geometry of physics -- categories and toposes

This entry is one chapter of *geometry of physics*.

next chapters: *smooth sets*, *supergeometry*

Context

Category theory

Topos Theory

Category theory and topos theory concern the general abstract structure underlying algebra, geometry and logic. They are ubiquitous in and indispensible for organizing conceptual mathematical frameworks.

We give here an introduction to the basic concepts and results, aimed at providing background for the synthetic higher supergeometry of relevance in formulations of fundamental physics, such as used in the chapters *on perturbative quantum field theory* and *on fundamental super p-branes*. For quick informal survey see *Introduction to Higher Supergeometry*.

This makes use of the following curious dictionary between category theory/topos theory and the geometry of generalized spaces, which we will explain in detail (following Grothendieck 65, Lawvere 86, p. 17, Lawvere 91):

category theory	**Rmk. 1.28**	**geometry of generalized spaces**
presheaf	Expl. 1.26	generalized space
representable presheaf	Expl. 1.27	model space regarded as generalized space
Yoneda lemma	Prop. 1.29	sets of probes of generalized spaces are indeed sets of maps from model spaces

category theory	**Rmk. 1.28**	**geometry of generalized spaces**
Yoneda embedding	Prop. 1.30	nature of model spaces is preserved when regarding them as generalized spaces
Yoneda embedding is free co-completion	Prop. 3.20	generalized spaces really are glued from ordinary spaces
topos theory	***Rmk. 4.1***	***local-global principle for generalized spaces***
coverage	Defn. 4.3	notion of locality
sheaf condition	Defn. 4.8 Prop. 4.29	plots of generalized spaces satisfy local-to-global principle
comparison lemma	Prop. 4.20	notion of generalized spaces independent under change of model space
gros topos theory	***Rmk. 5.1***	***generalized spaces at the foundations***
cohesion	Defn. 5.2	generalized spaces obey principles of differential topology
differential cohesion	Defn. 5.10	generalized spaces obey principles of differential geometry
super cohesion	Defn. 5.14	generalized spaces obey principles of supergeometry

The perspective is that of *functorial geometry* (Grothendieck 65). (For more exposition of this point see also at *motivation for sheaves, cohomology and higher stacks*.) This dictionary implies a wealth of useful tools for handling and reasoning about geometry:

We discuss below that sheaf toposes, regarded as categories of generalized spaces via the above disctionary, are “convenient contexts” for geometry (Prop. 4.23 below), in the technical sense that they provide just the right kind of generalization that makes all desireable constructions on spaces actually exist:

sheaf topos	**as category of generalized spaces**
Yoneda embedding:	contains and generalizes ordinary spaces
has all limits:	contains all Cartesian products and intersections

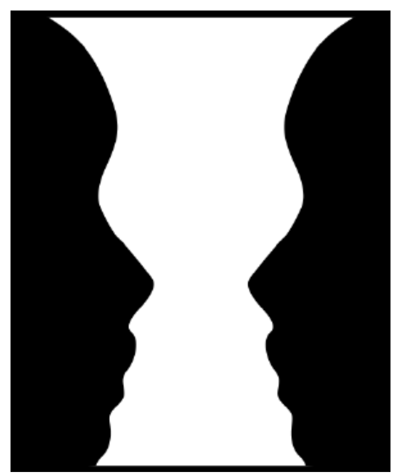
sheaf topos	**as category of generalized spaces**
has all colimits:	contains all disjoint unions and quotients
cartesian closure:	contains all mapping spaces
local cartesian closure:	contains all fiber-wise mapping spaces

Notably mapping spaces play a pivotal role in physics, in the guise of spaces of field histories, but fall outside the applicability of traditional formulations of geometry based on just manifolds. Topos theory provides their existence (Prop. 4.23 below) and the relevant infrastructure, for example for the construction of transgression of differential forms to mapping spaces of smooth sets, that is the basis for sigma-model-field theories. This is discussed in the following chapters *on smooth sets* and *on supergeometry*.

In conclusion, one motivation for category theory and topos theory is *a posteriori*: As a matter of experience, there is just no other toolbox that allows to deeply understand and handle the geometry of physics. Similar comments apply to a wealth of other topics of mathematics.

But we may offer also an *a priori* motivation:

Category theory is the theory of duality.

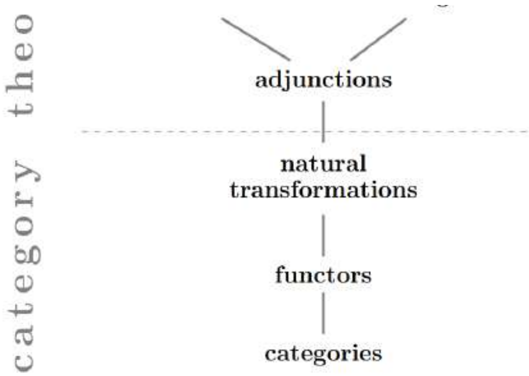


Duality is of course an ancient notion in philosophy. At least as a term, it makes a curious re-appearance in the conjectural theory of fundamental physics formerly known as *string theory*, in the guise of *duality in string theory*. In both cases, the literature left some room in delineating what precisely is meant. But the philosophically inclined mathematician could notice (see Lambek 82) that an excellent candidate to make precise the idea of *duality* is the mathematical concept of *adjunction*, from category theory. This is particularly pronounced for *adjoint triples* (Remark 1.34 below) and their induced adjoint modalities (Lawvere 91, see Def. 1.66 below), which exhibit a given "mode of being" of any object X as intermediate between two dual opposite extremes (Prop. 1.69 below):

$$\Box X \longrightarrow X \longrightarrow \bigcirc X$$

For example, *cohesive* geometric structure on generalized spaces is captured, this way, as modality in between the discrete and the codiscrete (Example 1.36, and Def. 5.2 below).

Historically, category theory was introduced in

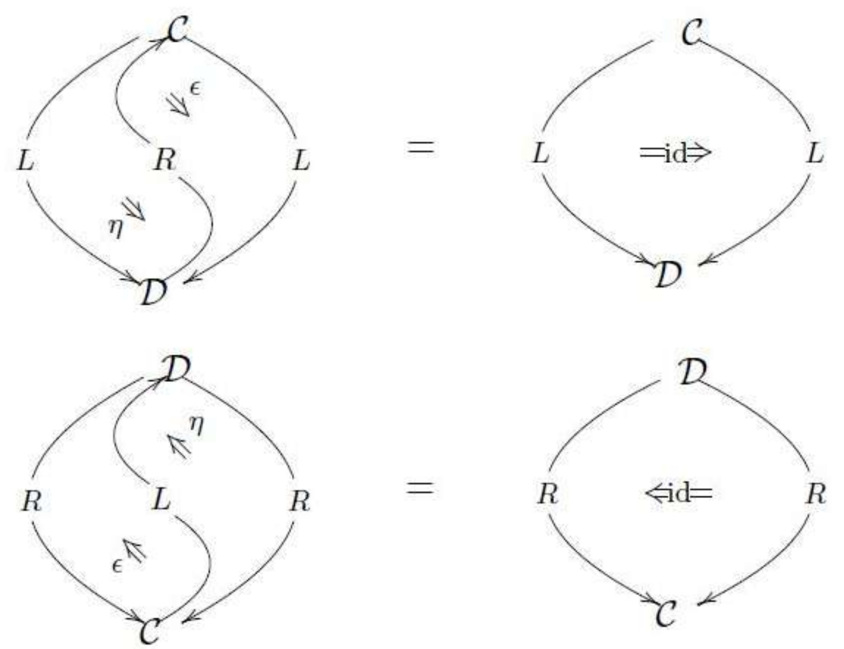


order to make precise the concept of *[natural transformation](#)*: The concept of *[functors](#)* was introduced just so as to support that of natural transformations, and the concept of *[categories](#)* only served that of functors (see Freyd 1964,, p 1).

But natural transformations are, in turn, exactly the basis for the concept of *[adjoint functors](#)* (Def. 1.32 below), equivalently *[adjunctions](#) between categories* (Prop. 1.39 below).

Shown below is the "Yin-Yang identity" (the *triangle identity*, cf. Prop. below) characterizing adjunctions.

All *universal constructions* — the heart of category theory — are special cases of adjoint functors, hence of dualities, if we follow Lambek 82: This includes the concepts of *limits* and *colimits* (Def. 3.1 below), ends and coends (Def. 3.13 below) Kan extensions (Prop. 3.29 below), and the behaviour of these constructions, such as for instance the free co-completion nature of the Yoneda embedding (Prop. 3.20 below).



Therefore it makes sense to regard category theory as the ***theory of adjunctions***, hence the ***theory of duality***:

hierarchy of concepts	**category theory**	**enriched**	**homotopical**
adjunction of adjunctions duality of dualities	Def. 1.52		Def. 6.59

hierarchy of concepts	**category theory**	**enriched**	**homotopical**
adjoint equivalence dual equivalence	Def. 1.56	Def. 2.53	Def. 6.55
adjunction duality	Def. 1.32	Def. 2.52	Def. 6.44
natural transformation	Def. 1.23	Def. 2.50	
functor	Def. 1.15	Def. 2.46	
category	Def. 1.1	Def. 2.40	Def. 6.1

The pivotal role of adjunctions in category theory (Lawvere 08) and in the foundations of mathematics (Lawvere 69, Lawvere 94) was particularly amplified by F. W. Lawvere[1]. Moreover, Lawvere saw the future of category theory (Lawvere 91) as concerned with adjunctions expressing systems of archetypical dualities that reveal foundations for geometry (Lawvere 07) and physics (Lawvere 97, see Def. 5.2 and Def. 5.10 below). He suggested (Lawvere 94) this as a precise formulation of core aspects of the *theory of everything* of early 19th century philosophy: Hegel's *Science of Logic*.

These days, of course, *theories of everything*, such as string theory, are understood less ambitiously than Hegel's ontological process, as mathematical formulations of fundamental theories of physics, that could conceptually unify the hodge-podge of currently available "standard models" of particle physics and of cosmology to a more coherent whole.

The idea of *duality in string theory* refers to different perspectives on physics that appear dual to each other while being *equivalent*. But one of the basic results of category theory (Prop. 1.58, below) is that equivalence is indeed a special case of adjunction. This allows to explore the possibility that there is more than a coincidence of terms.

Of course the usage of the term *duality in string theory* is too loose for one to expect to be able to refine each occurrence of the term in the literature to a mathematical adjunction. However, we will see mathematical formalizations of core aspects of key string-theoretic dualities, such as *topological T-duality* and the *duality between M-theory and type IIA string theory*, in terms of adjunctions. Indeed, at the heart of these *dualities in string theory* is the phenomenon of *double dimensional reduction*, which turns out to be formalized by one of the most fundamental adjunctions in (higher) category theory: *base change* along the point inclusion into a classifying space. All this is discussed in the chapter on *fundamental super p-branes*.

This suggests that there may be a deeper relation here between the superficially alien uses of the word "duality", that is worth exploring.

In this respect it is worth noticing that core structure of string/M-theory arises via universal constructions from the superpoint (as explained in the chapter *on fundamental super p-branes*), while the superpoint itself arises, in a sense made precise by category theory, "from nothing", by a system of twelve adjunctions (explained in the chapter on supergeometry).

Here we introduce the requisites for understanding these statements.

Contents

1. Basic notions of Category theory

We introduce here the basic notions of category theory, along with examples and motivation from geometry:

1. *Categories and functors*
2. *Natural transformations and presheaves*
3. *Adjunctions*
4. *Equivalences*
5. *Modalities*

This constitutes what is sometimes called the *language of categories*. While we state and prove some basic facts here, notably the notorious *Yoneda lemma* (Prop. 1.29 below), what makes category theory be a *mathematical theory* in the sense of a coherent collection of non-trivial theorems is all concerned with the topic of *universal constructions*, which may be formulated (only) in this language. This we turn to further below.

Categories and Functors

The notion of a *category* (Def. 1.1 below) embodies the idea of structuralism applied to concepts in mathematics: it collects, on top of the set (or generally: class) of mathematical objects that belong to it, also all the *structure-preserving maps* between them, hence the homomorphisms in the case of Bourbaki-style mathematical structures.

The first achievement of the notion of a category is to abstract away from such manifestly *concrete categories* (Examples 1.3, 1.21 below) to more indirectly defined mathematical objects whose “structure” is only defined, after the fact, by which maps, now just called *morphisms*, there are between them.

This structuralism-principle bootstraps itself to life by considering morphisms between categories themselves to be those “maps” that respect their structuralism, namely the connectivity and composition of the morphisms between their objects: These are the *functors* (Def. 1.15 below).

For the purpose of geometry, a key class of examples of functors are the assignments of *algebras of functions to spaces*, this is Example 1.22 below.

***Definition 1.1*. (*category*)**

A *category* $\mathcal{C}$ is

1. a class $\mathrm{Obj}_{\mathcal{C}}$, called the *class of objects*;
2. for each pair $X, Y \in \mathrm{Obj}_{\mathcal{C}}$ of objects, a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, called the *set of morphisms from X to Y*, or the *hom-set*, for short.
   We denote the elements of this set by arrows like this:
   $$X \xrightarrow{f} Y \quad \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \;.$$
3. for each object $X \in \mathrm{Obj}_{\mathcal{C}}$ a morphism
   $$X \xrightarrow{\mathrm{id}_X} X \quad \in \mathrm{Hom}_{\mathcal{C}}(X,X)$$
   called the *identity morphism* on X;
4. for each triple $X_1, X_2, X_3 \in \mathrm{Obj}$ of objects, a function

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) & \times & \mathrm{Hom}_{\mathcal{C}}(X_2, X_3) & \overset{\circ_{X_1, X_2, X_3}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_3) \\
X_1 \overset{f}{\to} X_2 & , & X_2 \overset{f}{\to} X_3 & \mapsto & X_1 \overset{g \circ f}{\longrightarrow} X_3
\end{array}
$$

called *composition*;

such that:

1. for all pairs of objects $X, Y \in \mathrm{Obj}_{\mathcal{C}}$ unitality holds: given

$$X \overset{f}{\to} Y \quad \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

then

$$X \overset{\mathrm{id}_Y \circ f}{\longrightarrow} Y \;=\; X \overset{f}{\longrightarrow} Y \;=\; X \overset{f \circ \mathrm{id}_X}{\longrightarrow} Y \,;$$

2. for all quadruples of objects $X_1, X_2, X_3, X_4 \in \mathrm{Obj}_{\mathcal{C}}$ composition satifies *associativity*: given

$$X_1 \overset{f_{12}}{\longrightarrow} X_2 \overset{f_{23}}{\longrightarrow} X_3 \overset{f_{34}}{\longrightarrow} X_4$$

then

$$X_1 \xrightarrow{f_{34} \circ (f_{23} \circ f_{12})} X_4 \;=\; X_1 \xrightarrow{(f_{34} \circ f_{23}) \circ f_{12}} X_4 \,.$$

The archetypical example of a category is the category of sets:

***Example 1.2*. (*category of all sets*)**

The class of all sets with functions between them is a category (Def. 1.1), to be denoted *Set*:

- $\mathrm{Obj}_{\mathrm{Set}}$ = class of all sets;
- $\mathrm{Hom}_{\mathrm{Set}}(X, Y)$ = set of functions from set X to set Y;
- $\mathrm{id}_X \in \mathrm{Hom}_{\mathrm{Set}}(X, X) =$ identity function on set X;
- $\circ_{X_1, X_2, X_3}$ = ordinary composition of functions.

More generally all kind of *sets with* *structure*, in the sense going back to Bourbaki, form categories, where the morphisms are the *homomorphisms* (whence the name "morphism"!). These are called *concrete categories* (we characterize them precisely in Example 1.21, further below):

Example 1.3*. *(basic examples of concrete categories)

For $\mathcal{S}$ a kind of mathematical structure, there is the category (Def. 1.1) $\mathcal{S}$ Set whose objects are the corresponding structured sets, and whose morphisms are the corresponding structure homomorphisms, hence the functions of underlying sets which respect the given structure.

Basic examples of concrete categories include the following:

concrete category	**objects**	**morphisms**
Set	sets	functions
Top	topological spaces	continuous functions
Mfd_k	differentiable manifolds	differentiable functions
Vect	vector spaces	linear functions
Grp	groups	group homomorphisms
Alg	algebras	algebra homomorphism

This is the motivation for the terminology "categories", as the examples in Example 1.3 are literally *categories of mathematical structures*. But not all categories are "concrete" in this way.

Some terminology:

Definition 1.4*. *(commuting diagram)

Let $\mathcal{C}$ be a category (Def. 1.1), then a directed graph with edges labeled by morphisms of the category is called a *commuting diagram* if for any two vertices any two ways of passing along edges from one to the other yields the same composition of the corresponding morphisms.

For example, a *commuting triangle* is

$$f = h \circ g \qquad \begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Y \arrow[rr, "h"'] & & Z \end{tikzcd}$$

while a *commuting square* is

$$
g_1 \circ f_1 \;=\; g_2 \circ f_2 \qquad \begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ {}^{f_2}\downarrow && \downarrow^{g_1} \\ Y_2 & \underset{g_2}{\longrightarrow} & Z \end{array}
$$

***Definition 1.5*. (*initial object* and *terminal object*)**

Let $\mathcal{C}$ be a category (Def. 1.1). Then

1. an object $* \in \mathcal{C}$ is called a *terminal object* if for every other object $c \in \mathcal{C}$, there is a unique morphism from c to $*$

   $$c \overset{\exists !}{\longrightarrow} *$$

   hence if the hom-set is a singleton $* \in \mathrm{Set}$:

   $$\mathrm{Hom}_{\mathcal{C}}(c, *) \;\simeq\; * \,.$$

2. an object $\emptyset \in \mathcal{C}$ is called an *initial object* if for every other object $c \in \mathcal{C}$, there is a unique morphism from $\emptyset$ to c

   $$\emptyset \overset{\exists !}{\longrightarrow} c$$

   hence if the hom-set is a singleton $* \in \mathrm{Set}$:

   $$\mathrm{Hom}_{\mathcal{C}}(\emptyset, c) \;\simeq\; * \,.$$

***Definition 1.6*. (*small category*)**

If a category $\mathcal{C}$ (Def. 1.1) happens to have as class $\mathrm{Obj}_{\mathcal{C}}$ of objects an actual set (i.e. a small set instead of a proper class), then $\mathcal{C}$ is called a *small category*.

As usual, there are some trivial examples, that are however usefully made explicit for the development of the theory:

***Example 1.7*. (*initial category* and *terminal category*)**

1. The *terminal category* $*$ is the category (Def. 1.1) whose class of objects is the singleton set, and which has a single morphism on this object, necessarily the identity morphism.
2. The *initial category* or *empty category* $\emptyset$ is the category (Def. 1.1) whose class of objects is the empty set, and which, hence, has no morphism whatsoever.

Clearly, these are small categories (Def. 1.6).

Example 1.8. (preordered sets as thin categories)

Let $(S, \leq)$ be a preordered set. Then this induces a small category whose set of objects is S, and which has precisely one morphism $x \to y$ whenever $x \leq y$, and no such morphism otherwise:

$$x \overset{\exists!}{\to} y \qquad \text{precisely if} \qquad x \leq y \tag{1}$$

Conversely, every small category with at most one morphism from any object to any other, called a *thin category*, induces on its set of objects the structure of a partially ordered set via (1).

Here the axioms for preordered sets and for categories match as follows:

	reflexivity	**transitivity**
partially ordered sets	$x \leq x$	$(x \leq y \leq z) \Rightarrow (x \leq z)$
thin categories	identity morphisms	composition

Definition 1.9. (isomorphism)

For $\mathcal{C}$ a category (Def. 1.1), a morphism

$$X \overset{f}{\to} Y \quad \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

is called an *isomorphism* if there exists an inverse morphism

$$Y \overset{f^{-1}}{\longrightarrow} X \quad \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

namely a morphism such that the compositions with f are equal to the identity morphisms on X and Y, respectively

$$f^{-1} \circ f \;=\; \mathrm{id}_X \qquad f \circ f^{-1} \;=\; \mathrm{id}_Y$$

Definition 1.10. (groupoid)

If $\mathcal{C}$ is a category in which *every* morphism is an isomorphism (Def. 1.9), then $\mathcal{C}$ is called a *groupoid*.

Example 1.11. (delooping groupoid)

For G a group, there is a groupoid (Def. 1.10) $\mathbf{B}\, G$ with a single object, whose single hom-set is G, with identity morphism the neutral element and composition the group operation in G:

- $\mathrm{Obj}_{\mathbf{B}\, G} = *$
- $\mathrm{Hom}_{\mathcal{C}}(*, *) \;=\; G$

In fact every groupoid with precisely one object is of the form.

***Remark 1.12*. (groupoids and homotopy theory)**

Even though groupoids (Def. 1.10) are special cases of categories (Def. 1.1), the theory of groupoids in itself has a rather different flavour than that of category theory: Part of the homotopy hypothesis-theorem is that the theory of groupoids is really *homotopy theory* for the special case of homotopy 1-types.

(In applications in homotopy theory, groupoids are considered mostly in the case that the class $\mathrm{Obj}_{\mathcal{C}}$ of objects is in fact a set: *small groupoids*, Def. 1.6).

For this reason we will not have more to say about groupoids here, and instead relegate their discussion to the section on homotopy theory, further below.

There is a range of constructions that provide new categories from given ones:

***Example 1.13*. (opposite category and formal duality)**

Let $\mathcal{C}$ be a category. Then its *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, but the direction of the morphisms is reversed. Accordingly, composition in the opposite category $\mathcal{C}^{\mathrm{op}}$ is that in $\mathcal{C}$, but with the order of the arguments reversed:

- $\mathrm{Obj}_{\mathcal{C}^{\mathrm{op}}} \;\coloneqq\; \mathrm{Obj}_{\mathcal{C}}$;
- $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) \;\coloneqq\; \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Hence for every statementa about some category $\mathcal{C}$ there is a corresponding "dual" statement about its opposite category, which is "the same but with the direction of all morphisms reversed". This relation is known as *formal duality*.

***Example 1.14*. (product category)**

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories (Def. 1.1). Then their *product category* $\mathcal{C} \times \mathcal{D}$ has as objects pairs (c, d) with $c \in \mathrm{Obj}_{\mathcal{C}}$ and $d \in \mathrm{Obj}_{\mathcal{D}}$, and as morphisms pairs $(c_1 \overset{f}{\to} c_2) \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$, $(d_1 \overset{g}{\to} d_2) \in \mathrm{Hom}_{\mathcal{D}}(d_1, d_2)$, and composition is defined by composition in each entry:

- $\mathrm{Obj}_{\mathcal{C}\times\mathcal{D}} \coloneqq \mathrm{Obj}_{\mathcal{C}} \times \mathrm{Obj}_{\mathcal{D}}$;
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c_1,d_1),(c_2,d_2)) \coloneqq \mathrm{Hom}_{\mathcal{C}}(c_1,c_2) \times \mathrm{Hom}_{\mathcal{D}}(d_1,d_2)$
- $(f_2, g_2) \circ (f_1, g_1) \;\coloneqq\; (f_2 \circ f_1, g_2 \circ g_1)$

Definition 1.15. (functor)

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories (Def. 1.1). A *functor from $\mathcal{C}$ to $\mathcal{D}$*, to be denoted

$$\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D}$$

is

1. a function between the classes of objects:

$$F_{\mathrm{Obj}} \;:\; \mathrm{Obj}_{\mathcal{C}} \longrightarrow \mathrm{Obj}_{\mathcal{D}}$$

2. for each pair $X, Y \in \mathrm{Obj}_{\mathcal{C}}$ of objects a function

$$F_{X,Y} \;:\; \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_{\mathrm{Obj}}(X), F_{\mathrm{Obj}}(Y))$$

such that

1. For each object $X \in \mathrm{Obj}_{\mathcal{C}}$ the identity morphism is respected:

$$F_{X,X}(\mathrm{id}_X) \;=\; \mathrm{id}_{F_{\mathrm{Obj}}(X)}\,;$$

2. for each triple $X_1, X_2, X_3 \in \mathrm{Obj}_{\mathcal{C}}$ of objects, composition is respected: given

$$X_1 \overset{f}{\longrightarrow} X_2 \overset{g}{\longrightarrow} X_3$$

   we have

$$F_{X_1,X_3}(g \circ f) \;=\; F_{X_2,X_3}(g) \circ F_{X_1,X_2}(f)\,.$$

Example 1.16. (categories of small categories and of small groupoids)

It is clear that functors (Def. 1.15) have a composition operation given componentwise by the composition of their component functions. Accordingly, this composition is unital and associative. This means that there is

1. the category (Def. 1.1) *Cat* whose objects are small categories (Def. 1.6) and whose morphisms are functors (Def. 1.15) between these
2. the category (Def. 1.1) *Grpd* whose objects are small groupoids (Def. 1.10) and

whose morphisms are functors (Def. 1.15) between these.

***Example 1.17*. (*hom-functor*)**

Let $\mathcal{C}$ be a category (Def. 1.1). Then its *hom-functor*

$$\mathrm{Hom}_{\mathcal{C}} \;:\; \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

is the functor (Def. 1.15) out of the product category (Def. 1.14) of $\mathcal{C}$ with its opposite category to the category of sets, which sends a pair $X, Y \in \mathcal{C}$ of objects to the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ between them, and which sends a pair of morphisms, with one of them into X and the other out of Y, to the operation of composition with these morphisms:

$$\mathrm{Hom}_{\mathcal{C}} \;:\; \begin{array}{ccc} X_1 & & Y_1 \\ {}^{g}\big\uparrow & & \big\downarrow{}^{h} \\ X_2 & & Y_2 \end{array} \;\mapsto\; \begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(X_1, Y_1) \\ \big\downarrow{}^{f \mapsto h \circ f \circ g} \\ \mathrm{Hom}_{\mathcal{C}}(X_2, Y_2) \end{array}$$

***Definition 1.18*. (*monomorphism* and *epimorphism*)**

Let $\mathcal{C}$ be a category (Def. 1.1). Then a morphism $X \overset{f}{\to} Y$ in $\mathcal{C}$ is called

- a *monomorphism* if for every object $Z \in \mathcal{C}$ the hom-functor (Example 1.17) out of Z takes f to an injective function of hom-sets:

$$\mathrm{Hom}_{\mathcal{C}}(Z, f) \;:\; \mathrm{Hom}_{\mathcal{C}}(Z, X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Z, Y)\,;$$

- an *epimorphism* if for every object $Z \in \mathcal{Z}$ the hom-functor (Example 1.17) into Z takes f to an injective function:

$$\mathrm{Hom}_{\mathcal{C}}(f, Z) \;:\; \mathrm{Hom}_{\mathcal{C}}(Y, Z) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)\;.$$

***Definition 1.19*. (*full*, *faithful* and *fully faithful functors*)**

A functor $F \;:\; \mathcal{C} \to \mathcal{D}$ (Def. 1.15) is called

- a *full functor* if all its hom-functions are surjective functions

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \underset{\mathrm{surj}}{\overset{F_{X,Y}}{\longrightarrow}} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

- a *faithful functor* if all its hom-functions are injective functions

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \underset{\mathrm{inj}}{\overset{F_{X,Y}}{\longrightarrow}} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

- a *fully faithful functor* if all its hom-functions are bijective functions

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \underset{\text{bij}}{\overset{F_{X,Y}}{\longrightarrow}} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

A fully faithful functor is also called a *full subcategory-inclusion*. We will denote this situation by

$$\mathcal{C} \overset{F}{\hookrightarrow} \mathcal{D} \,.$$

***Example 1.20*. (*full subcategory* on a *sub-class* of *objects*)**

Let $\mathcal{C}$ be a category (Def. 1.1) and let $S \subset \mathrm{Obj}_{\mathcal{C}}$ be a sub-class of its class of objects. The there is a category $\mathcal{C}_S$ whose class of objects is S, and whose morphisms are precisely the morphisms of $\mathcal{C}$, between these given objects:

$$\mathrm{Hom}_{\mathcal{C}_S}(s_1,s_2) \;\coloneqq\; \mathrm{Hom}_{\mathcal{C}}(s_1,s_2)$$

with identity morphisms and composition defined as in $\mathcal{C}$. Then there is a fully faithful functor (Def. 1.19)

$$\mathcal{C}_S \;\hookrightarrow\; \mathcal{C}$$

which is the evident inclsuion on objects, and the identity function on all hom-sets.

This is called the *full subcategory of $\mathcal{C}$ on the objects in S*.

Beware that not every fully faithful functor is, in components, exactly of this form, but, assuming the axiom of choice, every fully faithful functor is so up to *equivalence of categories* (Def. 1.57).

The concept of *faithful functor* from Def. 1.19 allows to make precise the idea of *concrete category* from Example 1.3:

***Example 1.21*. (*structured sets* and *faithful functors*)**

Let $\mathcal{S}$ be a kind of mathematical structure and let $\mathcal{S}\,\mathrm{Set}$ be the category of $\mathcal{S}$-structured sets. Then there is the forgetful functor

$$\mathcal{S}\,\mathrm{Set} \longrightarrow \mathrm{Set}$$

which sends each structured set to the underlying set ("forgetting" the structure that it carries), and which sends functions of sets to themselves. That a homomorphism of structured sets is a function between the underlying sets satisfying a *special condition* implies that this is a *faithful functor* (Def. 1.19).

Conversely, it makes sense to *define* structured sets in general to be the objects of a category $\mathcal{C}$ which is equipped with a faithful functor $\mathcal{C} \xrightarrow{\text{faithful}} \text{Set}$ to the category of sets. See at *structure* for more on this.

***Example 1.22*. (spaces seen via their algebras of functions)**

In any given context of geometry, there is typically a functor which sends any space of the given kind to its algebra of functions, and which sends a map (i.e. homomorphism) between the given spaces to the algebra homomorphism given by precomposition with that map (a hom-functor, Def. 1.17). Schematically:

$$\begin{array}{ccc} \{\text{geometric spaces}\} & \xrightarrow{\text{algebra of functions}} & \{\text{algebras}\}^{\text{op}} \\ X_1 & \mapsto & \text{FunctionsOn}(X_1) \\ {}^{f}\downarrow & & \uparrow^{\phi \mapsto \phi \circ f} \\ X_2 & \mapsto & \text{FunctionsOn}(X_2) \end{array}$$

Since the precomposition operation reverses the direction of morphisms, as shown, these are functors from the given category of spaces to the *opposite* (Example 1.13) of the relevant category of algebras.

In broad generality, there is a duality ("Isbell duality") between geometry/spaces and algebra/algebras of functions) ("space and quantity", Lawvere 86).

We now mention some concrete examples of this general pattern:

topological spaces and C*-algebras

Consider

1. the category Top_{cpt} of compact topological Hausdorff spaces with continuous functions between them;
2. the category C*Alg of unital C*-algebras over the complex numbers

from Example 1.3.

Then there is a functor (Def. 1.15)

$$C(-) : \text{Top}_{H,\text{cpt}} \longrightarrow C^* \text{Alg}^{\text{op}}$$

from the former to the opposite category of the latter (Example 1.13) which sends any compact topological space X to its C*-algebra $C(X)$ of continuous functions $X \xrightarrow{\phi} \mathbb{C}$ with values in the complex numbers, and which sends every continuous function between compact spaces to the C*-algebra-homomorphism that is given by precomposition:

$$C(-) \;:\; \begin{array}{ccc} X & \mapsto & C(X) \\ {}^{f}\big\downarrow & & \big\uparrow^{f^\ast : \phi \mapsto \phi \circ f} \\ Y & \mapsto & C(Y) \end{array}$$

Part of the statement of *Gelfand duality* is that this is a fully faithful functor, hence exhibiting a full subcategory-inclusion (Def. 1.19), namely that of commutative C*-algebras:

$$\mathrm{Top}_{H,\mathrm{cpt}} \hookrightarrow C^\ast\mathrm{Alg}^{\mathrm{op}} \,.$$

affine schemes* and *commutative algebras

The starting point of algebraic geometry is to consider *affine schemes* as the formal duals (Example 1.13) of finitely generated commutative algebras over some algebraically closed ground field $\mathbb{K}$:

$$\mathrm{Aff}_{\mathbb{K}} \;\coloneqq\; \mathrm{CAlg}_{\mathbb{K}}^{\mathrm{fin}\,\mathrm{op}} \,. \tag{2}$$

Beware that the immediate identification (2) is often obscured by the definition of affine schemes as locally ringed spaces. While the latter is much more complicated, at face value, in the end it yields an equivalent category (Def. 1.57 below) to the simple formal dualization (Example 1.13) in (2), see here. Already in 1973 Alexander Grothendieck had urged to abandon, as a foundational concept, the more complicated definition in favor of the simpler one in (2), see Lawvere 03.

smooth manifolds* and *real associative algebras

Consider

1. the category SmthMfd of smooth manifolds with smooth functions between them;
2. the category $\mathrm{Alg}_{\mathbb{R}}$ of associative algebras over the real numbers

from Example 1.3.

Then there is a functor (Def. 1.15)

$$C^\infty(-) \;:\; \mathrm{SmthMfd} \longrightarrow \mathrm{Alg}_{\mathbb{R}}^{\mathrm{op}}$$

from the former to the opposite category of the latter (Def. 1.13), which sends any smooth manifold X to its associative algebra $C^\infty(X)$ of continuous functions $X \overset{\phi}{\to} \mathbb{R}$ to the real numbers, and which sends every smooth function between smooth manifolds to the algebra homomorphism that is given by precomposition:

$$C^\infty(-) \;:\; \begin{array}{ccc} X & \mapsto & C^\infty(X) \\ {}^{\mathllap{f}}\downarrow && \uparrow^{\mathrlap{f^\ast : \phi \mapsto \phi \circ f}} \\ Y & \mapsto & C^\infty(Y) \end{array}$$

The statement of *Milnor's exercise* is that this this is a fully faithful functor, hence exhibiting a full subcategory-inclusion (Def. 1.19):

$$\mathrm{SmthMfd} \hookrightarrow \mathrm{Alg}_{\mathbb{R}}^{\mathrm{op}} \,.$$

These two statements, expressing categories of spaces as full subcategories of opposite categories of categories of algebras, are the starting point for many developments in geometry, such as *algebraic geometry*, *supergeometry*, *noncommutative geometry* and *noncommutative topology*.

Since a fully faithful functor/full subcategory-embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ exhibits the objects of $\mathcal{D}$ as a consistent generalization of the objects of $\mathcal{C}$, one may turn these examples around and *define* more general kinds of spaces as *formal duals* (Example 1.13) to certain algebras:

infinitesimally thickened points* and *formal Cartesian spaces

The category of *infinitesimally thickened points* is, by definition, the full subcategory (Example 1.20) of the opposite category (Example 1.13) of that of commutative algebras (Example 1.3) over the real numbers

$$\begin{array}{ccc} \mathrm{InfThckPoint} & \hookrightarrow & \mathrm{Alg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{D} & \mapsto & C^\infty(\mathbb{D}) \\ && \coloneqq \mathbb{R} \oplus V \end{array}$$

on those with a unique maximal ideal V which is a finite-dimensional as an $\mathbb{R}$-vector space and a nilradical: for each $a \in V$ there exists $n \in \mathbb{N}$ such that $a^n = 0$.

The category of *formal Cartesian spaces* is, by definition, the full subcategory (Example 1.20) of the opposite category (Example 1.13) of that of commutative algebras (Example 1.3) over the real numbers

$$\begin{array}{ccc} \mathrm{FormalCartSp} & \hookrightarrow & \mathrm{Alg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^n \times \mathbb{D} & \mapsto & C^\infty(\mathbb{R}^n \times \mathbb{D}) \\ & & \coloneqq C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus V) \end{array}$$

on those which are tensor products of algebras, of an algebra of smooth functions on a Cartesian space $\mathbb{R}^n$, for some $n \in \mathbb{Z}$, and the algebra of functions on an infinitesimally thickened point.

Notice that the formal Cartesian spaces $\mathbb{R}^{n|q}$ are fully *defined* by this assignment.

super points* and *super Cartesian spaces

The category of ***super points*** is *by definition*, the full subcategory (Example 1.20) of the opposite category (Example 1.13) of that of supercommutative algebras (Example 1.3) over the real numbers

$$\begin{array}{ccc} \mathrm{SuperPoint} & \hookrightarrow & \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^{0|q} & \mapsto & \Lambda_q \end{array}$$

on the Grassmann algebras:

$$\Lambda_q \;\coloneqq\; \mathbb{R}[\theta_1, \cdots, \theta_q] \,/\, (\theta_i \theta_j = -\theta_j \theta_i) \qquad q \in \mathbb{N}\,.$$

More generally, the category of ***super Cartesian spaces*** is *by definition*, the full subcategory

$$\begin{array}{ccc} \mathrm{SuperCartSp} & \hookrightarrow & \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^{n|q} & \mapsto & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Lambda_q \end{array}$$

on the tensor product of algebras, over $\mathbb{R}$ of the algebra of smooth functions on a Cartesian space, and a Grassmann algebra, as above.

Notice that the super Cartesian spaces $\mathbb{R}^{n|q}$ are fully *defined* by this assignment. We discuss this in more detail in the chapter on supergeometry.

Natural transformations and presheaves

Given a system of (homo-)morphisms ("transformations") in some category (Def. 1.1)

$$F_X \xrightarrow{\eta_X} G_X$$

between objects that depend on some variable X, hence that are values of functors of X (Def. 1.15), one says that this is *natural*, hence a *natural transformation* (Def. 1.23 below) if it is compatible with (homo-)morphisms of the variable itself.

These natural transformations are the evident homomorphisms between functors

$$F \xrightarrow{\eta} G\,,$$

and hence there is a *category of functors* between any two categories (Example 1.25 below).

A key class of such functor categories are those between an opposite category $\mathcal{C}^{\mathrm{op}}$ and the base category of sets, these are also called *categories of presheaves* (Example 1.26 below). It makes good sense (Remark 1.28 below) to think of these as categories of "generalized objects of $\mathcal{C}$", a perspective which is made precise by the statement of the *Yoneda lemma* (Prop. 1.29 below) and the resulting *Yoneda embedding* (Prop. 1.30 below). This innocent-looking lemma is the heart that makes category theory tick.

***Definition 1.23*. (natural transformation and natural isomorphism)**

Given two categories $\mathcal{C}$ and $\mathcal{D}$ (Def. 1.1) and given two functors F and G from $\mathcal{C}$ to $\mathcal{D}$ (Def. 1.15), then a *natural transformation* from F to G

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow \eta"] \end{tikzcd}$$

is

- for each object $X \in \mathrm{Obj}_{\mathcal{C}}$ a morphism

$$F(X) \overset{\eta_X}{\longrightarrow} G(X) \qquad (3)$$

such that

- for each morphism $X \xrightarrow{f} Y$ we have a commuting square (Def. 1.4) of the form

$$\eta_Y \circ F(X) \;=\; G(Y) \circ \eta_X \qquad \begin{array}{ccc} F(X) & \overset{\eta_X}{\longrightarrow} & G(X) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(Y) & \underset{\eta_Y}{\longrightarrow} & G(Y) \end{array} \tag{4}$$

(sometimes called the *naturality square* of the natural transformation).

If all the component morphisms η_X are isomorphisms (Def. 1.9), then the natural transformation η is called a *natural isomorphism*.

For

$$\mathcal{C} \underoverset{G}{F}{\Downarrow \eta} \mathcal{D} \quad \text{and} \quad \mathcal{C} \underoverset{H}{G}{\Downarrow \rho} \mathcal{D}$$

two natural transformations as shown, their *composition* is the natural transformation

$$\mathcal{C} \underoverset{H}{F}{\Downarrow \rho \circ \eta} \mathcal{D}$$

whose components (3) are the compositions of the components of η and ρ:

$$(\rho \circ \eta)_X \;\coloneqq\; \rho_X \circ \eta_X \qquad \begin{array}{ccccc} F(X) & \overset{\eta_X}{\longrightarrow} & G(X) & \overset{\rho_X}{\longrightarrow} & H(X) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} && \downarrow^{\mathrlap{H(f)}} \\ F(Y) & \underset{\eta_Y}{\longrightarrow} & G(Y) & \underset{\rho_Y}{\longrightarrow} & H(Y) \end{array} \tag{5}$$

***Example 1.24*. (*reduction* of *formal Cartesian spaces*)**

On the category FormalCartSp of formal Cartesian spaces Example 1.22, consider the endofunctor

$$\begin{array}{ccc} \text{FormalCartSp} & \overset{\Re}{\longrightarrow} & \text{FormalCartSp} \\ \mathbb{R}^n \times \mathbb{D} & \mapsto & \mathbb{R}^n \end{array}$$

which sends each formal Cartesian space to the underlying ordinary Cartesian space, forgetting the infinitesimally thickened point-factor. Moreover, on morphisms this functor is defined via the retraction

$$
\begin{array}{ccccccc}
\mathrm{id}: & \mathbb{R}^n & \overset{i}{\longrightarrow} & \mathbb{R}^n \times \mathbb{D} & \overset{r}{\longrightarrow} & \mathbb{R}^n \\
& C^\infty(\mathbb{R}^n) & \underset{\text{quotient projection}}{\overset{i^*}{\longleftarrow}} & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (R \oplus V) & \underset{f \mapsto f \otimes 1}{\overset{r^*}{\longleftarrow}} & C^\infty(\mathbb{R}^n)
\end{array}
$$

as

$$
\begin{array}{ccccc}
C^\infty(\mathbb{R}^n \times \mathbb{D}) & & C^\infty(\mathbb{R}^n) & \overset{i^*}{\longleftarrow} & C^\infty(\mathbb{R}^n \times \mathbb{D}) \\
{}^{f^*}\uparrow & & {}^{\Re(f^*) \coloneqq i^* \circ f^* \circ r^*}\uparrow & & \uparrow^{f^*} \\
C^\infty(\mathbb{R}^{n'} \times \mathbb{D}') & & C^\infty(\mathbb{R}^{n'}) & \overset{r^*}{\longrightarrow} & C^\infty(\mathbb{R}^{n'} \times \mathbb{D}')
\end{array}
$$

This is indeed functorial due to the fact that any algebra homomorphism f^* needs to send nilpotent elements to nilpotent elements, so that the following identity holds:

$$
i^* \circ f^* \;=\; i^* \circ f^* \circ r^* \circ i^* \,. \qquad (6)
$$

Then there is a natural transformation (Def. 1.23) from this functor to the identity functor

$$
\Re \overset{\eta^{\Re}}{\longrightarrow} \mathrm{Id}
$$

whose components inject the underlying Cartesian space along the unit point inclusion of the infinitesimally thickened point:

$$
\begin{array}{cccc}
\Re(\mathbb{R}^n \times \mathbb{D}) \coloneqq & \mathbb{R}^n & \overset{\eta^{\Re}_{\mathbb{R}^n \times \mathbb{D}}}{\longrightarrow} & \mathbb{R}^n \times \mathbb{D} \\
& C^\infty(\mathbb{R}^n) & \overset{i^*}{\longleftarrow} & C^\infty(\mathbb{R}^n \times \mathbb{D}) \\
& {}^{i^* \circ f^* \circ r^*}\uparrow & & \uparrow^{f^*} \\
& C^\infty(\mathbb{R}^{n'}) & \overset{i^*}{\longleftarrow} & C^\infty(\mathbb{R}^{n'} \times \mathbb{D}')
\end{array}
$$

The commutativity of this naturality square is again the identity (6).

***Example 1.25*. (*functor category*)**

Let $\mathcal{C}$ and $\mathcal{D}$ be categories (Def. 1.1). Then the *category of functors* between them, to be denoted $[\mathcal{C}, \mathcal{D}]$, is the category whose objects are the functors $\mathcal{C} \overset{F}{\to} \mathcal{D}$ (Def. 1.15) and whose morphisms are the natural transformations $F \overset{\eta}{\Rightarrow} G$ between functors (Def. 1.23) and whose composition operation is the composition of natural transformations (5).

***Example 1.26*. (*category of presheaves*)**

Given a category $\mathcal{C}$ (Def. 1.1), a functor (Def. 1.15) of the form

$$F \;:\; \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}\,,$$

hence out of the opposite category of $\mathcal{C}$ (Def. 1.13), into the category of sets (Example 1.2) is also called a *presheaf* (for reasons discussed below) *on* $\mathcal{C}$ or *over* $\mathcal{C}$.

The corresponding functor category (Example 1.25)

$$\mathrm{PSh}(\mathcal{C}) \;\coloneqq\; [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

is hence called the *category of presheaves* over $\mathcal{C}$.

***Example 1.27*. (representable presheaves)**

Given a category $\mathcal{C}$ (Def. 1.1), the hom-functor (Example 1.17) induces the following functor (Def. 1.15) from $\mathcal{C}$ to its category of presheaves (Def. 1.26):

$$y \;:\; \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \tag{7}$$

$$\begin{array}{ccccccc}
 & & & & c_1 & \overset{g}{\longrightarrow} & c_2 \\
X & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-,X) & : & \mathrm{Hom}_{\mathcal{C}}(c_1,X) & \overset{\mathrm{Hom}_{\mathcal{C}}(g,X)}{\longleftarrow} & \mathrm{Hom}_{\mathcal{C}}(c_2,X) \\
{}^{f}\big\downarrow & & \big\downarrow^{\mathrm{Hom}_{\mathcal{C}}(-,f)} & & \big\downarrow^{\mathrm{Hom}_{\mathcal{C}}(c_1,f)} & & \big\downarrow^{\mathrm{Hom}_{\mathcal{C}}(c_2,f)} \\
Y & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-,Y) & : & \mathrm{Hom}_{\mathcal{C}}(c_1,Y) & \overset{\mathrm{Hom}_{\mathcal{C}}(g,Y)}{\longleftarrow} & \mathrm{Hom}_{\mathcal{C}}(c_2,Y)
\end{array}$$

The presheaves $y(X) \coloneqq \mathrm{Hom}_{\mathcal{C}}(-,X)$ in the image of this functor are called the *representable presheaves* and $X \in \mathrm{Obj}_{\mathcal{C}}$ is called their representing object.

The functor (7) is also called the *Yoneda embedding*, due to Prop. 1.30 below.

***Remark 1.28*. (presheaves as generalized spaces)**

If a given category $\mathcal{C}$ (Def. 1.1) is thought of as a category of *spaces* of sorts, as those in Example 1.22, then it will be most useful to think of the corresponding category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Def. 1.26) as a category of *generalized spaces* *probe-able by* the test spaces in $\mathcal{C}$ (Lawvere 86, p. 17).

Namely, imagine a generalized space $\mathbf{X}$ which is at least probe-able by spaces in $\mathcal{C}$. This should mean that for each object $c \in \mathcal{C}$ there is some set of geometric maps "$c \to \mathbf{X}$". Here the quotation marks are to warn us that, *at this point*, $\mathbf{X}$ is not defined yet; and even if it were, it is not expected to be an object of $\mathcal{C}$, so that, at this point, an actual morphism from c to $\mathbf{X}$ is not definable. But we may anyway consider some *abstract set*

$$\mathbf{X}(c) \text{ "=}\mathrm{Hom}(c, \mathbf{X})\text{"} \tag{8}$$

whose elements we do want to think of maps (homomorphisms of spaces) from c to $\mathbf{X}$.

That this is indeed consistent, in that we may actually remove the quotation remarks on the right of (8), is the statement of the *Yoneda lemma*, which we discuss as Prop. 1.29 below.

A minimum consistency condition for this to make sense (we will consider further conditions later on when we discuss *sheaves*) is that we may consistently pre-compose the would-be maps from c to $\mathbf{X}$ with actual morphisms $d \xrightarrow{f} c$ in $\mathcal{C}$. This means that for every such morphism there should be a function between these sets of would-be maps

$$\begin{array}{ccc} c & & \mathbf{X}(c) \\ {}^{f}\downarrow & & \uparrow^{\mathbf{X}(f) \text{ "=}(-)\circ f\text{"}} \\ d & & \mathbf{X}(d) \end{array}$$

which respects composition and identity morphisms. But in summary, this says that what we have defined thereby is actually a *presheaf* on $\mathcal{C}$ (Def. 1.26), namely a functor

$$\mathbf{X} \,:\, \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set} \,.$$

For consistency of regarding this presheaf as a *presheaf of sets of plots of a generalized space*, it ought to be true that every "ordinary space", hence every object $X \in \mathcal{C}$, is also an example of a "generalized space probe-able by" object of $\mathcal{C}$, since, after all, these are the spaces which may manifestly be probed by objects $c \in \mathcal{C}$, in that morphisms $c \to X$ are already defined.

Hence the incarnation of $X \in \mathcal{C}$ as a generalized space probe-able by objects of $\mathcal{C}$ should be the presheaf $\mathrm{Hom}_{\mathcal{C}}(-, X)$, hence the presheaf represented by X (Example 1.27), via the Yoneda functor (7).

