge \nu_{ab} + \bar{e} \varepsilon_{AB} \bar{\psi}_B \wedge \Gamma_a \nu_A + \mathscr{D}\nu_A \quad \text{(III.5.66b)}$$

$$\nabla\nu_\otimes = - \frac{3}{2} \bar{e} \varepsilon_{AB} \bar{\psi}_B \wedge \nu_A + \mathscr{D}\nu_\otimes \quad \text{(III.5.66c)}$$

$$\nabla \nu_A = 2\,\bar{e}\,\varepsilon_{AB}\,\Gamma^{ab}\,\psi_B \wedge \nu_{ab} - i\,\Gamma^a\,\psi_A \wedge \nu_a - 2i\,\psi_A \wedge \nu_\otimes -$$

$$- \bar{e}\,V^a \wedge \varepsilon_{AB}\,\Gamma_a\,\nu_B + D\,\nu_A\,. \qquad \text{(III.5.66d)}$$

The covariant derivatives $\mathscr{D}$ and D appearing in the r.h.s. of Eqs. (III.5.66) are easily evaluated at $R^\Lambda = 0$ by use of Eqs. (III.5.31) at $R^\Lambda = 0$:

$$\mathscr{D}\,\nu_{ab}\,(R^\Lambda = 0) = \frac{i}{2}\,\bar{\psi}_A \wedge \Gamma^i\,\psi_A \wedge V^j \wedge V^k\,\varepsilon_{abijk} \qquad \text{(III.5.67a)}$$

$$\mathscr{D}\,\nu_a\,(R^\Lambda = 0) = i\,c_1\,\bar{\psi}_A \wedge \psi_A \wedge \frac{i}{2}\,\bar{\psi}_B \wedge \Gamma_a\,\psi_B \qquad \text{(III.5.67b)}$$

$$\mathscr{D}\,\nu_\otimes\,(R^\Lambda = 0) = -\,2i\,\bar{e}\,d_1\,\bar{\psi}_A \wedge \Gamma_{ab}\,\psi_B\,\varepsilon_{AB} \wedge V^a \wedge V^b -$$

$$-\,(d_2 + \frac{d_1}{2})\bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \psi_B \qquad \text{(III.5.67c)}$$

$$\mathscr{D}\,\nu_a\,(R^\Lambda = 0) = -\,k_1\,\bar{e}\,\Gamma_{abc}\,\psi_B\,\varepsilon_{AB} \wedge V^a \wedge V^b \wedge V^c +$$

$$+\,k_1\,\Gamma_{abc}\,\psi_A \wedge \bar{\psi}_B \wedge \Gamma^a\,\psi_B \wedge V^b\,. \qquad \text{(III.5.67d)}$$

Here we made repeated use of the Fierz identities (III.5.50).

Inserting Eqs. (III.5.67) into Eqs. (III.5.66) one finds that the Eqs. (III.5.63) imply the following set of relations among the parameters:

$$\begin{cases} \frac{i}{2} - \frac{i}{4}\,k_1 = 0 \\ ic_1 - \frac{i}{2}\,k_1 = 0 \end{cases};\quad \begin{cases} d_1 + \frac{3}{2}\,k_1 = 0 \\ 3\gamma + \frac{d_1}{2} = 0 \end{cases};\quad \begin{cases} a_3 - c_{1/2} = 0 \\ \frac{8}{3} + 5a_1 = 0 \\ \frac{i}{2}\,k_1 + 3a_2 = 0 \end{cases}$$

$$\begin{cases} k_1 + 2(d_1 + a_3) = 0 \\ -\frac{4}{5}\,k_1 + c_1 + 2a_3 + \frac{2}{5}\,(d_1 + a_3) = 0 \end{cases}. \qquad \text{(III.5.68)}$$

The last two equations have been obtained with the aid of Table II.8.IX. The solution of Eqs. (III.5.68) is:

$$c_1 = 1\,;\qquad k_1 = 2\,;\qquad d_1 = -\frac{3}{2}$$

$$a_1 = -\frac{8}{15}\,;\qquad a_2 = -\frac{i}{3}\,;\qquad a_3 = \frac{1}{2}$$

$$\gamma = -\frac{1}{4}\,. \qquad \text{(III.5.69)}$$

Inserting these values into (III.5.61) one gets the following Lagrangian:

$$\mathscr{L} = -\frac{8}{15}\,\bar{e}^2\,\varepsilon_{abijk}\,V^a \wedge V^b \wedge V^i \wedge V^j \wedge V^k -$$

$$-\,\frac{i\bar{e}}{3}\,\bar{\psi}_A \wedge \Gamma^{ab}\,\psi_B\,\varepsilon_{AB}\,\varepsilon_{abijk}\,V^i \wedge V^j \wedge V^k +$$

$$+\,\frac{1}{2}\,\bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \Gamma_a\,\psi_B \wedge V^a + \frac{1}{4}\,\bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \psi_B \wedge B$$

$$+\,\frac{1}{3}\,R^{ab} \wedge V^i \wedge V^j \wedge V^k\,\varepsilon_{abijk} + i\,R^a \wedge \bar{\psi}_A \wedge \psi_A \wedge V_a -$$

$$-\,\frac{i}{2}\,R^\otimes \wedge \bar{\psi}_A \wedge \psi_A \wedge B + 2i\,\bar{\rho}_A \wedge \Gamma_{ab}\,\psi_A \wedge V^a \wedge V^b -$$

$$-\,\frac{1}{4}\,R^\otimes \wedge R^\otimes \wedge B + \eta\,R^a \wedge R^\otimes \wedge V_a \qquad \text{(III.5.70)}$$

where only the coefficient η of the quadratic term is left undetermined.