At this point, however, a serious consistency condition arises: The "ordinary spaces" now exist as objects of two different categories: on the one hand there is the original $X \in \mathcal{C}$, on the other hand there is its Yoneda image $y(X) \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ in the category of generalized spaces. Hence we need to know that these two perspectives are compatible, notably that maps $X \to Y$ between ordinary spaces are the same whether viewed in $\mathcal{C}$ or in the more general context of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$.

That this, too, holds true, is the statement of the *Yoneda embedding*, which we discuss as Prop. 1.30 below.

Eventually one will want to impose one more consistency condition, namely that plots are determined by their *local behaviour*. This is the *sheaf condition* (Def. 4.8 below) and is what leads over from category theory to topos theory below.

***Proposition 1.29*. (*Yoneda lemma*)**

Let $\mathcal{C}$ be a category (Def. 1.1), $X \in \mathcal{C}$ any object, and $\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ a presheaf over $\mathcal{C}$ (Def. 1.26).

Then there is a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(y(X), (Y)) & \overset{\simeq}{\longrightarrow} & \mathbf{Y}(X) \\ \eta & \mapsto & \eta_X(\mathrm{id}_X) \end{array}$$

between the hom-set of the category of presheaves from the presheaf represented by X (7) to $\mathbf{Y}$, and the set which is assigned by $\mathbf{Y}$ to X.

***Proof*.** By Example 1.25, an element in the set on the left is a natural transformation (Def. 1.23) of the form

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "y(X)"{name=U}] \arrow[r, bend right, "\mathbf{Y}"'{name=D}] & \mathrm{Set} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$$

hence given by component functions (3)

$$\mathrm{Hom}_{\mathcal{C}}(c, X) \xrightarrow{\eta_c} \mathbf{Y}(X)$$

for each $c \in \mathcal{C}$. In particular there is the component at $c = X$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, X) & \overset{\eta_X}{\longrightarrow} & \mathbf{Y}(X) \\ \mathrm{id}_X & \mapsto & \eta_X(\mathrm{id}_X) \end{array}$$

and the identity morphism id_X on X is a canonical element in the set on the left. The statement to be proven is hence equivalently that for every element in $\mathbf{Y}(X)$ there is precisely one η such that this element equals $\eta_X(\mathrm{id}_X)$.

Now the condition to be satisfied by η is that it makes its naturality squares (4) commute (Def. 1.4). This includes those of the form

$$
\begin{array}{ccccccc}
\mathrm{id}_X \in & \mathrm{Hom}_{\mathcal{C}}(X,X) & \overset{\eta_X}{\longrightarrow} & \mathbf{Y}(X) & \{\mathrm{id}_X\} & \longrightarrow & \{\eta_X(\mathrm{id}_X)\} \\
{}^{\mathrm{Hom}_{\mathcal{C}}(f,X)}\downarrow & & & \downarrow^{\mathbf{Y}(f)} & \downarrow & & \downarrow \\
& \mathrm{Hom}_{\mathcal{C}}(Y,X) & \underset{\eta_Y}{\longrightarrow} & \mathbf{Y}(Y) & \{f\} & \longrightarrow & \left\{\eta_Y(f) = \mathbf{Y}(f)(\eta_X(\mathrm{id}_X))\right\}
\end{array}
$$

for any morphism

$$
(Y \overset{f}{\longrightarrow} X) \;\in\; \mathrm{Hom}_{\mathcal{C}}(Y,X)\;.
$$

As the diagram chase of elements on the right shows, this commutativity (Def. 1.4) fixes $\eta_Y(f)$ for all $Y \in \mathcal{C}$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ uniquely in terms of the element $\eta_X(\mathrm{id}_X)$.

It remains only to see that there is no condition on the element $\eta_X(\mathrm{id}_X)$, hence that with $\eta_Y(f)$ defined this way, the commutativity of all the remaining naturality squares is implies: The general naturality square for a morphism $Y_2 \overset{g}{\longrightarrow} Y_1$ is of the form

$$
\begin{array}{cccccccc}
& \mathrm{Hom}_{\mathcal{C}}(Y_1,X) & \overset{\eta_{Y_1}}{\longrightarrow} & \mathbf{Y}(Y_1) & \{f_1\} & \longrightarrow & \{\mathbf{Y}(f_1)(\eta_X(\mathrm{id}_X))\} \\
{}^{\mathrm{Hom}_{\mathcal{C}}(g,X)}\downarrow & & & \downarrow^{\mathbf{Y}(g)} & \downarrow & & \downarrow \\
& \mathrm{Hom}_{\mathcal{C}}(Y_2,X) & \underset{\eta_{Y_2}}{\longrightarrow} & \mathbf{Y}(Y_2) & \{f_2 = f_1 \circ g\} & \longrightarrow & \{\mathbf{Y}(f_2)(\eta_X(\mathrm{id}_X)) = \mathbf{Y}(g) \circ \mathbf{Y}(f_1)(\eta_X
\end{array}
$$

As shown on the right, the commutativity of this diagram now follows from the functoriality $\mathbf{Y}(f_2) = \mathbf{Y}(f_1 \circ g)$ of the presheaf $\mathbf{Y}$. ▮

As a direct corollary, we obtain the statement of the Yoneda embedding:

***Proposition 1.30**. (**Yoneda embedding**)*

> *The assignment (7) of represented presheaves (Example 1.27) is a fully faithful functor (Def. 1.19), hence exhibits a full subcategory inclusion*
>
> $$
> y \;:\; \begin{array}{ccc} \mathcal{C} & \hookrightarrow & [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \\ X & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-,X) \end{array}
> $$
>
> *of the given category $\mathcal{C}$ into its category of presheaves.*

***Proof**.* We need to show that for all $X_1, X_2 \in \mathrm{Obj}_{\mathcal{C}}$ the function

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(X_1,X_2) & \longrightarrow & \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}\big(\mathrm{Hom}_{\mathcal{C}}(-,X_1)\,,\;\mathrm{Hom}_{\mathcal{C}}(-,X_2)\big) \\
f & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-,f)
\end{array}
\qquad (9)
$$

is a bijection. But the Yoneda lemma (Prop. 1.29) states a bijection the other way around

$$\begin{array}{ccc} \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}\big(\mathrm{Hom}_{\mathcal{C}}(-,X_1)\,,\,\mathrm{Hom}_{\mathcal{C}}(-,X_2)\big) & \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(-,X_2)(X_1) & = & \mathrm{Hom}_{\mathcal{C}}(X \\ \eta & \mapsto & & \eta_{X_1}(\mathrm{id} \\ \mathrm{Hom}_{\mathcal{C}}(-,f) & \mapsto & & \mathrm{Hom}_{\mathcal{C}}(X_1,f) \end{array}$$

and hence it is sufficient to see that this is a left inverse to (9). This follows by inspection, as shown in the third line above. ▮

As a direct corollary we obtain the following alternative characterization of isomorphisms, to be compared with the definition of epimorphisms/monomorphisms in Def. 1.18:

***Example 1.31*. (*isomorphism* via *bijection* of *hom-sets*)**

Let $\mathcal{C}$ be a category (Def. 1.1), let $X, Y \in \mathrm{Obj}_X$ be a pair of objects, and let $X \overset{f}{\to} Y \quad \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ be a morphism between them. Then the following are equivalent:

1. $X \overset{f}{\to} Y$ is an isomorphism (Def. 1.9),
2. the hom-functors into and out of f take values in bijections of hom-sets: i.e. for all objects $A \in \mathrm{Obj}_{\mathcal{C}}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(A,f)\;:\;\mathrm{Hom}_{\mathcal{C}}(A,X) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(A,Y)$$

and

$$\mathrm{Hom}_{\mathcal{C}}(f,A)\;:\;\mathrm{Hom}_{\mathcal{C}}(Y,A) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(X,A)$$

Adjunctions

The concepts of categories, functors and natural transformations constitute the "language of categories". This language now allows to formulate the concept of *adjoint functors* (Def. 1.32) and more generally that of *adjunctions* (Def. 1.50 below. This is concept that category theory, as a theory, is all about.

Part of the data involved in an adjunction is its *adjunction unit* and *adjunction counit* (Def. 1.33 below) and depending on their behaviour special cases of adjunctions are identified (Prop. 1.46 below), which we discuss in detail in following sections:

adjunction		unit is iso:

Def. 1.32, Def. 1.50		
		coreflection Def. 1.60
counit is iso:	reflection Def. 1.60	adjoint equivalence Def. 1.56

We now discuss four equivalent definitions of adjoint functors:

1. via hom-isomorphism (Def. 1.32 below);
2. via adjunction unit and -counit satisfying triangle identities (Prop. 1.39);
3. via representing objects (Prop. 1.40);
4. via universal morphisms (Prop. 1.42 below).

Then we discuss some key properties:

1. uniqueness of adjoints (Prop. 1.45 below),
2. epi/mono/iso-characterization of adjunction (co-)units (Prop. 1.46 below).

***Definition 1.32*. (*adjoint functors*)**

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories (Def. 1.1), and let

$$\mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{C}$$

be a pair of functors between them (Def. 1.15), as shown. Then this is called a *pair of adjoint functors* (or an *adjoint pair of functors*) with L *left adjoint* and R *right adjoint*, denoted

$$\mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

if there exists a natural isomorphism (Def. 1.23) between the hom-functors (Example 1.17) of the following form:

$$\mathrm{Hom}_{\mathcal{D}}(L(-), -) \;\simeq\; \mathrm{Hom}_{\mathcal{C}}(-, R(-))\;. \tag{10}$$

This means that for all objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there is a bijection of hom-sets

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(L(c), d) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(c, R(d)) \\
(L(c) \xrightarrow{f} d) & \mapsto & (c \xrightarrow{\widetilde{f}} R(d))
\end{array}
$$

which is natural in c and d. This isomorphism is called the *adjunction hom-isomorphism* and the image $\widetilde{f}$ of a morphism f under this bijections is called the *adjunct* of f. Conversely, f is called the *adjunct* of $\widetilde{f}$.

Naturality here means that for every pair of morphisms $g : c_2 \to c_1$ in $\mathcal{C}$ and $h : d_1 \to d_2$ in $\mathcal{D}$, the resulting square

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(L(c_1), d_1) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c_1, R(d_1)) \\
{}^{\mathrm{Hom}_{\mathcal{D}}(L(g), h)}\big\downarrow & & \big\downarrow^{\mathrm{Hom}_{\mathcal{C}}(g, R(h))} \\
\mathrm{Hom}_{\mathcal{D}}(L(c_2), d_2) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c_2, R(d_2))
\end{array}
\tag{11}
$$

commutes (Def. 1.4), where the vertical morphisms are given by the hom-functor (Example 1.17).

Explicitly, this commutativity, in turn, means that for every morphism $f : L(c_1) \to d_1$ with adjunct $\widetilde{f} : c_1 \to R(d_1)$, the adjunct of the composition is

$$
\widetilde{
\begin{array}{ccc}
L(c_1) & \xrightarrow{f} & d_1 \\
{}^{L(g)}\uparrow & & \downarrow^{h} \\
L(c_2) & & d_2
\end{array}
}
\;=\;
\begin{array}{ccc}
c_1 & \xrightarrow{\widetilde{f}} & R(d_1) \\
{}^{g}\uparrow & & \downarrow^{R(h)} \\
c_2 & & R(d_2)
\end{array}
$$

***Definition 1.33*. (*adjunction unit* and *counit*)**

Given a pair of adjoint functors

$$
\mathcal{D} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}
$$

according to Def. 1.32, one says that

1. for any $c \in \mathcal{C}$ the adjunct of the identity morphism on $L(c)$ is the *unit morphism* of the adjunction at that object, denoted

$$
\eta_c \coloneqq \widetilde{\mathrm{id}_{L(c)}} \;:\; c \longrightarrow R(L(c))
$$

2. for any $d \in \mathcal{D}$ the adjunct of the identity morphism on $R(d)$ is the *counit morphism* of

the adjunction at that object, denoted

$$\epsilon_d \,:\, L(R(d)) \longrightarrow d$$

***Remark 1.34*. (adjoint triples)**

It happens that there are sequences of adjoint functors:

If two functors are adjoint to each other as in Def. 1.32, we also say that we have an *adjoint pair*:

$$L \dashv R\,.$$

It may happen that one functor C participates on the right and on the left of two such adjoint pairs $L \dashv C$ and $C \dashv R$ (not the same "L" and "R" as before!) in which case one may speak of an *adjoint triple*:

$$L \dashv C \dashv R\,. \tag{12}$$

Below in Example 1.52 we identify adjoint triples as *adjunctions of adjunctions*.

Similarly there are adjoint quadruples, etc.

Notice that in the case of an adjoint triple (12), the adjunction unit of $C \dashv R$ and the adjunction counit of $L \dashv C$ (Def. 1.33) provide, for each object X in the domain of C, a diagram

$$L\big(C(X)\big) \xrightarrow{\;\;\epsilon_X\;\;} X \xrightarrow{\;\;\eta_X\;\;} R\big(C(X)\big) \tag{13}$$

which is usefully thought of as exhibiting the nature of X as being in between two *opposite extreme aspects* $L\big(C(X)\big)$ and $R\big(C(X)\big)$ of X. This is illustrated by the following examples, and formalized by the concept of *modalities* that we turn to in Def. 1.62 below.

***Example 1.35*. (floor and ceiling as adjoint functors)**

Consider the canonical inclusion

$$\mathbb{Z}_{\leq} \overset{\iota}{\hookrightarrow} \mathbb{R}_{\leq}$$

of the integers into the real numbers, both regarded as preorders in the standard way ("lower or equal"). Regarded as full subcategory-inclusion (Def. 1.19) of the corresponding thin categories, via Example 1.8, this inclusion functor has both a left and right adjoint functor (Def. 1.32):

- the left adjoint to ι is the ceiling function;

- the right adjoint to ι is the floor function;

forming an adjoint triple (Def. 1.34)

$$\lceil(-)\rceil \;\dashv\; \iota \;\dashv\; \lfloor(-)\rfloor \,. \tag{14}$$

The adjunction unit and adjunction counit express that each real number is in between its "opposite extreme integer aspects" (13) given by floor and ceiling

$$\iota\lfloor x\rfloor \overset{\epsilon_x}{\leq} x \overset{\eta_x}{\leq} \iota\lceil x\rceil \,.$$

Proof. First of all, observe that we indeed have functors (Def. 1.15)

$$\lfloor(-)\rfloor\,,\; \lceil(-)\rceil \;:\; \mathbb{R} \longrightarrow \mathbb{Z}$$

since floor and ceiling preserve the ordering relation.

Now in view of the identification of preorders with thin categories in Example 1.8, the hom-isomorphism (10) defining adjoint functors of the form $\iota \dashv \lfloor(-)\rfloor$ says for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, that we have

$$\underset{\in \mathbb{Z}}{\underbrace{n \leq \lfloor x\rfloor}} \quad\Leftrightarrow\quad \underset{\in \mathbb{R}}{\underbrace{n \leq x}} \,.$$

This is clearly already the defining condition on the floor function $\lfloor x\rfloor$.

Similarly, the hom-isomorphism defining adjoint functors of the form $\lceil(-)\rceil \dashv \iota$ says that for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have

$$\underset{\in \mathbb{Z}}{\underbrace{\lceil x\rceil \leq n}} \quad\Leftrightarrow\quad \underset{\in \mathbb{R}}{\underbrace{x \leq n}} \,.$$

This is evidently already the defining condition on the floor function $\lfloor x\rfloor$.

Notice that in both cases the condition of a *natural isomorphism* in both variables, as required for an adjunction, is automatically satisfied: For let $x \leq x'$ and $n' \leq n$, then naturality as in (11) means, again in view of the identifications in Example 1.8, that

$$\begin{array}{ccc} (n \leq \lfloor x\rfloor) & \Leftrightarrow & (n \leq x) \\ \Downarrow && \Downarrow \\ (n' \leq \lfloor x'\rfloor) & \Leftrightarrow & (n' \leq x') \\ \\ \in \mathbb{Z} && \in \mathbb{R} \end{array}$$

Here the logical implications are equivalently functions between sets that are either empty or singletons. But Functions between such sets are unique, when they exist. ■

***Example 1.36*. (discrete and codiscrete topological spaces)**

Consider the "forgetful functor" $\mathrm{Top} \overset{U}{\longrightarrow} \mathrm{Set}$ from the category Top of topological spaces (Example 1.3) to the category of sets (Def. 1.2) which sends every topological space to its underlying set.

This has

- a left adjoint (Def. 1.32) Disc which equips a set with its discrete topology,
- a right adjoint coDisc which equips a set with the codiscrete topology.

These hence form an adjoint triple (Remark 1.34)

$$\mathrm{Disc} \;\dashv\; U \;\dashv\; \mathrm{coDisc}\,.$$

Hence the adjunction counit of $\mathrm{Disc} \dashv U$ and the adjunction unit of $U \dashv \mathrm{coDisc}$ exhibit every topology on a given set as "in between the opposite extremes" (13) of the discrete and the co-discrete

$$\mathrm{Disc}(U(X)) \overset{\epsilon}{\longrightarrow} X \overset{\eta}{\longrightarrow} \mathrm{coDisc}(U(X))\,.$$

Lemma 1.37*. *(pre/post-composition with (co-)unit followed by adjunct is adjoint functor)
If a functor C is the right adjoint

$$L \dashv C \;:\; \mathcal{C} \underoverset{\underset{C}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{D}$$

in a pair of adjoint functors (Def. 1.32), then its application to any morphism $X \overset{f}{\to} Y \;\;\in \mathcal{C}$ is equal to the joint operation of pre-composition with the $(L \dashv C)$-adjunction counit $\epsilon^\flat_X$ (Def. 1.33), followed by passing to the $(L \dashv C)$-adjunct:

$$C_{X,Y} \;=\; \widetilde{(-)\circ \epsilon^\flat_X}\,.$$

Dually, if C is a left adjoint

$$C \dashv R \;:\; \mathcal{C} \underoverset{\underset{R}{\longleftarrow}}{\overset{C}{\longrightarrow}}{} \mathcal{D}$$

then its action on any morphism $X \overset{f}{\to} Y \;\;\in \mathcal{C}$ equals the joint operation of post-composition *with the* $(C \dashv R)$-*adjunction unit* $\eta^\sharp_Y$ *(Def. 1.33), followed by passing to the* $(C \dashv R)$-*adjunct:*

$$\widetilde{\eta^\sharp_Y \circ (-)} \;=\; C_{X,Y} \,.$$

In particular, if C *is the middle functor in an adjoint triple (Remark 1.34)*

$$L \dashv C \dashv R \;:\; \mathcal{C} \underoverset{\underset{R}{\longleftarrow}}{\overset{L}{\longleftarrow}}{\xrightarrow{\;\;C\;\;}} \mathcal{D}$$

then these two operations coincide:

$$\widetilde{\eta^\sharp_Y \circ (-)} \;=\; C_{X,Y} \;=\; \widetilde{(-) \circ \epsilon^\flat_X} \,. \qquad (15)$$

Proof. For the first equality, consider the following naturality square (4) for the adjunction hom-isomorphism (10):

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}\big(C(X),\, C(X)\big) & \overset{\widetilde{(-)}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}\big(LC(X),\, X\big) \\ {}^{\mathllap{\big(C(\mathrm{id}_X),\, C(f)\big)}}\big\downarrow && \big\downarrow^{\mathrlap{\mathrm{Hom}_X\big(LC(\mathrm{id}_X),\, f\big)}} \\ \mathrm{Hom}_{\mathcal{D}}\big(C(X),\, C(Y)\big) & \overset{\widetilde{(-)}}{\longleftarrow} & \mathrm{Hom}_{\mathcal{C}}\big(LC(X),\, Y\big) \end{array} \qquad \begin{array}{ccc} \{CX \overset{\mathrm{id}_{CX}}{\longrightarrow} CX\} & \longrightarrow & \{LCX \overset{\epsilon^\flat_X}{\to} X\} \\ \big\downarrow && \big\downarrow \\ \{CX \overset{C(f)}{\longrightarrow} C(Y)\} & \longleftarrow & \{LCX \overset{f \circ \epsilon^\flat_X}{\longrightarrow} Y\} \end{array}$$

Chasing the identity morphism id_{CY} through this diagram yields the claimed equality, as shown on the right. Here we use that the left adjunct? of the identity morphism is the adjunction counit, as shown.

The second equality is fomally dual:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}\big(CY,\, CY\big) & \overset{\widetilde{(-)}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}\big(Y,\, RCY\big) \\ {}^{\mathllap{{}_{\mathcal{D}}\big(C(f), C(\mathrm{id}_Y)\big)}}\big\downarrow && \big\downarrow^{\mathrlap{\mathrm{Hom}_{\mathcal{C}}\big(f, RC(\mathrm{id}_Y)\big)}} \\ \mathrm{Hom}_{\mathcal{D}}\big(C(X),\, C(Y)\big) & \overset{\widetilde{(-)}}{\longleftarrow} & \mathrm{Hom}_{\mathcal{C}}\big(X, RC(Y)\big) \end{array} \qquad \begin{array}{ccc} \{CY \overset{\mathrm{id}_{CY}}{\longrightarrow} CY\} & \longrightarrow & \{Y \overset{\eta^\sharp_Y}{\to} RCY\} \\ \big\downarrow && \big\downarrow \\ \{CX \overset{C(f)}{\longrightarrow} CY\} & \longleftarrow & \{X \overset{\eta^\sharp_Y \circ f}{\longrightarrow} RCY\} \end{array}$$

■

We now turn to a ***sequence of equivalent reformulations*** of the condition of adjointness.

Proposition 1.38. (general adjuncts in terms of unit/counit)

Consider a pair of adjoint functors

$$\mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

according to Def. 1.32, with adjunction units η_c *and adjunction counits* ϵ_d *according to Def. 1.38.*

Then

1. *The adjunct* $\widetilde{f}$ *of any morphism* $L(c) \overset{f}{\to} d$ *is obtained from* R *and* η_c *as the composite*

   $$\widetilde{f} \;:\; c \overset{\eta_c}{\longrightarrow} R(L(c)) \overset{R(f)}{\longrightarrow} R(d) \qquad (16)$$

   Conversely, the adjunct f *of any morphism* $c \overset{\widetilde{f}}{\longrightarrow} R(d)$ *is obtained from* L *and* ϵ_d *as*

   $$f \;:\; L(c) \overset{L(\widetilde{f})}{\longrightarrow} R(L(d)) \overset{\epsilon_d}{\longrightarrow} d \qquad (17)$$

2. *The adjunction units* η_c *and adjunction counits* ϵ_d *are components of natural transformations of the form*

   $$\eta \;:\; \mathrm{Id}_{\mathcal{C}} \Rightarrow R \circ L$$

   and

   $$\epsilon \;:\; L \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

3. *The adjunction unit and adjunction counit satisfy the triangle identities, saying that*

   $$\mathrm{id}_{L(c)} \;:\; L(c) \overset{L(\eta_c)}{\longrightarrow} L(R(L(c))) \overset{\epsilon_{L(c)}}{\longrightarrow} L(c) \qquad (18)$$

   and

   $$\mathrm{id}_{R(d)} \;:\; R(d) \overset{\eta_{R(d)}}{\longrightarrow} R(L(R(d))) \overset{R(\epsilon_d)}{\longrightarrow} R(d)$$

Proof. For the first statement, consider the naturality square (11) in the form

$$
\begin{array}{ccccc}
\mathrm{id}_{L(c)} \in & \mathrm{Hom}_{\mathcal{D}}(L(c), L(c)) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c, R(L(c))) \\
& {}^{\mathrm{Hom}_{\mathcal{D}}(L(\mathrm{id}), f)}\downarrow && \downarrow^{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}, R(f))} \\
& \mathrm{Hom}_{\mathcal{D}}(L(c), d) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c, R(d))
\end{array}
$$

and consider the element $\mathrm{id}_{L(c_1)}$ in the top left entry. Its image under going down and then right in the diagram is $\widetilde{f}$, by Def. 1.32. On the other hand, its image under going right and then down is $R(f) \circ \eta_c$, by Def. 1.33. Commutativity of the diagram means that these two morphisms agree, which is the statement to be shown, for the adjunct of f.

The converse formula follows analogously.

The third statement follows directly from this by applying these formulas for the adjuncts twice and using that the result must be the original morphism:

$$
\begin{aligned}
\mathrm{id}_{L(c)} & = \widetilde{\widetilde{\mathrm{id}_{L(c)}}} \\
& = \overline{c \overset{\eta_c}{\to} R(L(c))} \\
& = L(c) \overset{L(\eta_c)}{\longrightarrow} L(R(L(c))) \overset{\epsilon_{L(c)}}{\longrightarrow} L(c)
\end{aligned}
$$

For the second statement, we have to show that for every moprhism $f : c_1 \to c_2$ the following square commutes:

$$
\begin{array}{ccc}
c_1 & \overset{f}{\longrightarrow} & c_2 \\
{}^{\eta_{c_1}}\downarrow && \downarrow^{\eta_{c_2}} \\
R(L(c_1)) & \underset{R(L(f))}{\longrightarrow} & R(L(c_2))
\end{array}
$$

To see this, consider the naturality square (11) in the form

$$
\begin{array}{ccccc}
\mathrm{id}_{L(c_2)} \in & \mathrm{Hom}_{\mathcal{D}}(L(c_2), L(c_2)) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c_2, R(L(c_2))) \\
& {}^{\mathrm{Hom}_{\mathcal{D}}(L(f), \mathrm{id}_{L(c_2)})}\downarrow && \downarrow^{\mathrm{Hom}_{\mathcal{C}}(f, R(\mathrm{id}_{L(c_2)}))} \\
& \mathrm{Hom}_{\mathcal{D}}(L(c_1), L(c_2)) & \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} & \mathrm{Hom}_{\mathcal{C}}(c_1, R(L(c_1)))
\end{array}
$$

The image of the element $\mathrm{id}_{L(c_2)}$ in the top left along the right and down is $f \circ \eta_{c_2}$, by Def. 1.33, while its image down and then to the right is $\widetilde{L(f)} = R(L(f)) \circ \eta_{c_1}$, by the previous statement. Commutativity of the diagram means that these two morphisms agree, which is the statement to be shown.

The argument for the naturality of ϵ is directly analogous. ■

***Proposition 1.39**. (**adjoint functors** equivalent to **adjunction** in **Cat**)*

Two functors

$$\mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{C}$$

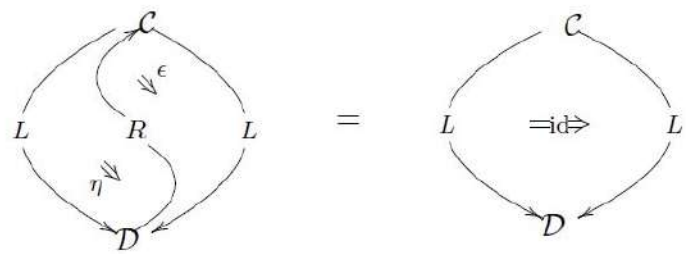
are an adjoint pair in the sense that there is a natural isomorphism (10) according to Def. 1.32, precisely if they participate in an adjunction in the 2-category Cat, meaning that

1. *there exist natural transformations*

   $$\eta \,:\, \mathrm{Id}_{\mathcal{C}} \Rightarrow R \circ L$$

   and

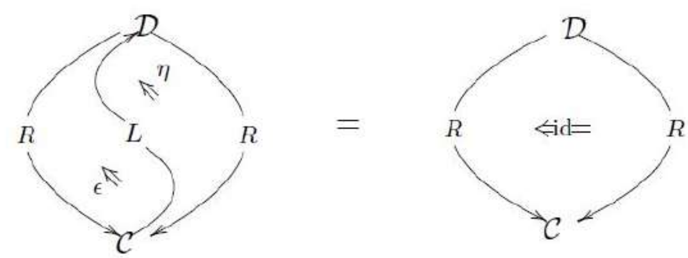
   $$\epsilon \,:\, L \circ R \Rightarrow \mathrm{Id}_{\mathcal{D}}$$



2. *which satisfy the triangle identities*

   $$\mathrm{id}_{L(c)} \,:\, L(c) \xrightarrow{L(\eta_c)} L(R(L(c))) \xrightarrow{\epsilon_{L(c}}$$

   and



$$\mathrm{id}_{R(d)} \,:\, R(d) \xrightarrow{\eta_{R(d)}} R(L(R(d))) \xrightarrow{R(\epsilon_d)} R(d)$$

Proof. That a hom-isomorphism (10) implies units/counits satisfying the triangle identities is the statement of the second two items of Prop. 1.38.

Hence it remains to show the converse. But the argument is along the same lines as the proof of Prop. 1.38: We now *define* forming of adjuncts by the formula (16). That the resulting assignment $f \mapsto \widetilde{f}$ is an isomorphism follows from the computation

$$
\begin{aligned}
\widetilde{\widetilde{f}} &= \widetilde{c \overset{\eta_c}{\to} R(L(c)) \overset{R(f)}{\longrightarrow} R(d)} \\
&= L(c) \overset{L(\eta_c)}{\longrightarrow} L(R(L(c))) \overset{L(R(f))}{\longrightarrow} L(R(d)) \overset{\epsilon_d}{\to} d \\
&= L(c) \overset{L(\eta_c)}{\longrightarrow} L(R(L(c))) \overset{\epsilon_{L(c)}}{\longrightarrow} L(c) \overset{f}{\longrightarrow} d \\
&= L(c) \overset{f}{\longrightarrow} d
\end{aligned}
$$

where, after expanding out the definition, we used naturality of ϵ and then the triangle identity.

Finally, that this construction satisfies the naturality condition (11) follows from the functoriality of the functors involved, and the naturality of the unit/counit:

$$
\begin{tikzcd}
c_2 \arrow[r, "\eta_{c_2}"] \arrow[d, "g"'] & R(L(c_2)) \arrow[d, "R(L(g))"] \arrow[dr, "R(L(g)\circ f)"] & \\
c_1 \arrow[r, "\eta_{c_1}"] & R(L(c_1)) \arrow[r, "R(f)"] \arrow[dr, "R(h\circ f)"'] & R(d_1) \arrow[d, "R(h)"] \\
& & R(d_2)
\end{tikzcd}
$$

■

The condition (10) on adjoint functors $L \dashv R$ in Def. 1.32 implies in particular that for every object $d \in \mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(L(-), d)$ is a *representable functor* with *representing object* $R(d)$. The following Prop. 1.40 observes that the existence of such representing objects for all d is, in fact, already sufficient to imply that there is a right adjoint functor.

This equivalent perspective on adjoint functors makes manifest that adjoint functors are, if they exist, unique up to natural isomorphism, this is Prop. 1.45 below.

***Proposition 1.40**. (adjoint functor from objectwise representing objects)*

A functor $L : \mathcal{C} \longrightarrow \mathcal{D}$ has a right adjoint $R : \mathcal{D} \to \mathcal{C}$, according to Def. 1.32, already if for all objects $d \in \mathcal{D}$ there is an object $R(d) \in \mathcal{C}$ such that there is a natural isomorphism

$$
\mathrm{Hom}_{\mathcal{D}}(L(-), d) \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} \mathrm{Hom}_{\mathcal{C}}(-, R(d)) \,,
$$

hence for each object $c \in \mathcal{C}$ a bijection

$$
\mathrm{Hom}_{\mathcal{D}}(L(c), d) \underset{\simeq}{\overset{\widetilde{(-)}}{\longrightarrow}} \mathrm{Hom}_{\mathcal{C}}(c, R(d))
$$

such that for each morphism $g : c_2 \to c_1$, the following diagram commutes

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(c_1), d) \arrow[r, "\widetilde{(-)}", "\simeq"'] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(L(g), \mathrm{id}_d)"'] & \mathrm{Hom}_{\mathcal{C}}(c_1, R(d)) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f, \mathrm{id}_{R(d)})"] \\
\mathrm{Hom}_{\mathcal{D}}(L(c_2), d) \arrow[r, "\widetilde{(-)}", "\simeq"'] & \mathrm{Hom}_{\mathcal{C}}(c_2, R(d))
\end{tikzcd}
\tag{19}
$$

(This is as in (11), except that only naturality in the first variable is required.)

In this case there is a unique way to extend R from a function on objects to a function on morphisms such as to make it a functor $R : \mathcal{D} \to \mathcal{C}$ which is right adjoint to L. , and hence the statement is that with this, naturality in the second variable is already implied.

Proof. Notice that

1. in the language of presheaves (Example 1.26) the assumption is that for each $d \in \mathcal{D}$ the presheaf

$$\mathrm{Hom}_{\mathcal{D}}(L(-), d) \;\in\; [\mathcal{D}^{\mathrm{op}}, \mathrm{Set}]$$

   is represented (7) by the object $R(d)$, and naturally so.

2. In terms of the Yoneda embedding (Prop. 1.30)

$$y \,:\, \mathcal{D} \hookrightarrow [\mathcal{D}^{\mathrm{op}}, \mathrm{Set}]$$

   we have

$$\mathrm{Hom}_{\mathcal{C}}(-, R(d)) = y(R(d)) \tag{20}$$

The condition (11) says equivalently that R has to be such that for all morphisms $h : d_1 \to d_2$ the following diagram in the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ commutes

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(-), d_1) \arrow[r, "\widetilde{(-)}", "\simeq"'] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(L(-), h)"'] & \mathrm{Hom}_{\mathcal{C}}(-, R(d_1)) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-, R(h))"] \\
\mathrm{Hom}_{\mathcal{D}}(L(-), d_2) \arrow[r, "\widetilde{(-)}", "\simeq"'] & \mathrm{Hom}_{\mathcal{C}}(-, R(d_2))
\end{tikzcd}
$$

This manifestly has a unique solution

$$y(R(h)) \;=\; \mathrm{Hom}_{\mathcal{C}}(-, R(h))$$

for every morphism $h: d_1 \to d_2$ under $y(R(-))$ (20). But the Yoneda embedding y is a fully faithful functor (Prop. 1.30), which means that thereby also $R(h)$ is uniquely fixed. ▮

We consider one more equivalent characterization of adjunctions:

***Definition 1.41*. (universal morphism)**

Let $\mathcal{C}, \mathcal{D}$ be two categories (Def. 1.1) and let $R : \mathcal{D} \to \mathcal{C}$ be a functor (Def. 1.15)

Then for $c \in \mathcal{C}$ an object, a *universal morphism from c to R* is

1. an object $L(c) \in \mathcal{D}$,
2. a morphism $\eta_c : c \to R(L(c))$, to be called the *unit*,

such that for any $d \in \mathcal{D}$, any morphism $f: c \to R(d)$ factors through this unit η_c as

$$f \;=\; R(\widetilde{f}) \circ \eta_c \qquad \begin{array}{ccccc} & & c & & \\ & {}^{\eta_c}\swarrow & & \searrow^{f} & \\ R(L(c)) & & \underset{R(\widetilde{f})}{\longrightarrow} & & R(d) \\ \\ L(c) & & \underset{\widetilde{f}}{\longrightarrow} & & d \end{array} \tag{21}$$

for a *unique* morphism $\widetilde{f} : L(c) \longrightarrow d$, to be called the adjunct of f.

***Proposition 1.42*. (collection of universal morphisms equivalent to adjoint functor)**

Let $R : \mathcal{D} \to \mathcal{C}$ be a functor (Def. 1.15). Then the following are equivalent:

1. *R has a left adjoint functor $L: \mathcal{C} \to \mathcal{D}$ according to Def. 1.32.*
2. *For every object $c \in \mathcal{C}$ there is a universal morphism $c \overset{\eta_c}{\longrightarrow} R(L(c))$, according to Def. 1.41.*

***Proof*.** In one direction, assume a left adjoint L is given. Define the would-be universal arrow at $c \in \mathcal{C}$ to be the unit of the adjunction η_c via Def. 1.33. Then the statement that this really is a universal arrow is implied by Prop. 1.38.

In the other direction, assume that universal arrows η_c are given. The uniqueness clause in Def. 1.41 immediately implies bijections

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(L(c), d) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(c, R(d)) \\ \left(L(c) \overset{\widetilde{f}}{\to} d \right) & \mapsto & \left(c \overset{\eta_c}{\to} R(L(c)) \overset{R(\widetilde{f})}{\longrightarrow} R(d) \right) \end{array}$$

Hence to satisfy (10) it remains to show that these are natural in both variables. In fact, by Prop. 1.40 it is sufficient to show naturality in the variable d. But this is immediate from the functoriality of R applied in (21): For $h : d_1 \to d_2$ any morphism, we have

$$\begin{tikzcd} & c \arrow[dl, "\eta_c"'] \arrow[dr, "f"] & \\ R(L(c)) \arrow[rr, "R(\widetilde{f})"'] \arrow[drr, "R(h\circ\widetilde{f})"'] & & R(d_1) \arrow[d, "R(h)"] \\ & & R(d_2) \end{tikzcd}$$

■

The following equivalent formulation (Prop. 1.44) of universal morphisms is often useful:

Example 1.43*. *(comma category)

Let $\mathcal{C}$ be a category, let $c \in \mathcal{C}$ be any object, and let $F : \mathcal{D} \to \mathcal{C}$ be a functor.

1. The *comma category* c / F is the category whose objects are pairs consisting of an object $d \in \mathcal{D}$ and morphisms $X \overset{f}{\to} F(d)$ in $\mathcal{C}$, and whose morphisms $(d_1, X_1, f_1) \to (d_2, X_2, f_2)$ are the morphisms $X_1 \overset{g}{\longrightarrow} X_2$ in $\mathcal{C}$ that make a commuting triangle (Def. 1.4):

$$f_2 \circ F(g) \;=\; f_1 \qquad \begin{tikzcd} X_1 \arrow[rr, "g"] & & X_2 \\ F(X_1) \arrow[rr, "F(g)"] \arrow[dr, "f_1"'] & & F(X_2) \arrow[dl, "f_2"] \\ & c & \end{tikzcd}$$

There is a canonical functor

$$F / c \;\longrightarrow\; \mathcal{D} \,.$$

2. The *comma category* F / c is the category whose objects are pairs consisting of an object $d \in \mathcal{D}$ and a morphism $F(d) \overset{f}{\to} X$ in $\mathcal{C}$, and whose morphisms $(d_1, X_1, f_1) \to (d_2, X_2, f_2)$ are the morphisms $X_1 \overset{g}{\longrightarrow} X_2$ in $\mathcal{C}$ that make a commuting triangle (Def. 1.4):

$$f_2 \circ F(g) \;=\; f_1 \qquad \begin{tikzcd} & c \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ F(X_1) \arrow[rr, "F(g)"'] & & F(X_2) \\ X_1 \arrow[rr, "g"'] & & X_2 \end{tikzcd}$$

Again, there is a canonical functor

$$c \,/\, F \;\longrightarrow\; \mathcal{D}] \tag{22}$$

With this definition, the following is evident:

***Proposition 1.44*. (universal morphisms are initial objects in the comma category)**

Let $\mathcal{C} \overset{R}{\longrightarrow} \mathcal{D}$ be a functor and $d \in \mathcal{D}$ an object. Then the following are equivalent:

1. *$d \overset{\eta_d}{\to} R(c)$ is a universal morphism into $R(c)$ (Def. 1.41);*
2. *(d, η_d) is the initial object (Def. 1.5) in the comma category $d \,/\, R$ (Example 1.43).*

After these equivalent characterizations of adjoint functors, we now consider some of their main properties:

***Proposition 1.45*. (adjoint functors are unique up to natural isomorphism)**

The left adjoint or right adjoint to a functor (Def. 1.32), if it exists, is unique up to natural isomorphism (Def. 1.23).

***Proof*.** Suppose the functor $L : \mathcal{D} \to \mathcal{C}$ is given, and we are asking for uniqueness of its right adjoint, if it exists. The other case is directly analogous.

Suppose that $R_1, R_2 \,:\, \mathcal{C} \to \mathcal{D}$ are two functors which both are right adjoint to L. Then for each $d \in \mathcal{D}$ the corresponding two hom-isomorphisms (10) combine to say that there is a natural isomorphism/

$$\Phi_d \;:\; \mathrm{Hom}_{\mathcal{C}}(-, R_1(d)) \;\simeq\; \mathrm{Hom}_{\mathcal{C}}(-, R_2(d))$$

As in the proof of Prop. 1.40, the Yoneda lemma implies that

$$\Phi_d \;=\; y(\phi_d)$$

for some isomorphism

$$\phi_d \;:\; R_1(d) \overset{\simeq}{\to} R_2(d) \;.$$

But then the uniqueness statement of Prop. 1.40 implies that the collection of these isomorphisms for each object constitues a natural isomorphism between the functors (Def. 1.23). ▮

Proposition 1.46. (characterization of epi/mono/iso (co-)unit of adjunction)

Let

$$L \dashv R \;:\; \mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

be a pair of adjoint functors (Def. 1.32).

Recall the definition of

1. *adjunction unit/counit, from Def. 1.33)*
2. *faithful/fully faithful functor from Def. 1.19*
3. *mono/epi/isomorphism from Def. 1.9 and Def. 1.18.*

The following holds:

- *R is faithful precisely if all components of the counit are epimorphisms $LR(c) \underset{\text{epi}}{\overset{\eta_c}{\longrightarrow}} c$;*
- *L is faithful precisely if all components of the unit are monomorphisms $d \underset{\text{mono}}{\overset{\eta_d}{\longrightarrow}} RL(d)$*
- *R is full and faithful (exhibits a reflective subcategory, Def. 1.60) precisely if all components of the counit are isomorphisms $LR(c) \underset{\text{iso}}{\overset{\eta_c}{\longrightarrow}} c$*
- *L is full and faithful (exhibits a coreflective subcategory, def. 1.60) precisely if all component of the unit are isomorphisms $d \underset{\text{iso}}{\overset{\eta_d}{\longrightarrow}} RL(d)$.*

Proof. This follows directly by Lemma 1.37, using the definition of epi/monomorphism (Def. 1.18) and the characterization of isomorphism from Example 1.31. ▮

To complete this pattern, we will see below in Prop. 1.58 that following are equivalent:

- the unit and counit are both natural isomorphism, hence L and R are both fully faithful;
- L is an equivalence (Def. 1.57);

- R is an equivalence (Def. 1.57)
- $L \dashv R$ is an adjoint equivalence (Def. 1.56).

***Proposition 1.47*. (right/left adjoint functors preserve monomorphism/epimorphisms and terminal/initial objects)**

Every right adjoint functor (Def. 1.32) preserves

1. *terminal objects (Def. 1.5),*
2. *monomorphisms (Def. 1.18)*

Every left adjoint functor (Def. 1.32) preserves

1. *initial objects (Def. 1.5),*
2. *epimorphisms (Def. 1.18).*

***Proof*.** This is immediate from the adjunction hom-isomorphism (10), but we spell it out:

We consider the first case, the second is formally dual (Example 1.13). So let $R : \mathcal{C} \to \mathcal{D}$ be a right adjoint functor with left adjoint L.

Let $* \in \mathcal{C}$ be a terminal object (Def. 1.5). We need to show that for every object $d \in \mathcal{D}$ the hom-set $\mathrm{Hom}_{\mathcal{D}}(d, R(*)) \simeq *$ is a singleton. But by the hom-isomorphism (10) we have a bijection

$$\begin{aligned} \mathrm{Hom}_{d}(d, R(*)) & \simeq \mathrm{Hom}_{\mathcal{C}}(L(d), *) \\ & \simeq * , \end{aligned}$$

where in the last step we used that * is a terminal object, by assumption.

Next let $c_1 \overset{f}{\hookrightarrow} c_2$ be a monomorphism. We need to show that for $d \in \mathcal{D}$ any object, the hom-functor out of d yields a monomorphism

$$\mathrm{Hom}_{\mathcal{D}}(d, R(f)) \;:\; \mathrm{Hom}_{\mathcal{D}}(d, R(c_1)) \hookrightarrow \mathrm{Hom}_{\mathcal{D}}(d, R(c_2)) \;.$$

Now consider the following naturality square (11) of the adjunction hom-isomorphism (10):

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(d, R(c_1)) & \simeq & \mathrm{Hom}_{\mathcal{C}}(L(d), c_1) \\ {}^{\mathllap{\mathrm{Hom}_{\mathcal{D}}(d, R(f))}}\big\downarrow && \big\downarrow^{\mathrlap{\mathrm{Hom}_{\mathcal{C}}(L(d), f)}}_{\mathrlap{\mathrm{mono}}} \\ \mathrm{Hom}_{\mathcal{D}}(d, R(c_2)) & \simeq & \mathrm{Hom}_{\mathcal{C}}(L(d), c_2) \end{array}$$

Here the right vertical function is an injective function, by assumption on f and the definition of monomorphism. Since the two horizontal functions are bijections, this implies that also $\mathrm{Hom}_d(d, R(f))$ is an injection. ▮

But the main preservation property of adjoint functors is that *adjoints preserve (co-)limits*. This we discuss as Prop. 3.8 below, after introducing limits and colimits in Def. 3.1 below.

Prop. 1.39 says that adjoint functors are equivalenty "adjunctions in Cat", as defined there. This is a special case of a general more abstract concept of adjunction, that is useful:

***Definition 1.48*. (*strict 2-category*)**

A *strict category* $\mathcal{C}$ is

1. a class $\mathrm{Obj}_{\mathcal{C}}$, called the *class of objects*;
2. for each pair $X, Y \in \mathrm{Obj}_{\mathcal{C}}$ of objects, a small category $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Cat}$ (Def. 1.6), called the *hom-category from X to Y*.
   We denote the objects of this hom-category by arrows like this:

   $$X \overset{f}{\longrightarrow} Y \quad \in \mathrm{Obj}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$$

   and call them the *1-morphisms* of $\mathcal{C}$,
   and we denote the morphisms in the hom-category by double arrows, like this:

   $$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \phi"] & Y \end{tikzcd}$$

   and call these the *2-morphisms* of $\mathcal{C}$;
3. for each object $X \in \mathrm{Obj}_{\mathcal{C}}$ a 1-morphism

   $$X \overset{\mathrm{id}_X}{\longrightarrow} X \quad \in \mathrm{Hom}_{\mathcal{C}}(X, X)$$

   called the *identity morphism* on X;
4. for each triple $X_1, X_2, X_3 \in \mathrm{Obj}$ of objects, a functor (Def. 1.15)

   $$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) & \times & \mathrm{Hom}_{\mathcal{C}}(X_2, X_3) & \overset{\circ_{X_1, X_2, X_3}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_3) \\ X_1 \overset{f}{\to} X_2 & , & X_2 \overset{f}{\to} X_3 & \mapsto & X_1 \overset{g \circ f}{\longrightarrow} X_3 \end{array}$$

   from the product category (Example 1.14) of hom-categories, called *composition*;

such that:

1. for all pairs of objects $X, Y \in \mathrm{Obj}_{\mathcal{C}}$ unitality holds:
   the functors of composition with identity morphisms are identity functors

$$(-) \circ \mathrm{id}_X \;=\; \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \qquad \mathrm{id}_Y \circ (-) \;=\; \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$$

2. for all quadruples of objects $X_1, X_2, X_3, X_4 \in \mathrm{Obj}_{\mathcal{C}}$ composition satisfies *associativity*, in that the following two composite functors are equal:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X_1,X_2) \times \mathrm{Hom}_{\mathcal{C}}(X_2,X_3) \times \mathrm{Hom}_{\mathcal{C}}(X_3,X_4) & \overset{((-)\circ(-))\circ(-)}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X_1,X_3) \times \mathrm{H} \\ {}^{(-)\circ((-)\circ(-))}\big\downarrow & & \big\downarrow {}^{(-)\circ(} \\ \mathrm{Hom}_{\mathcal{C}}(X_1,X_2) \times \mathrm{Hom}_{\mathcal{C}}(X_2,X_4) & \underset{(-)\circ(-)}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X_1, \end{array}$$

The archetypical example of a strict 2-category is the category of categories:

***Example 1.49**. (2-category of categories)*

There is a strict 2-category (Def. 1.48) Cat whose

- objects are small categories (Def. 1.6);
- 1-morphisms are functors (Def. 1.15);
- 2-morphisms are natural transformations (Def. 1.23)

with the evident composition operations.

With a concept of 2-category in hand, we may phrase Prop. 1.39 more abstractly:

***Definition 1.50**. (adjunction in a 2-category)*

Let $\mathcal{C}$ be a strict 2-category (Def. 1.48). Then an *adjunction* in $\mathcal{C}$ is

1. a pair of objects $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}_{\mathcal{C}}$;
2. 1-morphisms

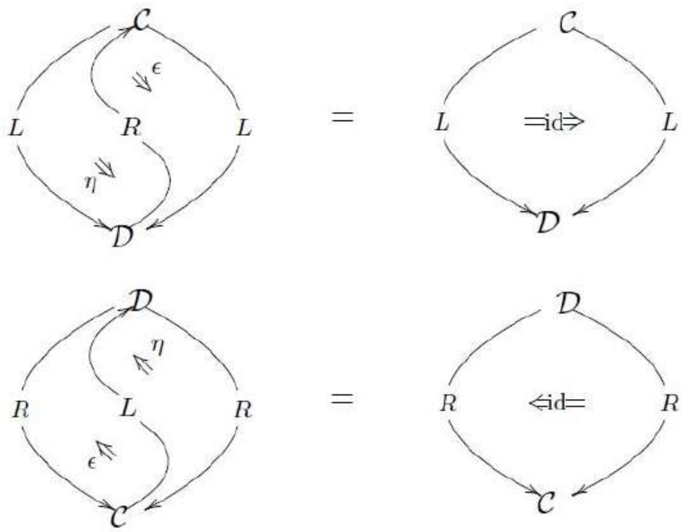
$$\mathcal{D} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{C}$$

   called the *left adjoint* L and *right adjoint* R;

3. 2-morphisms
   $\mathrm{id}_{\mathcal{C}} \overset{\eta}{\Rightarrow} R \circ L$, called the adjunction unit

$L \circ R \overset{\epsilon}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$, called the adjunction counit

such that the following *triangle identities* hold:



We denote this situation by

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "L"']
\end{tikzcd}
$$

Hence via Example 1.49, Prop. 1.39 says that an adjoint pair of functors is equivalente an adjunction in the general sense of Def. 1.50, realized in the very large strict 2-category Cat of categories.

This more abstract perspecive on adjunctions allow us now to understand "duality of dualities" as adjunction in a 2-category of adjunctions:

***Example 1.51*. (*strict 2-category of categories with adjoint functors between them*)**

Let $\mathrm{Cat}_{\mathrm{Adj}}$ be the strict 2-category which is defined just as Cat (Def. 1.49) but with the 1-morphisms being functors that are required to be left adjoints (Def. 1.32).

Since adjoints are unique up to natural isomorphism (Prop. 1.45), this may be thought of as a 2-category whose 1-morphisms are adjoint pairs of functors.

***Example 1.52*. (*adjunctions of adjoint pairs are adjoint triples*)**

An adjunction (Def. 1.50) in the 2-category $\mathrm{Cat}_{\mathrm{Adj}}$ of categories with adjoint functors between them (Example 1.51) is equivalently an adjoint triple of functors (Remark 1.34):

The adjunction says that two left adjoint functors L_1 and L_2, which, hence each participate in an adjoint pair

$$L_1 \dashv R_1 \qquad L_2 \dashv R_2$$

form themselves an adjoint pair

$$L_1 \dashv L_2 \ .$$

By essentiall uniqueness of adjoints (Prop. 1.45) this implies a natural isomorphism $R_1 \simeq L_2$ and hence an adjoint triple:

$$\mathcal{D} \begin{array}{c} \overset{L_1}{\longleftarrow} \\ \bot \\ \overset{R_1 \simeq L_2}{\longrightarrow} \\ \bot \\ \overset{R_2}{\longleftarrow} \end{array} \mathcal{C}$$

Example 1.52 suggest to consider a slight variant of the concept of strict 2-categories which allows to make the duality between left adjoints and right adjoints explicit:

***Definition 1.53*. (*double category*)**

A *double category* $\mathcal{C}$ is

1. a pair of categories $\mathcal{C}_h$, $\mathcal{C}_v$ (Def. 1.1) which share the same class of objects: $\mathrm{Obj}_{\mathcal{C}_1} = \mathrm{Obj}_{\mathcal{C}_2}$, to be called the class $\mathrm{Obj}_{\mathcal{C}}$ of *objects of* $\mathcal{C}$
   where the morphisms of $\mathcal{C}_h$ are to be called the *horizontal morphisms* of $\mathcal{C}$,
   while the morphisms of $\mathcal{C}_v$ are to be called the *vertical morphisms* of $\mathcal{C}$,
2. for each quadruple of objects $a, b, c, d, e \in \mathrm{Obj}_{\mathcal{C}}$ and pairs of pairs of horizontal/vertical morphisms of the form

   $$\begin{array}{ccc} a & \overset{f \in \mathcal{C}_h}{\longrightarrow} & b \\ {}^{\mathllap{h \in \mathcal{C}_v}}\downarrow && \downarrow^{\mathrlap{k \in \mathcal{C}_v}} \\ c & \underset{g \in \mathcal{C}_h}{\longrightarrow} & \end{array}$$

   a set $2\mathrm{Hom}(f, g, h, k)$, to be called the set of *2-morphisms* of $\mathcal{C}$ between the given 1-morphisms, whose elements we denote by

$$\begin{tikzcd} a \arrow[r, "f \in \mathcal{C}_h"] \arrow[d, "h \in \mathcal{C}_v"'] \arrow[dr, phantom, "\swArrow"] & b \arrow[d, "k \in \mathcal{C}_v"] \\ c \arrow[r, "g \in \mathcal{C}_h"'] & d \end{tikzcd}$$

3. a horizontal and a vertical composition operation of 2-morphisms which is unitality and associative in both directions in the evident way, which respects composition in $\mathcal{C}_h$ and $\mathcal{C}_v$, and such that horizontal and vertical composition commute over each other in the evident way.

***Example 1.54*. (*double category of squares* of a *strict 2-category*)**

Let $\mathcal{C}$ be a strict 2-category (Def. 1.48). Then its *double category of squares* $\mathrm{Sq}(\mathcal{C})$ is the double category (Def. 1.53) whose

- objects are those of $\mathcal{C}$;
- horizontal morphisms and vertical morphisms are both the 1-morphisms of $\mathcal{C}$;
- 2-morphisms

$$\begin{tikzcd} a \arrow[r, "f \in \mathcal{C}_h"] \arrow[d, "h \in \mathcal{C}_v"'] \arrow[dr, phantom, "\phi \swArrow"] & b \arrow[d, "k \in \mathcal{C}_v"] \\ c \arrow[r, "g \in \mathcal{C}_h"'] & d \end{tikzcd}$$

  are the 2-morphisms of $\mathcal{C}$ between the evident composites of 1-morphisms:

$$k \circ f \overset{\phi}{\Rightarrow} g \circ h$$

and composition is given by the evident compositions in $\mathcal{C}$.

***Remark 1.55*. (*strict* and *weak 2-functors*)**

Given two strict 2-categories (Def. 1.48) or double categories (Def. 1.53), $\mathcal{C}, \mathcal{D}$, there is an evident notion of *2-functor* or *double functor*

$$\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D}$$

between them, namely functions on objects, 1-morphisms and 2-morphisms which respect all the composition operations and identity morphisms.

These are also called *strict 2-functors*.

This is in contrast to a more flexible concept of *weak 2-functors*, often called *pseudofunctors*, which respect composition of 1-morphisms only up to invertible 2-morphisms (which themselves are required to satisfy some coherence condition):

$$
\begin{array}{ccccc}
 & & Y & & \\
 & {}^{F(f)}\nearrow & \Downarrow^{\rho}\simeq & \searrow^{F(G)} & \\
X & & \underset{F(g\circ f)}{\longrightarrow} & & Z
\end{array}
$$

We will see an important example of a weak double functor in the construction of derived functors of Quillen functors, below in Prop. 6.50.

Equivalences

We have seen functors (Def. 1.15) as the homomorphisms between categories (Def. 1.1). But functors themselves are identified only up to natural isomorphism (Def. 1.23), reflective the fact that they are the 1-morphisms in a 2-category of categories (Example 1.49). This means that in identifying two categories, we should not just ask for *isomorphisms* between them, hence for a functor between them that has a strict inverse morphism, but just for an inverse up to natural isomorphism.

This is called an *equivalence of categories* (Def. 1.57 below). A particularly well-behaved equivalence of categories is an equivalence exhibited by an adjoint pair of functors, called an *adjoint equivalence of categories* (Def. 1.56 below). In fact every equivalence of categories may be improved to an adjoint equivalence (Prop. 1.58).

***Definition 1.56*. (*adjoint equivalence of categories*)**

Let $\mathcal{C}$, $\mathcal{D}$ be two categories (Def. 1.1). Then an *adjoint equivalence of categories* between them is a pair adjoint functors (Def. 1.32)

$$
\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_{\perp}} \mathcal{D}
$$

such that their unit η and counit ϵ (Def. 1.33) are natural isomorphisms (as opposed to just being natural transformations)

$$
\eta \;:\; \mathrm{id}_{\mathcal{D}} \overset{\simeq}{\Rightarrow} R \circ L \qquad \text{and} \qquad \epsilon \;:\; L \circ R \overset{\simeq}{\Rightarrow} \mathrm{id}_{\mathcal{C}} \;.
$$

There is also the following, seemingly weaker, notion:

***Definition 1.57**. (equivalence of categories)*

Let $\mathcal{C}, \mathcal{D}$ be two categories (Def. 1.1). Then an *equivalence of categories*

$$
\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathcal{D}
$$

is a pair of functors back and forth, as shown (Def. 1.15), together with natural isomorphisms (Def. 1.23) between their composition and the identity functors:

$$
\mathrm{id}_{\mathcal{D}} \overset{\simeq}{\Rightarrow} R \circ L \qquad \text{and} \qquad L \circ R \overset{\simeq}{\Rightarrow} \mathrm{id}_{\mathcal{C}} \;.
$$

If a functor participates in an equivalence of categories, that functor alone is usually already called an equivalence of categories. If there is any equivalence of categories between two categories, these categories are called *equivalent*.

***Proposition 1.58**. (every equivalence of categories comes from an adjoint equivalence of categories)*

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories (Def. 1.1). Then the they are equivalent (Def. 1.57) precisely if there exists an adjoint equivalence of categories between them (Def. 1.56).

Moreover, let $R : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor (Def. 1.15) which participates in an equivalence of categories (Def. 1.57). Then for every functor $L : \mathcal{D} \to \mathcal{C}$ equipped with a natural isomorphism

$$
\eta \;:\; \mathrm{id}_{\mathcal{D}} \overset{\simeq}{\Rightarrow} R \circ L
$$

there exists a natural isomorphism

$$
\epsilon \;:\; L \circ R \overset{\simeq}{\Rightarrow} \mathrm{id}_{\mathcal{C}}
$$

which completes this to an adjoint equivalence of categories (Def. 1.56).

Inside every adjunction sits its maximal adjoint equivalence:

***Proposition 1.59**. (fixed point equivalence of an adjunction)*

Let

$$
\mathcal{D} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}
$$

be a pair of adjoint functors (Def. 1.32). Say that

1. *an object $c \in \mathcal{C}$ is a fixed point of the adjunction if its adjunction unit (Def. 1.33) is an isomorphism (Def. 1.9)*

$$c \underset{\simeq}{\overset{\eta_c}{\longrightarrow}} RL(c)$$

and write

$$\mathcal{C}_{\text{fix}} \hookrightarrow \mathcal{C}$$

for the full subcategory on these fixed objects (Example 1.20)

2. *an object $d \in \mathcal{D}$ is a fixed point of the adjunction if its adjunction counit (Def. 1.33) is an isomorphism (Def. 1.9)*

$$LR(d) \underset{\simeq}{\overset{\epsilon_d}{\longrightarrow}}$$

and write

$$\mathcal{D}_{\text{fix}} \hookrightarrow \mathcal{D}$$

for the full subcategory on these fixed objects (Example 1.20)

Then the adjunction (co-)restricts to an adjoint equivalence (Def. 1.56) on these full subcategories of fixed points

$$\mathcal{D}_{\text{fix}} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_\perp}}}} \mathcal{C}_{\text{fix}}$$

Proof. It is sufficient to see that the functors (co-)restrict as claimed, for then the restricted adjunction unit/counit are isomorphisms by definition, and hence exhibit an adjoint equivalence.

Hence we need to show that

1. for $c \in \mathcal{C}_{\text{fix}} \hookrightarrow \mathcal{C}$ we have that $\eta_{R(d)}$ is an isomorphism;
2. for $d \in \mathcal{D}_{\text{fix}} \hookrightarrow \mathcal{D}$ we have that $\epsilon_{L(c)}$ is an isomorphism.

For the first case we claim that $R(\eta_d)$ provides an inverse: by the triangle identity (18) it is a right inverse, but by assumption it is itself an invertible morphism, which implies that $\eta_{R(d)}$ is an isomorphism.

The second claim is formally dual. ■

Modalities

Generally, a full subcategory-inclusion (Def. 1.19) may be thought of as a consistent proposition about objects in a category: The objects in the full subcategory are those that have the given property.

This basic situation becomes particularly interesting when the inclusion functor has a left adjoint or a right adjoint (Def. 1.32), in which case one speaks of a *reflective subcategory*, or a *coreflective subcategory*, respectively (Def. 1.60 below). The adjunction now implies that each object is *reflected* or *coreflected* into the subcategory, and equipped with a comparison morphism to or from its (co-)reflection (the adjunction (co-)unit, Def. 1.33). This comparison morphism turns out to always be an idempotent (co-)projection, in a sense made precise by Prop. 1.64 below.

This means that, while any object may not fully enjoy the property that defines the subcategory, one may ask for the "aspect" of it that does, which is what is (co-)projected out. Regarding objects only via these aspects of them hence means to regard them only *locally* (where they exhibit that aspect) or only in the *mode* of focus on this aspect. Therefore one also calls the (co-)reflection operation into the given subcategory a *(co-)localization* or *(co-)modal operator*, or *modality*, for short (Def. 1.62 below).

One finds that (co-)modalities are a fully equivalent perspective on the (co-)reflective subcategories of their fully *(co-)modal objects* (Def. 1.65 below), this is the statement of Prop. 1.63 below.

Another alternative perspective on this situation is given by the concept of *localization of categories* (Def. 1.76 below), which is about universally forcing a given collection of morphisms ("weak equivalences", Def. 1.75 below) to become invertible. A *reflective localization* is equivalently a reflective subcategory-inclusion (Prop. 1.77 below), and this exhibits the modal objects (Def. 1.65 below) as equivalently forming the full subcategory of *local objects* (Def. 1.78 below).

Conversely, every reflection onto full subcategories of S-local objects (Def. 1.79 below) satisfies the universal property of a localization at S with respect to left adjoint functors (Prop. 1.82 below).

In conclusion, we have the following three equivalent perspectives on modalities.

reflective subcategory	**modal operator**	**reflective localization**
object in reflective full subcategory	modal object	local object

***Definition 1.60*. (reflective subcategory and coreflective subcategory)**

Let $\mathcal{D}$ be a category (Def. 1.1) and

$$\mathcal{C} \overset{\iota}{\hookrightarrow} \mathcal{D}$$

a full subcategory-inclusion (hence a fully faithful functor Def. 1.19). This is called:

1. a *reflective subcategory inclusion* if the inclusion functor ι has a left adjoint L def. 1.32)

$$\mathcal{C} \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D},$$

then called the *reflector*;

2. a *coreflective subcategory*-inclusion if the inclusion functor ι has a right adjoint R (def. 1.32)

$$\mathcal{C} \underoverset{\underset{R}{\longleftarrow}}{\overset{\iota}{\hookrightarrow}}{\bot} \mathcal{D},$$

then called the *coreflector*.

***Example 1.61*. (reflective subcategory inclusion of sets into small groupoids)**

There is a reflective subcategory-inclusion (Def. 1.60)

$$\mathrm{Set} \underoverset{\underset{}{\hookrightarrow}}{\overset{\pi_0}{\longleftarrow}}{\bot} \mathrm{Grpd}$$

of the category of sets (Example 1.2) into the category Grpd (Example 1.16) of small groupoids (Example 1.10) where

- the right adjoint full subcategory inclusion (Def. 1.19) sends a set S to the groupoid with set of objects being S, and the only morphisms being the identity morphisms on these objects (also called the *discrete groupoid* on S, but this terminology is ambiguous)
- the left adjoint reflector sends a small groupoid $\mathcal{G}$ to its set of connected components, namely to the set of equivalence classes under the equivalence relation on the set of objects, which regards two objects as equivalent, if there is any

morphism between them.

We now re-consider the concept of reflective subcategories from the point of view of modalities:

Definition 1.62. (modality)

Let $\mathcal{D}$ be a category (Def. 1.1). Then

1. a *modal operator on* $\mathcal{D}$ is
   1. an endofunctor

      $$\bigcirc \;:\; \mathcal{D} \to \mathcal{D}$$

      whose full essential image we denote by

      $$\mathrm{Im}(\bigcirc) \overset{\iota}{\hookrightarrow} \mathcal{D}\,,$$

   2. a natural transformation (Def. 1.23)

      $$X \overset{\eta_X}{\longrightarrow} \bigcirc X \qquad (23)$$

      for all objects $X \in \mathcal{D}$, to be called the *unit morphism*;

   such that:

   - for every object $Y \in \mathrm{Im}(\bigcirc) \hookrightarrow \mathcal{D}$ in the essential image of $\bigcirc$, every morphism f into Y factors *uniquely* through the unit (23)

     $$\begin{array}{ccccc} & & X & & \\ & {}^{\eta_X}\swarrow & & \searrow^{f} & \\ \bigcirc X & & \underset{\exists !}{\longrightarrow} & & Y & \in \mathrm{Im}(\bigcirc) \end{array}$$

     which equivalently means that if $Y \in \mathrm{Im}(\bigcirc)$ the operation of precomposition with the unit η_X yields a bijection of hom-sets

     $$(-)\circ\eta_X \;:\; \mathrm{Hom}_{\mathcal{D}}(\bigcirc X, Y) \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(X, Y)\,, \qquad (24)$$

2. a *comodal operator on* $\mathcal{D}$ is
   1. an endofunctor

$$\Box \;:\; \mathcal{D} \to \mathcal{D}$$

whose full essential image we denote by

$$\mathrm{Im}(\Box) \overset{\iota}{\hookrightarrow} \mathcal{D}$$

2. a natural transformation (Def. 1.23)

$$\Box X \overset{\epsilon_X}{\longrightarrow} X \qquad (25)$$

for all objects $X \in \mathcal{D}$, to be called the *counit morphism*;

such that:

- for every object $Y \in \mathrm{Im}(\Box) \hookrightarrow \mathcal{D}$ in the essential image of $\Box$, every morphism f out of Y factors *uniquely* through the counit (23)

$$\begin{array}{ccc} & & X \\ & {}^{\epsilon_X}\nearrow & & \nwarrow^{f} \\ \Box X & & \underset{\exists !}{\longleftarrow} & & Y \in \mathrm{Im}(\Box) \end{array}$$

which equivalently means that if $Y \in \mathrm{Im}(\bigcirc)$ the operation of postcomposition with the counit ϵ_X yields a bijection of hom-sets

$$\epsilon_X \circ (-) \;:\; \mathrm{Hom}_{\mathcal{D}}(Y, \Box X) \overset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(Y, X)\,, \qquad (26)$$

***Proposition 1.63**. (**modal operators** **equivalent to** **reflective subcategories**)*

If

$$\mathcal{C} \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

is a reflective subcategory-inclusion (Def. 1.60). Then the composite

$$\bigcirc \;\coloneqq\; \iota \circ L \;:\; \mathcal{D} \longrightarrow \mathcal{D}$$

equipped with the adjunction unit natural transformation (Def. 1.33)

$$X \overset{\eta_X}{\longrightarrow} \bigcirc X$$

is a modal operator on $\mathcal{D}$ (Def. 1.62).

Dually, if

$$\mathcal{C} \underoverset{\underset{R}{\longleftarrow}}{\overset{\iota}{\hookrightarrow}}{\bot} \mathcal{D}$$

is a coreflective subcategory-inclusion (Def. 1.60). Then the composite

$$\Box \;\coloneqq\; \iota \circ R \;:\; \mathcal{D} \longrightarrow \mathcal{D}$$

equipped with the adjunction counit natural transformation (Def. 1.33)

$$\Box X \overset{\epsilon_X}{\longrightarrow} X$$

is a comodal operator on $\mathcal{D}$ (Def. 1.62).

Conversely:

If an endofunctor $\bigcirc : \mathcal{D} \to \mathcal{D}$ with natural transformation $X \overset{\eta_X}{\to} \bigcirc X$ is a modal operator on a category $\mathcal{D}$ (Def. 1.62), then the inclusion of its full essential image is a reflective subcategory inclusion (Def. 1.60) with reflector given by the corestriction of $\bigcirc$ to its image:

$$\mathrm{Im}(\bigcirc) \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{\bigcirc}{\longleftarrow}}{} \mathcal{D}\,.$$

Dually, if an endofunctor $\Box : \mathcal{D} \to \mathcal{D}$ with natural transformation $\Box X \overset{\epsilon_X}{\longrightarrow} X$ is a comodal operator (Def. 1.62), then the inclusion of its full essential image is a coreflective subcategory inclusion (Def. 1.60) with coreflector given by the corestriction of $\Box$ to its image

$$\mathrm{Im}(\Box) \underoverset{\underset{\Box}{\longleftarrow}}{\overset{\iota}{\hookrightarrow}}{} \mathcal{D}\,.$$

Proof. The first two statements are immedialy a special case of the characterization of adjunctions via universal morphisms in Prop. 1.42: Using that $R = \iota$ is here assumed to be fully faithful, the uniqueness of $\tilde f$ in the universal morphism-factorization condition (21)

$$\begin{array}{ccccc} & & c & & \\ & {}^{\eta_c}\swarrow & & \searrow^{f} & \\ R(L(c)) & & \underset{R(\tilde f)}{\longrightarrow} & & R(d) \\ \\ L(c) & & \underset{\exists! \tilde f}{\longrightarrow} & & d \end{array}$$

implies that also $R(\tilde f) = \iota(\tilde f)$ is the unique morphism making that triangle commute.

Similarly for the converse: The assumption on a modal operator $\bigcirc$ is just so as to make its unit η be a universal morphism (Def. 1.41) into the inclusion functor ι of its essential image. ∎

***Proposition 1.64*. (modal operator is idempotent)**

Let $\mathcal{D}$ be a category (Def. 1.1).

For $\bigcirc$ a modal operator on $\mathcal{D}$, with unit η (Def. 1.63), it is idempotent, in that it is naturally isomorphic (Def. 1.23) to the composition with itself:

$$\bigcirc \;\simeq\; \bigcirc\,\bigcirc\,.$$

In fact, the image under $\bigcirc$ of its unit is such an isomorphism

$$\bigcirc\left(X \overset{\eta_X}{\to} \bigcirc X\right) \;:\; \bigcirc X \overset{\simeq}{\longrightarrow} \bigcirc(\bigcirc X)$$

as is its unit on its image

$$\eta_{\bigcirc X} \;:\; \bigcirc X \overset{\simeq}{\longrightarrow} \bigcirc(\bigcirc X)\,.$$

Formally dually, for $\Box$ a comodal operator on $\mathcal{D}$, with counit ϵ (Def. 1.63), it is idempotent, in that it is naturally isomorphic (Def. 1.23) to the composition with itsef:

$$\Box \circ \Box \;\simeq\; \Box\,.$$

In fact, the image under $\Box$ of its counit is such an isomorphism

$$\Box\left(\Box X \overset{\epsilon_X}{\to} X\right) \;:\; \Box(\Box X) \overset{\simeq}{\longrightarrow} \Box X$$

as is its counit on its image

$$\epsilon_{\Box X} \;:\; \Box(\Box X) \overset{\simeq}{\longrightarrow} \Box X\,.$$

***Proof*.** We discuss the first case, the second is formally dual (Example 1.13).

By Prop. 1.63, the modal operator is equivalent to the composite $\iota \circ L$ obtained from the reflective subcategory-inclusion (Def. 1.60) of its essential image of modal objects:

$$\mathrm{Im}(\bigcirc) \underset{\iota}{\overset{L}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}\,.$$

and its unit is the corresponding adjunction unit (Def. 1.33)

$$X \xrightarrow{\eta_X} \iota(L(X)) \,.$$

Hence it is sufficient to show that the morphisms and $L(\eta_X)$ and $\eta_{\iota Y}$ are isomorphisms.

Now, the triangle identities (18) for the adjunction $L \dashv \iota$, which hold by Prop. 1.38, say that their composition with the adjunction counit is the identity morphism

$$\epsilon_{L(\eta_X)} \circ L(\eta_X) \;=\; \mathrm{id}_{L(X)} \quad \text{and} \quad \iota(\epsilon_Y) \circ \eta_{\iota(Y)} \;=\; \mathrm{id}_{\iota(Y)} \,.$$

But by Prop. 1.46, the counit ϵ is a natural isomorphism, since ι is fully faithful. Hence we may cancel it on both sides of the triangle identities and find that $L(\eta_X)$ and $\eta_{\iota(Y)}$ are indeed isomorphisms. ▮

***Definition 1.65*. (*modal objects*)**

Let $\mathcal{D}$ be a category (Def. 1.1).

For $\bigcirc$ a modal operator on $\mathcal{D}$ (Def. 1.62), we say:

1. a $\bigcirc$-*modal object* is an object $X \in \mathcal{D}$ such that the following conditions hold (which are all equivalent, by Prop. 1.64):
   - it is in the $\bigcirc$-essential image: $X \in \mathrm{Im}(\bigcirc) \hookrightarrow \mathcal{D}$,
   - it is isomorphic to its own $\bigcirc$-image: $X \simeq \bigcirc X$,
   - specifically its $\bigcirc$-unit is an isomorphism $\eta_X \;:\; X \xrightarrow{\simeq} \bigcirc X$.
2. a $\bigcirc$-*submodal object* is an object $X \in \mathcal{D}$, such that
   - its $\bigcirc$-unit is a monomorphism (Def. 1.18): $\eta_X \;:\; X \hookrightarrow \bigcirc X$.

Dually (Example 1.13):

For $\Box$ a comodal operator on $\mathcal{D}$ (Def. 1.62), we say:

1. a $\Box$-*comodal object* is an object $X \in \mathcal{D}$ such that the following conditions hold (which are all equivalent, by Prop. 1.64):
   - it is in the $\Box$-essential image: $X \in \mathrm{Im}(\Box) \hookrightarrow \mathcal{D}$,
   - it is isomorphic to its own $\Box$-image: $\Box X \simeq X$,
   - specifically its $\Box$-counit is an isomorphism $\epsilon_X \;:\; \Box X \xrightarrow{\simeq} X$
2. a $\Box$-*supcomodal object* is an object $X \in \mathcal{D}$, such that
   - its $\Box$-counit is an epimorphism (Def. 1.18): $\epsilon_X \;:\; \Box X \xrightarrow{\mathrm{epi}} X$.

Definition 1.66. (adjoint modality)

Let

$$L \dashv C \dashv R : \mathcal{C} \underoverset{\underset{R}{\hookleftarrow}}{\overset{L}{\hookleftarrow}}{\overset{C}{\longrightarrow}} \mathcal{D}$$

be an adjoint triple (Remark 1.34) such that L and R are fully faithful functors (necessarily both, by Prop. 1.67). By Prop. 1.63, there are induced modal operators

$$\bigcirc \;\coloneqq\; L \circ C \qquad \Box \;\coloneqq\; R \circ C$$

which themselves form am adjoint pair

$$\Box \;\dashv\; \bigcirc \;,$$

hence called an *adjoint modality*. The adjunction unit and adjunction counit as in (13) may now be read as exhibiting each object X in the domain of C as "in between the opposite extremes of its $\bigcirc$-modal aspect and its $\Box$-modal aspect"

$$\Box X \overset{\epsilon^{\Box}_X}{\longrightarrow} X \overset{\eta^{\bigcirc}_X}{\longrightarrow} \bigcirc X \;.$$

A formally dual situation (Example 1.13) arises when C is fully faithful.

$$L \dashv C \dashv R : \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longrightarrow}}{\overset{C}{\hookleftarrow}} \mathcal{D}$$

with

$$\left(\bigcirc \;\coloneqq\; C \circ L \right) \dashv \left(\Box \;\coloneqq\; C \circ R \right)$$

and canonical natural transformation between opposite extreme aspects given by

$$\Box X \overset{\epsilon^{\Box}_X}{\longrightarrow} X \overset{\eta^{\bigcirc}_X}{\longrightarrow} \bigcirc X \qquad (27)$$

Proposition 1.67. (fully faithful adjoint triple)

Let $L \dashv C \dashv R$ be an adjoint triple (Remark 1.34). Then the following are equivalent:

1. *L is a fully faithful functor;*
2. *R is a fully faithful functor,*
3. *$(\Box \coloneqq L \circ C) \dashv (\bigcirc \coloneqq R \circ C)$ is an adjoint modality (Def. 1.67).*

For ***proof*** see this prop..

In order to analyze (in Prop. 1.69 below) the comparison morphism of opposite extreme aspects (27) induced by an adjoint modality (Def. 1.66), we need the following technical Lemma:

Lemma 1.68. *Let*

$$\mathcal{C} \underoverset{\xrightarrow{R}}{\xrightarrow{L}}{\hookleftarrow C} \mathcal{D}$$

be an adjoint triple with induced adjoint modality (Def. 1.66) to be denoted

$$(\bigcirc \coloneqq C \circ L) \dashv (\Box \coloneqq C \circ R)$$

Denoting the adjunction units/counits (Def. 1.33) as

adjunction	unit	counit
$(L \dashv C)$	$\eta^{\bigcirc}$	$\epsilon^{\bigcirc}$
$(C \dashv R)$	$\eta^{\Box}$	$\epsilon^{\Box}$

we have that the following composites of unit/counit components are equal:

$$\left(\eta^{\Box}_{LX}\right) \circ \left(L\epsilon^{\Box}_{X}\right) \;=\; \left(R\eta^{\bigcirc}_{X}\right) \circ \left(\epsilon^{\bigcirc}_{RX}\right) \qquad \begin{tikzcd} LCRX \arrow[r, "\epsilon^{\bigcirc}_{RX}"] \arrow[d, "L\epsilon^{\Box}_{X}"'] & RX \arrow[d, "R\eta^{\bigcirc}_{X}"] \\ LX \arrow[r, "\eta^{\Box}_{LX}"'] & RCLX \end{tikzcd} \tag{28}$$

(Johnstone 11, lemma 2.1)

Proof. We claim that the following diagram commutes (Def. 1.4):

$$
\begin{tikzcd}
 & & RX \arrow[dr, "R\eta^{\bigcirc}_X"] & & \\
 & LCRX \arrow[ur, "\epsilon^{\bigcirc}_{RX}"] \arrow[dl, "L\epsilon^{\square}_X"'] \arrow[dr, "LCR\eta^{\bigcirc}_X"] & & RCLX & \\
LX \arrow[dr, "L\eta^{\bigcirc}_X"'] & & LCRCLX \arrow[ur, "\epsilon^{\bigcirc}_{RCLX}"] \arrow[dl, "\text{iso}", "L\epsilon^{\square}_{CLX}"'] & & LX \arrow[ul, "\eta^{\square}_{LX}"'] \\
 & LCLX & & LCLX \arrow[ll, "\mathrm{id}_{LCLX}"] \arrow[ul, "\text{iso}"', "LC\eta^{\square}_{LX}"] \arrow[ur, "\epsilon^{\bigcirc}_{LX}"'] &
\end{tikzcd}
$$

This commutes, because:

1. the left square is the image under L of naturality (4) for $\epsilon^{\square}$ on $\eta^{\bigcirc}_X$;
2. the top square is naturality (4) for $\epsilon^{\bigcirc}$ on $R\eta^{\bigcirc}_X$;
3. the right square is naturality (4) for $\epsilon^{\bigcirc}$ on $\eta^{\square}_{LX}$;
4. the bottom commuting triangle is the image under L of the triangle identity (18) for $(C \dashv R)$ on LX.

Moreover, notice that

1. the total bottom composite is the identity morphism id_{LX}, due to the triangle identity (18) for $(C \dashv R)$;
2. also the other two morphisms in the bottom triangle are isomorphisms, as shown, due to the idempoency of the $(C - R)$-adjunction (Prop. 1.64.)

Therefore the total composite from $LCRX \to R \,/\, CLX$ along the bottom part of the diagram equals the left hand side of (28), while the composite along the top part of the diagram clearly equals the right hand side of (28). ▮

***Proposition 1.69**. (comparison transformation between opposite extremes of adjoint modality)*

Consider an adjoint triple of the form

$$
L \dashv C \dashv R \;:\; \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longrightarrow}}{\overset{C}{\longleftarrow}} \mathcal{B}
$$

with induced adjoint modality (Def. 1.66) to be denoted

$$(\bigcirc \;\coloneqq\; C \circ L) \dashv (\square \;\coloneqq\; C \circ R)$$

Denoting the adjunction units/counits (Def. 1.33) as

adjunction	unit	counit
$(L \dashv C)$	$\eta^{\bigcirc}$	$\epsilon^{\bigcirc}$
$(C \dashv E)$	$\eta^{\square}$	$\epsilon^{\square}$

Then for all $X \in \mathcal{C}$ the following two natural transformations, constructed from the adjunction units/counits (Def. 1.33) and their inverse morphisms (using idempotency, Prop. 1.64), are equal:

$$\mathrm{comp}_{\mathcal{B}} \;\coloneqq\; (L\epsilon^{\square}_X) \circ \left(\eta^{\bigcirc}_{RX}\right)^{-1} \;=\; \left(\eta^{\square}_{LX}\right)^{-1} \circ \left(\Gamma\eta^{\bigcirc}_X\right) \qquad \begin{tikzcd} RX \arrow[r, "\Gamma\eta^{\bigcirc}_X"] \arrow[d, "(\eta^{\bigcirc}_{RX})^{-1}"'] \arrow[dr, "\mathrm{comp}_{\mathcal{B}}"] & RC \arrow[d] \\ LCRX \arrow[r, "L\epsilon^{\square}_X"'] & L. \end{tikzcd} \tag{29}$$

Moreover, the image of these morphisms under C equals the following composite:

$$\mathrm{comp}_{\mathcal{C}} \;:\; \square X \xrightarrow{\epsilon^{\square}_X} X \xrightarrow{\eta^{\bigcirc}_X} \bigcirc X\,, \tag{30}$$

hence

$$\mathrm{comp}_{\mathcal{C}} \;=\; C(\mathrm{comp}_{\mathcal{B}})\;. \tag{31}$$

Proof. The first statement follows directly from Lemma 1.68.

For the second statement, notice that the $(C \dashv R)$-adjunct (Prop. 1.38) of

$$\mathrm{comp}_{\mathcal{C}} \;:\; CRX \xrightarrow{\epsilon^{\square}_X} X \xrightarrow{\eta^{\bigcirc}_X} CLX$$

is

$$\widetilde{\mathrm{comp}_{\mathcal{C}}} \;=\; \underbrace{\Gamma X \underset{\mathrm{iso}}{\xrightarrow{\eta^{\square}_{RX}}} RCRX \underset{\mathrm{iso}}{\xrightarrow{\Gamma\epsilon^{\square}_X}} RX}_{=\mathrm{id}_{RX}} \xrightarrow{R\eta^{\bigcirc}_X} RCLX\,, \tag{32}$$

where under the braces we uses the triangle identity (Prop. 1.39).

(As a side remark, for later usage, we observe that the morphisms on the left in (32) are isomorphisms, as shown, by idempotency of the adjunctions.)

From this we obtain the following commuting diagram:

$$
\begin{tikzcd}
CRX \arrow[r, "CR\eta^{\bigcirc}_X"] \arrow[dr, "\mathrm{comp}_{\mathcal{C}}"'] & CRCLX \arrow[r, "C(\eta^{\Box}_{LX})^{-1}", "\mathrm{iso}"'] \arrow[d, "\epsilon^{\Box}_{CLX}"', "\simeq"] & CLX \\
& CLX \arrow[ur, "\mathrm{id}_{LX}"'] &
\end{tikzcd}
$$

Here:

1. on the left we identified $\widetilde{\widetilde{\mathrm{comp}_{\mathcal{C}}}} = \mathrm{comp}_{\mathcal{C}}$ by applying the formula (Prop. 1.38) for $(C \dashv R)$-adjuncts to $\widetilde{\mathrm{comp}_{\mathcal{C}}} = R\eta^{\bigcirc}_X$ (32);
2. on the right we used the triangle identity (Prop. 1.38) for $(C \dashv R)$.

This proves the second statement. ▮

***Definition 1.70*. (preorder on modalities)**

Let $\bigcirc_1$ and $\bigcirc_2$ be two modal operators on a category $\mathcal{C}$. By Prop. 1.63 these are equivalently characterized by their reflective full subcategories $\mathcal{C}_{\bigcirc_1}, \mathcal{C}_{\bigcirc 2} \hookrightarrow \mathcal{C}$ of modal objects.

There is an evident preorder on full subcategories of $\mathcal{C}$, given by full inclusions of full subcategories into each other. We write $\mathcal{C}_{\bigcirc_1} \subset \mathcal{C}_{\bigcirc_2}$ if the full subcategory on the left is contained, as a full subcategory of $\mathcal{C}$, in that on the right. Via prop. 1.63 there is the induced preorder on modal operators, and we write

$$
\bigcirc_1 \;<\; \bigcirc_2 \quad \text{iff} \quad \mathcal{C}_{\bigcirc_1} \subset \mathcal{C}_{\bigcirc_2}\,.
$$

There is an analogous preorder on comodal operators (Def. 1.62).

If we have two adjoint modalities (Def. 1.66) of the same type (both modal left adjoint or both comodal left adjoint) such that both the modalities and the comodalities are compatibly ordered in this way, we denote this situation as follows:

$$
\begin{array}{ccc}
\bigcirc_2 & \dashv & \Box_2 \\
\vee & & \vee \\
\bigcirc_1 & \dashv & \Box_1
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccc}
\Box_2 & \dashv & \bigcirc_2 \\
\vee & & \vee \\
\Box_1 & \dashv & \bigcirc_1
\end{array}
$$

etc.

***Example 1.71*. (bottom and top adjoint modality)**

Let $\mathcal{C}$ be a category with both an initial object $\emptyset$ and a terminal object $*$ (Def. 1.5). Then, by Example 3.7 there is an adjoint triple between $\mathcal{C}$ and the terminal category $*$ (Example 1.7) of the form

$$\mathcal{C} \underoverset{\underset{\mathrm{const}_*}{\hookleftarrow}}{\overset{\mathrm{const}_\emptyset}{\hookleftarrow}}{\longrightarrow} * \,.$$

The induced adjoint modality (Def. 1.66) is

$$\mathrm{const}_\emptyset \;\dashv\; \mathrm{const}_* \;\;:\;\; \mathcal{C} \to \mathcal{C} \,.$$

By slight abuse of notation, we will also write this as

$$\emptyset \;\dashv\; * \;\;:\;\; \mathcal{C} \to \mathcal{C} \,. \tag{33}$$

On the other extreme, for $\mathcal{C}$ any category whatsoever, the identity functor on it is adjoint functor to itself, and constitutes an adjoint modality (Def. 1.66)

$$\mathrm{id}_{\mathcal{C}} \;\dashv\; \mathrm{id}_{\mathcal{C}} \;\;:\;\; \mathcal{C} \to \mathcal{C} \,. \tag{34}$$

Here

1. (33) is the *bottom* (or *ground*)
2. (34) is the *top*

in the preorder on adjoint modalities according to Def. 1.70, in that for every adjoint modality of the form $\bigcirc \dashv \Box$ we have the following:

$$\begin{array}{ccc} \mathrm{id} & \dashv & \mathrm{id} \\ \vee & & \vee \\ \Box & \dashv & \bigcirc \\ \vee & & \vee \\ \emptyset & \dashv & * \end{array}$$

***Definition 1.72*. (Aufhebung)**

On some category $\mathcal{C}$, consider an inclusion of adjoint modalities, according to Def. 1.70:

$$\begin{array}{ccc} \Box_2 & \dashv & \bigcirc_2 \\ \vee & & \vee \\ \Box_1 & \dashv & \bigcirc_1 \end{array}$$

We say:

1. This provides *right Aufhebung of the opposition* exhibited by $\Box_1 \dashv \bigcirc_1$ if there is also the diagonal inclusion

$$\Box_1 < \bigcirc_2 \qquad \text{equivalently} \qquad \mathcal{C}_{\Box_1} \subset \mathcal{C}_{\bigcirc_2}$$

   We indicate this situation by

$$\begin{array}{ccc} \Box_2 & \dashv & \bigcirc_2 \\ \vee & / & \vee \\ \Box_1 & \dashv & \bigcirc_1 \end{array}$$

2. This provides *left Aufhebung of the opposition* exhibited by $\Box_1 \dashv \bigcirc_1$ if there is also the diagonal inclusion

$$\bigcirc_1 < \Box_2 \qquad \text{equivalently} \qquad \mathcal{C}_{\bigcirc_1} \subset \mathcal{C}_{\Box_2}$$

   We indicate this situation by

$$\begin{array}{ccc} \Box_2 & \dashv & \bigcirc_2 \\ \vee & \backslash & \vee \\ \Box_1 & \dashv & \bigcirc_1 \end{array}$$

Remark 1.73. For a progression of adjoint modalities of the form

$$\begin{array}{ccc} \bigcirc_2 & \dashv & \Box_2 \\ \vee & & \vee \\ \bigcirc_1 & \dashv & \Box_1 \end{array}$$

the analog of Aufhebung (Def. 1.72) is automatic, since, by Prop. 1.63, in this situation the full subcategories modal objects at each stage coincide already.

For emphasis we may denote this situation by

$$
\begin{array}{ccc}
\bigcirc_2 & \dashv & \Box_2 \\
\vee & | & \vee \\
\bigcirc_1 & \dashv & \Box_1
\end{array}
.
$$

***Example 1.74*. (top adjoint modality provides Aufhebung of all oppositions)**

For $\mathcal{C}$ any category, the top adjoint modality id ⊣ id (Def. 1.71) provides Aufhebung (Def. 1.72) of every other adjoint modality.

But already Aufhebung of the bottom adjoint modality is a non-trivial and interesting condition. We consider this below in Prop. 5.7.

We now re-consider the concept of reflective subcategories from the point of view of localization of categories:

***Definition 1.75*. (category with weak equivalences)**

A *category with weak equivalences* is

1. a category $\mathcal{C}$ (Def. 1.1)
2. a subcategory $W \subset \mathcal{C}$ (i.e. sub-class of objects and morphisms that inherits the structure of a category)

such that the morphisms in W

1. include all the isomorphisms of $\mathcal{C}$,
2. satisfy *two-out-of-three*:
   If for g, f any two composable morphisms in $\mathcal{C}$, two out of the set $\{g, f, g \circ f\}$ are in W, then so is the third.

$$
\begin{array}{ccc}
 & & \\
{}^{f}\nearrow & & \searrow^{g} \\
 & \underset{g \circ f}{\longrightarrow} &
\end{array}
$$

***Definition 1.76*. (localization of a category)**

Let $W \subset \mathcal{C}$ be a category with weak equivalences (Def. 1.75). Then the *localization* of $\mathcal{C}$ at W is, if it exsists

1. a category $\mathcal{C}[W^{-1}]$,
2. a functor $\gamma \,:\, \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ (Def. 1.15)

such that

1. γ sends all morphisms in $W \subset \mathcal{C}$ to isomorphisms (Def. 1.9),
2. γ is universal with this property: If $F \,:\, \mathcal{C} \longrightarrow \mathcal{D}$ is any functor with this property, then it factors through γ, up to natural isomorphism (Def. 1.23):

$$F \;\simeq\; DF \circ \gamma \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\rho \Downarrow_{\simeq}"] & \mathcal{D} \\ & \mathcal{C}[W^{-1}] \arrow[ur, "DF"'] & \end{tikzcd}$$

and any two such factorizations DF and $D'F$ are related by a unique natural isomorphism κ compatible with ρ and ρ':

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\rho \Downarrow_{\simeq}"] & \mathcal{D} \arrow[dr, "\mathrm{id}"] & \\ & \mathcal{C}[W^{-1}] \arrow[ur, "DF"'] \arrow[dr, "\mathrm{id}"'] & \arrow[r, phantom, "\simeq \Downarrow^{\kappa}"] & \mathcal{D} \\ & & \mathcal{C}[W^{-1}] \arrow[ur, "D'F"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\rho' \Downarrow_{\simeq}"] & \mathcal{D} \\ & \mathcal{C}[W^{-1}] \arrow[ur, "D'F"'] & \end{tikzcd} \tag{35}$$

Such a localization is called a *reflective localization* if the localization functor has a fully faithful right adjoint, exhibiting it as the reflection functor of a reflective subcategory-inclusion (Def. 1.60)

$$\mathcal{C}[W^{-1}] \underoverset{\hookrightarrow}{\overset{\gamma}{\longleftarrow}}{\bot} \mathcal{C} \,.$$

Proposition 1.77. ***(reflective subcategories are localizations)***

Every reflective subcategory-inclusion (Def. 1.60)

$$\mathcal{C}_L \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

is the reflective localization (Def. 1.76) at the class $W \coloneqq L^{-1}(\mathrm{Isos})$ *of morphisms that are sent to isomorphisms by the reflector L.*

Proof. Let $F \,:\, \mathcal{C} \to \mathcal{D}$ be a functor which inverts morphisms that are inverted by L.

First we need to show that it factors through L, up to natural isomorphism. But consider the following whiskering of the adjunction unit η (Def. 1.33) with F:

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\Downarrow"] & \mathcal{D} \\
& \mathcal{C}_L \arrow[ur, "DF"'] &
\end{tikzcd}
\quad \coloneqq \quad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{id}"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\
& \mathcal{C}_L \arrow[ur, "\iota"'] & &
\end{tikzcd}
$$

By idempotency (Prop. 1.64), the components of the adjunction unit η are inverted by L, and hence by assumption they are also inverted by F, so that on the right the natural transformation $F(\eta)$ is indeed a natural isomorphism.

It remains to show that this factorization is unique up to unique natural isomorphism. So consider any other factorization $D'F$ via a natural isomorphism ρ. Pasting this now with the adjunction counit

$$
\begin{tikzcd}
& \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] \arrow[d, phantom, "\epsilon\Downarrow"] & {} \arrow[d, phantom, "\Downarrow\rho"] & \mathcal{D} \\
\mathcal{C}_L \arrow[ur, "\iota"] \arrow[r, phantom] & {} & \mathcal{C}_L \arrow[ur, "D'F"'] &
\end{tikzcd}
$$

with the bottom arrow $\mathcal{C}_L \xrightarrow{\mathrm{id}} \mathcal{C}_L$,

exhibits a natural isomorphism $\epsilon \cdot \rho$ between $DF \simeq D'F$. Moreover, this is compatible with $F(\eta)$ according to (35), due to the triangle identity (Prop. 1.39):

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{id}"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] \arrow[d, phantom, "\epsilon\Downarrow"] & {} \arrow[d, phantom, "\Downarrow\rho"] & \mathcal{D} \\
& \mathcal{C}_L \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}"'] & {} & \mathcal{C}_L \arrow[ur, "D'F"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow\rho"] & \mathcal{D} \arrow[dl] \\
& \mathcal{C}_L &
\end{tikzcd}
$$

Finally, since L is essentially surjective functor, by idempotency (Prop. 1.39), it is clear that this is the unique such natural isomorphism. ■

Definition 1.78. (local object)

Let $\mathcal{C}$ be a category (Def. 1.1) and let $S \subset \mathrm{Mor}_{\mathcal{C}}$ be a set of morphisms. Then an object $X \in \mathcal{C}$ is called an S-*local object* if for all $A \overset{s}{\to} B \ \in S$ the hom-functor (Def. 1.17) from s into X yields a bijection

$$
\mathrm{Hom}_{\mathcal{C}}(s, X) \;:\; \mathrm{Hom}_{\mathcal{C}}(B, X) \xrightarrow{\quad \simeq \quad} \mathrm{Hom}_{\mathcal{C}}(A, X) \,,
$$

hence if every morphism $A \overset{f}{\longrightarrow} X$ extends uniquely along w to B:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "w"'] & X \\
B \arrow[ur, "\exists !"'] &
\end{tikzcd}
$$

We write

$$
\mathcal{C}_S \overset{\iota}{\hookrightarrow} \mathcal{C} \qquad (36)
$$

for the full subcategory (Example 1.20) of S-local objects.

***Definition 1.79*. (reflection onto full subcategory of local objects)**

Let $\mathcal{C}$ be a category and set $S \subset \mathrm{Mor}_{\mathcal{C}}$ be a sub-class of its morphisms. Then the *reflection onto local S-objects* (often just called “localization at the collection S” is, if it exists, a left adjoint (Def. 1.32) L to the full subcategory-inclusion of the S-local objects (36):

$$
\begin{tikzcd}
\mathcal{C}_S \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "\iota"'] & \mathcal{C} \arrow[l, bend right, "L"']
\end{tikzcd} .
$$

A class of examples is the following, which comes to its full nature (only) after passage to homotopy theory (Example below):

***Definition 1.80*. (homotopy localization of 1-categories)**

Let $\mathcal{C}$ be a category, let $\mathbb{A} \in \mathcal{C}$ be an object, and consider the class of morphisms given by projection out of the Cartesian product with $\mathbb{A}$, of all objects $X \in \mathcal{C}$:

$$
X \times \mathbb{A} \xrightarrow{p_1} X .
$$

If the corresponding reflection onto the full subcategory of local objects (Def. 1.79) exists, we say this is *homotopy localization* at that object , and denote the modal operator corresponding to this (via Prop. 1.63) by

$$
Ⓐ : \mathcal{C} \longrightarrow \mathcal{C} .
$$

***Proposition 1.81*. (reflective localization reflects onto full subcategory of local objects)**

Let $W \subset \mathcal{C}$ be a category with weak equivalences (Def. 1.75). If its reflective localization (Def. 1.76) exists

$$
\begin{tikzcd}
\mathcal{C}[W^{-1}] \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "\iota"'] & \mathcal{C} \arrow[l, bend right, "L"']
\end{tikzcd}
$$

then $\mathcal{C}[W^{-1}] \overset{\iota}{\hookrightarrow} \mathcal{C}$ is equivalently the inclusion of the full subcategory (Example 1.20) on the W-local objects (Def. 1.78), and hence L is equivalently reflection onto the W-local objects, according to Def. 1.79.