III.5.4 - Superspace equations of motion and on-shell supersymmetry

The value of η will now be determined by the requirement of rheonomy (building rule E). Let us write down the equations of motion derived from (III.5.70). With by now straightforward computations we obtain:

$\delta\omega^{ab}$-variation:

$$\varepsilon_{abijk}\, R^{i} \wedge V^{j} \wedge V^{k} + R^{\otimes} \wedge V_{a} \wedge V_{b} = 0 \qquad \text{(III.5.71a)}$$

δV^{a}-variation:

$$\varepsilon_{abijk}\, R^{bi} \wedge V^{j} \wedge V^{k} + 2\,i\,R_{a} \wedge \bar{\psi}_{A} \wedge \psi_{A} - \frac{i}{2}(3-\eta)\, R^{\otimes} \wedge \bar{\psi}_{A} \wedge \Gamma_{a}\psi_{A} - 4\,i\,\varepsilon_{AB}\,\bar{\rho}_{A} \wedge \Gamma_{ab}\psi_{B} \wedge V^{b} + 2\,i\,(1-\eta)\,\bar{\rho}_{A} \wedge \psi_{A} \wedge V_{a} + 2\eta\, R_{a} \wedge R^{\otimes} = 0 \qquad \text{(III.5.71b)}$$

δB-variation:

$$\eta\, R^{ab} \wedge V_{a} \wedge V_{b} - \frac{i}{2}(3-\eta)\, R^{a} \wedge \bar{\psi}_{A} \wedge \Gamma_{a}\psi_{A} + \frac{3}{2}\,i\,R^{\otimes} \wedge \bar{\psi}_{A} \wedge \psi_{A} + i(3+\eta)\,\bar{\rho}_{A} \wedge \Gamma_{a}\psi_{A} \wedge V^{a} + \eta\, R^{a} \wedge R_{a} + \frac{3}{4} R^{\otimes} \wedge R^{\otimes} = 0 \qquad \text{(III.5.71c)}$$

$\delta\bar{\psi}_{A}$-variation:

$$2\,i\,\Gamma_{ab}\,\rho_{A} \wedge V^{a} \wedge V^{b} - 2\,i\,\Gamma_{ab}\,\psi_{A}\, R^{a} \wedge V^{b} + i(1-\eta)\,\psi_{A} \wedge R^{a} \wedge V^{a} - \frac{i}{2}(3+\eta)\,\Gamma_{a}\psi_{A} \wedge R^{\otimes} \wedge V^{a} = 0 \qquad \text{(III.5.71d)}$$

From (III.5.71a) it easily follows that both R^{a} and $R^{\otimes}$ cannot have outer components; expanding then the l.h.s. on the vielbein basis, one obtains the general solution:

$$R^{\otimes} = F_{ab}\, V^{a} \wedge V^{b} \qquad \text{(III.5.72a)}$$

$$R^{a} = -\frac{1}{4}\,\eta\,\varepsilon^{abcrs}\, F_{rs}\, V_{b} \wedge V_{c}\,. \qquad \text{(III.5.72b)}$$

The new feature here, with respect to the previously studied D=4 supergravities, is the non-vanishing value of the torsion on space-time.

It is easy to find the rheonomic parametrization of ρ_{A} using the parametrizations (III.5.72) in the gravitino equation (III.5.71d). Indeed the $VV\psi\psi$ content of (III.5.71d) implies that the $\psi \wedge \psi$ component of ρ_{A} is zero.

Therefore, setting quite generally:

$$\rho_{A} = \rho_{A|ab}\, V^{a} \wedge V^{b} + H^{c}_{AB}\,\psi_{B} \wedge V_{c} \qquad \text{(III.5.73)}$$

we find the following $VVV\psi$ content of Eq. (III.5.71d):

$$\Big\{2\,i\,\Gamma^{ab} H^{c}_{AB}\,\psi_{B} + \frac{i}{4}\,\eta\,(2\,\Gamma^{t}{}_{a} - (1-\eta)\,\delta^{t}{}_{a})\,\varepsilon_{tbcrs}\, F^{rs}\,\psi_{A} - \frac{i}{2}(3+\eta)\,\Gamma_{a} F_{bc}\,\psi_{A}\Big\} \wedge V^{a} \wedge V^{b} \wedge V^{c} = 0\,. \qquad \text{(III.5.74)}$$

Equation (III.5.74) implies

$$H^{c}_{AB} = \delta_{AB}\, H^{c} \qquad \text{(III.5.75)}$$

and, according to rheonomy and scale invariance, H^{c} must be expressed in terms of the F_{ab} inner components. Setting

$$H^{c} = p\,\Gamma_{\ell}\, F^{\ell c} + q\,\varepsilon^{\ell mrsc}\,\Gamma_{\ell m}\, F_{rs} \qquad \text{(III.5.76)}$$

we find, after some Γ-matrix algebra, that (III.5.75) is satisfied only if

$$\eta^{2} = 1 \;\Rightarrow\; \eta = \pm 1\,. \qquad \text{(III.5.77)}$$

In this case we also obtain

$$p = \frac{1}{2}\,; \qquad q = \frac{1}{16}(1-\eta)\,. \qquad \text{(III.5.78)}$$

We still have to determine the outer components of the Lorentz curvature. We set

$$R^{ab} = R^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\psi}_A\, \Theta^{ab}_{A|c} \wedge V^c + \bar{\psi}_A \wedge K^{ab}{}_{AB}\, \psi_B \qquad \text{(III.5.79)}$$

and we analyze the $VVV\psi$ projection of the Einstein equation (III.5.71b). With a computation analogous to that given in Sect. III.3.5 for the $\Theta^{ab|c}$ of the N=1, D=4 theory (Eqs. (III.3.64), (III.3.73b)), one finds:

$$\Theta^{ab}_{A|c} = -\, i\, \varepsilon_{AB} (\Gamma_{ck}\, \varepsilon^{ijkab} + \frac{2}{3}\, \Gamma_{k\ell}\, \varepsilon^{ijk\ell[a}\, \delta^{b]}_c)\, \rho_{B|ij} + \frac{i}{2}\, (1-\eta)\, \varepsilon^{ijab\cdot}{}_c\, \rho_{A|ij}\, . \qquad \text{(III.5.80)}$$