Proof. We need to show that

1. every $X \in \mathcal{C}[W^{-1}] \overset{\iota}{\hookrightarrow} \mathcal{C}$ is W-local,
2. every $Y \in \mathcal{C}$ is W-local precisely if it is isomorphic to an object in $\mathcal{C}[W^{-1}] \overset{\iota}{\hookrightarrow} \mathcal{C}$.

The first statement follows directly with the adjunction isomorphism (10):

$$\mathrm{Hom}_{\mathcal{C}}(w, \iota(X)) \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(L(w), X)$$

and the fact that the hom-functor takes isomorphisms to bijections (Example 1.31).

For the second statement, consider the case that Y is W-local. Observe that then Y is also local with respect to the class

$$W_{\mathrm{sat}} \;\coloneqq\; L^{-1}(\mathrm{Isos})$$

of *all* morphisms that are inverted by L (the "saturated class of morphisms"): For consider the hom-functor $\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,Y)} \mathrm{Set}^{\mathrm{op}}$ to the opposite of the category of sets. By assumption on Y this takes elements in W to isomorphisms. Hence, by the defining universal property of the localization-functor L, it factors through L, up to natural isomorphism.

Since, by idempotency (Prop. 1.64), the adjunction unit η_Y is in W_{sat}, this implies that we have a bijection of the form

$$\mathrm{Hom}_{\mathcal{C}}(\eta_Y, Y) \;:\; \mathrm{Hom}_{\mathcal{C}}(\iota L(Y), Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(Y, Y) \;.$$

In particular the identity morphism id_Y has a preimage η_Y^{-1} under this function, hence a left inverse to η:

$$\eta_Y^{-1} \circ \eta_Y \;=\; \mathrm{id}_Y \;.$$

But by 2-out-of-3 this implies that $\eta_Y^{-1} \in W_{\mathrm{sat}}$. Since the first item above shows that $\iota L(Y)$ is W_{sat}-local, this allows to apply this same kind of argument again,

$$\mathrm{Hom}_{\mathcal{C}}(\eta_Y^{-1}, \iota L(Y)) \;:\; \mathrm{Hom}_{\mathcal{C}}(Y, \iota L(Y)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(\iota L(Y), \iota L(Y))\,,$$

to deduce that also η_Y^{-1} has a left inverse $(\eta_Y^{-1})^{-1} \circ \eta_Y^{-1}$. But since a left inverse that itself has a left inverse is in fact an inverse morphisms (this Lemma), this means that η_Y^{-1} is an inverse morphism to η_Y, hence that $\eta_Y : Y \to \iota L(Y)$ is an isomorphism and hence that Y is isomorphic to an object in $\mathcal{C}[W^{-1}] \overset{\iota}{\hookrightarrow} \mathcal{C}$.

Conversely, if there is an isomorphism from Y to a morphism in the image of ι hence, by the first item, to a W-local object, it follows immediatly that also Y is W-local, since the hom-functor takes isomorphisms to bijections and since bijections satisfy 2-out-of-3. ▮

***Proposition 1.82.* (reflection onto local objects is localization with respect to left adjoints)**

Let $\mathcal{C}$ be a category (Def. 1.1) and let $S \subset \mathrm{Mor}_{\mathcal{C}}$ be a class of morphisms in $\mathcal{C}$. Then the reflection onto the S-local objects (Def. 1.79) satisfies, if it exists, the universal property of a localization of categories (Def. 1.76) with respect to left adjoint functors inverting S.

***Proof*.** Write

$$\mathcal{C}_S \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{C}$$

for the reflective subcategory-inclusion of the S-local objects.

Say that a morphism f in $\mathcal{C}$ is an S-*local morphism* if for every S-local object $A \in \mathcal{C}$ the hom-functor (Example 1.17) from f to A yields a bijection $\mathrm{Hom}_{\mathcal{C}}(f, A)$. Notice that, by the Yoneda embedding for $\mathcal{C}_S$ (Prop. 1.30), the S-local morphisms are precisely the morphisms that are taken to isomorphisms by the reflector L (via Example 1.31).

Now let

$$(F \dashv G) : \mathcal{C} \underoverset{\underset{G}{\longleftarrow}}{\overset{F}{\longrightarrow}}{\bot} \mathcal{D}$$

be a pair of adjoint functors, such that the left adjoint F inverts the morphisms in S. By the adjunction hom-isomorphism (10) it follows that G takes values in S-local objects. This in turn implies, now via the Yoneda embedding for $\mathcal{D}$, that F inverts all S-local morphisms, and hence all morphisms that are inverted by L.

Thus the essentially unique factorization of F through L now follows by Prop. 1.77. ▮

2. Basic notions of Categorical algebra

We have seen that the existence of Cartesian products in a category $\mathcal{C}$ equips is with a functor of the form

$$\mathcal{C} \times \mathcal{C} \xrightarrow{(-)\times(-)} \mathcal{C}$$

which is directly analogous to the operation of multiplication in an associative algebra or even just in a semigroup (or monoid), just "categorified" (Example 2.2 below). This is made precise by the concept of a *monoidal category* (Def. 2.1 below).

This relation between category theory and algebra leads to the fields of *categorical algebra* and of *universal algebra*.

Here we are mainly interested in monoidal categories as a foundations for enriched category theory, to which we turn below.

Monoidal categories

Definition 2.1. (monoidal category)

An_monoidal category_ is a category $\mathcal{C}$ (Def. 1.1) equipped with

1. a functor (Def. 1.15)

   $$\otimes \;:\; \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

   out of the product category of $\mathcal{C}$ with itself (Example 1.14), called the ***tensor product***,

2. an object

   $$1 \in \mathrm{Obj}_{\mathcal{C}}$$

   called the ***unit object*** or ***tensor unit***,

3. a natural isomorphism (Def. 1.23)

   $$a \;:\; ((-) \otimes (-)) \otimes (-) \xrightarrow{\simeq} (-) \otimes ((-) \otimes (-))$$

   called the ***associator***,

4. a natural isomorphism

   $$\ell \;:\; (1 \otimes (-)) \xrightarrow{\simeq} (-)$$

   called the ***left unitor***, and a natural isomorphism

$$r \,:\, (-) \otimes 1 \overset{\simeq}{\longrightarrow} (-)$$

called the ***right unitor***,

such that the following two kinds of diagrams commute, for all objects involved:

1. ***triangle identity***:

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\rho_x \otimes 1_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "1_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

2. the ***pentagon identity***:

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dr, "\alpha_{w,x,y \otimes z}"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "\alpha_{w \otimes x, y, z}"] \arrow[d, "\alpha_{w,x,y} \otimes \mathrm{id}_z"'] & & (w \otimes (x \otimes (y \otimes z))) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "\alpha_{w, x \otimes y, z}"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_w \otimes \alpha_{x,y,z}"']
\end{tikzcd}$$

***Example 2.2*. (*cartesian monoidal category*)**

Let $\mathcal{C}$ be a category in which all finite products exist. Then $\mathcal{C}$ becomes a monoidal category (Def. 2.1) by

1. taking the tensor product to be the Cartesian product

$$X \otimes Y \;\coloneqq\; X \times Y$$

2. taking the unit object to be the terminal object (Def. 1.5)

$$I \;\coloneqq\; *$$

Monoidal categories of this form are called *cartesian monoidal categories*.

***Lemma 2.3*. (*Kelly 64*)**

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category, def. 2.1. Then the left and right unitors ℓ and r satisfy the following conditions:

1. $\ell_1 = r_1 \,:\, 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1$;

2. for all objects $x, y \in \mathcal{C}$ the following diagrams commutes:

$$
\begin{array}{ccc}
(1 \otimes x) \otimes y & & \\
{}^{\mathllap{\alpha_{1,x,y}}}\downarrow & \searrow^{\ell_x \otimes \mathrm{id}_y} & \\
1 \otimes (x \otimes y) & \underset{\ell_{x \otimes y}}{\longrightarrow} & x \otimes y
\end{array}
;
$$

and

$$
\begin{array}{ccc}
x \otimes (y \otimes 1) & & \\
{}^{\mathllap{\alpha^{-1}_{1,x,y}}}\downarrow & \searrow^{\mathrm{id}_x \otimes r_y} & \\
(x \otimes y) \otimes 1 & \underset{r_{x \otimes y}}{\longrightarrow} & x \otimes y
\end{array}
;
$$

For ***proof*** see at *monoidal category* this lemma and this lemma.

Remark 2.4. Just as for an associative algebra it is sufficient to demand $1a = a$ and $a1 = a$ and $(ab)c = a(bc)$ in order to have that expressions of arbitrary length may be re-bracketed at will, so there is a *coherence theorem for monoidal categories* which states that all ways of freely composing the unitors and associators in a monoidal category (def. 2.1) to go from one expression to another will coincide. Accordingly, much as one may drop the notation for the bracketing in an associative algebra altogether, so one may, with due care, reason about monoidal categories without always making all unitors and associators explicit.

(Here the qualifier "freely" means informally that we must not use any non-formal identification between objects, and formally it means that the diagram in question must be in the image of a strong monoidal functor from a *free* monoidal category. For example if in a particular monoidal category it so happens that the object $X \otimes (Y \otimes Z)$ is actually *equal* to $(X \otimes Y) \otimes Z$, then the various ways of going from one expression to another using only associators *and* this equality no longer need to coincide.)

***Definition 2.5*. (*braided monoidal category*)**

A ***braided monoidal category***, is a monoidal category $\mathcal{C}$ (def. 2.1) equipped with a natural isomorphism (Def. 1.23)

$$
\tau_{x,y} : x \otimes y \to y \otimes x \tag{37}
$$

called the ***braiding***, such that the following two kinds of diagrams commute for all objects involved ("hexagon identities"):

$$
\begin{array}{ccccc}
(x \otimes y) \otimes z & \overset{a_{x,y,z}}{\longrightarrow} & x \otimes (y \otimes z) & \overset{\tau_{x,y \otimes z}}{\longrightarrow} & (y \otimes z) \otimes x \\
\downarrow^{\tau_{x,y} \otimes \mathrm{Id}} &&&& \downarrow^{a_{y,z,x}} \\
(y \otimes x) \otimes z & \overset{a_{y,x,z}}{\longrightarrow} & y \otimes (x \otimes z) & \overset{\mathrm{Id} \otimes \tau_{x,z}}{\longrightarrow} & y \otimes (z \otimes x)
\end{array}
$$

and

$$
\begin{array}{ccccc}
x \otimes (y \otimes z) & \overset{a^{-1}_{x,y,z}}{\longrightarrow} & (x \otimes y) \otimes z & \overset{\tau_{x \otimes y, z}}{\longrightarrow} & z \otimes (x \otimes y) \\
\downarrow^{\mathrm{Id} \otimes \tau_{y,z}} &&&& \downarrow^{a^{-1}_{z,x,y}} \\
x \otimes (z \otimes y) & \overset{a^{-1}_{x,z,y}}{\longrightarrow} & (x \otimes z) \otimes y & \overset{\tau_{x,z} \otimes \mathrm{Id}}{\longrightarrow} & (z \otimes x) \otimes y
\end{array}
\,,
$$

where $a_{x,y,z} \colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ denotes the components of the associator of $\mathcal{C}^{\otimes}$.

Definition 2.6. A ***symmetric monoidal category*** is a braided monoidal category (def. 2.5) for which the braiding

$$
\tau_{x,y} \colon x \otimes y \to y \otimes x
$$

satisfies the condition:

$$
\tau_{y,x} \circ \tau_{x,y} = 1_{x \otimes y}
$$

for all objects x, y

Remark 2.7. In analogy to the coherence theorem for monoidal categories (remark 2.4) there is a coherence theorem for symmetric monoidal categories (def. 2.6), saying that every diagram built freely (see remark 2.7) from associators, unitors and braidings such that both sides of the diagram correspond to the same permutation of objects, coincide.

Definition 2.8. ***(symmetric closed monoidal category)***

Given a symmetric monoidal category $\mathcal{C}$ with tensor product $\otimes$ (def. 2.6) it is called a *closed monoidal category* if for each $Y \in \mathcal{C}$ the functor $Y \otimes (-) \simeq (-) \otimes Y$ has a right adjoint, denoted $\hom(Y, -)$

$$
\mathcal{C} \underoverset{\underset{[Y,-]}{\longrightarrow}}{\overset{(-)\otimes Y}{\longleftarrow}}{\bot} \mathcal{C} \,, \tag{38}
$$

hence if there are natural bijections

$$
\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \;\simeq\; \mathrm{Hom}_{\mathcal{C}}\, C(X, [Y, Z])
$$

for all objects $X, Z \in \mathcal{C}$.

Since for the case that $X = 1$ is the tensor unit of $\mathcal{C}$ this means that

$$\mathrm{Hom}_{\mathcal{C}}(1, [Y, Z]) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, Z)\,,$$

the object $[Y, Z] \in \mathcal{C}$ is an enhancement of the ordinary hom-set $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ to an object in $\mathcal{C}$. Accordingly, it is also called the ***internal hom*** between Y and Z.

The adjunction counit (Def. 1.33) in this case is called the *evaluation morphism*

$$X \otimes [X, Y] \overset{\mathrm{ev}}{\longrightarrow} Y \qquad (39)$$

***Example 2.9*. (Set is a cartesian closed category)**

The category Set of all sets (Example 1.2) equipped with its cartesian monoidal category-structure (Example 2.2) is a closed monoidal category (Def. 2.8), hence a *cartesian closed category*. The Cartesian product is the original Cartesian product of sets, and the internal hom is the function set $[X, Y]$ of functions from X to Y

***Example 2.10*. (tensor product of abelian groups is closed monoidal category symmetric monoidal category-structure)**

The category Ab of abelian groups (as in Example 1.3) becomes a symmetric monoidal category (Def. 2.6) with tensor product the actual tensor product of abelian groups $\otimes_{\mathbb{Z}}$ and with tensor unit the additive group $\mathbb{Z}$ of integers. Again the associator, unitor and braiding isomorphism are the evident ones coming from the underlying sets.

This is a closed monoidal category with internal hom $\mathrm{hom}(A, B)$ being the set of homomorphisms $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ equipped with the pointwise group structure for $\phi_1, \phi_2 \in \mathrm{Hom}_{\mathrm{Ab}}(A, B)$ then $(\phi_1 + \phi_2)(a) \coloneqq \phi_1(a) + \phi_2(b) \ \in B$.

This is the archetypical case that motivates the notation "$\otimes$" for the pairing operation in a monoidal category.

***Example 2.11*. (Cat and Grpd are cartesian closed categories)**

The category Cat (Example 1.16) of all small categories (Example 1.6) is a cartesian monoidal category-structure (Example 2.2) with Cartesian product given by forming product categories (Example 1.14).

Inside this, the full subcategory (Example 1.20) Grpd (Example 1.16) of all small groupoids (Example 1.10) is itself a cartesian monoidal category-structure (Example 2.2) with Cartesian product given by forming product categories (Example 1.14).

In both cases this yields a closed monoidal category (Def. 2.8), hence a cartesian closed category: the internal hom is given by the functor category construction (Example 1.25).

***Example 2.12*. (*categories of presheaves are cartesian closed*)**

Let $\mathcal{C}$ be a category and write $[\mathcal{C}^{op}, \mathrm{Set}]$ for its category of presheaves (Example 1.26).

This is

1. a cartesian monoidal category (Example 2.2), whose Cartesian product is given objectwise in $\mathcal{C}$ by the Cartesian product in Set:
for $\mathbf{X}, \mathbf{Y} \in [\mathcal{C}^{op}, \mathrm{Set}]$, their Cartesian product $\mathbf{X} \times \mathbf{Y}$ exists and is given by

$$\mathbf{X} \times \mathbf{Y} \;:\; \begin{array}{ccc} c_1 & \mapsto & \mathbf{X}(c_1) \times \mathbf{Y}(c_1) \\ {}^{f}\downarrow & & \uparrow^{\mathbf{X}(f) \times \mathbf{Y}(f)} \\ c_2 & \mapsto & \mathbf{X}(c_2) \times \mathbf{Y}(c_2) \end{array}$$

2. a cartesian closed category (Def. 2.8), whose internal hom is given for $\mathbf{X}, \mathbf{Y} \in [\mathcal{C}^{op}, \mathrm{Set}]$ by

$$[\mathbf{X}, \mathbf{Y}] \;:\; \begin{array}{ccc} c_1 & \mapsto & \mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(y(c_1) \times \mathbf{X}, \mathbf{y}) \\ {}^{f}\downarrow & & \uparrow^{\mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(y(f) \times \mathbf{X}, \mathbf{y})} \\ c_2 & \mapsto & \mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(y(c_2) \times \mathbf{X}, \mathbf{y}) \end{array}$$

Here $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Set}]$ denotes the Yoneda embedding and $\mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(-,-)$ is the hom-functor on the category of presheaves.

***Proof*.** The first statement is a special case of the general fact that limits of presheaves are computed objectwise (Example 3.5).

For the second statement, first assume that $[\mathbf{X}, \mathbf{Y}]$ does exist. Then by the adjunction hom-isomorphism (10) we have for any other presheaf $\mathbf{Z}$ a natural isomorphism of the form

$$\mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]) \;\simeq\; \mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(\mathbf{Z} \times \mathbf{X}, \mathbf{Y}) \,. \qquad (40)$$

This holds in particular for $\mathbf{Z} = y(c)$ a representable presheaf (Example 1.27) and so the Yoneda lemma (Prop. 1.29) implies that if it exists, then $[\mathbf{X}, \mathbf{Y}]$ must have the claimed form:

$$\begin{aligned} [\mathbf{X}, \mathbf{Y}](c) & \simeq \mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(y(c), [\mathbf{X}, \mathbf{Y}]) \\ & \simeq \mathrm{Hom}_{[\mathcal{C}^{op}, \mathrm{Set}]}(y(c) \times \mathbf{X}, \mathbf{Y}) \,. \end{aligned}$$

Hence it remains to show that this formula does make (40) hold generally.

For this we use the equivalent characterization of adjoint functors from Prop. 1.42, in terms of the adjunction counit providing a system of universal arrows (Def. 1.41).

Define a would-be adjunction counit, hence a would-be *evaluation* morphism (39), by

$$
\begin{array}{ccc}
\mathbf{X} \times [\mathbf{X}, \mathbf{Y}] & \overset{\mathrm{ev}}{\longrightarrow} & \mathbf{Y} \\
\mathbf{X}(c) \times \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]}(y(c) \times \mathbf{X}, \mathbf{Y}) & \overset{\mathrm{ev}_c}{\longrightarrow} & \mathbf{Y}(c) \\
(x, \phi) & \mapsto & \phi_c(\mathrm{id}_c, x)
\end{array}
$$

Then it remains to show that for every morphism of presheaves of the form $\mathbf{X} \times \mathbf{A} \overset{f}{\longrightarrow} \mathbf{Y}$ there is a *unique* morphism $\widetilde{f} \;:\; \mathbf{A} \longrightarrow [\mathbf{X}, \mathbf{Y}]$ such that

$$
\begin{array}{ccc}
\mathbf{X} \times \mathbf{A} & \overset{\mathbf{X} \times \widetilde{f}}{\longrightarrow} & \mathbf{X} \times [\mathbf{X}, \mathbf{Y}] \\
& {}_{f}\searrow \quad \swarrow_{\mathrm{ev}} & \\
& \mathbf{Y} &
\end{array}
\tag{41}
$$

The commutativity of this diagram means in components at $c \in \mathcal{C}$ that, that for all $x \in \mathbf{X}(c)$ and $a \in \mathbf{A}(c)$ we have

$$
\begin{aligned}
\mathrm{ev}_c(x, \widetilde{f}_c(a)) & \coloneqq (\widetilde{f}_c(a))_c(\mathrm{id}_c, x) \\
& = f_c(x, a)
\end{aligned}
$$

Hence this fixes the component $\widetilde{f}_c(a)_c$ when its first argument is the identity morphism id_c. But let $g \;:\; d \to c$ be any morphism and chase (id_c, x) through the naturality diagram for $\widetilde{f}_c(a)$:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(c,c) \times \mathbf{X}(c) & \overset{(\widetilde{f}_c(a))_c}{\longrightarrow} & \mathbf{Y}(c) \\
{}^{g^\ast}\downarrow && \downarrow^{g^\ast} \\
\mathrm{Hom}_{\mathcal{C}}(d,c) \times \mathbf{X}(d) & \overset{(\widetilde{f}_c(a))_d}{\longrightarrow} & \mathbf{Y}(d)
\end{array}
\qquad
\begin{array}{ccc}
\{(\mathrm{id}_c, x)\} & \longrightarrow & \{f_c(x,a)\} \\
\downarrow && \downarrow \\
\{(g, g^\ast(x))\} & \longrightarrow & \{f_d(g^\ast(x), g^\ast(a))\}
\end{array}
$$

This shows that $(\widetilde{f}_c(a))_d$ is fixed to be given by

$$
(\widetilde{f}_c(a))_d(g, x') \;=\; f_d(x', g^\ast(a))
\tag{42}
$$

at least on those pairs (g, x') such that x' is in the image of $g^\ast$.

But, finally, $(\widetilde{f}_c(a))_d$ is also natural in c

$$
\begin{array}{ccc}
\mathbf{A}(c) & \overset{\widetilde{f}_c}{\longrightarrow} & [\mathbf{X}, \mathbf{Y}](c) \\
{}^{g^*}\downarrow & & \downarrow^{g^*} \\
\mathbf{A}(d) & \underset{}{\overset{\widetilde{f}_d}{\longrightarrow}} & [\mathbf{X}, \mathbf{Y}](d)
\end{array}
$$

which implies that (42) must hold generally. Hence naturality implies that (41) indeed has a unique solution. ▮

The internal hom (Def. 2.8) turns out to share all the abstract properties of the ordinary (external) hom-functor (Def. 1.17), even though this is not completely manifest from its definition. We make this explicit by the following three propositions.

***Proposition 2.13*. (internal hom bifunctor)**

For $\mathcal{C}$ a closed monoidal category (Def. 2.8), there is a unique functor (Def. 1.15) out of the product category (Def. 1.14) of $\mathcal{C}$ with its opposite category (Def. 1.13)

$$
[-,-] \,:\, \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}
$$

such that for each $X \in \mathcal{C}$ it coincides with the internal hom $[X,-]$ (38) as a functor in the second variable, and such that there is a natural isomorphism

$$
\mathrm{Hom}(X, [Y,Z]) \;\simeq\; \mathrm{Hom}(X \otimes Y, Z)
$$

which is natural not only in X and Z, but also in Y.

***Proof*.** We have a natural isomorphism for each fixed Y, and hence in particular for fixed Y and fixed Z by (38). With this the statement follows by Prop. 1.40. ▮

In fact the 3-variable adjunction from Prop. 2.13 even holds internally:

***Proposition 2.14*. (internal tensor/hom-adjunction)**

In a symmetric closed monoidal category (def. 2.8) there are natural isomorphisms

$$
[X \otimes Y, Z] \;\simeq\; [X, [Y,Z]]
$$

whose image under $\mathrm{Hom}_{\mathcal{C}}(1,-)$ (see also Example 2.38 below) are the defining natural bijections of Prop. 2.13.

***Proof*.** Let $A \in \mathcal{C}$ be any object. By applying the natural bijections from Prop. 2.13, there are composite natural bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A, [X \otimes Y, Z]) & \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes (X \otimes Y), Z) \\
& \simeq \mathrm{Hom}_{\mathcal{C}}((A \otimes X) \otimes Y, Z) \\
& \simeq \mathrm{Hom}_{\mathcal{C}}(A \otimes X, [Y, Z]) \\
& \simeq \mathrm{Hom}_{\mathcal{C}}(A, [X, [Y, Z]])
\end{aligned}
$$

Since this holds for all A, the fully faithfulness of the Yoneda embedding (Prop. 1.30) says that there is an isomorphism $[X \otimes Y, Z] \simeq [X, [Y, Z]]$. Moreover, by taking $A = 1$ in the above and using the left unitor isomorphisms $A \otimes (X \otimes Y) \simeq X \otimes Y$ and $A \otimes X \simeq X$ we get a commuting diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(1, [X \otimes Y, Z]) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(1, [X, [Y, Z]]) \\
{}^{\simeq}\downarrow && \downarrow^{\simeq} \\
\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) & \overset{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{C}}(X, [Y, Z])
\end{array}
\,.
$$

∎

Also the key respect of the hom-functor for limits is inherited by internal hom-functors

***Proposition 2.15*. (internal hom preserves limits)**

Let $\mathcal{C}$ be a symmetric closed monoidal category with internal hom-bifunctor $[-,-]$ (Prop. 2.13). Then this bifunctor preserves limits in the second variable, and sends colimits in the first variable to limits:

$$
[X, \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} Y(j)] \;\simeq\; \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} [X, Y(j)]
$$

and

$$
[\underset{\underset{j \in \mathcal{J}}{\longrightarrow}}{\lim} Y(j), X] \;\simeq\; \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} [Y(j), X]
$$

***Proof*.** For $X \in \mathcal{X}$ any object, $[X, -]$ is a right adjoint by definition, and hence preserves limits by Prop. 3.8.

For the other case, let $Y : \mathcal{L} \to \mathcal{C}$ be a diagram in $\mathcal{C}$, and let $C \in \mathcal{C}$ be any object. Then there are isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(C, \underset{\underset{j \in \mathcal{J}}{\longrightarrow}}{\lim} Y(j), X) & \simeq \mathrm{Hom}_{\mathcal{C}}(C \otimes \underset{\underset{j \in \mathcal{J}}{\longrightarrow}}{\lim} Y(j), X) \\
& \simeq \mathrm{Hom}_{\mathcal{C}}(\underset{\underset{j \in \mathcal{J}}{\longrightarrow}}{\lim}(C \otimes Y(j)), X) \\
& \simeq \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} \mathrm{Hom}_{\mathcal{C}}((C \otimes Y(j)), X) \\
& \simeq \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} \mathrm{Hom}_{\mathcal{C}}(C, [Y(j), X]) \\
& \simeq \mathrm{Hom}_{\mathcal{C}}(C, \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} [Y(j), X])
\end{aligned}
$$

which are natural in $C \in \mathcal{C}$, where we used that the ordinary hom-functor preserves limits (Prop. 3.6), and that the left adjoint $C \otimes (-)$ preserves colimits, since left adjoints preserve colimits (Prop. 3.8).

Hence by the fully faithfulness of the Yoneda embedding, there is an isomorphism

$$
\left[\underset{\underset{j \in \mathcal{J}}{\longrightarrow}}{\lim} Y(j), X \right] \xrightarrow{\simeq} \underset{\underset{j \in \mathcal{J}}{\longleftarrow}}{\lim} [Y(j), X] \,.
$$

▮

Now that we have seen monoidal categories with various extra properties, we next look at functors which preserve these:

Definition 2.16. (monoidal functors)

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories (def. 2.1). A *lax monoidal functor* between them is

1. a functor (Def. 1.15)

$$
F \,:\, \mathcal{C} \longrightarrow \mathcal{D} \,,
$$

2. a morphism

$$
\epsilon \,:\, 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}}) \tag{43}
$$

3. a natural transformation (Def. 1.23)

$$
\mu_{x,y} \,:\, F(x) \otimes_{\mathcal{D}} F(y) \longrightarrow F(x \otimes_{\mathcal{C}} y) \tag{44}
$$

for all $x, y \in \mathcal{C}$

satisfying the following conditions:

1. ***(associativity)*** For all objects $x, y, z \in \mathcal{C}$ the following diagram commutes

$$
\begin{array}{ccc}
(F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) & \underoverset{\simeq}{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \\
{}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id} \otimes \mu_{y,z}} \\
F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) && F(x) \otimes_{\mathcal{D}} (F(x \otimes_{\mathcal{C}} y)) \\
{}^{\mu_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\mu_{x, y \otimes_{\mathcal{C}} z}} \\
F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z))
}
\end{array}
,
$$

   where $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ denote the associators of the monoidal categories;

2. ***(unitality)*** For all $x \in \mathcal{C}$ the following diagrams commutes

$$
\begin{array}{ccc}
1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) & \overset{\epsilon \otimes \mathrm{id}}{\longrightarrow} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \\
{}^{\ell^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{1_{\mathcal{C}}, x}} \\
F(x) & \overset{F(\ell^{\mathcal{C}}_x)}{\longleftarrow} & F(1 \otimes_{\mathcal{C}} x)
\end{array}
$$

   and

$$
\begin{array}{ccc}
F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} & \overset{\mathrm{id} \otimes \epsilon}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \\
{}^{r^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{x, 1_{\mathcal{C}}}} \\
F(x) & \overset{F(r^{\mathcal{C}}_x)}{\longleftarrow} & F(x \otimes_{\mathcal{C}} 1)
\end{array}
,
$$

   where $\ell^{\mathcal{C}}, \ell^{\mathcal{D}}, r^{\mathcal{C}}, r^{\mathcal{D}}$ denote the left and right unitors of the two monoidal categories, respectively.

If ϵ and alll $\mu_{x,y}$ are isomorphisms, then F is called a ***strong monoidal functor***.

If moreover $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are equipped with the structure of braided monoidal categories (def. 2.5) with braidings $\tau^{\mathcal{C}}$ and $\tau^{\mathcal{D}}$, respectively, then the lax monoidal functor F is called a ***braided monoidal functor*** if in addition the following diagram commutes for all objects $x, y \in \mathcal{C}$

$$
\begin{array}{ccc}
F(x) \otimes_{\mathcal{C}} F(y) & \overset{\tau^{\mathcal{D}}_{F(x),F(y)}}{\longrightarrow} & F(y) \otimes_{\mathcal{D}} F(x) \\
{}^{\mu_{x,y}}\downarrow && \downarrow^{\mu_{y,x}} \\
F(x \otimes_{\mathcal{C}} y) & \underset{F(\tau^{\mathcal{C}}_{x,y})}{\longrightarrow} & F(y \otimes_{\mathcal{C}} x)
\end{array}
\,.
$$

A homomorphism $f : (F_1, \mu_1, \epsilon_1) \longrightarrow (F_2, \mu_2, \epsilon_2)$ between two (braided) lax monoidal functors is a ***monoidal natural transformation***, in that it is a natural transformation $f_x : F_1(x) \longrightarrow F_2(x)$ of the underlying functors

compatible with the product and the unit in that the following diagrams commute for all objects $x, y \in \mathcal{C}$:

$$
\begin{array}{ccc}
F_1(x) \otimes_{\mathcal{D}} F_1(y) & \overset{f(x) \otimes_{\mathcal{D}} f(y)}{\longrightarrow} & F_2(x) \otimes_{\mathcal{D}} F_2(y) \\
{}^{(\mu_1)_{x,y}}\downarrow && \downarrow^{(\mu_2)_{x,y}} \\
F_1(x \otimes_{\mathcal{C}} y) & \underset{f(x \otimes_{\mathcal{C}} y)}{\longrightarrow} & F_2(x \otimes_{\mathcal{C}} y)
\end{array}
$$

and

$$
\begin{array}{ccc}
& 1_{\mathcal{D}} & \\
{}^{\epsilon_1}\swarrow && \searrow^{\epsilon_2} \\
F_1(1_{\mathcal{C}}) & \underset{f(1_{\mathcal{C}})}{\longrightarrow} & F_2(1_{\mathcal{C}})
\end{array}
\,.
$$

We write $\mathrm{MonFun}(\mathcal{C}, \mathcal{D})$ for the resulting category of lax monoidal functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, similarly $\mathrm{BraidMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between braided monoidal categories, and $\mathrm{SymMonFun}(\mathcal{C}, \mathcal{D})$ for the category of braided monoidal functors between symmetric monoidal categories.

Remark 2.17. In the literature the term “monoidal functor” often refers by default to what in def. 2.16 is called a *strong monoidal functor*. But for the purpose of the discussion of functors with smash product below, it is crucial to admit the generality of lax monoidal functors.

If $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are symmetric monoidal categories (def. 2.6) then a braided monoidal functor (def. 2.16) between them is often called a ***symmetric monoidal functor***.

Proposition 2.18. *For $\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D} \overset{G}{\longrightarrow} \mathcal{E}$ two composable lax monoidal functors (def. 2.16) between monoidal categories, then their composite $F \circ G$ becomes a lax monoidal functor with structure morphisms*

$$\epsilon^{G \circ F} \;:\; 1_{\mathcal{E}} \overset{\epsilon^G}{\longrightarrow} G(1_{\mathcal{D}}) \overset{G(\epsilon^F)}{\longrightarrow} G(F(1_{\mathcal{C}}))$$

and

$$\mu^{G \circ F}_{c_1, c_2} \;:\; G(F(c_1)) \otimes_{\mathcal{E}} G(F(c_2)) \overset{\mu^G_{F(c_1), F(c_2)}}{\longrightarrow} G(F(c_1) \otimes_{\mathcal{D}} F(c_2)) \overset{G(\mu^F_{c_1, c_2})}{\longrightarrow} G(F(c_1 \otimes_{\mathcal{C}} c_2)) \,.$$

Algebras and modules

Definition 2.19. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (Def. 2.1), then a ***monoid internal to*** $(\mathcal{C}, \otimes, 1)$ is

1. an object $A \in \mathcal{C}$;
2. a morphism $e \;:\; 1 \longrightarrow A$ (called the *unit*)
3. a morphism $\mu \;:\; A \otimes A \longrightarrow A$ (called the *product*);

such that

1. (associativity) the following diagram commutes

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \underoverset{\simeq}{a_{A,A,A}}{\longrightarrow} & A \otimes (A \otimes A) & \overset{A \otimes \mu}{\longrightarrow} & A \otimes A \\ {}^{\mu \otimes A}\downarrow && && \downarrow^{\mu} \\ A \otimes A & \longrightarrow & & \overset{\mu}{\longrightarrow} & A \end{array} \,,$$

   where a is the associator isomorphism of $\mathcal{C}$;

2. (unitality) the following diagram commutes:

$$\begin{array}{ccccc} 1 \otimes A & \overset{e \otimes \mathrm{id}}{\longrightarrow} & A \otimes A & \overset{\mathrm{id} \otimes e}{\longleftarrow} & A \otimes 1 \\ & {}_{\ell}\searrow & \downarrow^{\mu} & \swarrow_{r} \\ && A \end{array} \,,$$

   where ℓ and r are the left and right unitor isomorphisms of $\mathcal{C}$.

Moreover, if $(\mathcal{C}, \otimes, 1)$ has the structure of a symmetric monoidal category (def. 2.6) $(\mathcal{C}, \otimes, 1, B)$ with symmetric braiding τ, then a monoid (A, μ, e) as above is called a ***commutative monoid in*** $(\mathcal{C}, \otimes, 1, B)$ if in addition

- (commutativity) the following diagram commutes

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\tau_{A,A}", "\simeq"'] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
.
$$

A homomorphism of monoids $(A_1, \mu_1, e_1) \longrightarrow (A_2, \mu_2, f_2)$ is a morphism

$$f : A_1 \longrightarrow A_2$$

in $\mathcal{C}$, such that the following two diagrams commute

$$
\begin{tikzcd}
A_1 \otimes A_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_1"'] & A_2 \otimes A_2 \arrow[d, "\mu_2"] \\
A_1 \arrow[r, "f"'] & A_2
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1_c \arrow[r, "e_1"] \arrow[dr, "e_2"'] & A_1 \arrow[d, "f"] \\
& A_2
\end{tikzcd}
.
$$

Write $\mathrm{Mon}(\mathcal{C}, \otimes, 1)$ for the ***category of monoids*** in $\mathcal{C}$, and $\mathrm{CMon}(\mathcal{C}, \otimes, 1)$ for its full subcategory of commutative monoids.

Example 2.20. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (Def. 2.1), the tensor unit 1 is a monoid in $\mathcal{C}$ (def. 2.19) with product given by either the left or right unitor

$$\ell_1 = r_1 : 1 \otimes 1 \overset{\simeq}{\longrightarrow} 1 \,.$$

By lemma 2.3, these two morphisms coincide and define an associative product with unit the identity $\mathrm{id} : 1 \to 1$.

If $(\mathcal{C}, \otimes, 1)$ is a symmetric monoidal category (def. 2.6), then this monoid is a commutative monoid.

Example 2.21. Given a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 2.6), and given two commutative monoids (E_i, μ_i, e_i) $i \in \{1, 2\}$ (def. 2.19), then the tensor product $E_1 \otimes E_2$ becomes itself a commutative monoid with unit morphism

$$e : 1 \overset{\simeq}{\longrightarrow} 1 \otimes 1 \overset{e_1 \otimes e_2}{\longrightarrow} E_1 \otimes E_2$$

(where the first isomorphism is, $\ell_1^{-1} = r_1^{-1}$ (lemma 2.3)) and with product morphism given by

$$E_1 \otimes E_2 \otimes E_1 \otimes E_2 \xrightarrow{\mathrm{id} \otimes \tau_{E_2,E_1} \otimes \mathrm{id}} E_1 \otimes E_1 \otimes E_2 \otimes E_2 \xrightarrow{\mu_1 \otimes \mu_2} E_1 \otimes E_2$$

(where we are notationally suppressing the associators and where τ denotes the braiding of $\mathcal{C}$).

That this definition indeed satisfies associativity and commutativity follows from the corresponding properties of (E_i, μ_i, e_i), and from the hexagon identities for the braiding (def. 2.5) and from symmetry of the braiding.

Similarly one checks that for $E_1 = E_2 = E$ then the unit maps

$$E \simeq E \otimes 1 \xrightarrow{\mathrm{id} \otimes e} E \otimes E$$

$$E \simeq 1 \otimes E \xrightarrow{e \otimes 1} E \otimes E$$

and the product map

$$\mu \,:\, E \otimes E \longrightarrow E$$

and the braiding

$$\tau_{E,E} \,:\, E \otimes E \longrightarrow E \otimes E$$

are monoid homomorphisms, with $E \otimes E$ equipped with the above monoid structure.

Definition 2.22. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 2.1), and given (A, μ, e) a monoid in $(\mathcal{C}, \otimes, 1)$ (def. 2.19), then a ***left module object*** in $(\mathcal{C}, \otimes, 1)$ over (A, μ, e) is

1. an object $N \in \mathcal{C}$;
2. a morphism $\rho \,:\, A \otimes N \longrightarrow N$ (called the *action*);

such that

1. (unitality) the following diagram commutes:

$$\begin{array}{ccc} 1 \otimes N & \xrightarrow{e \otimes \mathrm{id}} & A \otimes N \\ & {}_{\ell}\searrow & \downarrow^{\rho} \\ & & N \end{array},$$

   where ℓ is the left unitor isomorphism of $\mathcal{C}$.

2. (action property) the following diagram commutes

$$
\begin{array}{ccccc}
(A \otimes A) \otimes N & \underset{\simeq}{\overset{a_{A,A,N}}{\longrightarrow}} & A \otimes (A \otimes N) & \overset{A \otimes \rho}{\longrightarrow} & A \otimes N \\
{}^{\mu \otimes N}\downarrow && && \downarrow^{\rho} \\
A \otimes N & \longrightarrow & & \overset{\rho}{\longrightarrow} & N
\end{array}
,
$$

A homomorphism of left A-module objects

$$
(N_1, \rho_1) \longrightarrow (N_2, \rho_2)
$$

is a morphism

$$
f \;:\; N_1 \longrightarrow N_2
$$

in $\mathcal{C}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
A \otimes N_1 & \overset{A \otimes f}{\longrightarrow} & A \otimes N_2 \\
{}^{\rho_1}\downarrow && \downarrow^{\rho_2} \\
N_1 & \underset{f}{\longrightarrow} & N_2
\end{array}
\,.
$$

For the resulting ***category of modules*** of left A-modules in $\mathcal{C}$ with A-module homomorphisms between them, we write

$$
A\,\mathrm{Mod}(\mathcal{C}) \,.
$$

Example 2.23. Given a monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 2.1) with the tensor unit 1 regarded as a monoid in a monoidal category via example 2.20, then the left unitor

$$
\ell_C \;:\; 1 \otimes C \longrightarrow C
$$

makes every object $C \in \mathcal{C}$ into a left module, according to def. 2.22, over C. The action property holds due to lemma 2.3. This gives an equivalence of categories

$$
\mathcal{C} \simeq 1\mathrm{Mod}(\mathcal{C})
$$

of $\mathcal{C}$ with the category of modules over its tensor unit.

Example 2.24. The archetypical case in which all these abstract concepts reduce to the basic familiar ones is the symmetric monoidal category Ab of abelian groups from example 2.10.

1. A monoid in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 2.19) is equivalently a ring.

2. A commutative monoid in in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 2.19) is equivalently a commutative ring R.
3. An R-module object in $(\mathrm{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (def. 2.22) is equivalently an R-module;
4. The tensor product of R-module objects (def. 2.27) is the standard tensor product of modules.
5. The category of module objects $R\,\mathrm{Mod}(\mathrm{Ab})$ (def. 2.27) is the standard category of modules $R\,\mathrm{Mod}$.

Example 2.25. Closely related to the example 2.24, but closer to the structure we will see below for spectra, are monoids in the category of chain complexes $(\mathrm{Ch}_\bullet, \otimes, \mathbb{Z})$ from example . These monoids are equivalently differential graded algebras.

Proposition 2.26. *In the situation of def. 2.22, the monoid (A, μ, e) canonically becomes a left module over itself by setting $\rho \coloneqq \mu$. More generally, for $C \in \mathcal{C}$ any object, then $A \otimes C$ naturally becomes a left A-module by setting:*

$$\rho \;:\; A \otimes (A \otimes C) \underset{\simeq}{\overset{a^{-1}_{A,A,C}}{\longrightarrow}} (A \otimes A) \otimes C \overset{\mu \otimes \mathrm{id}}{\longrightarrow} A \otimes C \;.$$

The A-modules of this form are called ***free modules****.*

The free functor F constructing free A-modules is left adjoint to the forgetful functor U which sends a module (N, ρ) to the underlying object $U(N, \rho) \coloneqq N$.

$$A\,\mathrm{Mod}(\mathcal{C}) \underoverset{\underset{U}{\longrightarrow}}{\overset{F}{\longleftarrow}}{\bot} \mathcal{C} \;.$$

Proof. A homomorphism out of a free A-module is a morphism in $\mathcal{C}$ of the form

$$f \;:\; A \otimes C \longrightarrow N$$

fitting into the diagram (where we are notationally suppressing the associator)

$$\begin{array}{ccc} A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} \;.$$

Consider the composite

$$\tilde f \;:\; C \underset{\simeq}{\overset{\ell_C}{\longrightarrow}} 1 \otimes C \overset{e \otimes \mathrm{id}}{\longrightarrow} A \otimes C \overset{f}{\longrightarrow} N \,,$$

i.e. the restriction of f to the unit "in" A. By definition, this fits into a commuting square of the form (where we are now notationally suppressing the associator and the unitor)

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \underset{\mathrm{id} \otimes f}{\longrightarrow} & A \otimes N \end{array} \,.$$

Pasting this square onto the top of the previous one yields

$$\begin{array}{ccc} A \otimes C & \overset{\mathrm{id} \otimes \tilde f}{\longrightarrow} & A \otimes N \\ {}^{\mathrm{id} \otimes e \otimes \mathrm{id}}\downarrow && \downarrow^{=} \\ A \otimes A \otimes C & \overset{A \otimes f}{\longrightarrow} & A \otimes N \\ {}^{\mu \otimes \mathrm{id}}\downarrow && \downarrow^{\rho} \\ A \otimes C & \underset{f}{\longrightarrow} & N \end{array} \,,$$

where now the left vertical composite is the identity, by the unit law in A. This shows that f is uniquely determined by $\tilde f$ via the relation

$$f = \rho \circ (\mathrm{id}_A \otimes \tilde f) \,.$$

This natural bijection between f and $\tilde f$ establishes the adjunction. ▮

Definition 2.27. Given a closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 2.6, def. 2.8), given (A, μ, e) a commutative monoid in $(\mathcal{C}, \otimes, 1)$ (def. 2.19), and given (N_1, ρ_1) and (N_2, ρ_2) two left A-module objects (def.2.19), then

1. the ***tensor product of modules*** $N_1 \otimes_A N_2$ is, if it exists, the coequalizer

$$N_1 \otimes A \otimes N_2 \underoverset{\rho_1 \circ (\tau_{N_1, A} \otimes N_2)}{N_1 \otimes \rho_2}{\rightrightarrows} N_1 \otimes N_1 \overset{\mathrm{coeq}}{\longrightarrow} N_1 \otimes_A N_2$$

   and if $A \otimes (-)$ preserves these coequalizers, then this is equipped with the left A-action induced from the left A-action on N_1

2. the ***function module*** $\mathrm{hom}_A(N_1, N_2)$ is, if it exists, the equalizer

$$\mathrm{hom}_A(N_1, N_2) \overset{\mathrm{equ}}{\longrightarrow} \mathrm{hom}(N_1, N_2) \underoverset{\mathrm{hom}(A \otimes N_1, \rho_2) \circ (A \otimes (-))}{\mathrm{hom}(\rho_1, N_2)}{\rightrightarrows} \mathrm{hom}(A \otimes N_1, N_2) \,.$$

equipped with the left A-action that is induced by the left A-action on N_2 via

$$\frac{A \otimes \hom(X, N_2) \longrightarrow \hom(X, N_2)}{A \otimes \hom(X, N_2) \otimes X \overset{\mathrm{id} \otimes \mathrm{ev}}{\longrightarrow} A \otimes N_2 \overset{\rho_2}{\longrightarrow} N_2} \,.$$

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2 and lemma 2.2.8)

Proposition 2.28. *Given a closed symmetric monoidal category* $(\mathcal{C}, \otimes, 1)$ *(def. 2.6, def. 2.8), and given* (A, μ, e) *a commutative monoid in* $(\mathcal{C}, \otimes, 1)$ *(def. 2.19). If all coequalizers exist in* $\mathcal{C}$*, then the tensor product of modules* $\otimes_A$ *from def. 2.27 makes the category of modules* $A\,\mathrm{Mod}(\mathcal{C})$ *into a symmetric monoidal category,* $(A\,\mathrm{Mod}, \otimes_A, A)$ *with tensor unit the object* A *itself, regarded as an* A*-module via prop. 2.26.*

If moreover all equalizers exist, then this is a closed monoidal category (def. 2.8) with internal hom given by the function modules $\hom_A$ *of def. 2.27.*

(e.g. Hovey-Shipley-Smith 00, lemma 2.2.2, lemma 2.2.8)

Proof sketch. The associators and braiding for $\otimes_A$ are induced directly from those of $\otimes$ and the universal property of coequalizers. That A is the tensor unit for $\otimes_A$ follows with the same kind of argument that we give in the proof of example 2.29 below. ■

Example 2.29. For (A, μ, e) a monoid (def. 2.19) in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (def. 2.1), the tensor product of modules (def. 2.27) of two free modules (def. 2.26) $A \otimes C_1$ and $A \otimes C_2$ always exists and is the free module over the tensor product in $\mathcal{C}$ of the two generators:

$$(A \otimes C_1) \otimes_A (A \otimes C_2) \simeq A \otimes (C_1 \otimes C_2) \,.$$

Hence if $\mathcal{C}$ has all coequalizers, so that the category of modules is a monoidal category $(A\,\mathrm{Mod}, \otimes_A, A)$ (prop. 2.28) then the free module functor (def. 2.26) is a strong monoidal functor (def. 2.16)

$$F \;:\; (\mathcal{C}, \otimes, 1) \longrightarrow (A\,\mathrm{Mod}, \otimes_A, A) \,.$$

Proof. It is sufficient to show that the diagram

$$A \otimes A \otimes A \underoverset{\mathrm{id} \otimes \mu}{\mu \otimes \mathrm{id}}{\rightrightarrows} A \otimes A \overset{\mu}{\longrightarrow} A$$

is a coequalizer diagram (we are notationally suppressing the associators), hence that $A \otimes_A A \simeq A$, hence that the claim holds for $C_1 = 1$ and $C_2 = 1$.

To that end, we check the universal property of the coequalizer:

First observe that μ indeed coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$, since this is just the associativity clause in def. 2.19. So for $f : A \otimes A \longrightarrow Q$ any other morphism with this property, we need to show that there is a unique morphism $\phi : A \longrightarrow Q$ which makes this diagram commute:

$$\begin{array}{ccc} A \otimes A & \overset{\mu}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow & \swarrow_{\mathrlap{\phi}} & \\ Q & & \end{array} \,.$$

We claim that

$$\phi \;:\; A \underoverset{\simeq}{r^{-1}}{\longrightarrow} A \otimes 1 \overset{\mathrm{id} \otimes e}{\longrightarrow} A \otimes A \overset{f}{\longrightarrow} Q \,,$$

where the first morphism is the inverse of the right unitor of $\mathcal{C}$.

First to see that this does make the required triangle commute, consider the following pasting composite of commuting diagrams

$$\begin{array}{ccc} A \otimes A & \overset{\mu}{\longrightarrow} & A \\ {}^{\mathllap{\mathrm{id} \otimes r^{-1}}}_{\mathllap{\simeq}}\downarrow && \downarrow^{\mathrlap{r^{-1}}}_{\mathrlap{\simeq}} \\ A \otimes A \otimes 1 & \overset{\mu \otimes \mathrm{id}}{\longrightarrow} & A \otimes 1 \\ {}^{\mathllap{\mathrm{id} \otimes e}}\downarrow && \downarrow^{\mathrlap{\mathrm{id} \otimes e}} \\ A \otimes A \otimes A & \overset{\mu \otimes \mathrm{id}}{\longrightarrow} & A \otimes A \\ {}^{\mathllap{\mathrm{id} \otimes \mu}}\downarrow && \downarrow^{\mathrlap{f}} \\ A \otimes A & \underset{f}{\longrightarrow} & Q \end{array}$$

Here the the top square is the naturality of the right unitor, the middle square commutes by the functoriality of the tensor product $\otimes \;:\; \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and the definition of the product category (Example 1.14), while the commutativity of the bottom square is the assumption that f coequalizes $\mathrm{id} \otimes \mu$ with $\mu \otimes \mathrm{id}$.

Here the right vertical composite is ϕ, while, by unitality of (A, μ, e), the left vertical composite is the identity on A, Hence the diagram says that $\phi \circ \mu = f$, which we needed to show.

It remains to see that ϕ is the unique morphism with this property for given f. For that let $q : A \to Q$ be any other morphism with $q \circ \mu = f$. Then consider the commuting diagram

$$\begin{array}{ccc} A \otimes 1 & \overset{\simeq}{\longleftarrow} & A \\ {}^{\mathrm{id} \otimes e}\downarrow & \searrow^{\simeq} & \downarrow^{=} \\ A \otimes A & \overset{\mu}{\longrightarrow} & A \\ {}^{f}\downarrow & \swarrow_{q} & \\ Q & & \end{array},$$

where the top left triangle is the unitality condition and the two isomorphisms are the right unitor and its inverse. The commutativity of this diagram says that $q = \phi$. ▮

Definition 2.30. Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ as in prop. 2.28, then a monoid (E, μ, e) in $(A\,\mathrm{Mod}, \otimes_A, A)$ (def. 2.19) is called an A-***algebra***.

Proposition 2.31. *Given a monoidal category of modules $(A\,\mathrm{Mod}, \otimes_A, A)$ in a monoidal category $(\mathcal{C}, \otimes, 1)$ as in prop. 2.28, and an A-algebra (E, μ, e) (def. 2.30), then there is an equivalence of categories*

$$A\,\mathrm{Alg}_{\mathrm{comm}}(\mathcal{C}) \coloneqq \mathrm{CMon}(A\,\mathrm{Mod}) \simeq \mathrm{CMon}(\mathcal{C})^{A/}$$

between the category of commutative monoids in $A\,\mathrm{Mod}$ and the coslice category of commutative monoids in $\mathcal{C}$ under A, hence between commutative A-algebras in $\mathcal{C}$ and commutative monoids E in $\mathcal{C}$ that are equipped with a homomorphism of monoids $A \longrightarrow E$.

(e.g. EKMM 97, VII lemma 1.3)

Proof. In one direction, consider a A-algebra E with unit $e_E : A \longrightarrow E$ and product $\mu_{E/A} : E \otimes_A E \longrightarrow E$. There is the underlying product μ_E

$$\begin{array}{ccccc} E \otimes A \otimes E & \underset{\longrightarrow}{\overset{\longrightarrow}{}} & E \otimes E & \overset{\mathrm{coeq}}{\longrightarrow} & E \otimes_A E \\ & & {}_{\mu_E}\searrow & & \downarrow^{\mu_{E/A}} \\ & & & & E \end{array}.$$

By considering a diagram of such coequalizer diagrams with middle vertical morphism $e_E \circ e_A$, one find that this is a unit for μ_E and that $(E, \mu_E, e_E \circ e_A)$ is a commutative monoid in $(\mathcal{C}, \otimes, 1)$.

Then consider the two conditions on the unit $e_E : A \longrightarrow E$. First of all this is an A-module homomorphism, which means that

$$(\star) \qquad \begin{array}{ccc} A \otimes A & \overset{\mathrm{id} \otimes e_E}{\longrightarrow} & A \otimes E \\ {}^{\mu_A}\downarrow && \downarrow^{\rho} \\ A & \underset{e_E}{\longrightarrow} & E \end{array}$$

commutes. Moreover it satisfies the unit property

$$\begin{array}{ccc} A \otimes_A E & \overset{e_A \otimes \mathrm{id}}{\longrightarrow} & E \otimes_A E \\ & {}_{\simeq}\searrow & \downarrow^{\mu_{E/A}} \\ && E \end{array}$$

By forgetting the tensor product over A, the latter gives

$$\begin{array}{ccc} A \otimes E & \overset{e \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ \downarrow && \downarrow \\ A \otimes_A E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes_A E \\ {}^{\simeq}\downarrow && \downarrow^{\mu_{E/A}} \\ E & = & E \end{array} \quad \simeq \quad \begin{array}{ccc} A \otimes E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ {}^{\rho}\downarrow && \downarrow^{\mu_E} \\ E & \underset{\mathrm{id}}{\longrightarrow} & E \end{array} \,,$$

where the top vertical morphisms on the left the canonical coequalizers, which identifies the vertical composites on the right as shown. Hence this may be pasted to the square $(\star)$ above, to yield a commuting square

$$\begin{array}{ccccc} A \otimes A & \overset{\mathrm{id} \otimes e_E}{\longrightarrow} & A \otimes E & \overset{e_E \otimes \mathrm{id}}{\longrightarrow} & E \otimes E \\ {}^{\mu_A}\downarrow && {}^{\rho}\downarrow && \downarrow^{\mu_E} \\ A & \underset{e_E}{\longrightarrow} & E & \underset{\mathrm{id}}{\longrightarrow} & E \end{array} \quad = \quad \begin{array}{ccc} A \otimes A & \overset{e_E \otimes e_E}{\longrightarrow} & E \otimes E \\ {}^{\mu_A}\downarrow && \downarrow^{\mu_E} \\ A & \underset{e_E}{\longrightarrow} & E \end{array} \,.$$

This shows that the unit e_A is a homomorphism of monoids $(A, \mu_A, e_A) \longrightarrow (E, \mu_E, e_E \circ e_A)$.

Now for the converse direction, assume that (A, μ_A, e_A) and (E, μ_E, e'_E) are two commutative monoids in $(\mathcal{C}, \otimes, 1)$ with $e_E \,:\, A \to E$ a monoid homomorphism. Then E inherits a left A-module structure by

$$\rho \,:\, A \otimes E \overset{e_A \otimes \mathrm{id}}{\longrightarrow} E \otimes E \overset{\mu_E}{\longrightarrow} E \,.$$

By commutativity and associativity it follows that μ_E coequalizes the two induced morphisms $E \otimes A \otimes E \underset{\longrightarrow}{\overset{\longrightarrow}{}} E \otimes E$. Hence the universal property of the coequalizer gives a factorization through some $\mu_{E/A} \colon E \otimes_A E \longrightarrow E$. This shows that $(E, \mu_{E/A}, e_E)$ is a commutative A-algebra.

Finally one checks that these two constructions are inverses to each other, up to isomorphism. ▮

Definition 2.32*. *(lax monoidal functor)

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories (def. 2.1). A *lax monoidal functor* between them is

1. a functor

$$F \;:\; \mathcal{C} \longrightarrow \mathcal{D}\,,$$

2. a morphism

$$\epsilon \;:\; 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}})$$

3. a natural transformation

$$\mu_{x,y} \;:\; F(x) \otimes_{\mathcal{D}} F(y) \longrightarrow F(x \otimes_{\mathcal{C}} y)$$

for all $x, y \in \mathcal{C}$

satisfying the following conditions:

1. ***(associativity)*** For all objects $x, y, z \in \mathcal{C}$ the following diagram commutes

$$\begin{array}{ccc} (F(x) \otimes_{\mathcal{D}} F(y)) \otimes_{\mathcal{D}} F(z) & \underoverset{\simeq}{a^{\mathcal{D}}_{F(x),F(y),F(z)}}{\longrightarrow} & F(x) \otimes_{\mathcal{D}} (F(y) \otimes_{\mathcal{D}} F(z)) \\ {}^{\mu_{x,y} \otimes \mathrm{id}}\downarrow && \downarrow^{\mathrm{id}\otimes \mu_{y,z}} \\ F(x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{D}} F(z) && F(x) \otimes_{\mathcal{D}} (F(x \otimes_{\mathcal{C}} y)) \\ {}^{\mu_{x \otimes_{\mathcal{C}} y, z}}\downarrow && \downarrow^{\mu_{x, y \otimes_{\mathcal{C}} z}} \\ F((x \otimes_{\mathcal{C}} y) \otimes_{\mathcal{C}} z) & \underset{F(a^{\mathcal{C}}_{x,y,z})}{\longrightarrow} & F(x \otimes_{\mathcal{C}} (y \otimes_{\mathcal{C}} z)) \end{array} \,,$$

where $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ denote the associators of the monoidal categories;

2. ***(unitality)*** For all $x \in \mathcal{C}$ the following diagrams commutes

$$\begin{array}{ccc} 1_{\mathcal{D}} \otimes_{\mathcal{D}} F(x) & \overset{\epsilon \otimes \mathrm{id}}{\longrightarrow} & F(1_{\mathcal{C}}) \otimes_{\mathcal{D}} F(x) \\ {}^{\ell^{\mathcal{D}}_{F(x)}}\downarrow && \downarrow^{\mu_{1_{\mathcal{C}}, x}} \\ F(x) & \overset{F(\ell^{\mathcal{C}}_x)}{\longleftarrow} & F(1 \otimes_{\mathcal{C}} x) \end{array}$$

and

$$
\begin{tikzcd}
F(x) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "\mathrm{id} \otimes \epsilon"] \arrow[d, "r^{\mathcal{D}}_{F(x)}"'] & F(x) \otimes_{\mathcal{D}} F(1_{\mathcal{C}}) \arrow[d, "\mu_{x, 1_{\mathcal{C}}}"] \\
F(x) & F(x \otimes_{\mathcal{C}} 1) \arrow[l, "F(r^{\mathcal{C}}_x)"']
\end{tikzcd}
,
$$

where $\ell^{\mathcal{C}}, \ell^{\mathcal{D}}, r^{\mathcal{C}}, r^{\mathcal{D}}$ denote the left and right unitors of the two monoidal categories, respectively.

If ϵ and alll $\mu_{x,y}$ are isomorphisms, then F is called a ***strong monoidal functor***.

If moreover $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are equipped with the structure of braided monoidal categories (def. 2.5) with braidings $\tau^{\mathcal{C}}$ and $\tau^{\mathcal{D}}$, respectively, then the lax monoidal functor F is called a ***braided monoidal functor*** if in addition the following diagram commutes for all objects $x, y \in \mathcal{C}$

$$
\begin{tikzcd}
F(x) \otimes_{\mathcal{C}} F(y) \arrow[r, "\tau^{\mathcal{D}}_{F(x), F(y)}"] \arrow[d, "\mu_{x,y}"'] & F(y) \otimes_{\mathcal{D}} F(x) \arrow[d, "\mu_{y,x}"] \\
F(x \otimes_{\mathcal{C}} y) \arrow[r, "F(\tau^{\mathcal{C}}_{x,y})"'] & F(y \otimes_{\mathcal{C}} x)
\end{tikzcd}
.
$$

A homomorphism $f : (F_1, \mu_1, \epsilon_1) \longrightarrow (F_2, \mu_2, \epsilon_2)$ between two (braided) lax monoidal functors is a ***monoidal natural transformation***, in that it is a natural transformation $f_x : F_1(x) \longrightarrow F_2(x)$ of the underlying functors

compatible with the product and the unit in that the following diagrams commute for all objects $x, y \in \mathcal{C}$:

$$
\begin{tikzcd}
F_1(x) \otimes_{\mathcal{D}} F_1(y) \arrow[r, "f(x) \otimes_{\mathcal{D}} f(y)"] \arrow[d, "(\mu_1)_{x,y}"'] & F_2(x) \otimes_{\mathcal{D}} F_2(y) \arrow[d, "(\mu_2)_{x,y}"] \\
F_1(x \otimes_{\mathcal{C}} y) \arrow[r, "f(x \otimes_{\mathcal{C}} y)"'] & F_2(x \otimes_{\mathcal{C}} y)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& 1_{\mathcal{D}} \arrow[dl, "\epsilon_1"'] \arrow[dr, "\epsilon_2"] & \\
F_1(1_{\mathcal{C}}) \arrow[rr, "f(1_{\mathcal{C}})"'] & & F_2(1_{\mathcal{C}})
\end{tikzcd}
.
$$

We write $\mathrm{MonFun}(\mathcal{C},\mathcal{D})$ for the resulting category of lax monoidal functors between monoidal categories $\mathcal{C}$ and $\mathcal{D}$, similarly $\mathrm{BraidMonFun}(\mathcal{C},\mathcal{D})$ for the category of braided monoidal functors between braided monoidal categories, and $\mathrm{SymMonFun}(\mathcal{C},\mathcal{D})$ for the category of braided monoidal functors between symmetric monoidal categories.

Remark 2.33. In the literature the term "monoidal functor" often refers by default to what in def. 2.16 is called a *strong monoidal functor*. But for the purpose of the discussion of functors with smash product below, it is crucial to admit the generality of lax monoidal functors.

If $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are symmetric monoidal categories (def. 2.6) then a braided monoidal functor (def. 2.16) between them is often called a ***symmetric monoidal functor***.

Proposition 2.34. *For $\mathcal{C} \overset{F}{\longrightarrow} \mathcal{D} \overset{G}{\longrightarrow} \mathcal{E}$ two composable lax monoidal functors (def. 2.16) between monoidal categories, then their composite $F \circ G$ becomes a lax monoidal functor with structure morphisms*

$$\epsilon^{G\circ F} \;:\; 1_{\mathcal{E}} \overset{\epsilon^G}{\longrightarrow} G(1_{\mathcal{D}}) \overset{G(\epsilon^F)}{\longrightarrow} G(F(1_{\mathcal{C}}))$$

and

$$\mu^{G\circ F}_{c_1,c_2} \;:\; G(F(c_1)) \otimes_{\mathcal{E}} G(F(c_2)) \overset{\mu^G_{F(c_1),F(c_2)}}{\longrightarrow} G(F(c_1) \otimes_{\mathcal{D}} F(c_2)) \overset{G(\mu^F_{c_1,c_2})}{\longrightarrow} G(F(c_1 \otimes_{\mathcal{C}} c_2)) \,.$$

Proposition 2.35. ***(lax monoidal functors preserve monoids)***

Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be two monoidal categories (def. 2.1) and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a lax monoidal functor (def. 2.16) between them.

Then for (A, μ_A, e_A) a monoid in $\mathcal{C}$ (def. 2.19), its image $F(A) \in \mathcal{D}$ becomes a monoid $(F(A), \mu_{F(A)}, e_{F(A)})$ by setting

$$\mu_{F(A)} \;:\; F(A) \otimes_{\mathcal{C}} F(A) \longrightarrow F(A \otimes_{\mathcal{C}} A) \overset{F(\mu_A)}{\longrightarrow} F(A)$$

(where the first morphism is the structure morphism of F) and setting

$$e_{F(A)} \;:\; 1_{\mathcal{D}} \longrightarrow F(1_{\mathcal{C}}) \overset{F(e_A)}{\longrightarrow} F(A)$$

(where again the first morphism is the corresponding structure morphism of F).

This construction extends to a functor

$$\mathrm{Mon}(F) \;:\; \mathrm{Mon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{Mon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$$

from the category of monoids of $\mathcal{C}$ (def. 2.19) to that of $\mathcal{D}$.

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories (def. 2.6) and F is a braided monoidal functor (def. 2.16) and A is a commutative monoid (def. 2.19) then so is $F(A)$, and this construction extends to a functor between categories of commutative monoids:

$$\mathrm{CMon}(F) \;:\; \mathrm{CMon}(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}) \longrightarrow \mathrm{CMon}(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}) \,.$$

Proof. This follows immediately from combining the associativity and unitality (and symmetry) constraints of F with those of A. ▮

Enriched categories

The plain definition of categories in Def. 1.1 is phrased in terms of sets. Via Example 1.2 this assigns a special role to the category Set of all sets, as the "base" on top, or the "cosmos" inside which category theory takes place. For instance, the fact that hom-sets in a plain category are indeed sets, is what makes the hom-functor (Example 1.17) take values in Set, and this, in turn, governs the form of the all-important Yoneda lemma (Prop. 1.29) and Yoneda embedding (Prop. 1.30) as statements about presheaves of sets (Example 1.26).

At the same time, category theory witnesses the utility of abstracting away from concrete choices to their abstract properties that are actually used in constructions. This makes it natural to ask if one could replace the category Set by some other category $\mathcal{V}$ which could similarly serve as a "cosmos" inside which category theory may be developed.

Indeed, such $\mathcal{V}$-enriched category theory (see Example 2.43 below for the terminology) exists, beginning with the concept of $\mathcal{V}$-enriched categories (Def. 2.40 below) and from there directly paralleling, hence generalizing, plain category theory, as long as one assumes the "cosmos" category $\mathcal{V}$ to share a minimum of abstract properties with Set (Def. 2.36 below).

This turns out to be most useful. In fact, the perspective of enriched categories is helpful already when $\mathcal{V} =$ Set, in which case it reproduces plain category theory (Example 2.41 below), for instance in that it puts the (co)limits of the special form of (co)ends (Def. 3.13 below) to the forefront (discussed below).

***Definition 2.36*. (*cosmos*)**

A *Bénabou cosmos for enriched category theory*, or just *cosmos*, for short, is a symmetric (Def. 2.6) closed monoidal category (Def. 2.8) $\mathcal{V}$ which has all limits and colimits.

***Example 2.37**. (examples of cosmoi for enriched category theory)*

The following are examples of cosmoi (Def. 2.36):

1. $\mathrm{Sh}(\mathcal{C})$ the sheaf topos (Def. 4.8) over any site (Def. 4.3) – by Prop. 4.23 below.
   In particular:
   1. Set (Def. 1.2) equipped with its cartesian closed category-structure (Example 2.9)
   2. sSet $\simeq [\Delta^{\mathrm{op}}, \mathrm{Set}]$ (Def. , Prop.)
2. Grpd (Def. 1.16) equipped with its cartesian closed category-structure (Example 2.11).
3. Cat (Def. 1.16) equipped with its cartesian closed category-structure (Example 2.11).

***Example 2.38**. underlying set of an object in a cosmos*

Let $\mathcal{V}$ be a cosmos (Def. 2.36), with $1 \in \mathcal{V}$ its tensor unit (Def. 2.1). Then the hom-functor (Def. 1.17) out of 1

$$\mathrm{Hom}_{\mathcal{V}}(1, -) \;:\; \mathcal{V} \longrightarrow \mathrm{Set}$$

admits the structure of a lax monoidal functor (Def. 2.16) to Set, with the latter regarded with its cartesian monoidal structure from Example 2.9.

Given $V \in \mathcal{V}$, we call

$$\mathrm{Hom}_{\mathcal{V}}(1, V) \;\in\; \mathrm{Set}$$

also the *underlying* set of V.

***Proof**.* Take the monoidal transformations (eq“MonoidalComponentsOfMonoidalFunctor) to be

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{V}}(1, V_1) \times \mathrm{Hom}_{\mathcal{V}}(1, V_2) \longrightarrow \mathrm{Hom}_{\mathcal{V}}(1, V_1 \otimes V_2) & & \\ \left(1 \overset{f_1}{\to} V_1 \;,\; 1 \overset{f_2}{\to} V_2\right) & \mapsto & \left(1 \overset{\simeq}{\to} 1 \otimes 1 \xrightarrow{f_1 \otimes f_2} V_1 \otimes V_2\right) \end{array}$$

and take the unit transformation (43)

$$* \;\longrightarrow\; \mathrm{Hom}_{\mathcal{V}}(1, 1)$$

to pick $\mathrm{id}_1 \in \mathrm{Hom}_{\mathcal{V}}(1, 1)$. ▮

***Example 2.39**. (underlying set of internal hom is hom-set)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $X, Y \in \mathrm{Obj}_{\mathcal{V}}$ be two objects. Then the underlying set (Def. 2.38) of their internal hom $[X, Y] \in \mathcal{V}$ (Def. 2.8) is the hom-set (Def. 1.1):

$$\mathcal{H}om_{\mathcal{V}}(1, [X, Y]) \simeq \mathrm{Hom}_{\mathcal{V}}(X, Y) \,.$$

This identification is the adjunction isomorphism (10) for the internal hom adjunction (38) followed composed with a unitor (Def. 2.1).

Definition 2.40*. *(enriched category)

For $\mathcal{V}$ a cosmos (Def. 2.36), a $\mathcal{V}$-***enriched category*** $\mathcal{C}$ is:

1. a class $\mathrm{Obj}_{\mathcal{C}}$, called the ***class of objects***;
2. for each $a, b \in \mathrm{Obj}_{\mathcal{C}}$, an object

   $$\mathcal{C}(a, b) \in \mathcal{V} \,,$$

   called the $\mathcal{V}$-***object of morphisms*** between a and b;
3. for each $a, b, c \in \mathrm{Obj}(\mathcal{C})$ a morphism in $\mathcal{V}$

   $$\circ_{a,b,c} \;:\; \mathcal{C}(a, b) \times \mathcal{C}(b, c) \longrightarrow \mathcal{C}(a, c)$$

   out of the tensor product of hom-objects, called the *composition* operation;
4. for each $a \in \mathrm{Obj}(\mathcal{C})$ a morphism $\mathrm{Id}_a : * \to \mathcal{C}(a, a)$, called the *identity* morphism on a

such that the composition is associative and unital.

If the class $\mathrm{Obj}_{\mathcal{C}}$ happens to be a set (hence a small set instead of a proper class) then we say the $\mathcal{V}$-enriched category $\mathcal{C}$ is *small*, as in Def. 1.6.

Example 2.41*. *(Set-enriched categories are plain categories)

An enriched category (Def. 2.40) over the cosmos $\mathcal{V} =$ Set, as in Example 2.37, is the same as a plain category (Def. 1.1).

Example 2.42*. *(Cat-enriched categories are strict 2-categories)

An enriched category (Def. 2.40) over the cosmos $\mathcal{V} =$ Cat, as in Example 2.37, is the same as a strict 2-category (Def. 1.48).

Example 2.43*. *(underlying category of an enriched category)

Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40).

Using the lax monoidal structure (Def. 2.16) on the hom functor (Example 2.38)

$$\mathrm{Hom}_{\mathcal{V}}(1, -) \,:\, \mathcal{V} \longrightarrow \mathrm{Set}$$

out of the tensor unit $1 \in \mathcal{C}$ this induces a Set-enriched category $|\mathcal{C}|$ with hence an ordinary category (Example 2.41), with

- $\mathrm{Obj}_{|\mathcal{C}|} \;\coloneqq\; \mathrm{Obj}_{\mathcal{C}}$;
- $\mathrm{Hom}_{|\mathcal{C}|}(X,Y) \;\coloneqq\; \mathrm{Hom}_{\mathcal{V}}(1, \mathcal{C}(X,Y))$.

It is in this sense that $\mathcal{C}$ is a plain category $|\mathcal{C}|$ equipped with extra structure, and hence an "enriched category".

The archetypical example is $\mathcal{V}$ itself:

***Example 2.44**. ($\mathcal{V}$ as a $\mathcal{V}$-enriched category)*

Evert cosmos $\mathcal{C}$ (Def. 2.36) canonically obtains the structure of a $\mathcal{V}$-enriched category, def. 2.40:

the hom-objects are the internal homs

$$\mathscr{v}(X,Y) \coloneqq [X,Y]$$

and with composition

$$[X,Y] \times [Y,Z] \longrightarrow [X,Z]$$

given by the adjunct under the (Cartesian product⊣ internal hom)-adjunction of the evaluation morphisms

$$X \otimes [\mathrm{XmY}] \otimes [Y,Z] \xrightarrow{(\mathrm{ev},\mathrm{id})} Y \otimes [Y,Z] \xrightarrow{\mathrm{ev}} Z \;.$$

The usual construction on categories, such as that of opposite categories (Def. 1.13) and product categories (Def. 1.14) have evident enriched analogs

***Definition 2.45**. (enriched opposite category and product category)*

For $\mathcal{V}$ a cosmos, let $\mathcal{C}, \mathcal{D}$ be $\mathcal{V}$-enriched categories (Def. 2.40).

1. The ***opposite enriched category*** $\mathcal{C}^{\mathrm{op}}$ is the enriched category with the same objects as $\mathcal{C}$, with hom-objects

$$\mathcal{C}^{\mathrm{op}}(X,Y) \coloneqq \mathcal{C}(Y,X)$$

and with composition given by braiding (37) followed by the composition in $\mathcal{C}$:

$$\mathcal{C}^{\mathrm{op}}(X,Y) \otimes \mathcal{C}^{\mathrm{op}}(Y,Z) = \mathcal{C}(Y,X) \otimes \mathcal{C}(Z,Y) \underset{\simeq}{\overset{\tau}{\longrightarrow}} \mathcal{C}(Z,Y) \otimes \mathcal{C}(Y,X) \xrightarrow{\circ_{Z,Y,X}} \mathcal{C}(Z,X) = \mathcal{(}$$

2. the ***enriched product category*** $\mathcal{C} \times \mathcal{D}$ is the enriched category whose objects are pairs of objects (c,d) with $c \in \mathcal{C}$ and $d \in \mathcal{D}$, whose hom-spaces are the tensor product of the separate hom objects

$$(\mathcal{C} \times \mathcal{D})((c_1,d_1),\ (c_2,d_2)) \coloneqq \mathcal{C}(c_1,c_2) \otimes \mathcal{D}(d_1,d_2)$$

and whose composition operation is the braiding (37) followed by the tensor product of the separate composition operations:

$$\begin{array}{c} (\mathcal{C} \times \mathcal{D})((c_1,d_1),\ (c_2,d_2)) \otimes (\mathcal{C} \times \mathcal{D})((c_2,d_2),\ (c_3,d_3)) \\ {}^{=}\downarrow \\ (\mathcal{C}(c_1,c_2) \otimes \mathcal{D}(d_1,d_2)) \otimes (\mathcal{C}(c_2,c_3) \otimes \mathcal{D}(d_2,d_3)) \\ \downarrow^{\tau}_{\simeq} \\ (\mathcal{C}(c_1,c_2) \otimes \mathcal{C}(c_2,c_3)) \otimes (\mathcal{D}(d_1,d_2) \otimes \mathcal{D}(d_2,d_3)) \quad \xrightarrow{(\circ_{c_1,c_2,c_3}) \otimes (\circ_{d_1,d_2,d_3})} \quad (\\ \\ (\mathcal{C} \end{array}$$

***Definition 2.46.* (*enriched functor*)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{V}$-enriched categories (Def. 2.40).

A $\mathcal{V}$-*enriched functor* from $\mathcal{C}$ to $\mathcal{D}$

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is

1. a function

$$F_{\mathrm{Obj}} : \mathrm{Obj}_{\mathcal{C}} \longrightarrow \mathrm{Obj}_{\mathcal{D}}$$

of objects;

2. for each $a, b \in \mathrm{Obj}_{\mathcal{C}}$ a morphism in $\mathcal{V}$

$$F_{a,b} : \mathcal{C}(a,b) \longrightarrow \mathcal{D}(F_0(a), F_0(b))$$

between hom-objects

such that this preserves composition and identity morphisms in the evident sense.

***Example 2.47*. (enriched hom-functor)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40). Then there is a $\mathcal{V}$-enriched functor out of the enriched product category of $\mathcal{C}$ with its enriched opposite category (Def. 2.45)

$$\mathcal{C}(-,-) \,:\, \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$$

to $\mathcal{V}$, regarded as a $\mathcal{V}$-enriched category (Example 2.44), which sends a pair of objects $X, Y \in \mathcal{C}$ to the hom-object $\mathcal{C}(X,Y) \in \mathcal{V}$, and which acts on morphisms by composition in the evident way.

***Example 2.48*. (enriched presheaves)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40). Then a $\mathcal{V}$-enriched functor (Def. 2.46)

$$F \,:\, \mathcal{C} \longrightarrow \mathcal{V}$$

to the archetypical $\mathcal{V}$-enriched category from Example 2.44 is:

1. an object $F_a \in \mathrm{Obj}_{\mathcal{V}}$ for each object $a \in \mathrm{Obj}_{\mathcal{C}}$;
2. a morphism in $\mathcal{V}$ of the form

   $$F_a \otimes \mathcal{C}(a,b) \longrightarrow F_b$$

   for all pairs of objects $a, b \in \mathrm{Obj}(\mathcal{C})$
   (this is the adjunct of $F_{a,b}$ under the adjunction (38) on $\mathcal{V}$)

such that composition is respected, in the evident sense.

For every object $c \in \mathcal{C}$, there is an enriched representable functor, denoted

$$y(c) \;\coloneqq\; \mathcal{C}(c,-)$$

(where on the right we have the enriched hom-functor from Example 2.47)

which sends objects to

$$y(c)(d) = \mathcal{C}(c,d) \in \mathcal{V}$$

and whose action on morphisms is, under the above identification, just the composition operation in $\mathcal{C}$.

More generally, the following situation will be of interest:

***Example 2.49**. (enriched functor on enriched product category with opposite category)*

An $\mathcal{V}$-enriched functor (Def. 2.46) into $\mathcal{V}$ (Example 2.44) out of an enriched product category (Def. 2.45)

$$F \,:\, \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{V}$$

(an "enriched bifunctor") has component morphisms of the form

$$F_{(c_1, d_1), (c_2, d_2)} \,:\, \mathcal{C}(c_1, c_2) \otimes \mathcal{D}(d_1, d_2) \longrightarrow \left[F_0((c_1, d_1)), F_0((c_2, d_2)) \right] \,.$$

By functoriality and under passing to adjuncts (Def. 1.32) under (38) this is equivalent to two commuting *actions*

$$\rho_{c_1, c_2}(d) \,:\, \mathcal{C}(c_1, c_2) \otimes F_0((c_1, d)) \longrightarrow F_0((c_2, d))$$

and

$$\rho_{d_1, d_2}(c) \,:\, \mathcal{D}(d_1, d_2) \otimes F_0((c, d_1)) \longrightarrow F_0((c, d_2)) \,.$$

In the special case of a functor out of the enriched product category of some $\mathcal{V}$-enriched category $\mathcal{C}$ with its enriched opposite category (def. 2.45)

$$F \,:\, \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$$

then this takes the form of a "pullback action" in the first variable

$$\rho_{c_2, c_1}(d) \,:\, \mathcal{C}(c_1, c_2) \otimes F_0((c_2, d)) \longrightarrow F_0((c_1, d))$$

and a "pushforward action" in the second variable

$$\rho_{d_1, d_2}(c) \,:\, \mathcal{C}(d_1, d_2) \otimes F_0((c, d_1)) \longrightarrow F_0((c, d_2)) \,.$$

***Definition 2.50**. (enriched natural transformation)*

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{V}$-enriched categories (Def. 2.40) and let

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$$

be two $\mathcal{V}$-enriched functors (Def. 2.46) from $\mathcal{C}$ to $\mathcal{D}$.

Then a $\mathcal{V}$-enriched natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{D} \end{tikzcd}$$

is

- for each $c \in \mathrm{Obj}_{\mathcal{C}}$ a choice of morphism

$$\eta_c \;:\; I \longrightarrow \mathcal{D}(F(c), G(c))$$

such that for each pair of objects $c, d \in \mathcal{C}$ the two morphisms (in $\mathcal{V}$)

$$\eta_d \circ F(-) \;:\; \mathcal{C}(c,d) \overset{r}{\simeq} \mathcal{C}(c,d) \otimes I \xrightarrow{G_{c,d} \otimes \eta_c} \mathcal{D}(G(c), G(d)) \otimes \mathcal{D}(F(c), G(c)) \xrightarrow{\circ_{F(c),G(c),G(}} \tag{45}$$

and

$$G(-) \circ \eta_c \;:\; \mathcal{C}(c,d) \overset{\ell}{\simeq} I \otimes \mathcal{C}(c,d) \xrightarrow{\eta_d \otimes F_{c,d}} \mathcal{D}(F(d), G(d)) \otimes \mathcal{D}(F(c), F(d)) \xrightarrow{\circ_{F(c),F(d),G(}} \tag{46}$$

agree.

***Example 2.51*. (*functor category of enriched functors*)**

For $\mathcal{V}$ a cosmos (Def. 2.36) let $\mathcal{C}, \mathcal{D}$ be two $\mathcal{V}$-enriched categories (Def. 2.40). Then there is a category (Def. 1.1) of enriched functors (Def. 2.46), to be denoted

$$[\mathcal{C}, \mathcal{D}]$$

whose objects are the enriched functors $\mathcal{C} \overset{F}{\to} \mathcal{D}$ and whose morphisms are the enriched natural transformations between these (Def. 2.50).

In the case that $\mathcal{V} =$ Set, via Def. 2.37, with Set-enriched categories identified with plain categories via Example 2.41, this coincides with the functor category from Example 1.25.

Notice that, at this point, $[\mathcal{C}, \mathcal{D}]$ is a plain category, not itself a $\mathcal{V}$-enriched category, unless $\mathcal{V} =$ Set. But it may be enhanced to one, this is Def. 3.16 below.

There is now the following evident generalization of the concept of *adjoint functors* (Def. 1.32) from plain category theory to enriched category theory:

***Definition 2.52*. (*enriched adjunction*)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$, $\mathcal{D}$ be two $\mathcal{V}$-enriched categories (Def. 2.40). Then an *adjoint pair of $\mathcal{V}$-enriched functors* or *enriched adjunction*

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

is a pair of $\mathcal{V}$-enriched functors (Def. 2.46), as shown, such that there is a $\mathcal{V}$-enriched natural isomorphism (Def. 2.50) between enriched hom-functors (Def. 2.47) of the form

$$\mathcal{C}(L(-), -) \;\simeq\; \mathcal{D}(-, R(-)) \; . \tag{47}$$

Definition 2.53. **(*enriched equivalence of categories*)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$, $\mathcal{D}$ be two $\mathcal{V}$-enriched categories (Def. 2.40). Then an *equivalence of enriched categories*

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq} \mathcal{D}$$

is a pair of $\mathcal{V}$-enriched functors back and forth, as shown (Def. 2.46), together with $\mathcal{V}$-enriched natural isomorphisms (Def. 2.50) between their composition and the identity functors:

$$\mathrm{id}_{\mathcal{D}} \overset{\simeq}{\Rightarrow} R \circ L \qquad \text{and} \qquad L \circ R \overset{\simeq}{\Rightarrow} \mathrm{id}_{\mathcal{C}} \; .$$

3. Universal constructions

What makes category theory be *theory*, as opposed to just a language, is the concept of *universal constructions*. This refers to the idea of objects with a prescribed property which are *universal* with this property, in that they “know about” or “subsume” every other object with that same kind of property. Category theory allows to make precise what this means, and then to discover and prove theorems about it.

Universal constructions are all over the place in mathematics. Iteratively finding the universal constructions in a prescribed situation essentially amounts to systematically following the unravelling of the given situation or problem or theory that one is studying.

There are several different formulations of the concept of universal constructions, discussed below:

- *Limits and colimits*
- *Ends and coends*

- *Left and right Kan extensions*

But these three kinds of constructions all turn out to be special cases of each other, hence they really reflect different perspectives on a single topic of universal constructions. In fact, all three are also special cases of the concept of *adjunction* (Def. 1.32), thus re-amplifying that category theory is really the theory of adjunctions and hence, if we follow (Lambek 82), of duality.

Limits and colimits

Maybe the most hands-on version of universal constructions are *limits* (Def. 3.1 below), which is short for *limiting cones* (Remark 3.2 below). The formally dual concept (Example 1.13) is called *colimits* (which are hence limits in an opposite category). Other terminology is in use, too:

$\underset{\longleftarrow}{\lim}$	$\underset{\longrightarrow}{\lim}$
limit	colimit
inverse limit	direct limit

There is a variety of different kinds of limits/colimits, depending on the diagram shape that they are limiting (co-)cones over. This includes universal constructions known as *equalizers*, *products*, *fiber products*/*pullbacks*, *filtered limits* and various others, all of which are basic tools frequently used whenever category theory applies.

A key fact of category theory, regarding limits, is that right adjoints preserve limits and left adjoints preserve colimits (Prop. 3.8 below). This will be used all the time. A partial converse to this statement is that if a functor preserves limits/colimits, then its adjoint functor is, if it exists, objectwise given by a limit/colimit over a comma category under/over the given functor (Prop. 3.11 below). Since these comma categories are in general not small, this involves set-theoretic size subtleties that are dealt with by the *adjoint functor theorem* (Remark 3.12 below). We discuss in detail a very special but also very useful special case of this in Prop. 3.29, further below.

***Definition 3.1*. (*limit* and *colimit*)**

Let $\mathcal{C}$ be a small category (Def. 1.6), and let $\mathcal{D}$ be any category (Def. 1.1). In this case one also says that a functor

$$F \,:\, \mathcal{C} \longrightarrow \mathcal{D}$$

is a *diagram of shape $\mathcal{C}$ in $\mathcal{D}$*.

Recalling the functor category (Example 1.25) $[\mathcal{C},\mathcal{D}]$, there is the *constant diagram-functor*

$$\mathrm{const} \,:\, \mathcal{D} \longrightarrow [\mathcal{C},\mathcal{D}]$$

which sends an object $X \in \mathcal{D}$ to the functor that sends every $c \in \mathcal{C}$ to X, and every morphism in $\mathcal{C}$ to the identity morphism on X. Accordingly, every morphism in $\mathcal{D}$ is sent by const to the natural transformation (Def. 1.23) all whose components are equal to that morphism.

Now:

1. if const has a right adjoint (Def. 1.32), this is called the construction of forming the *limiting cone of $\mathcal{C}$-shaped diagrams in $\mathcal{D}$*, or just *limit* (or *inverse limit*) for short, and denoted

$$\underset{\underset{\mathcal{C}}{\longleftarrow}}{\lim} \,:\, [\mathcal{C},\mathcal{D}] \longrightarrow \mathcal{D}$$

2. if const has a left adjoint (Def. 1.32), this is called the construction of forming the *colimiting cocone of $\mathcal{C}$-shaped diagrams in $\mathcal{D}$*, or just *colimit* (or *direct limit*) for short, and denoted

$$\underset{\underset{\mathcal{C}}{\longrightarrow}}{\lim} \,:\, [\mathcal{C},\mathcal{D}] \longrightarrow \mathcal{D}$$

$$[\mathcal{C},\mathcal{D}] \underoverset{\underset{\underset{\underset{\mathcal{C}}{\longleftarrow}}{\lim}}{\longrightarrow}}{\overset{\underset{\underset{\mathcal{C}}{\longrightarrow}}{\lim}}{\longrightarrow}}{\overset{\mathrm{const}}{\longleftarrow}} \mathcal{D} \,. \qquad (48)$$

If $\underset{\underset{\mathcal{C}}{\longleftarrow}}{\lim}$ ($\underset{\underset{\mathcal{C}}{\longrightarrow}}{\lim}$) exists for a given $\mathcal{D}$, one says that $\mathcal{D}$ *has all limits* (_has all colimits_) of shape $\mathcal{C}$_ or that all limits (colimits) of shape $\mathcal{D}$ *exist in* $\mathcal{D}$. If this is the case for all small diagrams $\mathcal{C}$, one says that $\mathcal{D}$ *has all limits* (_has all colimits_) or that *all limits exist in* $\mathcal{D}$, (_all colimits exist in $\mathcal{D}$.)

***Remark 3.2**. (limit cones)*

Unwinding Definition 3.1 of limits and colimits, it says the following.

First of all, for $d \in \mathcal{D}$ any object and $F \,:\, \mathcal{C} \longrightarrow \mathcal{D}$ any functor, a natural transformation (Def.

1.23) of the form

$$\mathrm{const}_d \overset{i}{\Rightarrow} F \tag{49}$$

has component morphisms

$$\begin{array}{c} d \\ \downarrow^{i_c} \\ F(c) \end{array}$$

in $\mathcal{D}$, for each $c \in \mathcal{C}$, and the naturality condition (4) says that these form a commuting diagram (Def. 1.4) of the form

$$\begin{array}{ccccc} & & d & & \\ & {}^{i_{c_1}}\swarrow & & \searrow^{i_{c_2}} & \\ F(c_1) & & \underset{F(f)}{\longrightarrow} & & F(c_2) \end{array} \tag{50}$$

for each morphism $c_1 \overset{f}{\to} c_2$ in $\mathcal{C}$. Due to the look of this diagram, one also calls such a natural transformation a *cone* over the functor F.

Now the counit (Def. 1.33) of the $(\mathrm{const} \dashv \underset{\longleftarrow}{\lim})$-adjunction (48) is a natural transformation of the form

$$\mathrm{const}_{\underset{\longleftarrow}{\lim} F} \overset{\epsilon_F}{\longrightarrow} F$$

and hence is, in components, a cone (50) over F:

$$\begin{array}{ccccc} & & \underset{\longleftarrow}{\lim} F & & \\ & {}^{\epsilon_F(c_1)}\swarrow & & \searrow^{\epsilon_F(c_2)} & \\ F(c_1) & & \underset{F(f)}{\longrightarrow} & & F(c_2) \end{array} \tag{51}$$

to be called the *limiting cone* over F

But the universal property of adjunctions says that this is a very special cone: By Prop. 1.42 the defining property of the limit is equivalently that for every natural transformation of the form (49), hence for every cone of the form (50), there is a *unique* natural transformation

$$\mathrm{const}_d \overset{\tilde i}{\Rightarrow} \mathrm{const}_{\underset{\longleftarrow}{\lim}}$$

which, due to constancy of the two functors applied in the naturality condition (4), has a constant component morphism

$$d \overset{\tilde i}{\longrightarrow} \underset{\longleftarrow}{\lim} F \tag{52}$$

such that

$$\begin{tikzcd} \mathrm{const}_d \arrow[rr, "\tilde i"] \arrow[dr, "\epsilon_F"'] & & \mathrm{const}_{\underset{\longleftarrow}{\lim} F} \arrow[dl, "i"] \\ & F & \end{tikzcd}$$

hence such that (52) factors the given cone (50) through the special cone (51):

$$\begin{tikzcd} & d \arrow[dl, "i_{c_1}"'] \arrow[dr, "i_{c_2}"] & \\ F(c_1) \arrow[rr, "F(f)"'] & & F(c_2) \end{tikzcd} \quad = \quad \begin{tikzcd} & d \arrow[d, "\tilde i"] & \\ & \underset{\longleftarrow}{\lim} F \arrow[dl, "\epsilon_F(c_1)"'] \arrow[dr, "\epsilon_F(c_2)"] & \\ F(c_1) \arrow[rr, "F(f)"'] & & F(c_2) \end{tikzcd}$$

In this case one also says that $\tilde i$ is a *morphism of cones*.

Hence a *limit cone* is a cone over F, such that every other cone factors through it in a unique way.

Of course this concept of (co)limiting cone over a functor $F : \mathcal{C} \to \mathcal{D}$ makes sense also when

1. $\mathcal{C}$ is not small,
2. and/or when a (co-)limiting cone exists only for some but not for all functors of this form.

***Example 3.3*. (terminal/initial object is empty limit/colimit)**

Let $\mathcal{C}$ be a category, and let $* \in \mathcal{C}$ be an object. The following are equivalent:

1. $*$ is a terminal object of $\mathcal{C}$ (Def. 1.5);
2. $*$ is the limit of the empty diagram.

And formally dual (example 1.13): Let $\emptyset \in \mathcal{C}$ be an object. The following are equivalent:

1. $\emptyset$ is an initial object of $\mathcal{C}$ (Def. 1.5);
2. $\emptyset$ is the colimit of the empty diagram.

Proof. We discuss the case of the terminal object, the other case is formally dual (Example 1.13).

It suffices to observe that a cone over the empty diagram (Remark 3.2) is clearly just a plain object of $\mathcal{C}$. Hence a morphism of such cones is just a plain morphism of $\mathcal{C}$. This way the condition on a limiting cone is now manifestly the same as the condition on a terminal object. ▮

Example 3.4. ***(initial object is limit over identity functor)***

Let $\mathcal{C}$ be a category, and let $\emptyset \in \mathcal{C}$ be an object. The following are equivalent:

1. *$\emptyset$ is an initial object of $\mathcal{C}$ (Def. 1.5);*
2. *$\emptyset$ is the tip of a limit cone (Remark 3.2) over the identity functor on $\mathcal{C}$.*

Proof. First let $\emptyset$ be an initial object. Then, by definition, it is the tip of a unique cone over the identity functor

$$\begin{array}{ccccccc} \mathrm{const}_{\emptyset} & & & & \emptyset & & \\ {}^{i^{\emptyset}}\Downarrow & & & {}^{i^{\emptyset}_{c_1}}\swarrow & & \searrow^{i^{\emptyset}_{c_2}} & \\ \mathrm{id}_{\mathcal{C}} & & c_1 & & \underset{f}{\longrightarrow} & & c_2 \end{array} \qquad (53)$$

We need to show that that every other cone i^x

$$\begin{array}{ccccccc} \mathrm{const}_{x} & & & & x & & \\ {}^{i^{x}}\Downarrow & & & {}^{i^{x}_{c_1}}\swarrow & & \searrow^{i^{x}_{c_2}} & \\ \mathrm{id}_{\mathcal{C}} & & c_1 & & \underset{f}{\longrightarrow} & & c_2 \end{array}$$

factors uniquely through $i^{\emptyset}$.

First of all, since the cones are over the identity functor, there is the component $i^x_{\emptyset} \,:\, x \to \emptyset$, and it is a morphism of cones.

To see that this is the unique morphism of cones, consider any morphism of cones $j^x_{\emptyset}$, hence a morphism in $\mathcal{C}$ such that $i^x_c = i^{\emptyset}_c \circ j^x_{\emptyset}$ for all $c \in \mathcal{C}$. Taking here $c = \emptyset$ yields

$$\begin{aligned} i^x_\emptyset &= \underbrace{i^\emptyset_\emptyset}_{=\mathrm{id}_\emptyset} \circ j^x_\emptyset \\ &= j^x_\emptyset \,, \end{aligned}$$

where under the brace we used that $\emptyset$ is initial. This proves that $i^\emptyset$ is the limiting cone.

For the converse, assume now that $i^\emptyset$ is a limiting cone over the identity functor, with labels as in (53). We need to show that its tip $\emptyset$ is an initial object.

Now the cone condition applied for any object $x \in \mathcal{C}$ over the morphims $f \coloneqq i^\emptyset_x$ says that

$$i^\emptyset_x \circ i^\emptyset_\emptyset = i^\emptyset_x$$

which means that $i^\emptyset_\emptyset$ constitutes a morphism of cones from $i^\emptyset$ to itself. But since $i^\emptyset$ is assumed to be a limiting cone, and since the identity morphism on $\emptyset$ is of course also a morphism of cones from $i^\emptyset$ to itsely, we deduce that

$$i^\emptyset_\emptyset \;=\; \mathrm{id}_\emptyset \,. \tag{54}$$

Now consider any morphism of the form $\emptyset \overset{f}{\to} x$. Since we already have the morphism $\emptyset \overset{i^\emptyset_x}{\to} x$, to show initiality of $\emptyset$ we need to show that $f = i^\emptyset_x$.

Indeed, the cone condition of $i^\emptyset_x$ applied to f now yields

$$\begin{aligned} i^\emptyset_x &= f \circ \underbrace{i^\emptyset_\emptyset}_{=\mathrm{id}_\emptyset} \\ &= f \,, \end{aligned}$$

where under the brace we used (54). ▮

***Example 3.5*. (*limits of presheaves are computed objectwise*)**

Let $\mathcal{C}$ be a category and write $[\mathcal{C}^{op}, \mathrm{Set}]$ for its category of presheaves (Example 1.26). Let moreover $\mathcal{D}$ be a small category and consider any functor

$$F \;:\; \mathcal{D} \longrightarrow [\mathcal{C}^{op}, \mathcal{D}] \,,$$

hence a $\mathcal{D}$-shaped diagram in the category of presheaves.