The $\psi\psi VV$ projection of the Einstein equation or, more easily, the same projection of the Maxwell equation (III.5.71c), determines $K^{ab}{}_{AB}$. One obtains

$$K^{ab}{}_{AB} = \delta_{AB}\, (\frac{i}{2}\, F^{ab}\, \mathbb{1} - \frac{i}{8}\, (1+\eta)\, \varepsilon^{abijk}\, F_{ij}\, \Gamma_k)\, . \qquad \text{(III.5.81)}$$

We leave to the reader the verification that all the other outer projections of Eqs. (III.5.71) are verified by the same rheonomic constraints (III.5.76) and (III.5.80-81) with $\eta^2 = 1$.

Finally the VVVV-projection of Eqs. (III.5.71) gives the space-time equations of motion.

They are:

$$R^{am}{}_{bm} - \frac{1}{2}\, \delta^a_b\, R^{mn}{}_{mn} = F^{am}\, F_{bm} + \frac{1}{2}\, F^{rs}\, F_{rs}\, \delta^a_b \qquad \text{(III.5.82a)}$$

$$\varepsilon_{abcdm}\, (\eta\, R^{ab}{}_{cd} + (\frac{3}{4} - \frac{1}{4}\, \eta)\, F^{ab}\, F^{cd}) = 0 \qquad \text{(III.5.82b)}$$

$$\varepsilon^{ijk\ell t}\, \Gamma_{ij}\, \rho_{A|k\ell} = 0 \qquad \text{(III.5.82c)}$$

$$R^a{}_{mn} = -\, \frac{\eta}{4}\, \varepsilon^a{}_{mnpq}\, F^{pq} \qquad \text{(III.5.82d)}$$

$$R^{\otimes}{}_{mn} = F_{mn} \qquad \text{(III.5.82e)}$$

where we have added to the VVVV projections of the Einstein, Maxwell and gravitino equations, i.e. Eqs. (III.5.71b,c,d), also the VVVV content of the torsion equation, i.e. (III.5.71a), in tensor form.

Notice that the use of the gravitino space-time equation (III.5.82c) simplifies the rheonomic constraint (III.5.80) since the second term is identically zero. The complete form of the rheonomic curvatures is therefore:

$$R^a = -\, \frac{\eta}{4}\, \varepsilon^{abcdf}\, V_b \wedge V_c\, F_{df} \qquad \text{(III.5.83a)}$$

$$R^{\otimes} = F_{ab}\, V^a \wedge V^b \qquad \text{(III.5.83b)}$$

$$R^{ab} = R^{ab}{}_{cd}\, V^c \wedge V^d - i\, \varepsilon_{AB}\, \bar{\psi}_A \wedge \Gamma_{c\ell}\, \rho_{B|ij}\, \varepsilon^{ij\ell ab}\, V^c + \frac{i}{2}\, (1-\eta)\, \bar{\psi}_A \wedge \rho_{A|ij}\, \varepsilon^{ijabc}\, V_c + \frac{i}{2}\, \bar{\psi}_A \wedge \psi_A\, F^{ab} - \frac{i}{8}\, (1+\eta)\, \varepsilon^{abijk}\, F_{ij}\, \bar{\psi}_A \wedge \Gamma_k\, \psi_A \qquad \text{(III.5.83c)}$$

$$\rho_A = \rho_{A|ij}\, V^i \wedge V^j + [\, \frac{1}{2}\, F_{\ell c}\, \Gamma^{\ell} + \frac{1}{16}\, (1-\eta)\, \varepsilon_{\ell mpqc}\, \Gamma^{\ell m}\, F^{pq}]\, \psi_A \wedge V^c\, . \qquad \text{(III.5.83d)}$$

We stress that the space-time equations of motion, Eqs. (III.5.82), are actually the integrability conditions of the rheonomic constraints (III.5.83) (and (III.5.77)).

That this must be so is clear from the general discussion of this property given in Sect. III.3.6 for the D=4 case and in the discussion of the building principles, given in Sect. III.3.9. The actual proof of the statement consists in inserting the rheonomic parametrization (III.5.83) into the Bianchi identities (III.5.32) and in working out the various projections, in analogy with the N=1, D=4 case. We leave this exercise to the reader.

Finally we explicitly write down the supersymmetry transformations, which, as we know from our general discussion of Sect. III.3.7, close an algebra of transformations only on-shell, that is on the equations of motion (III.5.82a,b,c). Using the general formula (III.3.170) or (III.3.175) and restricting the general tangent vector

$$\varepsilon = \varepsilon^{ab}\,\tilde{D}_{ab} + \varepsilon^{\otimes}\,\tilde{D}_{\otimes} + \varepsilon^{a}\,\tilde{D}_{a} + \bar{D}_{A}\,\varepsilon^{A} \qquad \text{(III.5.84)}$$

to the superspace directions, namely setting $\varepsilon^{ab} = \varepsilon^{\otimes} = \varepsilon^{a} = 0$, we get the first order form of the supersymmetry transformations:

$$\delta\omega^{ab} = -\,2i\,\bar{\varepsilon}_A\,\Gamma_{c\ell}\,\rho_{B|ij}\,\epsilon_{AB}\,\varepsilon^{ij\ell ab}\,V^c + i(1-\eta)\,\bar{\varepsilon}_A\,\rho_{A|ij}\,\varepsilon^{ijabc}\,V_c + \frac{i}{2}\,\bar{\varepsilon}_A\,\psi_A\,F^{ab} - \frac{i}{8}\,(1+\eta)\,\varepsilon^{abijk}\,F_{ij}\,\bar{\varepsilon}_A\,\Gamma_k\,\psi_A \qquad \text{(III.5.85a)}$$

$$\delta V^a = +\,i\,\bar{\varepsilon}_A\,\Gamma^a\,\psi_A \qquad \text{(III.5.85b)}$$

$$\delta B = 2i\,\bar{\varepsilon}_A\,\psi_A \qquad \text{(III.5.85c)}$$

$$\delta\psi_A = D\,\varepsilon_A + \left[\frac{1}{2}\,\Gamma_a\,\varepsilon_A\,F^{ab} + \frac{1}{16}\,(1-\eta)\,\varepsilon^{\ell mbrs}\,F_{\ell m}\,\Gamma_{rs}\,\varepsilon_A\right]V_b\,. \qquad \text{(III.5.85d)}$$

Comparison of the supersymmetry transformations (III.5.85) with those obtained in the N=2, D=4 supergravity shows the similarity of the two theories. One may therefore wonder why the same rheonomic conditions are obtained in one case in a purely geometrical way, while in the other case (D=4) one needs the introduction of a 0-form F_{ab} which essentially plays the double role of Lagrangian multiplier for the rheonomic constraint and device to propagate the spin 1-field. The answer is that N=2, D=4 supergravity is a subtheory of the theory obtained by dimensional reduction of the present one and the 0-form F_{ab} is actually one of the remainders, in D=4, of the ω^{ab} 1-form in D=5, exactly in the same way as it happens in ordinary Klein-Kaluza theory (see Part Five).

Let us conclude this section with two remarks:

a) The first order formulation developed so far depends on the two possible values of the η parameter, namely $\eta=\pm 1$. The rheonomic parametrization (III.5.83), the field equations (III.5.82) and the first order supersymmetry transformation laws (III.5.85) are actually different in the two cases. As we are going to show in the following, however, when we go to the second order formulation the dependence of the theory on the value of η disappears. Therefore the dichotomy $\eta=\pm 1$ does not seem to play any fundamental role from the physical point of view.

b) We notice that the Maxwell equation (III.5.82) seems to be, at first sight, an algebraic equation and therefore does not seem to describe the propagation of the spin 1 field B. The B-field propagation, however, becomes apparent as soon as we go to second order formalism, that is when we solve the torsion equation (III.5.82d) for the spin connection ω^{ab}, as we anticipated in the introduction. This will be the subject of the next section.

III.5.5 - The second order formulation and the contracted version of the theory

In this section we present the usual calculations needed to obtain the second order formalism: we solve the constraints (III.5.82d) for the Lorentz connection ω^{ab} in terms of the other fields and then we insert the result in the first order Lagrangian. By explicit calculation we shall demonstrate that the Hodge duality operator

$$* : F \rightarrow *F \qquad \text{(III.5.86)}$$

arises from the elimination of the ω^{ab}-connection field. This leads to the Maxwell term

$$F \wedge *F = 2\,F_{ab}\,F^{ab}\,\det V\,d^5x \qquad \text{(III.5.87)}$$

of the second order Lagrangian. Moreover we shall see that, surprisingly enough, the second order Lagrangian turns to be independent of the value of η.

We recall that because of rheonomy we may restrict all the equations among forms to physical space-time. This is obtained by expanding the differential form equations in the dx^μ space-time differentials only; the transition from flat Latin indices to Greek curved ones and vice versa is performed as usual via the components of the fünfbein V^a_μ and its inverse $V^\mu_a \equiv (V^{-1})^\mu_a$.

Using these notations, Eq. (III.5.82d) may be written:

$$R^a_{\ \mu\nu} \equiv \mathscr{D}_{[\mu} V^a_{\nu]} - \frac{i}{2} \bar\psi^A_{[\mu} \Gamma^a \psi^A_{\nu]} = = -\frac{1}{4} \eta\, \epsilon^{abcdf} V_{b|\mu} V_{c|\nu} F_{df} = = -\frac{1}{4} \eta\, V^a_{\ \lambda}\, \epsilon^{\lambda..\rho\sigma}_{\ \ \mu\nu} F_{\rho\sigma} \qquad \text{(III.5.88)}$$

where ψ^A_μ, $\mathscr{D}_\mu$ are the space-time components of the gravitino 1-form and of the Lorentz covariant derivative. F_{df} is defined by (III.5.82e) and we have:

$$F_{\rho\sigma} \equiv R^{\otimes}_{\rho\sigma} = V^d_{\ \rho} V^f_{\ \sigma} F_{df}\,. \qquad \text{(III.5.89)}$$

Notice that because of the definition (III.5.31c) we also have:

$$F_{\rho\sigma} = \mathscr{F}_{\rho\sigma} - i\, \bar\psi^A_{[\rho} \psi^A_{\sigma]} \qquad \text{(III.5.90)}$$

where $\mathscr{F}_{\rho\sigma}$ is given by:

$$\mathscr{F}_{\rho\sigma} = \frac{1}{2} (\partial_\rho B_\sigma - \partial_\sigma B_\rho)\,. \qquad \text{(III.5.91)}$$