Then

1. The limit (Def. 3.1) of F exists, and is the presheaf which over any object $c \in \mathcal{C}$ is given by the limit in Set of the values of the presheaves at c:

$$\left(\underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d) \right)(c) \;\simeq\; \underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c)$$

2. The colimit (Def. 3.1) of F exists, and is the presheaf which over any object $c \in \mathcal{C}$ is given by the colimit in Set of the values of the presheaves at c:

$$\left(\underset{\underset{d \in \mathcal{D}}{\longrightarrow}}{\lim} F(d) \right)(c) \;\simeq\; \underset{\underset{d \in \mathcal{D}}{\longrightarrow}}{\lim} F(d)(c)$$

Proof. We discuss the case of limits, the other case is formally dual (Example 1.13).

Observe that there is a canonical equivalence (Def. 1.57)

$$[\mathcal{D}, [\mathcal{C}^{op}, \mathrm{Set}]] \simeq [\mathcal{D} \times \mathcal{C}^{op}, \mathrm{Set}]$$

where $\mathcal{D} \times \mathcal{C}^{op}$ is the product category.

This makes manifest that a functor $F \,:\, \mathcal{D} \to [\mathcal{C}^{op}, \mathrm{Set}]$ is equivalently a diagram of the form

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F(d_1)(c_2) & \longrightarrow & F(d_2)(c_2) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F(d_1)(c_1) & \longrightarrow & F(d_2)(c_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Then observe that taking the limit of each “horizontal row” in such a diagram indead does yield a presheaf on $\mathcal{C}$, in that the construction extends from objects to morphisms, and uniquely so: This is because for any morphism $c_1 \overset{g}{\to} c_2$ in $\mathcal{C}$, a cone over $F(-)(c_2)$ (Remark 3.2) induces a cone over $F(-)(c_1)$, by vertical composition with $F(-)(g)$

$$
\begin{array}{ccccc}
 && \underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c_2) && \\
 & \swarrow && \searrow & \\
F(d_1)(c_2) && \longrightarrow && F(d_2)(c_2) \\
{}^{\mathllap{F(d_1)(g)}}\downarrow && && \downarrow^{\mathrlap{F(d_2)(g)}} \\
F(d_1)(c_1) && \longrightarrow && F(d_2)(c_1)
\end{array}
$$

From this, the universal property of limits of sets (as in Remark 3.2) implies that there is a *unique* morphism between the pointwise limits which constitutes a presheaf over $\mathcal{C}$

$$
\begin{array}{c}
\underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c_2) \\
\downarrow^{\mathrlap{\underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(g)}} \\
\underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c_1)
\end{array}
$$

and that is the tip of a cone over the diagram $F(-)$ in presheaves.

Hence it remains to see that this cone of presheaves is indeed universal.

Now if I is any other cone over F in the category of presheaves, then by the universal property of the pointswise limits, there is for each $c \in \mathcal{C}$ a unique morphism of cones in sets

$$
I(c) \longrightarrow \underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c) \,.
$$

Hence there is at most one morphisms of cones of presheaves, namely if these components make all their naturality squares commute.

$$
\begin{array}{ccc}
I(c_2) & \longrightarrow & \underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c_2) \\
\downarrow && \downarrow \\
I(c_1) & \longrightarrow & \underset{\underset{d \in \mathcal{D}}{\longleftarrow}}{\lim} F(d)(c_1)
\end{array}
\,.
$$

But since everything else commutes, the two ways of going around this diagram constitute two morphisms from a cone over $F(-)(c_1)$ to the limit cone over $F(-)(c_1)$, and hence they must be equal, by the universal property of limits. ▮

***Proposition 3.6*. (*hom-functor preserves limits*)**

Let $\mathcal{C}$ be a category and write

$$\mathrm{Hom}_{\mathcal{C}} \;:\; \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

for its hom-functor. This preserves limits (Def. 3.1) in both its arguments (recalling that a limit in the opposite category $\mathcal{C}^{\mathrm{op}}$ is a colimit in $\mathcal{C}$).

More in detail, let $X_\bullet : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram. Then:

1. *If the limit $\underset{\longleftarrow i}{\lim} X_i$ exists in $\mathcal{C}$ then for all $Y \in \mathcal{C}$ there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}\left(Y, \underset{\longleftarrow i}{\lim} X_i\right) \simeq \underset{\longleftarrow i}{\lim} (\mathrm{Hom}_{\mathcal{C}}(Y, X_i)) \,,$$

where on the right we have the limit over the diagram of hom-sets given by

$$\mathrm{Hom}_{\mathcal{C}}(Y, -) \circ X \;:\; \mathcal{I} \overset{X}{\longrightarrow} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(Y,-)} \mathrm{Set} \,.$$

2. *If the colimit $\underset{\longrightarrow i}{\lim} X_i$ exists in $\mathcal{C}$ then for all $Y \in \mathcal{C}$ there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}\left(\underset{\longrightarrow i}{\lim} X_i, Y\right) \simeq \underset{\longleftarrow i}{\lim} (\mathrm{Hom}_{\mathcal{C}}(X_i, Y)) \,,$$

where on the right we have the limit over the diagram of hom-sets given by

$$\mathrm{Hom}_{\mathcal{C}}(-, Y) \circ X \;:\; \mathcal{I}^{\mathrm{op}} \overset{X}{\longrightarrow} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,Y)} \mathrm{Set} \,.$$

Proof. We give the proof of the first statement, the proof of the second statement is formally dual (Example 1.13).

First observe that, by the very definition of limiting cones, maps out of some Y into them are in natural bijection with the set $\mathrm{Cones}(Y, X_\bullet)$ of cones over the diagram $X_\bullet$ with tip Y:

$$\mathrm{Hom}\left(Y, \underset{\longleftarrow i}{\lim} X_i\right) \;\simeq\; \mathrm{Cones}(Y, X_\bullet) \,.$$

Hence it remains to show that there is also a natural bijection like so:

$$\mathrm{Cones}(Y, X_\bullet) \;\simeq\; \underset{\longleftarrow i}{\lim} (\mathrm{Hom}(Y, X_i)) \,.$$

Now, again by the very definition of limiting cones, a single element in the limit on the right is equivalently a cone of the form

$$\left\{ \begin{tikzcd} & * \arrow[dl, "\mathrm{const}_{p_i}"'] \arrow[dr, "\mathrm{const}_{p_j}"] & \\ \mathrm{Hom}(Y, X_i) \arrow[rr, "X_\alpha \circ (-)"'] & & \mathrm{Hom}(Y, X_j) \end{tikzcd} \right\}_{i,j \in \mathrm{Obj}(\mathcal{I}), \alpha \in \mathrm{Hom}_{\mathcal{I}}(i,j)} .$$

This is equivalently for each object $i \in \mathcal{I}$ a choice of morphism $p_i : Y \to X_i$, such that for each pair of objects $i, j \in \mathcal{I}$ and each $\alpha \in \mathrm{Hom}_{\mathcal{I}}(i,j)$ we have $X_\alpha \circ p_i = p_j$. And indeed, this is precisely the characterization of an element in the set $\mathrm{Cones}(Y, X_\bullet\})$. ▮

***Example 3.7*. (initial and terminal object in terms of adjunction)**

Let $\mathcal{C}$ be a category (Def. 1.1).

1. The following are equivalent:

   1. $\mathcal{C}$ has a terminal object (Def. 1.5);

   2. the unique functor $\mathcal{C} \to *$ (Def. 1.15) to the terminal category (Example 1.7) has a right adjoint (Def. 1.32)

   $$* \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \mathcal{C}$$

   Under this equivalence, the terminal object is identified with the image under the right adjoint of the unique object of the terminal category.

2. Dually, the following are equivalent:

   1. $\mathcal{C}$ has an initial object (Def. 1.5);

   2. the unique functor $\mathcal{C} \to *$ to the terminal category has a left adjoint

   $$\mathcal{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} *$$

   Under this equivalence, the initial object is identified with the image under the left adjoint of the unique object of the terminal category.

***Proof*.** Since the unique hom-set in the terminal category is the singleton, the hom-isomorphism (10) characterizing the adjoint functors is directly the universal property of an initial object in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(L(*), X) \;\simeq\; \mathrm{Hom}_*(*, R(X)) = *$$

or of a terminal object

$$\mathrm{Hom}_{\mathcal{C}}(X, R(*)) \;\simeq\; \mathrm{Hom}_*(L(X), *) = * ,$$

respectively. ▮

Proposition 3.8. ***(left adjoints preserve colimits and right adjoints preserve limits)***

Let $(L \dashv R) : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors (Def. 1.32). Then

- *L preserves all colimits (Def. 3.1) that exist in $\mathcal{C}$,*
- *R preserves all limits (Def. 3.1) in $\mathcal{D}$.*

Proof. Let $y : I \to \mathcal{D}$ be a diagram whose limit $\lim_{\leftarrow i} y_i$ exists. Then we have a sequence of natural isomorphisms, natural in $x \in C$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(x, R \underset{\longleftarrow i}{\lim} y_i) &\simeq \mathrm{Hom}_{\mathcal{D}}(Lx, \underset{\longleftarrow i}{\lim} y_i) \\ &\simeq \underset{\longleftarrow i}{\lim} \mathrm{Hom}_{\mathcal{D}}(Lx, y_i) \\ &\simeq \underset{\longleftarrow i}{\lim} \mathrm{Hom}_{\mathcal{C}}(x, R y_i) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(x, \underset{\longleftarrow i}{\lim} R y_i)\,, \end{aligned}$$

where we used the hom-isomorphism (10) and the fact that any hom-functor preserves limits (Def. 3.6). Because this is natural in x the Yoneda lemma implies that we have an isomorphism

$$R \underset{\longleftarrow i}{\lim} y_i \simeq \underset{\longleftarrow i}{\lim} R y_i \;.$$

The argument that shows the preservation of colimits by L is analogous. ▮

Proposition 3.9. ***(limits commute with limits)***

Let $\mathcal{D}$ and $\mathcal{D}'$ be small categories (Def. 1.6) and let $\mathcal{C}$ be a category (Def. 1.1) which admits limits (Def. 3.1) of shape $\mathcal{D}$ as well as limits of shape $\mathcal{D}'$. Then these limits "commute" with each other, in that for $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{C}$ a functor (hence a diagram of shape the product category), with corresponding adjunct functors (via Example 2.11)

$$\mathcal{D}' \overset{F_{\mathcal{D}}}{\longrightarrow} [\mathcal{D}, \mathcal{C}] \qquad \mathcal{D} \overset{F_{\mathcal{D}'}}{\longrightarrow} [\mathcal{D}', \mathcal{C}]$$

we have that the canonical comparison morphism

$$\lim F \simeq \lim_{\mathcal{D}}(\lim_{\mathcal{D}'} F_{\mathcal{D}}) \simeq \lim_{\mathcal{D}'}(\lim_{\mathcal{D}} F_{\mathcal{D}'}) \qquad (55)$$

is an isomorphism.

Proof. Since the limit-construction is the right adjoint functor to the constant diagram-functor, this is a special case of *right adjoints preserve limits* (Prop. 3.8). ▮

See *limits and colimits by example* for what formula (55) says for instance for the special case $\mathcal{C} = \mathrm{Set}$.

Remark 3.10*. *(general non-commutativity of limits with colimits)

In general limits do *not* commute with colimits. But under a number of special conditions of interest they do. Special cases and concrete examples are discussed at *commutativity of limits and colimits*.

Proposition 3.11*. *(pointwise expression of left adjoints in terms of limits over comma categories)

A functor $R : \mathcal{C} \longrightarrow \mathcal{D}$ (Def. 1.15) has a left adjoint $L : \mathcal{D} \longrightarrow \mathcal{C}$ (Def. 1.32) precisely if

1. *R preserves all limits (Def. 3.1) that exist in $\mathcal{C}$;*
2. *for each object $d \in \mathcal{D}$, the limit (Def. 3.1) of the canonical functor (22) out of the comma category (Example 1.43)*

   $$d / R \longrightarrow \mathcal{C}$$

   exists.

In this case the value of the left adjoint L on d is given by that limit:

$$L(d) \;\simeq\; \underset{\underset{\left(c, \; \begin{array}{c} d \\ \downarrow^{f} \\ R(c) \end{array}\right) \in d / R}{\longleftarrow}}{\lim} \; c \qquad (56)$$

***Proof*.** First assume that the left adjoint exist. Then

1. R is a right adjoint and hence preserves limits since all right adjoints preserve limits (Prop. 3.8);
2. by Prop. 1.42 the adjunction unit provides a universal morphism η_d into $L(d)$, and hence, by Prop. 1.44, exhibits $(L(d), \eta_d)$ as the initial object of the comma category d / R. The limit over any category with an initial object exists, as it is given by that initial object.

Conversely, assume that the two conditions are satisfied and let $L(d)$ be given by (56). We need to show that this yields a left adjoint.

By the assumption that R preserves all limits that exist, we have

$$
\begin{aligned}
R(L(d)) & = R\left(\underset{\longleftarrow}{\lim}_{\left(c, \; \begin{matrix} d \\ \downarrow^{f} \\ R(c) \end{matrix}\right) \in d/R} c \right) \\
& \simeq \underset{\longleftarrow}{\lim}_{\left(c, \; \begin{matrix} d \\ \downarrow^{f} \\ R(c) \end{matrix}\right) \in d/R} R(c)
\end{aligned}
\tag{57}
$$

Since the $d \overset{f}{\to} R(d)$ constitute a cone over the diagram of the $R(d)$, there is universal morphism

$$d \xrightarrow{\eta_d} R(L(d)) \,.$$

By Prop. 1.42 it is now sufficient to show that η_d is a universal morphism into $L(d)$, hence that for all $c \in \mathcal{C}$ and $d \overset{g}{\longrightarrow} R(c)$ there is a unique morphism $L(d) \overset{\widetilde{f}}{\longrightarrow} c$ such that

$$
\begin{tikzcd}
 & d \arrow[dl, "\eta_d"'] \arrow[dr, "f"] & \\
R(L(d)) \arrow[rr, "R(\widetilde{f})"'] & & R(c) \\
L(d) \arrow[rr, "\widetilde{f}"'] & & c
\end{tikzcd}
$$

By Prop. 1.44, this is equivalent to $(L(d), \eta_d)$ being the initial object in the comma category c / R, which in turn is equivalent to it being the limit of the identity functor on c / R (by Example 3.4). But this follows directly from the limit formulas (56) and (57). ▮

***Remark 3.12*. (*adjoint functor theorem*)**

Beware the subtle point in Prop. 3.11, that the comma category c / F is in general not a small category (Def. 1.6): It has typically "as many" objects as $\mathcal{C}$ has, and $\mathcal{C}$ is not assumed to be small (while of course it may happen to be). But typical categories, such as notably the category of sets (Example 1.2) are generally guaranteed only to admit limits over small categories. For this reason, Prop. 3.11 is rarely useful for *finding* an adjoint functor which is not already established to exist by other means.

But there are good sufficient conditions known, on top of the condition that R preserves limits, which guarantee the existence of an adjoint functor, after all. This is the topic of the *adjoint functor theorem* (one of the rare instances of useful and non-trivial theorems in mathematics for which issues of set theoretic size play a crucial role for their statement and proof).

A very special but also very useful case of the adjoint functor theorem is the existence of adjoints of base change functors between categories of (enriched) presheaves via Kan extension. This we discuss as Prop. 3.29 below. Since this is most conveniently phrased in terms of special limits/colimits called ends/coends (Def. 3.13 below) we first discuss these.

Ends and coends

For working with enriched categories (Def. 2.40) , a certain shape of limits/colimits (Def. 3.1) is particularly relevant: these are called *ends* and *coends* (Def. 3.13 below). We here introduce these and then derive some of their basic properties, such as notably the expression for Kan extension in terms of (co-)ends (prop. 3.29 below).

***Definition 3.13*. ((co)end)**

Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category (Def. 2.40). Let

$$F \;:\; \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$$

be an enriched functor (Def. 2.46) out of the enriched product category of $\mathcal{C}$ with its opposite category (Def. 2.45). Then:

1. The ***coend*** of F, denoted

$$\int^{c \in \mathcal{C}} F(c,c) \;\in\; \mathcal{V}\,,$$

is the coequalizer in $\mathcal{V}$ of the two actions encoded in F via Example 2.49:

$$\coprod_{c,d \in \mathcal{C}} \mathcal{C}(c,d) \otimes F(d,c) \underset{\underset{c,d}{\sqcup} \rho_{(d,c)}(c)}{\overset{\underset{c,d}{\sqcup} \rho_{(c,d)}(d)}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c,c) \overset{\mathrm{coeq}}{\longrightarrow} \int^{c \in \mathcal{C}} F(c,c)\,.$$

2. The ***end*** of F, denoted

$$\int_{c \in \mathcal{C}} F(c,c) \;\in\; \mathcal{V}\,,$$

is the ***equalizer*** in $\mathcal{V}$ of the adjuncts of the two actions encoded in F via example 2.49:

$$\int_{c \in \mathcal{C}} F(c,c) \xrightarrow{\text{equ}} \prod_{c \in \mathcal{C}} F(c,c) \underset{\underset{c,d}{\sqcup} \tilde\rho_{(c,d)}(c)}{\overset{\underset{c,d}{\sqcup} \tilde\rho_{d,c}(d)}{\rightrightarrows}} \prod_{c \in \mathcal{C}} \big[\mathcal{C}(c,d),\; F(c,d)\big]\,.$$

Example 3.14. For $\mathcal{V}$ a cosmos, let $G \in \mathcal{V}$ be a group object. There is the n the one-object $\mathcal{V}$-enriched category $\mathbf{B}G$ as in Example 1.11.

Then a $\mathcal{V}$-enriched functor

$$(X, \rho_l) \;:\; \mathbf{B}G \longrightarrow \mathcal{V}$$

is an object $X \coloneqq F(*) \;\in \mathcal{V}$ equipped with a morphism

$$\rho_l \;:\; G \otimes X \longrightarrow X$$

satisfying the action property. Hence this is equivalently an action of G on X.

The opposite category (def. 2.45) $(\mathbf{B}G)^{\mathrm{op}}$ comes from the opposite group-object

$$(\mathbf{B}G)^{\mathrm{op}} = \mathbf{B}(G^{\mathrm{op}})\,.$$

(The isomorphism $G \simeq G^{\mathrm{op}}$ induces a canonical euqivalence of enriched categories $(\mathbf{B}G)^{\mathrm{op}} \simeq \mathbf{B}G$.)

So an enriched functor

$$(Y, \rho_r) \;:\; (\mathbf{B}G)^{\mathrm{op}} \longrightarrow \mathcal{V}$$

is equivalently a *right* action of G.

Therefore the coend of two such functors (def. 3.13) coequalizes the relation

$$(xg,\; y) \sim (x,\; gy)$$

(where juxtaposition denotes left/right action) and is the quotient of the plain tensor product by the diagonal action of the group G:

$$\int^{* \in \mathbf{B}(G_+)} (Y, \rho_r)(*) \otimes (X, \rho_l)(*) \;\simeq\; Y \otimes_G X \,.$$

***Example 3.15**. (enriched natural transformations as ends)*

Let $\mathcal{C}$ be a small enriched category (Def. 2.40). For $F, G \;:\; \mathcal{C} \longrightarrow \mathcal{V}$ two enriched presheaves (Example 2.48), the end (def. 3.13) of the internal-hom-functor

$$[F(-), G(-)] \;:\; \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$$

is an object of $\mathcal{V}$ whose underlying set (Example 2.38) is the set of enriched natural transformations $F \Rightarrow G$ (Def. 2.50)

$$\mathrm{Hom}_{\mathcal{V}}\left(1, \left(\int_{c \in \mathcal{C}} [F(c), G(c)] \right) \right) \;\simeq\; \mathrm{Hom}_{[\mathcal{C}, \mathcal{V}]}(F, G) \,.$$

***Proof*.** The underlying pointed set functor $\mathrm{Hom}_{\mathcal{V}}(1, -) : \mathcal{V} \to \mathrm{Set}$ preserves all limits, since hom-functors preserve limits (Prop. 3.6). Therefore there is an equalizer diagram in Set of the form

$$\mathrm{Hom}_{\mathcal{V}}\left(1, \left(\int_{c \in \mathcal{C}} [F(c), G(c)] \right) \right) \overset{\mathrm{equ}}{\longrightarrow} \prod_{c \in \mathcal{C}} \mathrm{Hom}_{\mathcal{V}}(F(c), G(c)) \underoverset{\underset{c,d}{\sqcup} U(\tilde \rho_{(c,d)}(c))}{\underset{c,d}{\sqcup} U(\tilde \rho_{d,c}(d))}{\rightrightarrows} \prod_{c,d \in \mathcal{C}} \mathrm{Hom}_{\mathcal{V}}(\mathcal{C}(c,d)$$

where we used Example 2.39 to identify underlying sets of internal homs with hom-sets.

Here the object in the middle is just the set of indexed sets of component morphisms $\left\{ F(c) \overset{\eta_c}{\to} G(c) \right\}_{c \in \mathcal{C}}$. The two parallel maps in the equalizer diagram take such a collection to the indexed set of composites (45) and (46). Hence that these two are equalized is precisely the condition that the indexed set of components constitutes an enriched natural transformation. ▮

Conversely, example 3.15 says that ends over bifunctors of the form $[F(-), G(-))]$ constitute hom-spaces between pointed topologically enriched functors:

***Definition 3.16**. (enriched presheaf category)*

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category (Def. 2.40).

Then the $\mathcal{V}$-enriched presheaf category $[\mathcal{C}, \mathcal{V}]$ is $\mathcal{V}$-enriched functor category from $\mathcal{C}$ to $\mathcal{V}$, hence is the following $\mathcal{V}$-enriched category (Def. 2.40)

1. the objects are the $\mathcal{C}$-enriched functors $\mathcal{C} \overset{F}{\to} \mathcal{V}$ (Def. 2.46);
2. the hom-objects are the ends

$$[\mathcal{C}, \mathcal{V}](F,G) \;\coloneqq\; \int_{c \in \mathcal{C}} [F(c), G(c)] \tag{58}$$

3. the composition operation on these is defined to be the one induced by the composite maps

$$\left(\int_{c \in \mathcal{C}} [F(c), G(c)] \right) \otimes \left(\int_{c \in \mathcal{C}} [G(c), H(c)] \right) \longrightarrow \prod_{c \in \mathcal{C}} [F(c), G(c)] \otimes [G(c), H(c)] \xrightarrow{(\circ_{F(c),(}}$$

where the first morphism is degreewise given by projection out of the limits that defined the ends. This composite evidently equalizes the two relevant adjunct actions (as in the proof of example 3.15) and hence defines a map into the end

$$\left(\int_{c \in \mathcal{C}} [F(c), G(c)] \right) \otimes \left(\int_{c \in \mathcal{C}} [G(c), H(c)] \right) \longrightarrow \int_{c \in \mathcal{C}} [F(c), H(c)] \;.$$

By Example 3.15, the underlying plain category (Example 2.43) of this enriched functor category is the plain functor category of enriched functors from Example 2.51.

***Proposition 3.17*. (*enriched Yoneda lemma*)**

For $\mathcal{V}$ a cosmos (Def. 2.36) let $\mathcal{C}$ be a small enriched category (Def. 2.40). For $F : \mathcal{C} \to \mathcal{V}$ an enriched presheaf (Example 2.48) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$[\mathcal{C}, \mathcal{V}](\mathcal{C}(c,-),\; F) \;\simeq\; F(c)$$

between the hom-object of the enriched functor category (Def. 3.16), from the functor represented by c to F, and the value of F on c.

In terms of the ends (def. 3.13) defining these hom-objects (58), this means that

$$\int_{d \in \mathcal{C}} [\mathcal{C}(c,d), F(d)] \;\simeq\; F(c) \;.$$

*In this form the statement is also known as **Yoneda reduction**.*

Now that natural transformations are expressed in terms of ends (example 3.15), as is the enriched Yoneda lemma (prop. 3.17), it is natural to consider the dual statement (Example 1.13) involving coends:

***Proposition 3.18*. (enriched co-Yoneda lemma)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category (Def. 2.40). For $F : \mathcal{C} \to \mathcal{V}$ an enriched presheaf (Def. 2.48) and for $c \in \mathcal{C}$ an object, there is a natural isomorphism

$$F(-) \simeq \int^{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes F(c) \;.$$

Moreover, the morphism that hence exhibits $F(c)$ as the coequalizer of the two morphisms in def. 3.13 is componentwise the canonical action

$$\mathcal{C}(c,d) \otimes F(c) \longrightarrow F(d)$$

which is adjunct to the component map $\mathcal{C}(d,c) \to [F(c), F(d)]$ of the enriched functor F.

(e.g. MMSS 00, lemma 1.6)

***Proof*.** By the definition of coends and the universal property of colimits, enriched natural transformations of the form

$$\left(\int^{c \in \mathcal{C}} \mathcal{C}(c, -) \otimes F(c) \right) \longrightarrow G$$

are in natural bijection with systems of component morphisms

$$\mathcal{C}(c,d) \otimes F(c) \longrightarrow G(d)$$

which satisfy some compatibility conditions in their dependence on c and d (natural in d and "extranatural" in c). By the internal hom adjunction, these are in natural bijection to systems of morphisms of the form

$$F(c) \longrightarrow [\mathcal{C}(c,d), G(d)]$$

satisfying the analogous compatibility conditions. By Example 3.15 these are in natural bijection with systems of morphisms

$$F(c) \longrightarrow [\mathcal{C}, \mathcal{V}](\mathcal{C}(c, -), G(-))$$

natural in c

By the enriched Yoneda lemma (Prop. 3.17), these, finally, are in natural bijection with systems of morphisms

$$F(c) \longrightarrow G(c)$$

natural in c. Moreover, all these identifications are also natural in G. Therefore, in summary, this shows that there is a natural isomorphism

$$\mathrm{Hom}_{[\mathcal{C},\mathcal{V}]}\left(\int^{c \in \mathcal{C}} \mathcal{C}(c,-) \otimes F(c) \;,\; (-) \right) \;\simeq\; \mathrm{Hom}_{[\mathcal{C},\mathcal{V}]}(F, (-)) \;.$$

With this, the ordinary Yoneda lemma (Prop. 1.29) in the form of the Yoneda embedding of $[\mathcal{C},\mathcal{V}]$ implies the required isomorphism. ▮

***Example 3.19*. (*co-Yoneda lemma* over *Set*)**

Consider the co-Yoneda lemma (Prop. 3.18) in the special case $\mathcal{V} = $ Set (Example 2.37).

In this case the coequalizer in question is the set of equivalence classes of pairs

$$(c \to c_0,\; x) \quad \in \mathcal{C}(c, c_0) \otimes F(c) \,,$$

where two such pairs

$$(c \overset{f}{\to} c_0,\; x \in F(c)) \,, \qquad (d \overset{g}{\to} c_0,\; y \in F(d))$$

are regarded as equivalent if there exists

$$c \overset{\phi}{\to} d$$

such that

$$f = g \circ \phi \,, \qquad \text{and} \qquad y = \phi(x) \;.$$

(Because then the two pairs are the two images of the pair (g, x) under the two morphisms being coequalized.)

But now considering the case that $d = c_0$ and $g = \mathrm{id}_{c_0}$, so that $f = \phi$ shows that any pair

$$(c \overset{\phi}{\to} c_0,\; x \in F(c))$$

is identified, in the coequalizer, with the pair

$$(\mathrm{id}_{c_0},\ \phi(x) \in F(c_0))\,,$$

hence with $\phi(x) \in F(c_0)$.

As a conceptually important corollary we obtain:

***Proposition 3.20*. (*category of presheaves* is *free co-completion*)**

For $\mathcal{C}$ a small category (Def. 1.6), its Yoneda embedding $\mathcal{C} \overset{y}{\hookrightarrow} [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Prop. 1.30) exhibits the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 1.26) as the free co-completion of $\mathcal{C}$ under forming colimits (Def. 3.1), in that it is a universal morphism, as in Def. 1.41 but "up to natural isomorphism", into a category with all colimits (by Example 3.5) in the following sense:

1. *for $\mathcal{D}$ any category with all colimits (Def. 3.1);*
2. *for $F : \mathcal{C} \longrightarrow \mathcal{D}$ any functor;*

there is a functor $\widetilde{F} : [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \longrightarrow \mathcal{D}$, unique up to natural isomorphism such that

1. *$\widetilde{F}$ preserves all colimits,*
2. *$\widetilde{F}$ extends F through the Yoneda embedding, in that the following diagram commutes, up to natural isomorphism (Def. 1.23):*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "y"'] \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow"] & \\ {[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]} \arrow[rr, "\widetilde{F}"'] & {} & \mathcal{D} \end{tikzcd}$$

Hence when interpreting presheaves as generalized spaces, this says that "generalized spaces are precisely what is obtained from allowing arbitrary gluings of ordinary spaces", see also Remark 4.16 below.

***Proof*.** The last condition says that $\widetilde{F}$ is fixed on representable presheaves by

$$\widetilde{F}(y(c)) \simeq F(c)\ . \tag{59}$$

and in fact naturally so:

$$\begin{array}{ccccc} c_1 & & \widetilde{F}(y(c_1)) & \simeq & F(c_1) \\ {}^{f}\big\downarrow & & \big\downarrow^{F(y(f))} & & \big\downarrow^{F(f)} \\ c_2 & & \widetilde{F}(y(c_2)) & \simeq & F(c_2) \end{array} \tag{60}$$

But the co-Yoneda lemma (Prop. 3.18) expresses every presheaf $\mathbf{X} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ as a colimit of representable presheaves (in the special case of enrichment over Set, Example 3.19)

$$\mathbf{X} \simeq \int^{c \in \mathcal{C}} y(c) \cdot \mathbf{X}(c) .$$

Since $\tilde{F}$ is required to preserve any colimit and hence these particular colimits, (59) implies that $\widetilde{F}$ is fixed to act, up to isomorphism, as

$$\widetilde{F}(\mathbf{X}) = \widetilde{F}\left(\int^{c \in \mathcal{C}} y(c) \cdot \mathbf{X}(c) \right) \coloneqq \int^{c \in \mathcal{C}} F(c) \cdot \mathbf{X}(c) \quad \in \mathcal{D}$$

(where the colimit on the right is computed in $\mathcal{D}$!). ▮

Remark 3.21. The statement of the co-Yoneda lemma in prop. 3.18 is a kind of categorification of the following statement in analysis (whence the notation with the integral signs):

For X a topological space, $f : X \to \mathbb{R}$ a continuous function and $\delta(-, x_0)$ denoting the Dirac distribution, then

$$\int_{x \in X} \delta(x, x_0) f(x) = f(x_0) .$$

It is this analogy that gives the name to the following statement:

Proposition 3.22. ***(Fubini theorem for (co)-ends)***

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}_1, \mathcal{C}_2$ be two $\mathcal{V}$-enriched categories (Def. 2.40) and

$$F : (\mathcal{C}_1 \times \mathcal{C}_2)^{\mathrm{op}} \times (\mathcal{C}_1 \times \mathcal{C}_2) \longrightarrow \mathcal{V}$$

a $\mathcal{V}$-enriched functor (Def. 2.46) from the product category with opposite categories (Def. 2.45), as shown.

Then its end and coend (def. 3.13) is equivalently formed consecutively over each variable, in either order:

$$\int^{(c_1, c_2)} F((c_1, c_2), (c_1, c_2)) \simeq \int^{c_1} \int^{c_2} F((c_1, c_2), (c_1, c_2)) \simeq \int^{c_2} \int^{c_1} F((c_1, c_2), (c_1, c_2))$$

and

$$\int_{(c_1,c_2)} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_1}\int_{c_2} F((c_1,c_2),(c_1,c_2)) \simeq \int_{c_2}\int_{c_1} F((c_1,c_2),(c_1,c_2)) \,.$$

Proof. Because limits commute with limits, and colimits commute with colimits. ▮

***Remark 3.23*. (*internal hom* preserves *ends*)**

Let $\mathcal{V}$ be a cosmos (Def. 2.36). Since the internal hom-functor in $\mathcal{V}$ (Def. 2.8) preserves limits in both variables (Prop. 2.15), in particular it preserves ends (Def. 3.13) in the second variable, and sends coends in the second variable to ends:

For all small $\mathcal{C}$-enriched categories, $\mathcal{V}$-enriched functors $F : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ (Def. 2.46) and all objects $X \in \mathcal{V}$ we have natural isomorphisms

$$\left[X, \int^{c \in \mathcal{C}} F(c,c)\right] \simeq \int^{c\in\mathcal{C}} [X, F(c,c)]$$

and

$$\left[\int_{c\in\mathcal{C}} F(c,c), X\right] \simeq \int^{c\in\mathcal{C}} [F(c,c), X] \,.$$

With this coend calculus in hand, there is an elegant proof of the defining universal property of the smash tensoring and powering enriched presheaves

***Definition 3.24*. (*tensoring* and *powering* of *enriched presheaves*)**

Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category, def. 2.40, with $[\mathcal{C},\mathcal{V}]$ its functor category of enriched functors (Example 2.51).

1. Define a functor

   $$(-)\cdot(-) : [\mathcal{C},\mathcal{V}] \times \mathcal{V} \longrightarrow [\mathcal{C},\mathcal{V}]$$

   by forming objectwise tensor products

   $$F \cdot X : c \mapsto F(c) \otimes X \,.$$

   This is called the ***tensoring*** of $[\mathcal{C},\mathcal{V}]$ over $\mathcal{V}$.

2. Define a functor

   $$(-)^{(-)} : \mathcal{V}^{\mathrm{op}} \times [\mathcal{C},\mathcal{V}] \longrightarrow [\mathcal{C},\mathcal{V}]$$

by forming objectwise internal homs (Def. 2.8)

$$F^X \,:\, c \mapsto [X, F(c)] \;.$$

This is called the ***powering*** of $[\mathcal{C}, \mathcal{V}]$ over $\mathcal{V}$.

***Proposition 3.25.* (universal property of tensoring and powering of enriched presheaves)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category (Def. 2.40), with $[\mathcal{C}, \mathcal{V}]$ the corresponding enriched presheaf category.

Then there are natural isomorphisms

$$[\mathcal{C}, \mathcal{V}](X \cdot K,\, Y) \;\simeq\; [K, \big([\mathcal{C}, \mathcal{V}](X, Y)\big)]$$

and

$$[\mathcal{C}, \mathcal{V}]\big(X,\, Y^K\big) \;\simeq\; [K, \big([\mathcal{C}, \mathcal{C}](X, Y)\big)]$$

for all $X, Y \in [\mathcal{C}, \mathcal{V}]$ and all $K \in \mathcal{C}$, where $(-)^K$ is the powering and $(-) \cdot K$ the tensoring from Def. 3.24.

In particular there is the composite natural isomorphism

$$[\mathcal{C}, \mathcal{V}](X \cdot K, Y) \;\simeq\; [\mathcal{C}, \mathcal{V}]\big(X, Y^K\big)$$

exhibiting a pair of adjoint functors

$$[\mathcal{C}, \mathcal{V}] \underoverset{\underset{(-)^K}{\longrightarrow}}{\overset{(-) \cdot K}{\longleftarrow}}{\bot} [\mathcal{C}, \mathcal{V}] \;.$$

***Proof*.** Via the end-expression for $[\mathcal{C}, \mathcal{V}](-,-)$ from Example 3.15, and the fact (remark 3.23) that the internal hom-functor ends in the second variable, this reduces to the fact that $[-,-]$ is the internal hom in the closed monoidal category $\mathcal{V}$ (Example 2.44) and hence satisfies the internal tensor/hom-adjunction isomorphism (prop. 2.14):

$$
\begin{aligned}
[\mathcal{C},\mathcal{V}](X\cdot K, Y) &= \int_c [(X\cdot K)(c), Y(c)] \\
&\simeq \int_c [X(c)\otimes K, Y(x)] \\
&\simeq \int_c [K, [X(c), Y(c)]] \\
&\simeq [K, \int_c [X(c), Y(c)]] \\
&= [K, ([\mathcal{C},\mathcal{V}](X,Y))]
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{C},\mathcal{V}](X, Y^K) &= \int_c [X(c), Y^K(c)] \\
&\simeq \int_c [X(c), [K, Y(c)]] \\
&\simeq \int_c [X(c)\otimes K, Y(c)] \\
&\simeq \int_c [K, [X(c), Y(c)]] \\
&\simeq [K, \int_c [X(c), Y(c)]] \\
&\simeq [K, [\mathcal{C},\mathcal{V}](X,Y] \,.
\end{aligned}
$$

▮

Tensoring and cotensoring

We make explicit the general concept of which Prpp. 3.25 provides a key class of examples:

***Definition 3.26*. (tensoring and cotensoring)**

For $\mathcal{V}$ a cosmos (Def. 2.36) let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40). Recall the enriched hom-functors (Example 2.47)

$$\mathcal{C}(-,-) \,:\, \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$$

and (via Example 2.44)

$$\mathcal{V}(-,-) = [-,-] \,:\, \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \longrightarrow \mathcal{V} \,.$$

1. A *powering* (or *cotensoring*) of $\mathcal{C}$ over $\mathcal{V}$ is
   1. a functor (Def. 1.15)
      $$[-,-] \,:\, \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$$
   2. for each $v \in \mathcal{V}$ a natural isomorphism (Def. 1.23) of the form
      $$\mathcal{V}(v, \mathcal{C}(c_1, c_2)) \;\simeq\; \mathcal{C}(c_1, [v, c_2]) \tag{61}$$
2. A *copowering* (or *tensoring*) of $\mathcal{C}$ over $\mathcal{V}$ is
   1. a functor (Def. 1.15)
      $$(-) \otimes (-) \,:\, \mathcal{V} \times \mathcal{C} \longrightarrow \mathcal{C}$$
   2. for each $v \in \mathcal{V}$ a natural isomorphism (Def. 1.23) of the form
      $$\mathcal{C}(v \otimes c_1, c_2) \;\simeq\; \mathcal{V}(v, \mathcal{C}(c_1, c_2)) \tag{62}$$

If $\mathcal{C}$ is equipped with a (co-)powering it is called *(co-)powered* over $\mathcal{V}$.

***Proposition 3.27*. (tensoring left adjoint to cotensoring)**

For $\mathcal{V}$ a cosmos (Def. 2.36) let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40).

If $\mathcal{C}$ is both tensored and cotensored over $\mathcal{V}$ (Def. 3.26), then for fixed $v \in \mathcal{V}$ the operations of tensoring with v and of cotensoring with $\mathcal{V}$ form a pair of adjoint functors (Def. 1.32)

$$\begin{tikzcd} \mathcal{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "{[v,-]}"'] & \mathcal{C} \arrow[l, bend right, "v \otimes (-)"'] \end{tikzcd}$$

Proof. The hom-isomorphism (10) characterizing the pair of adjoint functors is provided by the composition of the natural isomorphisms (61) and (62):

$$\mathcal{C}(v \otimes c_1, c_2) \;\simeq\; \mathcal{V}(v, \mathcal{C}(c_1, c_2)) \;\simeq\; \mathcal{C}(c_1, [v, c_2])$$

■

Proposition 3.28. (in tensored and cotensored categories initial/terminal objects are enriched initial/terminal)

For $\mathcal{V}$ a cosmos (Def. 2.36) let $\mathcal{C}$ be a $\mathcal{V}$-enriched category (Def. 2.40).

If $\mathcal{C}$ is both tensored and cotensored over $\mathcal{V}$ (Def. 3.26) then

1. *an initial object $\emptyset$ (Def. 1.5) of the underlying category of $\mathcal{C}$ (Example 2.43) is also enriched initial, in that the hom-object out of it is the terminal object $*$ of $\mathcal{V}$*

$$\mathcal{C}(\emptyset, c) \;\simeq\; *$$

2. *a terminal object $*$ (Def. 1.5) of the underlying category of $\mathcal{C}$ (Example 2.43) is also enriched terminal, in that the hom-object into it is the terminal object of $\mathcal{V}$:*

$$\mathcal{C}(c, *) \;\simeq\; *$$

Proof. We discuss the first claim, the second is formally dual.

By prop. 3.27, tensoring is a left adjoint. Since left adjoints preserve colimits (Prop. 3.8), and since an initial object is the colimit over the empty diagram (Example 3.3), it follows that

$$v \otimes \emptyset \;\simeq\; \emptyset$$

for all $v \in \mathcal{V}$, in particular for $\emptyset \in \mathcal{V}$. Therefore the natural isomorphism (62) implies for all $v \in \mathcal{V}$ that

$$\mathcal{C}(\emptyset, c) \;\simeq\; \mathcal{C}(\emptyset \otimes \emptyset, c) \;\simeq\; \mathcal{V}(\emptyset, \mathcal{C}(\emptyset, c)) \;\simeq\; *$$

where in the last step we used that the internal hom $\mathcal{V}(-,-) = [-,-]$ in $\mathcal{V}$ sends colimits in its first argument to limits (Prop. 2.15) and used that a terminal object is the limit over the empty diagram (Example 3.3). ■

Kan extensions

Proposition 3.29. (Kan extension)

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}, \mathcal{D}$ be small $\mathcal{V}$-enriched categories (Def. 2.40) and let

$$p : \mathcal{C} \longrightarrow \mathcal{D}$$

be a $\mathcal{V}$-enriched functor (Def. 2.46). Then precomposition with p constitutes a functor between the corresponding $\mathcal{V}$-enriched presheaf categories (Def. 3.16)

$$p^* : \begin{array}{ccc} [\mathcal{D}, \mathcal{V}] & \longrightarrow & [\mathcal{C}, \mathcal{V}] \\ G & \mapsto & G \circ p \end{array} \qquad (63)$$

1. *This enriched functor p^* (63) has an enriched left adjoint Lan_p (Def. 2.52), called left Kan extension along p*

$$[\mathcal{D}, \mathcal{V}] \underoverset{\underset{p^*}{\longrightarrow}}{\overset{\mathrm{Lan}_p}{\longleftarrow}}{\bot} [\mathcal{C}, \mathcal{V}]$$

   which is given objectwise by the coend (def. 3.13):

$$(\mathrm{Lan}_p F) : d \;\mapsto\; \int^{c \in \mathcal{C}} \mathcal{D}(p(c), d) \otimes F(c) \,. \qquad (64)$$

2. *The enriched functor p^* (63) has an enriched right adjoint Ran_p (Def. 2.52), called right Kan extension along p*

$$[\mathcal{C}, \mathcal{V}] \underoverset{\underset{\mathrm{Ran}_p}{\longrightarrow}}{\overset{p^*}{\longleftarrow}}{\bot} [\mathcal{D}, \mathcal{V}]$$

   which is given objectwise by the end (def. 3.13):

$$(\mathrm{Ran}_p F) : d \;\mapsto\; \int_{c \in \mathcal{C}} [\mathcal{D}(d, p(c)), F(c)] \,. \qquad (65)$$

In summary, this means that the enriched functor

$$\mathcal{C} \xrightarrow{p} \mathcal{D}$$

induces, via Kan extension, an adjoint triple (Remark 1.34) of enriched functors

$$\mathrm{Lan}_p \;\dashv\; p^* \;\dashv\; \mathrm{Ran}_p \;:\; [\mathcal{C}, \mathcal{V}] \leftrightarrow [\mathcal{D}, \mathcal{V}] \,. \qquad (66)$$

Proof. Use the expression of enriched natural transformations in terms of coends (example 3.15 and def. 3.16), then use the respect of $[-,-]$ for ends/coends (remark 3.23), use the internal-hom adjunction (38), use the Fubini theorem (prop. 3.22) and finally use Yoneda reduction (prop. 3.17) to obtain a sequence of natural isomorphisms as follows:

$$
\begin{aligned}
[\mathcal{D},\mathcal{V}](\mathrm{Lan}_p F,\, G) &= \int_{d\in\mathcal{D}} [(\mathrm{Lan}_p F)(d),\, G(d)] \\
&= \int_{d\in\mathcal{D}} \left[\int^{c\in\mathcal{C}} \mathcal{D}(p(c),d)\otimes F(c),\ G(d) \right] \\
&\simeq \int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} [\mathcal{D}(p(c),d)\otimes F(c)\,,\ G(d)] \\
&\simeq \int_{c\in\mathcal{C}} \int_{d\in\mathcal{D}} [F(c), [\mathcal{D}(p(c),d),\, G(d)]] \\
&\simeq \int_{c\in\mathcal{C}} [F(c),\ \int_{d\in\mathcal{D}} [\mathcal{D}(p(c),d),\, G(d)]] \\
&\simeq \int_{c\in\mathcal{C}} [F(c), G(p(c))] \\
&= [\mathcal{C},\mathcal{V}](F, p^* G)
\end{aligned}
.
$$

and similarly:

$$
\begin{aligned}
[\mathcal{D},\mathcal{V}](G, \mathrm{Ran}_p F) & \simeq \int_{d \in \mathcal{D}} [G(d), (\mathrm{Ran}_p F)(d),]
\\
& \simeq \int_{d \in \mathcal{D}} \left[G(d), \int_{c \in \mathcal{C}} [\mathcal{D}(d, p(c)), F(c)] \right]
\\
& \simeq \int_{d \in \mathcal{D}} \int_{c \in \mathcal{C}} [G(d) \otimes \mathcal{D}(d, p(c)), F(c)]
\\
& \simeq \int_{c \in \mathcal{C}} \left[\int^{d \in \mathcal{D}} G(d) \otimes \mathcal{D}(d, p(c)), F(c) \right]
\\
& \simeq \int_{c \in \mathcal{D}} [G(p(c)), F(c)]
\\
& \simeq [\mathcal{C},\mathcal{V}](p^* G, F)
\end{aligned}
$$

▮

***Example 3.30**. (coend formula for left Kan extension of ordinary presheaves)*

Consider the cosmos to be $\mathcal{V} = \mathrm{Set}$, via Example 2.37, so that small $\mathcal{V}$-enriched categories (Def. 2.40) are just a plain small category (Def. 1.1) by Example 2.41, and $\mathcal{V}$-enriched presheaves (Example 2.48) are just plain presheaves (Example 1.26).

Then for any plain functor (Def. 1.15)

$$
\mathcal{C}^{\mathrm{op}} \xrightarrow{p} (\mathcal{C}')^{\mathrm{op}}
$$

the general formula (64) for left Kan extension

$$
[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\mathrm{Lan}_p} [(\mathcal{C}')^{\mathrm{op}}, \mathrm{Set}]
$$

is

$$
(\mathrm{Lan}_p F)(c') \simeq \int^{c \in C} C'(c', p(c)) \times F(c) \; .
$$

Using here the Yoneda lemma (Prop. 1.29) to rewrite $F(c) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(c, F)$, this is

$$(\mathrm{Lan}_p F)(c') \simeq \int^{c \in C} \mathrm{Hom}_{C'}(c', p(c)) \times \mathrm{Hom}_{\mathrm{PSh}(C)}(c, F) \,.$$

Hence this coend-set consists of equivalence classes of pairs of morphisms

$$(c' \to p(c), c \to F)$$

where two such are regarded as equivalent whenever there is $f : c'_1 \to c'_2$ such that

$$\begin{array}{ccccc} & & c' & & \\ & \swarrow & & \searrow & \\ p(c_1) & & \overset{p(f)}{\longrightarrow} & & p(c_2) \\ c_1 & & \overset{f}{\longrightarrow} & & c_2 \\ & \searrow & & \swarrow & \\ & & F & & \end{array} \,.$$

This is particularly suggestive when p is a full subcategory inclusion (Def. 1.19). For in that case we may imagine that a representative pair $(c' \to p(c), c \to F)$ is a stand-in for the actual pullback of elements of F along the would-be composite "$c' \to c \to F$", only that this composite need not be defined. But the above equivalence relation is precisely that under which this composite would be invariant.

Further properties

We collect here further key properties of the various universal constructions considered above.

***Proposition 3.31**. (**left Kan extension** **preserves** **representable functors**)*

For $\mathcal{V}$ a cosmos (Def. 2.36), let

$$\mathcal{C} \overset{p}{\longrightarrow} \mathcal{D}$$

be a $\mathcal{V}$-enriched functor (Def. 2.46) between small $\mathcal{V}$-enriched categories (Def. 2.40).

Then the left Kan extension Lan_p (Prop. 3.29) takes representable enriched presheaves $\mathcal{C}(c, -) \,:\, \mathcal{C} \to \mathcal{V}$ to their image under p:

$$\mathrm{Lan}_p \, \mathcal{C}(c, -) \;\simeq\; \mathcal{D}(p(c), -)$$

for all $c \in C$.