The solution of (III.5.88) is:

$$\omega^{ab}_{\ \ \mu} = \overset{o}{\omega}{}^{ab}_{\ \ \mu} + h^{ab}_{\ \ \mu} \qquad \text{(III.5.92)}$$

where $\overset{o}{\omega}{}^{ab}_\mu$ is the usual 1-form satisfying the space time torsionless condition

$$\mathscr{D}(\overset{o}{\omega})_{[\mu} V^a_{\nu]} = \partial_{[\mu} V^a_{\nu]} - \overset{o}{\omega}{}^{ab}_{[\mu} V_{\nu]b} \qquad \text{(III.5.93)}$$

and h^{ab}_μ is given by

$$h^{ab}_{\ \ \mu} = V^{a|\lambda} V^{b|\nu} h_{\lambda\nu|\mu} \qquad \text{(III.5.94a)}$$

$$h_{\lambda\nu|\mu} = \overset{(1)}{h}_{\lambda\nu|\mu} + \overset{(2)}{h}_{\lambda\nu|\mu} \qquad \text{(III.5.94b)}$$

$$\overset{(1)}{h}_{\lambda\nu|\mu} = \frac{\eta}{4} \epsilon_{\lambda\mu\nu\rho\sigma} F^{\rho\sigma} \qquad \text{(III.5.94c)}$$

$$\overset{(2)}{h}_{\lambda\nu|\mu} = \frac{i}{4} (\bar\psi^A_\mu \Gamma_\lambda \psi^A_\nu + \bar\psi^A_\lambda \Gamma_\nu \psi^A_\mu + \bar\psi^A_\lambda \Gamma_\mu \psi^A_\nu - - [\lambda \leftrightarrow \nu])\,. \qquad \text{(III.5.94d)}$$

Note that the dependence of ω^{ab}_μ on $F^{\rho\sigma}$ and η is contained only in $\overset{(1)}{h}_{\lambda\nu|\mu}$.

If we introduce the decompositions (III.5.92,94) into the first order supersymmetry transformations (III.5.85), we see that the η-dependent terms in the r.h.s. of this latter are cancelled by the $h^{(1)}$-contributions arising from the definitions of the covariant derivative and the gravitino curvature $\rho_A(\omega)$.

Setting

$$\bar\omega^{ab} = \overset{o}{\omega}{}^{ab} + \overset{(2)}{h}{}^{ab} \rightarrow \omega^{ab} = \bar\omega^{ab} + \overset{(1)}{h}{}^{ab} \qquad \text{(III.5.95)}$$

one finds the following second order transformation laws:

$$\delta V^a_\mu = -i\, \bar\varepsilon^A \Gamma^a \psi^A_\mu \qquad \text{(III.5.96a)}$$

$$\delta B_\mu = 2i\, \bar\varepsilon_A \psi_A \qquad \text{(III.5.96b)}$$

$$\delta\psi_{A|\mu} = D(\bar\omega)\varepsilon_A - \bar\varepsilon\, V^a_\mu \Gamma_a \varepsilon_{AB} \varepsilon_B + + \left(\frac{1}{2}\Gamma^\rho F_{\rho\mu} + \frac{1}{16}\varepsilon_{\mu\nu\rho\sigma\tau} F^{\nu\rho} F^{\sigma\tau}\right)\varepsilon_A \qquad (III.5.96c)$$

$$\delta\omega^{ab}{}_\mu \equiv \text{chain rule.} \qquad (III.5.96d)$$

The spin connection transformation law has been omitted since in second order formalism it is a consequence of the transformations (III.5.96a,b,c) (see the discussion in the D=4 case, Eqs. (III.2.48,61) and Eqs. (III.3.126-127)). The η-independence of the second order transformations (III.5.96) indicates that the 2nd order Lagrangian and the equations of motion should also be η-independent.

Let us first see how it happens that the Maxwell equation (III.5.82b) becomes a propagation equation when we go to second order formalism. From the definition (III.5.95) it follows that:

$$R^{ab}(\omega) = R^{ab}(\bar\omega) + H^{ab} \qquad (III.5.97)$$

where we have set

$$H^{ab} = \mathcal{D}(\bar\omega)\, \overset{(1)}{h}{}^{ab} - \overset{(1)}{h}{}^{a}{}_{c} \wedge \overset{(1)}{h}{}^{cb} \;. \qquad (III.5.98)$$

Using the explicit form of $\overset{(1)}{h}{}^{ab}$, (Eq. (III.5.94c)), we find:

$$H^{ab} = -\frac{\eta}{4}\varepsilon^{abcrs}(\mathcal{D}(\bar\omega)F_{rs} \wedge V_c + F_{rs}\mathcal{D}(\bar\omega) \wedge V_c) - -\frac{1}{16}\varepsilon^{acdrs}\varepsilon_c{}^{bpqt} F_{qt} V_d \wedge V_p F_{rs} \;. \qquad (III.5.99)$$

Since

$$\mathcal{D}(\bar\omega)V_c = \mathcal{D}(\omega)V_c + \overset{(1)}{h}{}^{cb} \wedge V_b = = R^a + \frac{i}{2}\bar\psi_A \wedge \Gamma^a \psi_A - R^a \equiv \frac{i}{2}\bar\psi_A \wedge \Gamma^a \psi_A \qquad (III.5.100)$$

we obtain the following expression for H^{ab}:

$$H^{ab} = -\frac{\eta}{4}\varepsilon^{abcrs}\mathcal{D}(\bar\omega)F_{rs} V_c - \frac{1}{16}\varepsilon^{arscd}\varepsilon^{b}{}_{.pqtc} F_{rs} F^{qt} V_d \wedge V^p - -\frac{i}{8}\eta\,\varepsilon^{abcrs} F_{rs}\bar\psi_A \wedge \Gamma_c \psi_A \;. \qquad (III.5.101)$$

Therefore

$$R^{ab}{}_{cd}(\omega) = R^{ab}{}_{cd}(\bar\omega) + \frac{\eta}{4}\varepsilon^{abrs.}{}_{[c}\mathcal{D}_{d]}(\bar\omega)F_{rs} + + \frac{1}{16}\varepsilon^{ap}{}_{rs[c}\varepsilon_{d]pqt}{}^{b} F^{rs} F^{qt} \;. \qquad (III.5.102)$$