Proof. By the coend formula (64) we have, naturally in $d' \in \mathcal{D}$, the expression

$$\begin{aligned} \mathrm{Lan}_p\, \mathcal{C}(c,-) \;:\; d' \mapsto & \int^{c' \in C} \mathcal{D}(p(c'), d') \otimes \mathcal{C}(c,-)(c') \\ & \simeq \int^{c' \in C} \mathcal{D}(p(c'), d') \otimes \mathcal{C}(c,c') \\ & \simeq \mathcal{D}(p(c), d') \end{aligned},$$

where the last step is the co-Yoneda lemma (Prop. 3.18). ■

Example 3.32. ***(Kan extension of adjoint pair is adjoint quadruple)***

For $\mathcal{V}$ a cosmos (Def. 2.36), let $\mathcal{C}, \mathcal{D}$ be two small $\mathcal{V}$-enriched categories (Def. 2.40) and let

$$\mathcal{C} \underoverset{\underset{p}{\longrightarrow}}{\overset{q}{\longleftarrow}}{\bot} \mathcal{D}$$

be a $\mathcal{V}$-enriched adjunction (Def. 2.52). Then there are $\mathcal{V}$-enriched natural isomorphisms (Def. 2.50)

$$\begin{aligned} (q^{\mathrm{op}})^* & \simeq \mathrm{Lan}_{p^{\mathrm{op}}} \;:\; [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{D}^{\mathrm{op}}, \mathcal{V}] \\ (p^{\mathrm{op}})^* & \simeq \mathrm{Ran}_{q^{\mathrm{op}}} \;:\; [\mathcal{D}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \end{aligned}$$

between the precomposition on enriched presheaves with one functor and the left/right Kan extension of the other (Def. 3.29).

By essential uniqueness of adjoint functors, this means that the two adjoint triples (Remark 1.34) given by Kan extension (66) of q and p

$$\begin{array}{ccccccccc} \mathrm{Lan}_{q^{\mathrm{op}}} & \dashv & (q^{\mathrm{op}})^* & \dashv & \mathrm{Ran}_{q^{\mathrm{op}}} & & \\ & & \mathrm{Lan}_{p^{\mathrm{op}}} & \dashv & (p^{\mathrm{op}})^* & \dashv & \mathrm{Ran}_{p^{\mathrm{op}}} \end{array}$$

merge into an adjoint quadruple (Remark 1.34)

$$\mathrm{Lan}_{q^{\mathrm{op}}} \;\dashv\; (q^{\mathrm{op}})^* \;\dashv\; (p^{\mathrm{op}})^* \;\dashv\; \mathrm{Ran}_{p^{\mathrm{op}}} \;:\; [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrow [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$$

Proof. For every enriched presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ we have a sequence of $\mathcal{V}$-enriched natural isomorphism as follows

$$
\begin{aligned}
(\mathrm{Lan}_{p^{\mathrm{op}}} F)(d) & \simeq \int^{c \in \mathcal{C}} \mathcal{D}(d, p(c)) \otimes F(c) \\
& \simeq \int^{c \in \mathcal{C}} \mathcal{C}(q(d), c) \otimes F(c) \\
& \simeq F(q(d)) \\
& = ((q^{\mathrm{op}})^* F)(d) \;.
\end{aligned}
$$

Here the first step is the coend-formula for left Kan extension (Prop. 3.29), the second step if the enriched adjunction-isomorphism (47) for $q \dashv p$ and the third step is the co-Yoneda lemma.

This shows the first statement, which, by essential uniqueness of adjoints, implies the following statements. ▮

***Proposition 3.33*. (left Kan extension along fully faithful functor is fully faithful)**

For $\mathcal{V}$ a cosmos (Def. 2.36), let

$$
\mathcal{C} \overset{p}{\hookrightarrow} \mathcal{D}
$$

be a fully faithful $\mathcal{V}$-enriched functor (Def. 2.46) between small $\mathcal{V}$-enriched categories (Def. 2.40).

Then for all $c \in \mathcal{C}$

$$
p^*(\mathrm{Lan}_p\, c) \simeq c
$$

and in fact the $(\mathrm{Lan}_F \dashv F^*)$-unit of an adjunction is a natural isomorphism

$$
\mathrm{Id} \overset{\simeq}{\Rightarrow} p^* \circ \mathrm{Lan}_p \;.
$$

hence, by Prop. 1.46,

$$
[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \overset{\mathrm{Lan}_p}{\hookrightarrow} [\mathcal{D}^{\mathrm{op}}, \mathrm{Set}]
$$

is a fully faithful functor.

***Proof*.** By the coend formula (64) we have, naturally in $d' \in \mathcal{D}$, the left Kan extension of any $F : \mathcal{C} \to \mathcal{V}$ on the image of p is

$$
\begin{aligned}
\mathrm{Lan}_p F \,:\, p(c) \mapsto & \int^{c' \in C} \mathcal{D}(p(c'), p(c)) \cdot F(c') \\
& \simeq \int^{c' \in C} \mathcal{C}(c', c) \cdot F(c') \\
& \simeq F(c)
\end{aligned}
,
$$

where in the second step we used the assumption of fully faithfulness of p and in the last step we used the co-Yoneda lemma (Prop. 3.18). ▮

***Lemma 3.34*. (colimit of representable is singleton)**

Let $\mathcal{C}$ be a small category (Def. 1.6). Then the colimit of a representable presheaf (Def. 1.26), regarded as a functor

$$
y(c) \,:\, \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}
$$

is the singleton set.

$$
\underset{\mathcal{D}^{\mathrm{op}}}{\underset{\longrightarrow}{\lim}}\, y(c) \;\simeq\; * \,. \tag{67}
$$

***Proof*.** One way to see this is to regard the colimit as the left Kan extension (Prop. 3.29) along the unique functor $\mathcal{C}^{\mathrm{op}} \overset{p}{\to} *$ to the terminal category (Def. 1.7). By the formula (64) this is

$$
\begin{aligned}
\underset{\mathcal{D}^{\mathrm{op}}}{\underset{\longrightarrow}{\lim}}\, y(c) & \simeq \int^{c_1 \in \mathcal{C}} \underbrace{*\,(-, p(c_1))}_{\mathrm{const}_*(c_1)} \times y(c)(c_1) \\
& \simeq \int^{c_1 \in \mathcal{C}} \mathrm{const}_*(c_1) \times \mathcal{C}(c_1, c) \\
& \simeq \mathrm{const}_*(c) \\
& \simeq *
\end{aligned}
$$

where we made explicit the constant functor $\mathrm{const}_* \,:\, \mathcal{C} \to \mathrm{Set}$, constant on the singleton set $*$, and then applied the co-Yoneda lemma (Prop. 3.18). ▮

***Proposition 3.35*. (categories with finite products are cosifted**

Let $\mathcal{C}$ be a small category (Def. 1.6) which has finite products. Then $\mathcal{C}$ is a cosifted category, equivalently its opposite category $\mathcal{C}^{\mathrm{op}}$ is a sifted category, equivalently colimits over $\mathcal{C}^{\mathrm{op}}$

with values in Set are sifted colimits, equivalently colimits over $\mathcal{C}^{\mathrm{op}}$ with values in Set commute with finite products, as follows:

For $\mathbf{X}, \mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ to functors on the opposite category of $\mathcal{C}$ (hence two presheaves on $\mathcal{C}$, Example 1.26) we have a natural isomorphism (Def. 1.23)

$$\underset{\underset{\mathcal{C}^{\mathrm{op}}}{\longrightarrow}}{\lim}(\mathbf{X} \times \mathbf{Y}) \;\simeq\; \left(\underset{\underset{\mathcal{C}^{\mathrm{op}}}{\longrightarrow}}{\lim}\mathbf{X}\right) \times \left(\underset{\underset{\mathcal{C}^{\mathrm{op}}}{\longrightarrow}}{\lim}\mathbf{Y}\right)$$

between the colimit of their Cartesian product and the Cartesian product of their separate colimits.

Proof. First observe that for $\mathbf{X}, \mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ two presheaves, their Cartesian product is a colimit over presheaves represented by Cartesian products in $\mathcal{C}$. Explicity, using coend-notation, we have:

$$\mathbf{X} \times \mathbf{Y} \;\simeq\; \int^{c_1, c_2 \in \mathcal{C}} y(c_1 \times c_2) \times \mathbf{X}(c_1) \times \mathbf{Y}(c_2)\,, \tag{68}$$

where $y \,:\, \mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ denotes the Yoneda embedding.

This is due to the following sequence of natural isomorphisms:

$$\begin{aligned} (\mathbf{X} \times \mathbf{Y})(c) & \simeq \left(\int^{c_1 \in \mathcal{C}} \mathcal{C}(c, c_1) \times \mathbf{X}(c_1)\right) \times \left(\int^{c_2 \in \mathcal{C}} \mathcal{C}(c, c_2) \times \mathbf{Y}(c_2)\right) \\ & \simeq \int^{c_1 \in \mathcal{C}} \int^{c_2 \in \mathcal{C}} \underbrace{\mathcal{C}(c, c_1) \times \mathcal{C}(c, c_2)}_{\simeq \mathcal{C}(c, c_1 \times c_2)} \times (\mathbf{X}(c_1) \times \mathbf{X}(c_2)) \\ & \simeq \int^{c_1 \in \mathcal{C}} \int^{c_2 \in \mathcal{C}} \mathcal{C}(c, c_1 \times c_2) \times \mathbf{X}(c_1) \times \mathbf{X}(c_2)\,, \end{aligned}$$

where the first step expands out both presheaves as colimits of representables separately, via the co-Yoneda lemma (Prop. 3.18), the second step uses that the Cartesian product of presheaves is a two-variable left adjoint (by the symmetric closed monoidal structure on presheaves) and as such preserves colimits (in particular coends) in each variable separately (Prop. 3.8), and under the brace we use the defining universal property of the Cartesian products, assumed to exist in $\mathcal{C}$.

With this, we have the following sequence of natural isomorphisms:

$$
\begin{aligned}
\underset{\mathcal{D}^{\mathrm{op}}}{\underrightarrow{\lim}}(\mathbf{X}\times\mathbf{Y}) & \simeq \underset{\mathcal{D}^{\mathrm{op}}}{\underrightarrow{\lim}} \int^{c_1, c_2 \in \mathcal{C}} y(c_1 \times c_2) \times \mathbf{X}(c_1) \times \mathbf{Y}(c_2) \\
& \simeq \int^{c_1, c_2 \in \mathcal{C}} \underset{\mathcal{D}^{\mathrm{op}}}{\underrightarrow{\lim}}\left(y(c_1 \times c_2) \times \mathbf{X}(c_1) \times \mathbf{Y}(c_2)\right) \\
& \simeq \int^{c_1, c_2 \in \mathcal{C}} \left(\underbrace{\underset{\mathcal{D}^{\mathrm{op}}}{\underrightarrow{\lim}}\, y(c_1 \times c_2)}_{\simeq *} \right) \times \mathbf{X}(c_1) \times \mathbf{Y}(c_2) \\
& \simeq \int^{c_1, c_2 \in \mathcal{C}} \left(\mathbf{X}(c_1) \times \mathbf{Y}(c_2)\right) \\
& \simeq \left(\int^{c_1 \in \mathcal{C}} \mathbf{X}(c_1) \right) \times \left(\int^{c_2 \in \mathcal{C}} \mathbf{Y}(c_2) \right) \\
& \simeq \left(\underset{\mathcal{C}^{\mathrm{op}}}{\underrightarrow{\lim}} \mathbf{X} \right) \times \left(\underset{\mathcal{C}^{\mathrm{op}}}{\underrightarrow{\lim}} \mathbf{Y} \right)
\end{aligned}
$$

Here the first step is (68), the second uses that colimits commute with colimits (Prop. 3.9), the third uses again that the Cartesian product respects colimits in each variable separately, the fourth is by Lemma 3.34, the last step is again the respect for colimits of the Cartesian product in each variable separately. ■

4. Basic notions of Topos theory

We have explained in Remark 1.28 how presheaves on a category $\mathcal{C}$ may be thought of as *generalized spaces probe-able by the objects of* $\mathcal{C}$, and that two consistency conditions on this interpretation are provided by the Yoneda lemma (Prop. 1.29) and the resulting Yoneda embedding (Prop. 1.30). Here we turn to a third consistency condition that one will want to impose, namely a *locality* or *gluing condition* (Remark 4.1 below), to be called the *sheaf* condition (Def. 4.1 below).

More in detail, we had seen that any category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is the free cocompletion of the given small category $\mathcal{C}$ (Prop. 3.20) and hence exhibits generalized spaces $\mathbf{X} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ as being glued or *generated* form the "ordinary spaces" $X \in \mathcal{C}$. Further conditions to be imposed now will impose *relations* among these generators, such as the locality relation embodied by the sheaf-condition.

It turns out that these relations are reflected by special properties of an adjunction (Def. 1.32) that relates generalized spaces to ordinary spaces:

generalized spaces via generators and relations:

free cocompletion $=$ **presheaves**	**loc. presentable category**	**sheaf topos**
$\mathbf{H} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}} [\mathcal{C}^{op}, \mathrm{Set}]$	$\mathbf{H} \underset{\text{accessible}}{\overset{\longleftarrow}{\underset{\hookrightarrow}{\perp}}} [\mathcal{C}^{op}, \mathrm{Set}]$	$\mathbf{H} \underset{\text{accessible}}{\overset{\text{left exact}}{\overset{\longleftarrow}{\underset{\hookrightarrow}{\perp}}}} [\mathcal{C}^{op}, \mathrm{Set}]$
Prop. 3.20	Def. 4.30	Prop. 4.32
simplicial presheaves	***combinatorial model category***	***model topos***
$\mathbf{H} \underset{\longrightarrow}{\overset{\longleftarrow}{\simeq_{Qu}}} [\mathcal{C}^{op}, \mathrm{sSet}_{Qu}]_{proj}$	$\mathbf{H} \underset{\text{accessible}}{\overset{\longleftarrow}{\underset{\hookrightarrow}{\perp_{Qu}}}} [\mathcal{C}^{op}, \mathrm{sSet}_{Qu}]_{proj}$	$\mathbf{H} \underset{\text{accessible}}{\overset{\text{left exact}}{\overset{\longleftarrow}{\underset{\hookrightarrow}{\perp_{Qu}}}}} [\mathcal{C}^{op}, \mathrm{sSet}_{Qu}]_{proj}$
Example	Def.	Def.

***Remark 4.1*. (*sheaf condition as local-to-global principle for generalized spaces*)**

If the objects of $\mathcal{C}$ are thought of as spaces of sorts, as in Remark 1.28, then there is typically a notion of *locality* in these spaces, reflected by a notion of what it means to *cover* a given space by (“smaller”) spaces (a *coverage*, Def. 4.3 below).

But if a space $X \in \mathcal{C}$ is covered, say by two other spaces $U_1, U_2 \in \mathcal{C}$, via morphisms

$$\begin{array}{ccc} U_1 & & U_2 \\ & {}_{i_1}\searrow \quad \swarrow_{i_2} & \\ & X & \end{array}$$

then this must be reflected in the behaviour of the probes of any generalized space $\mathbf{Y}$ (in the sense of Remark 1.28) by these test spaces:

For ease of discussion, suppose that there is a sense in which these two patches above intersect in X to form a space $U_1 \cap_X U_2 \in \mathcal{C}$. Then locality of probes should imply that the ways of mapping U_1 and U_2 into $\mathbf{Y}$ such that these maps agree on the intersection

$U_1 \cap_X U_2$, should be equivalent to the ways of mapping all of X into $\mathbf{Y}$.

$$\text{locality} : \left\{ \begin{matrix} \text{maps from } U_1 \text{ and } U_2 \text{ to } \mathbf{Y} \\ \text{that coincide on } U_1 \cap_X U_2 \end{matrix} \right\} \simeq \{\text{maps from } X \text{ into } \mathbf{Y}\}$$

One could call this the condition of *locality of probes of generalized spaces probeable by objects of $\mathcal{C}$*. But the established terminology is that this is the *sheaf condition (74) on presheaves over $\mathcal{C}$*. Those presheaves which satisfy this condition are called the *sheaves* (Def. 4.8 below).

***Remark 4.2*. Warning**

Most (if not all) introductions to sheaf theory insist on motivating the concept from the special case of sheaves on topological spaces (Example 4.12 below). This is good motivation for what Grothendieck called "petit topos"-theory. The motivation above, instead, naturally leads to the "gros topos"-perspective, as in Example 4.15 below, which is more useful for discussing the synthetic higher supergeometry of physics. In fact, this is the perspective of *functorial geometry* that has been highlighted since Grothendieck 65, but which has maybe remained underappreciated.

We now first introduce the sheaf-condition (Def. 4.8) below in its traditional form via "matching families" (Def. 4.6 below). Then we show (Prop. 4.29 below) how this is equivalently expressed in terms of *Cech groupoids* (Example 4.28 below). This second formulation is convenient for understanding and handling various constructions in ordinary topos theory (for instance the definition of cohesive sites) and it makes immediate the generalization to higher topos theory.

Descent

Here we introduce the sheaf-condition (Def. 4.8 below) in its component-description via matching families (Def. 4.6 below). Then we consider some of the general key properties of the resulting categories of sheaves, such as notably their "convenience", in the technical sense of Prop. 4.23 below.

***Definition 4.3*. (coverage and site)**

Let $\mathcal{C}$ be a small category (Def. 1.6). Then a *coverage* on $\mathcal{C}$ is

- for each object $X \in \mathcal{C}$ a set of indexed sets of morphisms into X

$$\left\{ U_i \overset{\iota_i}{\to} X \right\}_{i \in I}$$

called the *coverings* of X,

such that

- for every covering $\left\{ U_i \overset{\iota_i}{\to} X \right\}_{i \in I}$ of X and every morphism $Y \overset{f}{\longrightarrow} X$ there exists a *refining covering* $\left\{ V_j \overset{\iota_j}{\to} Y \right\}_{j \in J}$ of Y, meaning that for each $j \in J$ there exists $i \in I$ and a morphism $V_j \overset{\iota_{j,i}}{\to} U_i$ such that

$$f \circ \iota_j \;=\; \iota_i \circ \iota_{j,i} \qquad \begin{array}{ccc} V_j & \overset{\iota_{j,i}}{\longrightarrow} & U_i \\ {}^{\mathllap{\iota_j}}\downarrow && \downarrow^{\mathrlap{\iota_i}} \\ Y & \underset{f}{\longrightarrow} & X \end{array} \tag{69}$$

A small category $\mathcal{C}$ equipped with a coverage is called a *site*.

Example 4.4*. *(canonical coverage on topological spaces)

The category Top of (small) topological spaces (Example 1.3) carries a coverage (Def. 4.3) whose coverings are the usal open covers of topological spaces.

The condition (69) on a coverage is met, since the preimages of open subsets under a continuous function f are again open subsets, so that the preimages of an open cover consistitute an open cover of the domain, such that the commuting diagram-condition (69) is immediage.

Similarly, for $X \in \mathrm{Top}$ a fixed topological space, there is the site $\mathrm{Op}(X)$ whose underlying category is the *category of opens* of X, which is the thin category (Example 1.8) of open subsets of X and subset inclusions, and whose coverings are again the open covers.

Example 4.5*. *(differentiably good open covers of smooth manifolds)

The category SmthMfd of smooth manifold (Example 1.3) carries a coverage (Def. 4.3), where for $X \in \mathrm{SmthMfd}$ any smooth manifold of dimension $D \in \mathbb{N}$, its coverings are collections of smooth functions from the Cartesian space $\mathbb{R}^D$ to X whose image is the inclusion of an open ball.

Hence these are the usual *open covers* of X, but with the extra condition that every patch is diffeomorphic to a Cartesian space (hence to a smooth open ball).

One may further constrain this and ask that also all the non-empty finite intersections of these open balls are diffeomorphic to open balls. These are the *differentiably good open covers*.

To see that these coverings satisfy the condition (69): The plain pullback of an open cover along any continuous function is again an open cover, just not necessarily by patches diffeomorphic to open balls. But every open cover may be *refined* by one that is (see at *good open cover*), and this is sufficient for (69).

Example 4.5 is further developed in the chapters *smooth sets* and *on smooth homotopy types*.

Definition 4.6. (matching family – descent object)

Let $\mathcal{C}$ be a small category equipped with a coverage, hence a site (Def. 4.3) and consider a presheaf $\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 1.26) over $\mathcal{C}$.

Given an object $X \in \mathcal{C}$ and a covering $\left\{U_i \overset{\iota_i}{\to} X\right\}_{i \in I}$ of it (Def. 4.3) we say that a *matching family* (of probes of $\mathbf{Y}$) is a tuple $(\phi_i \in \mathbf{Y}(U_i))_{i \in I}$ such that for all $i, j \in I$ and pairs of morphisms $U_i \overset{\kappa_i}{\leftarrow} V \overset{\kappa_j}{\to} U_j$ satisfying

$$\iota_i \circ \kappa_i \;=\; \iota_j \circ \kappa_j \qquad \begin{array}{ccccc} & & V & & \\ & {}^{\kappa_i}\swarrow & & \searrow^{\kappa_j} & \\ U_i & & & & U_j \\ & {}_{\iota_i}\searrow & & \swarrow_{\iota_j} & \\ & & X & & \end{array} \tag{70}$$

we have

$$\mathbf{Y}(\kappa_i)(\phi_i) \;=\; \mathbf{Y}(\kappa_j)(\phi_j)\;. \tag{71}$$

We write

$$\mathrm{Match}\big(\{U_i\}_{i \in I}, \mathbf{Y}\big) \subset \prod_i \mathbf{Y}(U_i) \;\in\; \mathrm{Set} \tag{72}$$

for the set of matching families for the given presheaf and covering.

This is also called the *descent object* of $\mathbf{Y}$ for *descent* along the covering $\{U_i \overset{\iota_i}{\to} X\}$.

Example 4.7. (matching families that glue)

Let $\mathcal{C}$ be a small category equipped with a coverage, hence a site (Def. 4.3) and consider a presheaf $\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 1.26) over $\mathcal{C}$.

Given an object $X \in \mathcal{C}$ and a covering $\left\{ U_i \overset{\iota_i}{\to} X \right\}_{i \in I}$ of it (Def. 4.3), then every element

$$\phi \;\in\; \mathbf{Y}(X)$$

induces a matching family (Def. 4.6) by

$$\big(\mathbf{Y}(\iota_i)(\phi) \big)_{i \in I} \,.$$

(That this indeed satisfies the matching condition follows immediately by the functoriality of $\mathbf{Y}$.)

This construction provides a function of the form

$$\mathbf{Y}(X) \longrightarrow \mathrm{Match}\big(\{U_i\}_{i \in I}, \mathbf{Y} \big) \tag{73}$$

The matching families in the image of this function are hence those tuples of probes of $\mathbf{Y}$ by the patches U_i of X which *glue* to a global probe out of X.

***Definition 4.8.* (sheaves and sheaf toposes)**

Let $\mathcal{C}$ be a small category equipped with a coverage, hence a site (Def. 4.3) and consider a presheaf $\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 1.26) over $\mathcal{C}$.

The presheaf $\mathbf{Y}$ is called a *sheaf* if for every object $X \in \mathcal{C}$ and every covering $\left\{ U_i \overset{\iota_i}{\to} X \right\}_{i \in I}$ of X *all matching families glue uniquely*, hence if the comparison morphism (73) is a bijection

$$\mathbf{Y}(X) \overset{\simeq}{\longrightarrow} \mathrm{Match}\big(\{U_i\}_{i \in I}, \mathbf{Y} \big) \,. \tag{74}$$

The full subcategory (Example 1.20) of the category of presheaves over a given site $\mathcal{C}$, on those that are sheaves is the *category of sheaves*, denoted

$$\mathrm{Sh}(\mathcal{C}) \xhookrightarrow{\;\;\iota\;\;} [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \,. \tag{75}$$

A category which is equivalent (Def. 1.57) to a category of sheaves is called a *sheaf topos*, or often just *topos*, for short.

For $\mathbf{H}_1$ and $\mathbf{H}_2$ two such sheaf toposes, a homomorphism $f : \mathbf{H}_1 \to \mathbf{H}_2$ between them, called a *geometric morphism* is an adjoint pair of functors (Def. 1.32)

$$\mathbf{H}_1 \underoverset{\underset{f_*}{\longrightarrow}}{\overset{f^*}{\longleftarrow}}{} \mathbf{H}_2 \tag{76}$$

such that

- the left adjoint f^*, called the *inverse image*, preserves finite products.

Hence there is a category *Topos*, whose objects are sheaf toposes and whose morphisms are geometric morphisms.

***Example 4.9*. (*global sections geometric morphism*)**

Let $\mathbf{H}$ be a sheaf topos (Def. 4.8). Then there is a geometric morphism (76) to the category of sets (Example 1.2), unique up to natural isomorphism (Def. 1.23):

$$\mathbf{H} \underoverset{\underset{\Gamma}{\longrightarrow}}{\overset{L}{\hookleftarrow}}{\bot} \mathrm{Set} \,.$$

Here Γ is called the *global sections*-*functor*.

***Proof*.** Notice that every set $S \in \mathrm{Set}$ is the coproduct, indexed by itself, of the terminal object $* \in \mathrm{Set}$ (the singleton):

$$S \;\simeq\; \underset{s \in S}{\coprod} * \,.$$

Since L is a left adjoint, it preserves this coproduct (Prop. 3.8). Moreover, since L is assumed to preserve finite products, and since the terminal object is the empty product (Example 3.3), it also preserves the terminal object. Therefore L is fixed, up to natural isomorphism, to act as

$$\begin{aligned} L(S) & \simeq L\left(\textstyle\coprod_{s \in S} *\right) \\ & \simeq \textstyle\coprod_{s \in S} L(*) \\ & \simeq \textstyle\coprod_{s \in S} * \end{aligned} \,.$$

This shows that L exists and uniquely so, up to natural isomorphism. This implies the essential uniqueness of Γ by uniqueness of adjoints (Prop. 1.45). ▮

***Example 4.10*. (*trivial coverage*)**

For $\mathcal{C}$ a small category (Def. 1.6), the *trivial coverage* on it is the coverage (Def. 4.3) with no covering families at all, meaning that the sheaf condition (Def. 4.8) over the resulting site is empty, in that *every* presheaf is a sheaf for this coverage.

Hence the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 1.26) over a site $\mathcal{C}_{\mathrm{triv}}$ with trivial coverage is already the corresponding category of sheaves, hence the corresponding sheaf topos:

$$\mathrm{Sh}(\mathcal{C}_{\mathrm{triv}}) \;\simeq\; [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]\;.$$

***Example 4.11*. (sheaves on the terminal category are plain sets)**

Consider the terminal category $*$ (Example 1.7) equipped with its trivial coverage (Example 4.10). Then there is a canonical equivalence of categories (Def. 1.57) between the category of sheaves on this site (Def. 4.8) and the category of sets (Example 1.2):

$$\mathrm{Sh}(*) \;\simeq\; \mathrm{Set}\;.$$

Hence the category of sets is a sheaf topos.

***Example 4.12*. (sheaves on a topological space – spatial petit toposes)**

In the literature, the concept of (pre-)sheaf (Def. 4.8) is sometimes not defined relative to a site, but relative to a topological space. But the latter is a special case: For X a topological space, consider its category of open subsets $\mathrm{Op}(X)$ from Example 4.4, with coverage given by the usual open covers. Then a "sheaf on this topological space" is a sheaf, in the sense of Def. 4.8, on this site of opens. One writes

$$\mathrm{Sh}(X) \;\coloneqq\; \mathrm{Sh}(\mathrm{Op}(X)) \hookrightarrow [\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Set}]\,,$$

for short. The sheaf toposes arising this way are also called *spatial toposes*.

***Proposition 4.13*. (localic reflection)**

The construction of categories of sheaves on a topological space (Example 4.12) extends to a functor from the category Top of topological spaces and continuous functions between them (Example 1.3) to the category Topos of sheaf toposes and geometric morphisms between them (Example 4.12).

$$\mathrm{Sh}(-) \;:\; \mathrm{Top} \longrightarrow \mathrm{Topos}\;.$$

Moreover, when restricted to sober topological spaces, this becomes a fully faithful functor, hence a full subcategory-inclusion (Def. 1.19)

$$\mathrm{Sh}(-) \;:\; \mathrm{SoberTop} \hookrightarrow \mathrm{Topos}\;.$$

More generally, this holds for locales (i.e. for "sober topological spaces not necessarily supported on points"), in which case it becomes a reflective subcategory-inclusion (Def. 1.60)

$$\mathrm{Locale} \underset{\mathrm{Sh}(-)}{\overset{\longleftarrow}{\underset{\hookrightarrow}{\bot}}} \mathrm{Topos}$$

This says that categories of sheaves on topological spaces are but a reflection of soper topological spaces (generally: locales) and nothing more, whence they are also called petit toposes.

***Example 4.14*. (abelian sheaves)**

In the literature, sometimes sheaves are understood by default as taking values not in the category of sets, but in the category of abelian groups. Combined with Example 4.12 this means that some authors really mean "sheaf of abelian groups of the site of opens of a topological space", when they write just "sheaf".

But for $\mathcal{S}$ any mathematical structure, a sheaf of $\mathcal{S}$-structured sets is equivalently an $\mathcal{S}$-structure internal to the category of sheaves according to Def. 4.8. In particular sheaves of abelian groups are equivalently abelian group objects in the category of sheaves of sets as discussed here.

***Example 4.15*. (smooth sets)**

Consider the site SmthMfd of *all* smooth manifolds, from Example 4.5. The category of sheaves over this (Def. 4.8) is equivalent to the category of *smooth sets*, discussed in the chapter *geometry of physics – smooth sets*:

$$\mathrm{Sh}(\mathrm{SmthMfd}) \;\simeq\; \mathrm{SmoothSet}\,.$$

This is a *gros topos*, in a sense made precise by Def. 5.2 below (a *cohesive topos*).

***Remark 4.16*. (ordinary spaces and their coverings are generators and relations for generalized spaces)**

Given a site $\mathcal{C}$ (Def. 4.3), then its presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (Example 4.10) is the free cocompletion of the category $\mathcal{C}$ (Prop. 3.20), hence the category obtained by freely forming colimits ("gluing") of objects of $\mathcal{C}$.

In contrast, the full subcategory inclusion $\mathrm{Sh}(\mathcal{C}) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ enforces *relations* between these free colimits.

Therefore in total we may think of a sheaf topos $\mathrm{Sh}(\mathcal{C})$ as obtained by generators and relations from the objects of its site $\mathcal{C}$:

- the objects of $\mathcal{C}$ are the generators;
- the coverings of $\mathcal{C}$ are the relations.

Proposition 4.17. (sheafification and plus construction)

Let $\mathcal{C}$ be a site (Def. 4.3). Then the full subcategory-inclusion (75) of the category of sheaves over $\mathcal{C}$ (Def. 4.8) into the category of presheaves (Example 1.26) has a left adjoint (Def. 1.32) called sheafification

$$\mathrm{Sh}(\mathcal{C}) \underoverset{\underset{\iota}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]\,.$$

An explicit formula for sheafification is given by applying the following "plus construction" twice:

$$L(\mathbf{Y}) \simeq (\mathbf{Y}^+)^+\,.$$

Here the plus construction

$$(-)^+ \;:\; [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

is given by forming equivalence classes of sets of matching families (Def. 4.6) for all possible covers (Def. 4.3)

$$\mathbf{Y}^+(X) \;\coloneqq\; \left\{ \{U_i \overset{\iota_i}{\to} X\} \text{ covering }, \phi \in \mathrm{Match}(\{U_i\}, \mathbf{Y}) \right\} / \;\sim$$

under the equivalence relation which identifies two such pairs if the two covers have a joint refinement such that the restriction of the two matching families to that joint refinement coincide.

Example 4.18. (induced coverage)

Let $\mathcal{C}$ be a site (Def. 4.3). Then a full subcategory (Def. 1.19)

$$\mathcal{D} \hookrightarrow \mathcal{C}$$

becomes a site itself, whose coverage consists of those coverings $\{U_i \overset{\iota_i}{\to} Y\}$ in $\mathcal{C}$ that happen to be in $\mathcal{D} \hookrightarrow \mathcal{C}$.

Definition 4.19. (dense subsite)

Let $\mathcal{C}$ and $\mathcal{D}$ be sites (Def. 4.3) with a a full subcategory-inclusion (Def. 1.19)

$$\mathcal{D} \hookrightarrow \mathcal{C}$$

and regard $\mathcal{D}$ as equipped with the induced coverage (Def. 4.18).

This is called a *dense subsite-inclusion* if every object $X \in \mathcal{C}$ has a covering $\{U_i \overset{\iota_i}{\to} X\}_i$ such that for all i the patches are in the subcategory:

$$U_i \;\in\; \mathcal{D} \hookrightarrow \mathcal{C} \,.$$

***Proposition 4.20*. (comparison lemma)**

Let $\mathcal{D} \overset{\iota}{\hookrightarrow} \mathcal{C}$ be a dense subsite inclusion (def. 4.19). Then precomposition with ι induces an equivalence of categories (Def. 1.57) between their categories of sheaves (Def. 4.8):

$$\iota^\ast \;:\; \mathrm{Sh}(\mathcal{C}) \overset{\simeq}{\longrightarrow} \mathrm{Sh}(\mathcal{D})$$

***Proposition 4.21*. (recognition of epi-/mono-/isomorphisms of sheaves)**

Let $\mathcal{C}$ be a site (Def. 4.3) with $\mathrm{Sh}(\mathcal{C})$ its category of sheaves (Def. 4.8).

Then a morphisms $f \;:\; \mathbf{X} \to \mathbf{Y}$ in $\mathrm{Sh}(\mathcal{C})$ is

1. *a monomorphism (Def. 1.18) or isomorphism (Def. 1.9) precisely if it is so globally in that for each object $U \in \mathcal{C}$ in the site, then the component $f_U \colon \mathbf{X}(U) \to \mathbf{Y}(U)$ is an injection or bijection of sets, respectively.*
2. *an epimorphism (Def. 1.18) precisely if it is so locally, in that: for all $U \in C$ there is a covering $\{p_i \colon U_i \to U\}_{i \in I}$ such that for all $i \in I$ and every element $y \in \mathbf{Y}(U)$ the element $f(p_i)(y)$ is in the image of $f(U_i) \colon \mathbf{X}(U_i) \to \mathbf{Y}(U_i)$.*

***Proposition 4.22*. (epi/mono-factorization through image)**

Let $\mathrm{Sh}(\mathcal{C})$ be a category of sheaves (Def. 4.8). Then every morphism $f \;:\; \mathbf{X} \to \mathbf{Y}$ factors as an epimorphism followed by a monomorphism (Def. 1.18) uniquely up to unique isomorphism:

$$f \;:\; \mathbf{X} \overset{\text{epi}}{\longrightarrow} \mathrm{im}(f) \overset{\text{mono}}{\longrightarrow} \mathbf{Y} \,.$$

Theobject $\mathrm{im}(f)$, as a subobject of $\mathbf{Y}$, is called the image of f.

In fact this is an orthogonal factorization system, in that for every commuting square where the left morphism is an epimorphism, and the right one a monomorphism, there exists a unique lift:

(77)

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\text{epi}"'] & B \arrow[d, "\text{mono}"] \\
C \arrow[r] \arrow[ur, "\exists !"] & D
\end{tikzcd}
$$

This implies that this is a functorial factorization, in that for every commuting square

$$
\begin{array}{ccc}
\mathbf{X}_1 & \overset{f_1}{\longrightarrow} & \mathbf{Y}_1 \\
\downarrow & & \downarrow \\
\mathbf{X}_2 & \underset{f_2}{\longrightarrow} & \mathbf{Y}_2
\end{array}
$$

there is an induced morphism of images such that the resulting rectangular diagram commutes:

$$
\begin{array}{ccccc}
\mathbf{X}_1 & \overset{\text{epi}}{\longrightarrow} & \mathrm{im}(f_1) & \overset{\text{mono}}{\longrightarrow} & \mathbf{Y}_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{X}_2 & \overset{\text{epi}}{\longrightarrow} & \mathrm{im}(f_2) & \overset{\text{mono}}{\longrightarrow} & \mathbf{Y}_2
\end{array}
$$

We discuss some of the key properties of sheaf toposes:

Proposition 4.23. (sheaf toposes are cosmoi)

Let $\mathcal{C}$ be a site (Def. 4.3) and $\mathrm{Sh}(\mathcal{C})$ its sheaf topos (Def. 4.8). Then:

1. *All limits exist in $\mathrm{Sh}(\mathcal{C})$ (Def. 3.1), and they are computed as limits of presheaves, via Example 3.5:*

$$
\iota\left(\underset{\underset{d}{\longleftarrow}}{\lim} \mathbf{X}_d \right) \simeq \underset{\underset{d}{\longleftarrow}}{\lim} \iota(\mathbf{X}_d)
$$

2. *All colimits exist in $\mathrm{Sh}(\mathcal{C})$ (Def. 3.1) and they are given by the sheafification (Def. 4.17) of the same colimits computed in the category of presheaves, via Example 3.5:*

$$
\underset{\underset{d}{\longrightarrow}}{\lim} \mathbf{X}_d \simeq L\left(\underset{\underset{d}{\longleftarrow}}{\lim} \iota(\mathbf{X}_d) \right)
$$

3. *The cartesian (Example 2.2) closed monoidal category-structure (Def. 2.8) on the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ from Example 2.12 restricts to sheaves:*

$$\mathrm{Sh}(\mathcal{C}) \underset{[\mathbf{X}, -]}{\overset{\mathbf{X} \times (-)}{\underset{\longrightarrow}{\longleftarrow}}} \mathrm{Sh}(\mathcal{C})$$

In particular, for $\mathbf{X}, \mathbf{Y} \in \mathrm{Sh}(\mathcal{C})$ two sheaves, their internal hom $[\mathbf{X}, \mathbf{Y}] \in \mathrm{Sh}(\mathcal{C})$ is a sheaf given by

$$[\mathbf{X}, \mathbf{Y}] \;:\; U \;\mapsto\; \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(y(U)\mathbf{X}, \mathbf{Y})\,,$$

where $y(U)$ is the presheaf represented by $U \in \mathcal{C}$ (Example 1.27).

This may be summarized by saying that every sheaf topos (in particular every category of presheaves, by Example 4.10) is a cosmos for enriched category theory (Def. 2.36).

Definition 4.24. (local epimorphism)

Let $\mathcal{C}$ be a site (Def. 4.3). Then a morphism of presheaves over $\mathcal{C}$ (Example 1.26)

$$\mathbf{Y} \overset{f}{\longrightarrow} \mathbf{X} \quad \in [\mathcal{S}^{\mathrm{op}}, \mathrm{Set}]$$

is called a *local epimorphism* if for every object $U \in \mathcal{C}$, every morphism $y(U) \longrightarrow \mathbf{X}$ out of its represented presheaf (Example 1.27) has the *local* lifting property through f in that there is a covering $\{U_i \overset{\iota_i}{\to} U\}$ (Def. 4.3) and a commuting diagram of the form

$$\begin{array}{ccc} y(U_i) & \overset{\exists}{\longrightarrow} & \mathbf{Y} \\ {}^{\mathllap{y(\iota_i)}}\downarrow & & \downarrow^{\mathrlap{f}} \\ y(U) & \longrightarrow & \mathbf{X} \end{array}$$

Codescent

In order to understand the sheaf condition (74) better, it is useful to consider Cech groupoids (Def. 4.28 below). These are really *presheaves of groupoids* (Def. 4.25 below), a special case of the general concept of enriched presheaves. The key property of the Cech groupoid is that it co-represents the sheaf condition (Prop. 4.29 below). It is in this incarnation that the concept of sheaf seamlessly generalizes to homotopy theory via “higher stacks”.

Definition 4.25. (presheaves of groupoids)

For $\mathcal{C}$ a small category (Def. 1.6) consider the functor category (Example 1.25) from the opposite category of $\mathcal{C}$ (Example 1.13) to the category Grpd of small groupoids (Example 1.16)

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}] \;.$$

By Example 2.37 we may regard Grpd as a cosmos for enriched category theory. Since the inclusion Set $\hookrightarrow$ Grpd (Example 1.61) is a strong monoidal functor (Def. 2.16) of cosmoi (Example 2.37), the plain category $\mathcal{C}$ may be thought of as a Grpd-enriched category (Def. 2.40) and hence a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Grpd}$ is equivalently a Grpd-enriched functor (Def. 2.46).

This means that the plain category of functors $[\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]$ enriches to Grpd-enriched category of Grpd-enriched presheaves (Example 2.48).

Hence we may speak of *presheaves of groupoids*.

***Remark 4.26.* (presheaves of groupoids as internal groupoids in presheaves)**

From every presheaf of groupoids $\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]$ (Def. 4.25), we obtain two ordinary presheaves of sets (Def. 1.26) called the

- *presheaf of objects*

$$\mathrm{Obj}_{\mathbf{Y}(-)} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

- the *presheaf of morphisms*

$$\mathrm{Mor}_{\mathbf{Y}(-)} \;\coloneqq\; \coprod_{x,y \in \mathrm{Obj}_{\mathbf{Y}(-)}} \mathrm{Hom}_{\mathbf{Y}(-)} \;:\; [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

In more abstract language this assignment constitutes an equivalence of categories

$$\begin{array}{ccc} [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}] & \overset{\simeq}{\longrightarrow} & \mathrm{Grpd}\big([\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]\big) \\ \mathbf{Y} & \mapsto & \left(\begin{array}{ccccc} & & \underbrace{\textstyle\coprod_{x,y \in \mathrm{Obj}_{\mathbf{Y}(-)}} \mathrm{Hom}_{\mathbf{Y}(-)}}_{\mathrm{Mor}_{\mathbf{Y}(-)}} & & \\ \begin{array}{c} (x \overset{f}{\to} y) \\ \mapsto \\ x \end{array} & \Big\downarrow & \Big\uparrow \begin{array}{c} x \\ \mapsto \\ x \overset{\mathrm{id}_x}{\longrightarrow} x \end{array} & \Big\downarrow & \begin{array}{c} (x \overset{f}{\to} y) \\ \mapsto \\ y \end{array} \\ & & \mathrm{Obj}_{\mathbf{Y}(-)} & & \end{array} \right) \end{array} \;. \tag{78}$$

from presheaves of groupoids to *internal groupoids- in the category of presheaves over $\mathcal{C}$ (Def. 1.26).*

***Example 4.27*. (*presheaves of sets form reflective subcategory of presheaves of groupoids*)**

Let $\mathcal{C}$ be a small category (Def. 1.6). There is the reflective subcategory-inclusion (Def. 1.60) of the category of presheaves over $\mathcal{C}$ (Example 1.26) into the category of presheaves of groupoids over $\mathcal{C}$ (Def. 4.25)

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \underset{\hookrightarrow}{\overset{\pi_0}{\underset{\perp}{\longleftarrow}}} [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]$$

which is given over each object of $\mathcal{C}$ by the reflective inclusion of sets into groupoids (Example 1.61).

***Example 4.28*. (*Cech groupoid*)**

Let $\mathcal{C}$ be a site (Def. 4.3), and $X \in \mathcal{C}$ an object of that site. For each covering family $\{U_i \overset{\iota_i}{\to} X\}$ of X in the given coverage, the *Cech groupoid* is the presheaf of groupoids (Def. 4.25)

$$C(\{U_i\}) \;\in\; [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}] \;\simeq\; \mathrm{Grpd}\big([\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]\big)$$

which, regarded as an internal groupoid in the category of presheaves over $\mathcal{C}$, via (78), has as presheaf of objects the coproduct

$$\mathrm{Obj}_{C(\{U_i\})} \;\coloneqq\; \coprod_i y(U_i)$$

of the presheaves represented (under the Yoneda embedding, Prop. 1.30) by the covering objects U_i, and as presheaf of morphisms the coproduct over all fiber products of these:

$$\mathrm{Mor}_{C(\{U_i\})} \;\coloneqq\; \coprod_{i,j} y(U_i) \times_{y(X)} y(U_j) \;.$$

This means equivalently that for any $V \in \mathcal{C}$ the groupoid assigned by $C(\{U_i\})$ has as set of objects pairs consisting of an index i and a morphism $V \overset{\kappa_i}{\to} U_i$ in $\mathcal{C}$, and there is a unique morphism between two such objects

$$\kappa_i \longrightarrow \kappa_j$$

precisely if

(79)

$$\iota_i \circ \kappa_i \;=\; \iota_j \circ \kappa_j \qquad \begin{tikzcd} & V \arrow[dl, "\kappa_i"'] \arrow[dr, "\kappa_j"] & \\ U_i \arrow[dr, "\iota_i"'] & & U_j \arrow[dl, "\iota_j"] \\ & X & \end{tikzcd}$$

Condition (79) for morphisms in the Cech groupoid to be well-defined is verbatim the condition (70) in the definition of matching families. Indeed, Cech groupoids serve to conveniently summarize (and then generalize) the sheaf condition (Def. 4.8):

***Proposition 4.29*. (Cech groupoid co-represents matching families – codescent)**

For Grpd regarded as a cosmos (Example 2.37), and $\mathcal{C}$ a site (Def. 4.3), let

$$\mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]$$

be a presheaf on $\mathcal{C}$ (Example 1.26), regarded as a Grpd-enriched presheaf via Example 4.27, let $X \in \mathcal{C}$ be any object and $\{U_i \xrightarrow{\iota_i} X\}_i$ a covering family (Def. 4.3) with induced Cech groupoid $C(\{U_i\}_i)$ (Example 4.28).

Then there is an isomorphism

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]\big(C(\{U_i\}_i), \mathbf{Y}\big) \;\simeq\; \mathrm{Match}(\{U_i\}_i, \mathbf{Y})$$

between the hom-groupoid of Grpd-enriched presheaves (Def. 3.16) and the set of matching families (Def. 4.6).

Since hence the Cech-groupoid co-represents the descent object, it is sometimes called the codescent object along the given covering.

Moreover, under this identification the canonical morphism

$$C(\{U_i\}_i) \xrightarrow{p_{\{U_i\}_i}} y(X) \qquad (80)$$

induces the comparison morphism (73)

$$\begin{array}{ccc} [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}](y(X), \mathbf{Y}) & \simeq & \mathbf{Y}(X) \\ {}^{[\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}](p_{\{U_i\}_i}, \mathbf{Y})}\big\downarrow & & \big\downarrow \\ [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]\big(C(\{U_i\}_i), \mathbf{Y}\big) & \simeq & \mathrm{Match}(\{U_i\}_i, \mathbf{Y}) \end{array} \,.$$

In conclusion, this means that the presheaf $\mathbf{Y}$ *is a sheaf (Def. 4.8) precisely if homming Cech groupoid projections into it produces an isomorphism:*

$$\mathbf{Y} \text{ is a sheaf} \qquad \Leftrightarrow \qquad \left[C(\{U_i\}_i) \xrightarrow{p_{\{U_i\}I}} y(X)\,,\; \mathbf{Y} \right] \text{ is iso, for all covering families} \tag{81}$$

One also says in this case that $\mathbf{Y}$ *is a local object with respect to Cech covers/*

Proof. By (58) the hom-groupoid is computed as the end

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]\big(C(\{U_i\}_i), \mathbf{Y}\big) \;=\; \int_{V \in \mathcal{C}} \big[C(\{U_i\}_i)(V), \mathbf{Y}(V) \big],$$

where, by Example 2.37, the “integrand” is the functor category (here: a groupoid) from the Cech groupoid at a given V to the set (regarded as a groupoid) assigned by $\mathbf{Y}$ to V.

Since $\mathbf{Y}(V)$ is just a set, that functor groupoid, too, is just a set, regarded as a groupoid. Its elements are the functors $C(\{U_i\}_i)(V) \longrightarrow \mathbf{Y}(V)$, which are equivalently those functions on sets of objects

$$\coprod_i y(U_i)(V) = \mathrm{Obj}_{C(\{U_i\}_i)(V)} \longrightarrow \mathrm{Obj}_{\mathbf{Y}(V)} = \mathbf{Y}(V)$$

which respect the equivalence relation induced by the morphisms in the Cech groupoid at V.

Hence the hom-groupoid is a subset of the end of these function sets:

$$\begin{aligned} \int_{V \in \mathcal{C}} \big[C(\{U_i\}_i)(V), \mathbf{Y}(V) \big] & \hookrightarrow \int_{V \in \mathcal{C}} \Big[\coprod_i y(U_i)(V), \mathbf{Y}(V) \Big] \\ & \simeq \int_{V \in \mathcal{C}} \prod_i [y(U_i)(V), \mathbf{Y}(V)] \\ & \simeq \prod_i \int_{V \in \mathcal{C}} [y(U_i)(V), \mathbf{Y}(V)] \\ & \simeq \prod_i \mathbf{Y}(U_i) \end{aligned}$$

Here we used: first that the internal hom-functor turns colimits in its first argument into limits (Prop. 2.15), then that limits commute with limits (Prop. 3.9), hence that in particular ends commute with products , and finally the enriched Yoneda lemma (Prop. 3.17), which here is, via Example 3.15, just the plain Yoneda lemma (Prop. 1.29). The end result is hence the same Cartesian product set that also the set of matching families is defined to be a subset of, in (72).