Introducing this result into (III.4.82b) we find

$$3\mathcal{D}_m(\bar\omega)F^{m\ell} + \frac{3}{4}\varepsilon_{abcd}{}^{\ell} F^{ab} F^{cd} = 0 \qquad (III.5.103)$$

since we have identically

$$R^{a.}{}_{[b|cd]}(\bar\omega) = 0 \;. \qquad (III.5.104)$$

The last equation is the cyclic identity of the Riemann tensor which follows from the Bianchi identity (III.5.32b) at the 3V-level when $R^a = 0$; indeed this is the case if we use $\bar\omega^{ab}$ instead of ω^{ab}. Equation (III.5.103) is the expected propagation equation for the spin 1 field B interacting with the spin 2 and spin 3/2 fields through the $\bar\omega^{ab}$ connection.

We also notice that the 2nd-order Maxwell equation (III.5.103) contains a self-interaction term $F \cdot F$ characteristic of the electromagnetic field in D=5 and which has no analogue in D=4. If the same decomposition (III.5.95) is used in the l.h.s. of the Einstein equation (III.5.82a) one obtains:

$$R^{am}{}_{bm}(\omega) - \frac{1}{2}\delta^a_b R^{mn}{}_{mn}(\omega) = R^{am}{}_{bm}(\bar\omega) - \frac{1}{2}\delta^a_b R^{mn}{}_{mn}(\bar\omega) + + H^{am}{}_{bm} - \frac{1}{2}\delta^a_b H^{mn}{}_{mn} \;. \qquad (III.5.105)$$

Using the above formula and the Bianchi identity following from (III.5.32c)

$$\varepsilon^{abrsc}\mathscr{D}_b(\omega)F_{rs} = 0 \qquad \text{(III.5.106)}$$

which implies

$$\varepsilon^{abrsc}\mathscr{D}_b(\bar\omega)F_{rs} = 2 \overset{(1)}{h}{}^{rt}{}_b F_{ts}\varepsilon^{abrsc} =$$

$$= 4\eta F_{t[a}F_{c]t} \equiv 0 \qquad \text{(III.5.107)}$$

equation (III.5.82a) becomes:

$$R^{am}{}_{bm}(\bar\omega) - \frac{1}{2}\delta^a_b R^{mn}{}_{mn}(\bar\omega) =$$

$$= \frac{3}{2}(F^{at}F_{tb} + \frac{1}{4}F^{rs}F_{rs}\delta^a_b) . \qquad \text{(III.5.108)}$$

Finally, since

$$\rho_A(\omega) = \rho_A(\bar\omega) - \frac{1}{4}F_{rs}\overset{(1)}{h}{}^{rs}\psi_A \qquad \text{(III.5.109)}$$

the 3V projection of the gravitino equation (III.5.71d) remains unchanged when $\bar\omega^{ab}$ is used in place of ω^{ab}. Therefore:

$$\varepsilon^{ijk\ell t}\Gamma_{ij}\rho_{A|k\ell}(\bar\omega) = 0 . \qquad \text{(III.5.110)}$$

Equations (III.5.103,108,110) are the 2nd-order equations of the theory.

Finally we derive the explicit form of the D=5, N=2 second order Lagrangian on space-time. Expanding all the 5-forms in (III.5.70) along the differentials dx^μ's and taking the coefficient of d^5x we get:

$$\mathscr{L}_{(\text{space-time})} = 4R^{ab}{}_{\mu\nu}(\omega)V^\mu_a V^\nu_b \det V +$$

$$+ \frac{i}{2}F_{\mu\nu}\bar\psi^A_\rho\psi^A_\sigma B_\tau\varepsilon^{\mu\nu\rho\sigma\tau} - \frac{3}{2}iF_{\mu\nu}\bar\psi^A_\rho\Gamma_a\psi_\sigma V^a_\tau\varepsilon^{\mu\nu\rho\sigma\tau} -$$

$$- 3i\eta F^{\mu\nu}\bar\psi^A_\mu\psi^A_\nu \det V + 2i\bar\rho^A_{\mu\nu}(\omega)\Gamma_{\ell m}\psi^A_\rho V^\ell_\sigma V^m_\tau\varepsilon^{\mu\nu\rho\sigma\tau} +$$

$$+ \frac{1}{2}\bar\psi^A_\mu\psi^A_\nu\bar\psi^B_\rho\Gamma_a\psi^B_\sigma V^a_\tau\varepsilon^{\mu\nu\rho\sigma\tau} + \frac{1}{2}F_{\mu\nu}F_{\rho\sigma}B_\tau\varepsilon^{\mu\nu\rho\sigma\tau} -$$

$$- 3F_{\rho\sigma}F^{\rho\sigma}\det V - 16\bar e^2\det V -$$

$$- \frac{i}{6}\varepsilon_{AB}\varepsilon_{abijk}\bar e\bar\psi^A_\mu\Gamma^{ab}\psi^A_\nu V^i_\rho V^j_\sigma V^k_\tau\varepsilon^{\mu\nu\rho\sigma\tau} \qquad \text{(III.5.111)}$$

where $R^{ab}{}_{\mu\nu}$ and $\rho^A_{\mu\nu}$ still depend on the full connection ω^{ab}_μ. We notice the generation of the Maxwell kinetic term as a consequence of the fact that the curvatures $R^\otimes$ and R^a given by (III.5.82d,e) have components related by a duality transformation. Another contribution to the Maxwell kinetic term comes also from the Einstein term. In fact using the decomposition (III.5.102) one obtains:

$$4R^{ab}{}_{\mu\nu}(\omega)V^\mu_a V^\nu_b = 4R^{\mu\nu}{}_{\mu\nu}(\bar\omega) + \frac{3}{2}F_{\mu\nu}F^{\mu\nu} +$$

$$+ i\frac{\eta}{2}F_{\rho\sigma}\bar\psi_{A|\mu}\Gamma_\lambda\psi_{A|\nu}\varepsilon^{\rho\sigma\mu\nu\lambda} \qquad \text{(III.5.112)}$$

so that once more we generate the Maxwell term. If one decomposes the gravitino kinetic term as in (III.5.109) and collects the various contributions one finds that all the η-dependent terms cancel, as expected. The final form of the second order space-time restricted Lagrangian is therefore:

$$\mathscr{L}^{(2\text{nd-order})} =$$

$$= [4\mathscr{R}^{\mu\nu}{}_{\mu\nu}(\bar\omega) - \frac{3}{2}\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} + 3i\mathscr{F}^{\mu\nu}\bar\psi^A_\mu\psi^A_\nu - 4(2\bar e)^2 +$$

$$+ 12i\bar e\bar\psi^A_\alpha\Gamma^{\alpha\beta}\psi^B_\beta e_{AB} - 6\bar\psi^A_\mu\psi^B_\nu\bar\psi^{A|\mu}\psi^{B|\nu}]\det V +$$

$$+ [\frac{1}{4}\mathscr{F}_{\mu\nu}\mathscr{F}_{\rho\sigma}B_\tau + \frac{3}{2}i\bar\psi^A_\mu\Gamma_\nu\psi^A_\rho\mathscr{F}_{\sigma\tau} +$$

$$+ \bar\psi^A_\mu\psi^A_\nu\bar\psi^B_\rho\Gamma_\sigma\psi^B_\tau + 2i(\mathscr{D}_\mu(\bar\omega)\bar\psi^A_\nu)\Gamma_{\sigma\tau}\psi^B_\rho\varepsilon_{AB} -$$

$$- 3i\bar e\varepsilon_{AB}\bar\psi^A_\mu\Gamma_{\rho\sigma}\psi^B_\nu B_\tau]\varepsilon^{\mu\nu\rho\sigma\tau} \qquad \text{(III.5.113)}$$

where $\mathscr{R}^{ab}$ has been defined in (III.5.33a).

This concludes our analysis of the D=5 N=2 theory in the non-contracted version.

The contracted version of the theory based on $\overline{SU(2,2|1)}$ can be immediately worked out simply by letting $\bar{e} \to 0$ in all the equations of the corresponding uncontracted case. In particular one finds:

a) The equations of motion are the same as in the uncontracted case if one replaces the uncontracted curvatures with the contracted ones (Eqs. (III.5.33)). The Lagrangian looses the 4 fermion terms proportional to the contraction parameter.

b) The only change in the transformations laws (III.5.85) is given by the absence of the term $\bar{e} V^a \wedge \Gamma^a$ in the r.h.s. of Eq. (III.5.96c). As a last point we also stress that in the contracted case the $SO(1,4) \times SO(2)$ covariant derivative reduces to the $SO(1,4)$ covariant derivative since the SO(2) gauge coupling constant is proportional to $\bar{e}$. Indeed in Minkowski space the group SO(2) can be realized only globally on the supersymmetry charges, exactly in the same way as it happens in the D=4 N=2 theory.

In Tables III.5.I-VI our results have been summarized for the reader's convenience. Let us make some observation:

i) All the formulae written there are in first order formalism while the corresponding ones in second order formalism are given in the text.

ii) We have set everywhere $\eta = +1$ since in any case the theory in second order formalism is independent from the value of η.

iii) The ψV component of R^{ab} has been computed from the field equations; if we had computed it from the Bianchi identities, we would have found the same expression as for $\overset{(II)}{\theta}{}^{ab|c}_{A}$ of Table III.4.I (with $\gamma_a \to \Gamma_a$).

iv) The supersymmetry transformations of Table III.5.VI are a symmetry (closed algebra) of the inner equations of Table III.5.V and an invariance (on-shell closed algebra) of the Lagrangian in the first-order formalism (in second order formalism if the chain rule for $\delta\omega^{ab}$ is assumed).

TABLE III.5.I

Summary of D=5 N=2 Supergravity

(Anti de Sitter supergravity if $\bar{e} \neq 0$, Poincaré supergravity if $\bar{e}=0$)

Definition of the SU(2,2|1) curvatures

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{c} \wedge \omega^{cb} + 4\,\bar{e}^2\, V^a \wedge V^b + i\,\bar{e}\,\epsilon_{AB}\,\bar{\psi}_A \wedge \Gamma^{ab}\,\psi_B$$

$$R^a = \mathscr{D} V^a - \frac{i}{2}\,\bar{\psi}_A \wedge \Gamma^a\,\psi_A$$

$$R^{\otimes} = dB - i\,\bar{\psi}_A \wedge \psi_A$$

$$\rho_A = \mathscr{D}\psi_A - \bar{e}\, V^a \wedge \Gamma_a\,\psi_B\,\epsilon_{AB} + \frac{3}{2}\,\bar{e}\,\epsilon_{AB}\, B \wedge \psi_B$$

TABLE III.5.II

Action of D=5 N=2 Supergravity

$$\mathscr{A} = \int_{M_5 \subset \tilde{G}/H} \mathscr{L}$$

where:

$$\begin{aligned}\mathscr{L} = &- \frac{8}{15}\,\bar{e}^2\,\epsilon_{abijk}\, V^a \wedge V^b \wedge V^i \wedge V^j \wedge V^k - \\ &- \frac{i}{3}\,\bar{e}\,\bar{\psi}_A \wedge \Gamma^{ab}\,\psi_B\,\epsilon_{AB}\,\epsilon_{abijk}\, V^i \wedge V^j \wedge V^k + \\ &+ \frac{1}{2}\,\bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \Gamma_a\,\psi_B \wedge V^a + \frac{1}{4}\,\bar{\psi}_A \wedge \psi_A \wedge \bar{\psi}_B \wedge \psi_B \wedge B + \\ &+ \frac{1}{3}\, R^{ab} \wedge V^i \wedge V^j \wedge V^k\,\epsilon_{abijk} + i\, R^a \wedge \bar{\psi}_A \wedge \psi_A \wedge V_a - \\ &- \frac{3}{2}\, i\, R^{\otimes} \wedge \bar{\psi}_A \wedge \Gamma_a\,\psi_A \wedge V^a - \frac{i}{2}\, R^{\otimes} \wedge \bar{\psi}_A \wedge \psi_A \wedge B + \\ &+ 2\,i\,\bar{\rho}_A \wedge \Gamma_{ab}\,\psi_A \wedge V^a \wedge V^b - \frac{1}{4}\, R^{\otimes} \wedge R^{\otimes} \wedge B + R^a \wedge R^{\otimes} \wedge V_a\end{aligned}$$

and

$$\tilde{G}/H = \widetilde{SU}(2,2|1)/SO(1,3) \times SO(2)$$

TABLE III.5.III

Superspace Field-Equations of D=5, N=2 Supergravity

$$\varepsilon_{abijk} R^{i} \wedge V^{j} \wedge V^{k} + R^{\otimes} \wedge V_a \wedge V_b = 0$$

$$\varepsilon_{abijk} R^{bi} \wedge V^{j} \wedge V^{k} + 2 i R_a \wedge \bar{\psi}_A \wedge \psi_A - i R^{\otimes} \wedge \bar{\psi}_A \wedge \Gamma_a \psi_A$$
$$- 4 i \varepsilon_{AB} \bar{\rho}_A \wedge \Gamma_{ab} \psi_B \wedge V^b + 2 R_a \wedge R^{\otimes} = 0$$

$$R^{ab} \wedge V^a \wedge V^b - i R^a \wedge \bar{\psi}_A \wedge \Gamma_a \psi_A + \frac{3}{2} i R^{\otimes} \wedge \bar{\psi}_A \wedge \psi_A -$$
$$- 4 i \bar{\rho}_A \wedge \Gamma_a \psi_A \wedge V^a + R^a \wedge R_a +$$
$$+ \frac{3}{4} R^{\otimes} \wedge R^{\otimes} = 0$$

$$2 i \Gamma_{ab} \psi_A \wedge V^a \wedge V^b - 2 i \Gamma_{ab} \psi_A \wedge R^a \wedge V^b -$$
$$- 2 i \Gamma_a \psi_A \wedge R^{\otimes} \wedge V^a = 0$$

TABLE III.5.IV

Rheonomic Parametrization of the SU(2,2|1) Curvatures from Field Equations

$$R^a = -\frac{1}{4} \varepsilon^{abcdf} V_b \wedge V_c F_{df}$$

$$R^{\otimes} = F_{ab} V^a \wedge V^b$$

$$\rho_A = \rho_{A|ij} V^i \wedge V^j + \frac{1}{2} F_{\ell c} \Gamma^{\ell} \psi_A \wedge V^c$$

$$R^{ab} = R^{ab}{}_{cd} V^c \wedge V^d - i \varepsilon_{AB} \bar{\psi}_A \wedge \Gamma_{c\ell} \rho_{B|ij} V^c \varepsilon^{ij\ell ab} +$$
$$+ \frac{i}{2} \bar{\psi}_A \wedge \psi_A F^{ab} - \frac{i}{4} \varepsilon^{abijk} F_{ij} \bar{\psi}_A \wedge \Gamma_k \psi_A$$

TABLE III.5.V

Inner Field Equations

$$R^a{}_{mn} = -\frac{1}{4} \varepsilon^a{}_{mnpq} F^{pq}$$

$$R^{\otimes}{}_{mn} = F_{mn}$$

$$R^{am}{}_{bm} - \frac{1}{2} \delta^a_b R^{mn}{}_{mn} = F^{am} F_{bm} + \frac{1}{2} F^{rs} F_{rs} \delta^a_b$$

$$\varepsilon^{ijk\ell t} \Gamma_{ij} \rho_{A|k\ell} = 0$$

$$(R^{ab|cd} + \frac{1}{2} F^{ab} F^{cd}) \varepsilon_{abcdm} = 0$$

TABLE III.5.VI

Supersymmetry Transformation Laws

$$\delta_\varepsilon V^a = + i \bar{\varepsilon}_A \Gamma^a \psi_A$$

$$\delta_\varepsilon B = 2 i \bar{\varepsilon}_A \psi_A$$

$$\delta_\varepsilon \psi_A = \mathscr{D} \varepsilon_A - \bar{e} V^a \Gamma_a \varepsilon_B \varepsilon_{AB} + \frac{1}{2} \Gamma_a \varepsilon_A F^{ab} V_b$$

$$\delta_\varepsilon \omega^{ab}(\text{first-order}) = - 2i \bar{\varepsilon}_A \Gamma_{c\ell} \rho_{B|ij} \varepsilon_{AB} \varepsilon^{ij\ell ab} V^c +$$
$$+ \frac{i}{2} \bar{\varepsilon}_A \psi_A F^{ab} - \frac{i}{4} \varepsilon^{abijk} F_{ij} \bar{\varepsilon}_A \Gamma_k \psi_A$$

$$\delta_\varepsilon \omega^{ab}(\text{second-order}) = \text{chain rule.}$$