This shows that an element in $\int_{V \in \mathcal{C}} \left[C(\{U_i\}_i)(V), \mathbf{Y}(V) \right]$ is a tuple $(\phi_i \in \mathbf{Y}(U_i))_i$, subject to some condition. This condition is that for each $V \in \mathcal{C}$ the assignment

$$\begin{array}{ccc} C(\{U_i\}_i)(V) & \longrightarrow & \mathbf{Y}(V) \\ (V \overset{\kappa_i}{\to} U_i) & \mapsto & \kappa_i^* \phi_i = \mathbf{Y}(\kappa_i)(\phi_i) \end{array}$$

constitutes a functor of groupoids.

By definition of the Cech groupoid, and since the codomain is a just set regarded as a groupoid, this is the case precisely if

$$\mathbf{Y}(\kappa_i)(\phi_i) \;=\; \mathbf{Y}(\kappa_j)(\phi_j) \qquad \text{for all } i, j \,,$$

which is exactly the condition (71) that makes $(\phi_i)_i$ a matching family. ▮

Local presentation

We now discuss a more abstract characterization of sheaf toposes, in terms of properties enjoyed by the adjunction that relates them to the corresponding categories of presheaves.

***Definition 4.30*. (*locally presentable category*)**

A category $\mathbf{H}$ (Def. 1.1) is called *locally presentable* if there exists a small category $\mathcal{C}$ (Def. 1.6) and a reflective subcategory-inclusion of $\mathcal{C}$ into its category of presheaves (Example 1.26)

$$\mathbf{H} \underoverset{\underset{\text{acc}}{\hookrightarrow}}{\overset{L}{\longleftarrow}}{\bot} [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

such that the inclusion functor is an *accessible functor* in that it preserves κ-filtered colimits for some regular cardinal κ.

***Proposition 4.31*. (*Giraud's theorem*)**

A sheaf topos (Def. 4.8) is equivalently a locally presentable category (Def. 4.30) with

1. *universal colimits,*
2. *effective quotients,*

3. *disjoint coproducts.*

Proposition 4.32. (sheaf toposes are equivalently the left exact reflective subcategories of presheaf toposes)

Let $(\mathcal{C}, \tau)$ be a site (Def. 4.3). Then the full subcategory inclusion $i : \mathrm{Sh}(\mathcal{C}, \tau) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ of its sheaf topos (Def. 4.8) into its category of presheaves is a reflective subcategory inclusion (Def. 1.60)

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{C}, \tau) \arrow[r, hook, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "L"'] \end{tikzcd}$$

such that:

1. *the inclusion ι is an accessible functor, thus exhibiting $\mathrm{Sh}(\mathcal{C}, \tau)$ as a locally presentable category (Def. 4.30)*
2. *the reflector $L : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ (which is sheafification, Prop. 4.17) is left exact ("lex") in that it preserves finite limits.*

Conversely, every sheaf topos arises this way. Hence sheaf toposes $\mathbf{H}$ are equivalently the left exact-reflectively full subcategories of presheaf toposes over some small category $\mathcal{C}$:

$$\begin{tikzcd} \mathbf{H} \arrow[r, hook, shift right, "\mathrm{acc}"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "\mathrm{lex}"'] \end{tikzcd} \qquad (82)$$

(e.g. Borceux 94, prop. 3.5.4, cor. 3.5.5, Johnstone, C.2.1.11)

Remark 4.33. (left exact reflections of categories of presheaves are locally presentable categories)

In the characterization of sheaf toposes as left exact reflections of categories of presheaves in Prop. 4.32, the accessibility of the inclusion, equivalently the local presentability (Def. 4.30) is automatically implied (using the adjoint functor theorem), as indicated in (82).

5. Gros toposes

We have seen roughly two different kinds of sheaf toposes:

- categories of sheaves on a given space$X$ (Example 4.12), which, by localic reflection (Prop. 4.13), really are just a reflection of the space X in the category of toposes, these are called *petit toposes*;

- categories of sheaves whose objects are generalized spaces (Example 4.15) these are called *gros toposes*.

***Remark 5.1*. (cohesive generalized spaces as foundations of geometry)**

If we aim to lay foundations for geometry, then we are interested in isolating those kinds of generalized spaces which have foundational *a priori* meaning, independent of an otherwise pre-configured notion of space. Hence we would like to first characterize suitable gros toposes, extract concepts of space from these, and only then, possibly, consider the petit topos-reflections of these (Prop. 4.13 below).

The gros toposes of such foundational generalized spaces ought to have an *internal logic* that knows about *modalities of geometry* such as *discreteness* or *concreteness*. Via the formalization of modalities in Def. 1.62 this leads to the definiton of cohesive toposes (Def. 5.2, Prop. 5.7 below, due to Lawvere 91, Lawvere 07).

gros topos		**generalized spaces obey...**	**example:**
cohesion	Def. 5.2	principles of differential topology	SmoothSet
elasticity	Def. 5.10	principles of differential geometry	FormallSmoothset
solidity	Def. 5.14	principles of supergeometry	SuperFormalSmoothSet

Cohesive toposes

***Definition 5.2*. (cohesive topos)**

A sheaf topos $\mathbf{H}$ (Def. 4.8) is called a *cohesive topos* if there is a quadruple (Remark 1.34) of adjoint functors (Def. 1.32) to the category of sets (Example 1.2)

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc} \;\;:\;\; \mathbf{H} \begin{array}{c} \xrightarrow{\;\;\Pi\;\;} \\ \xleftarrow{\;\;\mathrm{Disc}\;\;} \\ \xrightarrow{\;\;\Gamma\;\;} \\ \xleftarrow{\;\;\mathrm{coDisc}\;\;} \end{array} \mathrm{Set} \tag{83}$$

such that:

1. Disc and coDisc are full and faithful functors (Def. 1.19)
2. Π preserves finite products.

***Example 5.3*. (adjoint quadruple of presheaves over site with finite products)**

Let $\mathcal{C}$ be a small category (Def. 1.6) with finite products (hence with a terminal object $* \in \mathcal{C}$ and for any two objects $X, Y \in \mathcal{C}$ their Cartesian product $X \times Y \in \mathcal{C}$).

Then there is an adjoint quadruple (Remark 1.34) of functors between the category of presheaves over $\mathcal{C}$ (Example 1.26), and the category of sets (Example 1.2)

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \begin{array}{c} \xrightarrow{\quad \Pi \quad} \\ \xhookleftarrow{\quad \mathrm{Disc} \quad} \\ \xrightarrow{\quad \Gamma \quad} \\ \xhookleftarrow{\quad \mathrm{coDisc} \quad} \end{array} \mathrm{Set} \tag{84}$$

such that:

1. the functor Γ sends a presheaf $\mathbf{Y}$ to its set of global sections, which here is its value on the terminal object:

$$\begin{aligned} \Gamma \mathbf{Y} &= \underset{\underset{\mathcal{C}}{\longleftarrow}}{\lim} \mathbf{Y} \\ &\simeq \mathbf{Y}(*) \end{aligned} \tag{85}$$

2. Disc and coDisc are full and faithful functors (Def. 1.19).
3. Π preserves finite products:
   for $\mathbf{X}, \mathbf{Y} \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, we have a natural bijection

$$\Pi(\mathbf{X} \times \mathbf{Y}) \;\simeq\; \Pi(\mathbf{X}) \times \Pi(\mathbf{Y}) \;.$$

Hence the category of presheaves over a small category with finite products, hence the category of sheaves for the trivial coverage (Example 4.10) is a cohesive topos (Def. 5.2).

***Proof*.** The existence of the terminal object in $\mathcal{C}$ means equivalently (by Example 1.7) that there is an adjoint pair of functors between $\mathcal{C}$ and the terminal category (Example 1.7):

$$* \underset{\hookrightarrow}{\overset{\overset{p}{\longleftarrow}}{\bot}} \mathcal{C}$$

whose right adjoint takes the unique object of the terminal category to that terminal object.

From this it follows, by Example 3.32, that Kan extension produces an adjoint quadruple (Remark 1.34) of functors between the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ and $[\ast, \mathrm{Set}] \simeq \mathrm{Set}$, as shown, where

1. Γ is the operation of pre-composition with the terminal object inclusion $\ast \hookrightarrow \mathcal{C}$
2. Disc is the left Kan extension along the inclusion $\ast \hookrightarrow \mathcal{C}$ of the terminal object.

The former is manifestly the operation of evaluating on the terminal object. Moreover, since the terminal object inclusion is manifestly a fully faithful functor (Def. 1.19), it follows that also its left Kan extension Disc is fully faithful (Prop. 3.33). This implies that also coDisc is fully faithful, by (Prop. 1.67).

Equivalently, $\mathrm{Disc} \simeq p^{*}$ is the constant diagram-assigning functor. By uniqueness of adjoints (Prop. 1.45) implies that Π is the functor that sends a presheaf, regarded as a functor $\mathbf{Y} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, to its colimit

$$\Pi(\mathbf{Y}) \;=\; \underset{\underset{\mathcal{C}^{\mathrm{op}}}{\longrightarrow}}{\lim} \mathbf{Y}\,. \qquad (86)$$

The fact that this indeed preserves products follows from the assumption that $\mathcal{C}$ has finite products, since categories with finite products are cosifted (Prop. 3.35) ▮

Example 5.3 suggests to ask for coverages on categories with finite products which are such that the adjoint quadruple (107) on the category of presheaves (co-)restricts to the corresponding category of sheaves. The following Definition 5.4 states a sufficient condition for this to be the case:

***Definition 5.4*. (*cohesive site*)**

We call a site $\mathcal{C}$ (Def. 4.3) *cohesive* if the following conditions are satisfied:

1. The category $\mathcal{C}$ has finite products (as in Example 5.3).
2. For every covering family $\{U_i \to X\}_i$ in the given coverage on $\mathcal{C}$ the induced Cech groupoid $C(\{U_i\}_i) \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Grpd}]$ (Def. 4.28) satisfies the following two conditions:
   1. the set of connected components of the groupoid obtained as the colimit over the Cech groupoid is the singleton:
   $$\pi_0 \underset{\underset{\mathcal{C}^{\mathrm{op}}}{\longrightarrow}}{\lim} C(\{U_i\}) \;\simeq\; \ast$$
   2. the set of connected components of the groupoid obtained as the limit of the Cech groupoid is equivalent to the set of points of X, regarded as a groupoid:

$$\pi_0 \varprojlim_{\mathcal{C}^{\mathrm{op}}} C(\{U_i\}) \simeq \mathrm{Hom}_{\mathcal{C}}(\ast, X) \,.$$

This definition is designed to make the following true:

***Proposition 5.5*. (category of sheaves on a cohesive site is a cohesive topos)**

Let $\mathcal{C}$ be a cohesive site (Def. 5.4). Then the adjoint quadruple on the category of presheaves over $\mathcal{C}$, from Example 5.3 (given that a cohesive site by definition has finite products) (co-)restricts from the category of presheaves over $\mathcal{C}$, to the category of sheaves (Def. 4.8) and hence exhibits $\mathrm{Sh}(\mathcal{C})$ as a cohesive topos (Def. 5.2):

$$\mathrm{Sh}(\mathcal{C}) \; \begin{array}{c} \xrightarrow{\;\;\Pi\;\;} \\ \xleftarrow{\;\mathrm{Disc}\;} \\ \xrightarrow{\;\;\Gamma\;\;} \\ \xleftarrow{\mathrm{coDisc}} \end{array} \; \mathrm{Set} \qquad (87)$$

***Proof*.** By example 5.3 we already have the analogous statement for the categories of presheaves. Hence it is sufficient to show that the functors Disc and coDisc from Example 5.3 factor through the definition inclusion of the category of sheaves, hence that for each set S the presheaves $\mathrm{Disc}(S)$ and $\mathrm{coDisc}(S)$ are indeed sheaves (Def. 4.8).

By the formulaton of the sheaf condition via the Cech groupoid (Prop. 4.29), and using the adjunction hom-isomorphisms (here) this is readily seen to be equivalent to the two further conditions on a cohesive site (Def. 5.4):

Let $\{U_i \to X\}$ be a covering family.

The sheaf condition (81) for $\mathrm{Disc}(S)$ says that

$$\left[C(\{U_i\}) \overset{p_{\{U_i\}_i}}{\longrightarrow} y(X) \,,\, \mathrm{Disc}(S) \right]$$

is an isomorphism of groupoids, which by adjunction and using (86) means equivalently that

$$\left[\varinjlim_{\mathcal{C}^{\mathrm{op}}} (C(\{U_i\})) \to \ast \,,\, S \right]$$

is an isomorphism of groupoids, where we used that colimits of representables are singletons (Lemma 3.34) to replace $\varinjlim_{\mathcal{C}^{\mathrm{op}}} y(X) \simeq \ast$.

But now in this internal hom of groupoids, the set S is really a groupoid in the image of the reflective embedding of sets into groupoids, whose left adjoint is the connected components-functor π_0 (Example 1.61). Hence by another adjunction isomoprhism this is equivalent to

$$\left[\pi_0 \underset{\underset{\mathcal{C}^{op}}{\longrightarrow}}{\lim} (C(\{U_i\})) \to * \, , \, S \right]$$

being an isomorphism (a bijection of sets, now). This is true for all $S \in \mathrm{Set}$ precisely if (by the Yoneda lemma, if you wish) the morphism

$$\pi_0 \underset{\underset{\mathcal{C}^{op}}{\longrightarrow}}{\lim} (C(\{U_i\})) \to *$$

is already an isomorphism (here: bijection) itself.

Similarly, the sheaf condition (81) for $\mathrm{coDisc}(S)$ says that

$$\left[C(\{U_i\}) \xrightarrow{p_{\{U_i\}_i}} y(X) \, , \, \mathrm{coDisc}(S) \right]$$

is an isomorphism, and hence by adjunction and using (85), this is equivalent to

$$\left[\pi_0 \underset{\underset{\mathcal{C}^{op}}{\longleftarrow}}{\lim} C(\{U_i\}) \xrightarrow{p_{\{U_i\}_i}} \mathrm{Hom}_{\mathcal{C}}(* , X) \, , \, S \right]$$

being an isomorphism. This holds for all $S \in \mathrm{Set}$ if (by the Yoneda lemma, if you wish)

$$\pi_0 \underset{\underset{\mathcal{C}^{op}}{\longleftarrow}}{\lim} C(\{U_i\}) \xrightarrow{p_{\{U_i\}_i}} \mathrm{Hom}_{\mathcal{C}}(* , X)$$

is an isomorphism. ■

Definition 5.6. (adjoint triple of adjoint modal operators on cohesive topos)

Given a cohesive topos (Def. 5.2), its adjoint quadruple (Remark 1.34) of functors to and from Set

(88)

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc} \;\; : \;\; \mathbf{H} \begin{array}{c} \xrightarrow{\;\;\Pi\;\;} \\ \xleftarrow{\;\;\mathrm{Disc}\;\;} \\ \xrightarrow{\;\;\Gamma\;\;} \\ \xleftarrow{\;\;\mathrm{coDisc}\;\;} \end{array} \mathrm{Set}$$

induce, by composition of functors, an adjoint triple (Remark 1.34) of adjoint modalities (Def. 1.66):

$$\int \dashv \flat \dashv \sharp \;\; : \;\; \mathbf{H} \begin{array}{c} \xleftarrow{\;\int \,\coloneqq\, \mathrm{Disc}\circ\Pi\;} \\ \xrightarrow{\;\flat \,\coloneqq\, \mathrm{Disc}\circ\Gamma\;} \\ \xleftarrow{\;\sharp \,\coloneqq\, \mathrm{coDisc}\circ\Gamma\;} \end{array} \mathbf{H}\,.$$

Since Disc and coDisc are fully faithful functors by assumption, these are (co-)modal operators (Def. 1.62) on the cohesive topos, by (Prop. 1.63).

We pronounce these as follows:

shape modality	**flat modality**	**sharp modality**
$\int \;\coloneqq\; \mathrm{Disc}\circ\Pi$	$\flat \;\coloneqq\; \mathrm{Disc}\circ\Gamma$	$\sharp \;\coloneqq\; \mathrm{coDisc}\circ\Gamma$

and we refer to the corresponding modal objects (Def. 1.65) as follows:

- a flat-comodal object

  $$\flat X \underset{\simeq}{\overset{\epsilon^\flat_X}{\longrightarrow}} X$$

  is called a *discrete object*;

- a sharp-modal object

  $$X \underset{\simeq}{\overset{\eta^\sharp_X}{\longrightarrow}} \sharp X$$

  is called a *codiscrete object*;

- a sharp-submodal object

  $$X \underset{\mathrm{mono}}{\overset{\eta^\sharp_X}{\longrightarrow}} \sharp X$$

is a concrete object.

Proposition 5.7. ***(pieces have points ≃ discrete objects are concrete ≃ Aufhebung of bottom adjoint modality)***

Let $\mathbf{H}$ be a cohesive topos (Def. 5.2). Then the following conditions are equivalent:

1. ***pieces have points****: For every object $X \in \mathbf{H}$, comparison of extremes-transformation (27) for the $(\int, \dashv \flat)$-adjoint modality (27), hence the $\flat$-counit of an adjunction composed with the $\int$-unit*

$$\flat X \xrightarrow{\epsilon^{\flat}_X} X \xrightarrow{\epsilon^{\int}_X} \int X$$

is an epimorphism (Def. 1.18)

2. ***discrete objects are concrete****: For every object $X \in \mathbf{H}$, we have that its discrete object $\flat X$ is a concrete object (Def. 5.6).*

3. ***Aufhebung of bottom adjoint modality***
The adjoint modality $\flat \dashv \sharp$ exhibits Aufhebung (Def. 1.72) of the bottom adjoint modality (Example 1.71), i.e. the initial object (Def. 1.5) is codiscrete (Def. 5.6):

$$\sharp \emptyset \;\simeq\; \emptyset \,.$$

Proof. The comparison morphism $\mathrm{ptp}_{\mathbf{H}}$ is a special case of that discussed in Prop. 1.69. First observe, in the notation there, that

$$\mathrm{ptp}_{\mathbf{H}} \;\text{ is epi} \qquad \text{iff} \qquad \mathrm{ptp}_{\mathbf{B}} \;\text{ is epi} \,.$$

In one direction, assume that $\mathrm{ptp}_{\mathbf{B}}$ is an epimorphism. By (31) we have $\mathrm{ptp}_{\mathbf{H}} = \mathrm{Disc}(\mathrm{ptp}_{\mathbf{B}})$, but Disc is a left adjoint and left adjoints preserve monomorphisms (Prop. 1.47).

In the other direction, assume that $\mathrm{ptp}_{\mathbf{H}}$ is an epimorphism. By (29) and (32) we see that $\mathrm{ptp}_{\mathbf{B}}$ is re-obtained from this by applying Γ and then composition with isomorphisms. But Γ is again a left adjoint, and hence preserves epimorphism by Prop. 1.47, as does composition with isomorphisms.

By applying (29) again, we find in particular that *pieces have points* is also equivalent to $\Pi\epsilon^{\flat}_{\mathrm{Disc}\, S}$ being an epimorphism, for all $S \in \mathbf{B}$. But this is equivalent to

$$\mathrm{Hom}_{\mathbf{B}}(\Pi\epsilon^{\flat}_{\mathbf{X}}, S) = \mathrm{Hom}_{\mathbf{H}}(\epsilon^{\flat}_{\mathbf{X}}, \mathrm{Disc}(S))$$

being a monomorphism for all S (by adjunction isomorphism (10) and definition of epimorphism, Def. 1.18).

Now by Lemma 1.37, this is equivalent to

$$\mathrm{Hom}_{\mathbf{H}}(\mathbf{X}, \eta^{\sharp}_{\mathrm{Disc}(S)})$$

being an injection for all $\mathbf{X}$, which, by Def. 1.18, is equivalent to $\eta^{\sharp}_{\mathrm{Disc}(S)}$ being a monomorphism, hence to *discrete objects are concrete*.

This establishes the equivalence between the first two items. ▮

***Proposition 5.8*. (*cohesive site* such that *pieces have points*/*discrete objects are concrete*)**

Let $\mathcal{C}$ be a cohesive site (Def. 5.4), such that

- *for every object $X \in \mathcal{C}$, there is at least one morphism $* \overset{\exists}{\to} X$ from the terminal object to X, hence such that the hom set $\mathrm{Hom}_{\mathcal{C}}(*, X)$ is non-empty.*

Then the cohesive topos $\mathrm{Sh}(\mathcal{C})$, according to Prop. 5.5, satisfies the equivalent conditions from Prop. 5.7:

1. *pieces have points,*
2. *discrete objects are concrete.*

***Proof*.** By Prop. 5.7 it is sufficient to show the second condition, hence to check that for each set $S \in \mathrm{Set}$, the canonical morphism

$$\mathrm{Disc}(S) \longrightarrow \mathrm{coDisc}(S)$$

is a monomorphism. By Prop. 4.21 this means equivalently that for each object $X \in \mathcal{C}$ in the site, the component function

$$\mathrm{Disc}(S)(X) \longrightarrow \mathrm{coDisc}(S)(X)$$

is an injective function.

Now, by the proof of Prop. 5.5, this is the diagonal function

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}}\big(\mathrm{Hom}_{\mathcal{C}}(*, X), S\big) \\ s & \mapsto & \mathrm{const}_s \end{array}$$

This function is injective precisely if $\mathrm{Hom}_{\mathcal{C}}(*, X)$ is non-empty, which is true by assumption. ▮

***Proposition 5.9*. (*quasitopos* of *concrete objects* in a *cohesive topos*)**

For $\mathbf{H}$ *a cohesive topos (Def. 5.2), write*

$$\mathbf{H}_{\mathrm{conc}} \hookrightarrow \mathbf{H}$$

for its full subcategory (Example 1.20) of concrete objects (Def. 5.6).

Then there is a sequence of reflective subcategory-inclusions (Def. 1.60) that factor the $(\Gamma \dashv \mathrm{coDisc})$*-adjunction as*

$$\Gamma \dashv \mathrm{coDisc} \;:\; \mathbf{H} \underset{\iota_{\mathrm{conc}}}{\overset{\mathrm{conc}}{\rightleftarrows}} \mathbf{H}_{\mathrm{conc}} \underset{\mathrm{coDisc}}{\overset{\Gamma}{\rightleftarrows}} \mathrm{Set}$$

If in addition discrete objects are concrete (Prop. 5.7), then the full adjoint quadruple factors through the concrete objects:

$$\Gamma \dashv \mathrm{coDisc} \;:\; \mathbf{H} \underset{\iota_{\mathrm{conc}}}{\overset{\mathrm{conc}}{\rightleftarrows}} \mathbf{H}_{\mathrm{conc}} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{Disc}} \\ \xrightarrow{\Gamma} \\ \xleftarrow{\mathrm{coDisc}} \end{array} \mathrm{Set}$$

Proof. For the adjunction on the right, we just need to observe that for every set $S \in \mathrm{Set}$, the codiscrete object $\mathrm{coDisc}(S)$ is concrete, which is immediate by idempotency of $\sharp$ (Prop. 1.64) and the fact that every isomorphism is also a monomorphism. Similarly, the assumption that discrete objects are concrete says exactly that also Disc factors through $\mathbf{H}_{\mathrm{conc}}$.

For the adjunction on the left we claim that the left adjoint conc, (to be called *concretification*), is given by sending each object to the image (Def. 4.22) of its $(\Gamma \dashv \mathrm{coDisc})$ adjunction unit $\eta^\sharp$:

$$\mathrm{conc} \;:\; X \mapsto \mathrm{im}(\eta^\sharp_X)\,,$$

hence to the object which exhibits the epi/mono-factorization (Prop. 4.22) of $\eta^\sharp_X$

$$\eta^\sharp_X \;:\; X \underset{\mathrm{epi}}{\overset{\eta^{\mathrm{conc}}_X}{\longrightarrow}} \mathrm{conc}\,X \underset{\mathrm{mono}}{\longrightarrow} \sharp X\;. \tag{89}$$

First we need to show that $\mathrm{conc}\,X$, thus defined, is indeed concrete, hence that $\eta^\sharp_{\mathrm{im}(\eta^\sharp_X)}$ is a monomorphism (Def. 1.18). For this, consider the following naturality square (11) of the $\Gamma \dashv \mathrm{coDisc}$-adjunction hom-isomorphism

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathrm{Set}}(\Gamma\, \mathrm{im}(\eta^\sharp_X), \Gamma\, \mathrm{im}(\eta^\sharp_X)) & \simeq & \mathrm{Hom}_{\mathbf{H}}(\mathrm{im}(\eta^\sharp_X), \sharp\, \mathrm{im}(\eta^\sharp_X)) && \left\{ \mathrm{id}_{\Gamma\, \mathrm{im}(\eta^\sharp_X)} \right\} & \longrightarrow & (90) \\ {}^{(-)\circ \Gamma(\eta^{\mathrm{conc}}_X)}\big\downarrow && \big\downarrow^{(-)\circ \eta^{\mathrm{conc}}_X} && \big\downarrow \\ \mathrm{Hom}_{\mathrm{Set}}(\Gamma X, \Gamma\, \mathrm{im}(\eta^\sharp_X)) & \simeq & \mathrm{Hom}_{\mathbf{H}}(X, \sharp\, \mathrm{im}(\eta^\sharp_X)) && \left\{ \Gamma(\eta^{\mathrm{conc}}_X) \right\} & \longrightarrow \end{array}$$

By chasing the identity morphism on $\Gamma\, \mathrm{im}(\eta^\sharp_X)$ through this diagram, as shown by the diagram on the right, we obtain the equality displayed in the bottom right entry, where we used the general formula for adjuncts (Prop. 1.38) and the definition $\sharp \coloneqq \mathrm{coDisc} \circ \Gamma$ (Def. 5.6).

But observe that $\Gamma(\eta^{\mathrm{conc}}_X)$, and hence also $\sharp(\eta^{\mathrm{conc}}_X)$, is an isomorphism (Def. 1.9), as indicated above: Since Γ is both a left adjoint as well as a right adjoint, it preserves both epimorphisms as well as monomorphisms (Prop. 1.47), hence it preserves image factorizations (Prop. 4.22). This implies that $\Gamma \eta^{\mathrm{conc}}_X$ is the epimorphism onto the image of $\Gamma(\eta^\sharp_X)$. But by idempotency of $\sharp$, the latter is an isomorphism, and hence so is the epimorphism in its image factorization.

Therefore the equality in (90) says that

$$\begin{aligned} \eta^\sharp_X & = \left(\mathrm{iso} \circ \eta^\sharp_{\mathrm{im}(\eta^\sharp_X)} \right) \circ \eta^{\mathrm{conc}}_X \\ & = \mathrm{mono} \circ \eta^{\mathrm{conc}}_X \,, \end{aligned}$$

where in the second line we remembered that η^{conc}_X is, by definition, the epimorphism in the epi/mono-factorization of $\eta^\sharp_X$.

Now the defining property of epimorphisms (Def. 1.18) allows to cancel this commmon factor on both sides, which yields

$$\eta^\sharp_{\mathrm{im}(\eta^\sharp_X)} \;=\; \mathrm{iso} \circ \mathrm{mono} \;=\; \mathrm{mono} \,.$$

This shows that $\mathrm{conc}\, X \coloneqq \mathrm{im}(\eta^\sharp_X)$ is indeed concret.

It remains to show that this construction is left adjoint to the inclusion. We claim that the adjunction unit (Def. 1.33) of $(\mathrm{conc} \dashv \iota_{\mathrm{conc}})$ is provided by η^{conc} (89).

To see this, first notice that, since the epi/mono-factorization (Prop. 4.22) is orthogonal and hence functorial, we have commuting diagrams of the form

$$
\begin{array}{ccccc}
X_1 & \underset{\text{epi}}{\overset{\eta^{\text{conc}}_{X_1}}{\longrightarrow}} & \text{im}(\eta^\sharp_{X_1}) & \underset{\text{mono}}{\longrightarrow} & \sharp X_1 \\
\downarrow && \downarrow && \downarrow \\
X_2 & \underset{\text{epi}}{\overset{\eta^{\text{conc}}_{X_2}}{\longrightarrow}} & \text{im}(\eta^\sharp_{X_2}) & \underset{\text{mono}}{\longrightarrow} & \sharp X_2
\end{array}
\qquad (91)
$$

Now to demonstrate the adjunction it is sufficient, by Prop. 1.42, to show that η^{conc} is a universal morphism in the sense of Def. 1.41. Hence consider any morphism $f : X_1 \to X_2$ with $X_2 \in \mathbf{H}_{\text{conc}} \hookrightarrow \mathbf{H}$. Then we need to show that there is a unique diagonal morphism as below, that makes the following *top left triangle* commute:

$$
\begin{array}{ccc}
X_1 & \overset{f}{\longrightarrow} & X_2 \\
{}^{\text{epi}}\downarrow^{\eta^{\text{conc}}_{X_1}} & {}^{\exists !}\nearrow & \downarrow^{\text{mono}} \\
\text{im}(\eta^\sharp_{X_1}) & \longrightarrow & \sharp X_2
\end{array}
$$

Now, from (91), we have a commuting square as shown. Here the left morphism is an epimorphism by construction, while the right morphism is a monomorphism by assumption on X_2. With this, the epi/mono-factorization in Prop. 4.22 says that there is a diagonal lift which makes *both* triangles commute.

It remains to see that the lift is unique with just the property of making the top left triangle commute. But this is equivalently the statement that the left morphism is an epimorphism, by Def. 1.18. ■

The equivalence of the first two follows with (Johnstone, lemma 2.1, corollary 2.2). The equivalence of the first and the last is due to Lawvere-Menni 15, lemma 4.1, lemma 4.2.

Elastic toposes

***Definition 5.10*. (*elastic topos*)**

Let $\mathbf{H}_{\text{red}}$ be a cohesive topos (Def. 5.2). Then an *elastic topos* or *differentially cohesive topos* over $\mathbf{H}_{\text{red}}$ is a sheaf topos $\mathbf{H}$ which is

1. a cohesive topos over Set,
2. equipped with a quadruple of adjoint functors (Def. 1.32) to $\mathbf{H}_{\text{red}}$ of the form

$$\mathbf{H}_{\mathrm{red}} \begin{array}{c} \overset{\iota_{\mathrm{inf}}}{\hookrightarrow} \\ \overset{\Pi_{\mathrm{inf}}}{\longleftarrow} \\ \overset{\mathrm{Disc}_{\mathrm{inf}}}{\hookrightarrow} \\ \overset{\Gamma_{\mathrm{inf}}}{\longleftarrow} \end{array} \mathbf{H}$$

***Lemma 5.11**. (progression of (co-)reflective subcategories of elastic topos)*

Let $\mathbf{H}$ *be an elastic topos (Def. 5.10) over a cohesive topos* $\mathbf{H}_{\mathrm{red}}$ *(Def. 5.2):*

$$\mathrm{Set} \begin{array}{c} \overset{\Pi_{\mathrm{red}}}{\longleftarrow} \\ \overset{\mathrm{Disc}_{\mathrm{red}}}{\hookrightarrow} \\ \overset{\Gamma_{\mathrm{red}}}{\longleftarrow} \\ \overset{\mathrm{coDisc}_{\mathrm{red}}}{\hookrightarrow} \end{array} \mathbf{H}_{\mathrm{red}} \begin{array}{c} \overset{\iota_{\mathrm{inf}}}{\hookrightarrow} \\ \overset{\Pi_{\mathrm{inf}}}{\longleftarrow} \\ \overset{\mathrm{Disc}_{\mathrm{inf}}}{\hookrightarrow} \\ \overset{\Gamma_{\mathrm{inf}}}{\longleftarrow} \end{array} \mathbf{H}$$

and write

$$\mathrm{Set} \begin{array}{c} \overset{\Pi}{\longleftarrow} \\ \overset{\mathrm{Disc}}{\hookrightarrow} \\ \overset{\Gamma}{\longleftarrow} \\ \overset{\mathrm{coDisc}}{\hookrightarrow} \end{array} \mathbf{H}$$

for the adjoint quadruple exhibiting the cohesion of $\mathbf{H}$ *itself. Then these adjoint functors arrange and decompose as in the following diagram*

$$\begin{array}{ccccccccc} & & & & & \hookrightarrow\!\iota_{\mathrm{inf}}\!\longrightarrow & & & \\ & & & \longleftarrow\Pi_{\mathrm{red}}\!- & & \longleftarrow\Pi_{\mathrm{inf}}\!- & & : & \Pi \\ \mathrm{Disc} & : & \mathrm{Set} & \hookrightarrow\mathrm{Disc}_{\mathrm{red}}\!\rightarrow & \mathbf{H}_{\mathrm{red}} & \hookrightarrow\mathrm{Disc}_{\mathrm{inf}}\!\rightarrow & \mathbf{H} & & \\ & & & \longleftarrow\Gamma_{\mathrm{red}}\!- & & \longleftarrow\Gamma_{\mathrm{Disc}}\!- & & : & \Gamma \\ \mathrm{coDisc} & : & & \hookrightarrow & \!\!\mathrm{coDisc}\!\! & \longrightarrow & & & \end{array}$$

Proof. The identification

$$(\mathrm{Disc} \dashv \Gamma) \;\simeq\; (\mathrm{Disc}_{\mathrm{inf}} \circ \mathrm{Disc}_{\mathrm{red}} \;\dashv\; \Gamma_{\mathrm{red}} \circ \Gamma_{\mathrm{inf}})$$

follows from the essential uniqueness of the global section-geometric morphism (Example 4.9). This implies the identifications $\Pi \simeq \Pi_{\mathrm{red}} \circ \Pi_{\mathrm{inf}}$ by essential uniqueness of adjoints (Prop. 1.45). ▮

***Definition 5.12*. (adjoint modalities on elastic topos)**

Given an elastic topos (differentially cohesive topos) $\mathbf{H}$ over $\mathbf{H}_{\mathrm{red}}$ (Def. 5.10), composition of the functors in Lemma 5.11 yields, via Prop. 1.63, the following adjoint modalities (Def. 1.66)

$$\Re \dashv \Im \dashv \& \;\;:\;\; \mathbf{H} \underoverset{\underset{\& \coloneqq \mathrm{Disc}_{\mathrm{inf}} \circ \Gamma_{\mathrm{inf}}}{\longleftarrow}}{\overset{\Re \coloneqq \iota_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}}{\longleftarrow}}{\xrightarrow{\Im \coloneqq \mathrm{Disc}_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}}} \mathbf{H} \,.$$

Since ι_{inf} and $\mathrm{Disc}_{\mathrm{inf}}$ are fully faithful functors by assumption, these are (co-)modal operators (Def. 1.62) on the cohesive topos, by (Prop. 1.63).

We pronounce these as follows:

reduction modality	infinitesimal shape modality	infinitesimal flat modality
$\Re \;\coloneqq\; \iota_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}$	$\Im \;\coloneqq\; \mathrm{Disc}_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}$	$\& \;\coloneqq\; \mathrm{Disc}_{\mathrm{inf}} \circ \Gamma_{\mathrm{inf}}$

and we refer to the corresponding modal objects (Def. 1.65) as follows:

- a reduction-comodal object

  $$\Re X \underoverset{\simeq}{\epsilon^{\Re}_X}{\longrightarrow} X$$

  is called a *reduced object*;

- an infinitesimal shape-modal object

  $$X \underoverset{\simeq}{\eta^{\Im}_X}{\longrightarrow} \Im X$$

  is called a *coreduced object*.

***Proposition 5.13*. (progression of adjoint modalities on elastic topos)**

Let $\mathbf{H}$ be an elastic topos (Def. 5.10) and consider the corresponding adjoint modalities which it inherits

1. *for being a cohesive topos, from Def. 5.6,*
2. *for being an elastic topos, from Def. 5.12:*

shape modality	***flat modality***	***sharp modality***
$\int \;\coloneqq\; \mathrm{Disc} \circ \Pi$	$\flat \;\coloneqq\; \mathrm{Disc} \circ \Gamma$	$\sharp \;\coloneqq\; \mathrm{coDisc} \circ \Gamma$
reduction modality	*infinitesimal shape modality*	*infinitesimal flat modality*
$\Re \;\coloneqq\; \iota_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}$	$\Im \;\coloneqq\; \mathrm{Disc}_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}}$	$\& \;\coloneqq\; \mathrm{Disc}_{\mathrm{inf}} \circ \Gamma_{\mathrm{inf}}$

Then these arrange into the following progression, via the preorder on modalities from Def. 1.70

$$\begin{array}{ccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \vee & & \vee & & \\ & & \int & \dashv & \flat & \dashv & \sharp \\ & & & & \vee & & \vee \\ & & & & \emptyset & \dashv & \ast \end{array}$$

where we display also the bottom adjoint modality $\emptyset \dashv \ast$ (Example 1.71), for completeness.

Proof. We need to show, for all $X \in \mathbf{H}$, that

1. $\flat X$ is an $\&$-modal object (Def. 1.65), hence that

$$\& \flat X \;\simeq\; X$$

2. $\int X$ is an $\Im$-modal object (Def. 1.65), hence that

$$\Im \int X \;\simeq\; X$$

After unwinding the definitions of the modal operators Def. 5.6 and Def. 5.6, and using their re-identification from Lemma 5.11, this comes down to the fact that

$$\Pi_{\mathrm{inf}} \, \mathrm{Disc}_{\mathrm{inf}} \;\simeq\; \mathrm{id} \qquad \text{and} \qquad \Gamma_{\mathrm{inf}} \, \mathrm{Disc}_{\mathrm{inf}} \;\simeq\; \mathrm{id} \,,$$

which holds by Prop. 1.46, since $\mathrm{Disc}_{\mathrm{inf}}$ is a fully faithful functor and Π_{inf}, $\mathrm{Gamma}_{\mathrm{inf}}$ are (co-)reflectors for it, respectively:

$$
\begin{aligned}
\underbrace{\&}_{\mathrm{Disc}_{\mathrm{inf}}\,\Gamma_{\mathrm{inf}}}\underbrace{\flat}_{\mathrm{Disc}\,\Gamma} & = \mathrm{Disc}_{\mathrm{inf}}\,\Gamma_{\mathrm{inf}}\underbrace{\mathrm{Disc}}_{\mathrm{Disc}_{\mathrm{inf}}\mathrm{Disc}_{\mathrm{red}}}\,\Gamma \\
& = \underbrace{\mathrm{Disc}_{\mathrm{inf}}\underbrace{\Gamma_{\mathrm{inf}}\,\mathrm{Disc}_{\mathrm{inf}}}_{\simeq \mathrm{id}}\mathrm{Disc}_{\mathrm{red}}}_{\simeq \mathrm{Disc}}\,\Gamma \\
& \simeq \underbrace{\mathrm{Disc}_{\mathrm{inf}}\mathrm{Disc}_{\mathrm{red}}}_{\mathrm{Disc}}\Gamma\,\mathbf{X} \\
& = \mathrm{Disc}\,\Gamma \\
& = \flat
\end{aligned}
$$

and

$$
\begin{aligned}
\underbrace{\Im}_{\mathrm{Disc}_{\mathrm{inf}}\,\Pi_{\mathrm{inf}}}\underbrace{\int}_{\mathrm{Disc}\,\Pi} & = \mathrm{Disc}_{\mathrm{inf}}\,\Pi_{\mathrm{inf}}\underbrace{\mathrm{Disc}}_{\mathrm{Disc}_{\mathrm{inf}}\mathrm{Disc}_{\mathrm{red}}}\,\Pi \\
& = \underbrace{\mathrm{Disc}_{\mathrm{inf}}\underbrace{\Pi_{\mathrm{inf}}\,\mathrm{Disc}_{\mathrm{inf}}}_{\simeq \mathrm{id}}\mathrm{Disc}_{\mathrm{red}}}_{\simeq \mathrm{Disc}}\Pi \\
& \simeq \mathrm{Disc}\,\Pi \\
& = \int
\end{aligned}
$$

▮

Solid toposes

Definition 5.14. (solid topos)

Let $\mathbf{H}_{\mathrm{bos}}$ be an elastic topos (Def. 5.10) over a cohesive topos $\mathbf{H}_{\mathrm{red}}$ (Def. 5.2). Then a *solid topos* or *super-differentially cohesive topos* over $\mathbf{H}_{\mathrm{bos}}$ is a sheaf topos $\mathbf{H}$, which is

1. a cohesive topos over Set (Def. 5.2),
2. an elastic topos over $\mathbf{H}_{\mathrm{red}}$,
3. equipped with a quadruple of adjoint functors (Def. 1.32) to $\mathbf{H}_{\mathrm{bos}}$ of the form

$$
\mathbf{H}_{\mathrm{bos}}
\begin{array}{c}
\xleftarrow{\;\mathrm{even}\;} \\
\xhookrightarrow{\;\iota_{\mathrm{sup}}\;} \\
\xleftarrow{\;\Pi_{\mathrm{sup}}\;} \\
\xhookrightarrow{\;\mathrm{Disc}_{\mathrm{sup}}\;}
\end{array}
\mathbf{H}
$$

hence with ι_{sup} and $\mathrm{Disc}_{\mathrm{sup}}$ being fully faithful functors (Def. 1.19).

***Lemma 5.15**. (progression of (co-)reflective subcategories of solid topos)*

Let $\mathbf{H}$ be a solid topos (Def. 5.14) over an elastic topos $\mathbf{H}_{\mathrm{red}}$ (Def. 5.10):

$$
\mathrm{Set}
\underoverset{\underset{\mathrm{coDisc}_{\mathrm{red}}}{\hookrightarrow}}{\overset{\Pi_{\mathrm{red}}}{\longleftarrow}}{\overset{\overset{\mathrm{Disc}_{\mathrm{red}}}{\hookrightarrow}}{\underset{\Gamma_{\mathrm{red}}}{\longleftarrow}}}
\mathbf{H}_{\mathrm{red}}
\underoverset{\underset{\Gamma_{\mathrm{inf}}}{\longleftarrow}}{\overset{\overset{\iota_{\mathrm{inf}}}{\hookrightarrow}}{\overset{\Pi_{\mathrm{inf}}}{\longleftarrow}}}{\overset{\mathrm{Disc}_{\mathrm{inf}}}{\hookrightarrow}}
\mathbf{H}_{\mathrm{bos}}
\underoverset{\underset{\Gamma_{\mathrm{sup}}}{\longleftarrow}}{\overset{\overset{\overset{\mathrm{even}}{\longleftarrow}}{\overset{\iota_{\mathrm{sup}}}{\hookrightarrow}}}{\overset{\Pi_{\mathrm{sup}}}{\longleftarrow}}}{\overset{\mathrm{Disc}_{\mathrm{sup}}}{\hookrightarrow}}
\mathbf{H}
$$

Then these adjoint functors arrange and decompose as shown in the following diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & \overset{\iota_{\mathrm{inf}}}{\hookrightarrow} & & \overset{\overset{\mathrm{even}}{\longleftarrow}}{\overset{\iota_{\mathrm{sup}}}{\hookrightarrow}} & & & \\
 & & & \overset{\Pi_{\mathrm{red}}}{\longleftarrow} & & \overset{\Pi_{\mathrm{inf}}}{\longleftarrow} & & \overset{\Pi_{\mathrm{sup}}}{\longleftarrow} & & : & \Pi \\
\mathrm{Disc} & : & \mathrm{Set} & \overset{\mathrm{Disc}_{\mathrm{red}}}{\hookrightarrow} & \mathbf{H}_{\mathrm{red}} & \overset{\mathrm{Disc}_{\mathrm{inf}}}{\hookrightarrow} & \mathbf{H}_{\mathrm{bos}} & \overset{\mathrm{Disc}_{\mathrm{sup}}}{\hookrightarrow} & \mathbf{H} & & \\
 & & & \overset{\Gamma_{\mathrm{red}}}{\longleftarrow} & & \longleftarrow & \Gamma_{\mathrm{sup}} & \longrightarrow & & : & \Gamma \\
\mathrm{coDisc} & : & & \hookrightarrow & & \longrightarrow & \mathrm{coDisc} & \longrightarrow & & &
\end{array}
$$

Here the composite adjoint quadruple

$$
\mathrm{Set}
\underoverset{\underset{\mathrm{coDisc}}{\hookrightarrow}}{\overset{\Pi \simeq \Pi_{\mathrm{red}}\Pi_{\mathrm{inf}}\Pi_{\mathrm{sup}}}{\longleftarrow}}{\overset{\overset{\mathrm{Disc} = \mathrm{Disc}_{\mathrm{sup}}\mathrm{Disc}_{\mathrm{inf}}\mathrm{Disc}_{\mathrm{red}}}{\hookrightarrow}}{\underset{\Gamma = \Gamma_{\mathrm{sup}}\Gamma_{\mathrm{inf}}\Gamma_{\mathrm{red}}}{\longleftarrow}}}
\mathbf{H}
$$

exhibits the cohesion of $\mathbf{H}$ over Set, and the composite adjoint quadruple

$$\mathbf{H}_{\mathrm{red}} \begin{array}{c} \overset{\iota_{\mathrm{sup}}\iota_{\mathrm{inf}}}{\hookrightarrow} \\ \overset{\Pi_{\mathrm{inf}}\Pi_{\mathrm{sup}}}{\longleftarrow} \\ \overset{\mathrm{Disc}_{\mathrm{inf}}\mathrm{Disc}_{\mathrm{red}}}{\hookrightarrow} \\ \overset{\Gamma_{\mathrm{sup}}}{\longleftarrow} \end{array} \mathbf{H}$$

exhibits the elasticity of $\mathbf{H}$ *over* $\mathbf{H}_{\mathrm{red}}$.

Proof. As in the proof of Prop. 5.11, this is immediate by the essential uniqueness of adjoints (Prop. 1.45) and of the global section-geometric morphism (Example 4.9). ▮

Definition 5.16*. *(adjoint modalities on solid topos)

Given a solid topos $\mathbf{H}$ over $\mathbf{H}_{\mathrm{bos}}$ (Def. 5.14), composition of the functors in Lemma 5.15 yields, via Prop. 1.63, the following adjoint modalities (Def. 1.66)

$$\rightrightarrows \;\dashv\; \rightsquigarrow \;\dashv\; \mathrm{Rh} \;:\; \mathbf{H} \begin{array}{c} \xleftarrow{\rightrightarrows \;\coloneqq\; \iota_{\mathrm{sup}} \circ \mathrm{even}} \\ \xrightarrow{\rightsquigarrow \;\coloneqq\; \iota_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}} \\ \xleftarrow{\mathrm{Rh} \;\coloneqq\; \mathrm{Disc}_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}} \end{array} \mathbf{H} \,.$$

Since ι_{sup} and $\mathrm{Disc}_{\mathrm{sup}}$ are fully faithful functors by assumption, these are (co-)modal operators (Def. 1.62) on the cohesive topos, by (Prop. 1.63).

We pronounce these as follows:

fermionic modality	bosonic modality	rheonomy modality
$\rightrightarrows \;\coloneqq\; \iota_{\mathrm{sup}} \circ \mathrm{even}$	$\rightsquigarrow \;\coloneqq\; \iota_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}$	$\mathrm{Rh} \;\coloneqq\; \mathrm{Disc}_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}$

and we refer to the corresponding modal objects (Def. 1.65) as follows:

- a $\rightsquigarrow$-comodal object

  $$\overset{\rightsquigarrow}{X} \underset{\simeq}{\overset{\epsilon_X^{\rightsquigarrow}}{\longrightarrow}} X$$

  is called a *bosonic object*;

- a Rh-modal object

  $$X \underset{\simeq}{\overset{\eta_X^{\mathrm{Rh}}}{\longrightarrow}} \mathrm{Rh}\, X$$

is called a *rheonomic object*;

***Proposition 5.17*. (progression of adjoint modalities on solid topos)**

Let $\mathbf{H}$ be a solid topos (Def. 5.14) and consider the adjoint modalities which it inherits

1. *for being a cohesive topos, from Def. 5.6,*
2. *for being an elastic topos, from Def. 5.12,*
3. *for being a solid topos, from Def. 5.16:*

shape modality	***flat modality***	***sharp modality***
$\int \;\coloneqq\; \mathrm{Disc}\, \Pi$	$\flat \;\coloneqq\; \mathrm{Disc} \circ \Gamma$	$\sharp \;\coloneqq\; \mathrm{coDisc} \circ \Gamma$
reduction modality	***infinitesimal shape modality***	***infinitesimal flat modality***
$\Re \;\coloneqq\; \iota_{\mathrm{sup}} \iota_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}} \Pi_{\mathrm{sup}}$	$\Im \;\coloneqq\; \mathrm{Disc}_{\mathrm{sup}} \mathrm{Disc}_{\mathrm{inf}} \circ \Pi_{\mathrm{inf}} \Pi_{\mathrm{sup}}$	$\& \;\coloneqq\; \mathrm{Disc}_{\mathrm{sup}} \mathrm{Disc}_{\mathrm{inf}} \circ \Gamma_{\mathrm{inf}} \Gamma_{\mathrm{sup}}$
fermionic modality	***bosonic modality***	***rheonomy modality***
$\rightrightarrows \;\coloneqq\; \iota_{\mathrm{sup}} \circ \mathrm{even}$	$\rightsquigarrow \;\coloneqq\; \iota_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}$	$\mathrm{Rh} \;\coloneqq\; \mathrm{Disc}_{\mathrm{sup}} \circ \Pi_{\mathrm{sup}}$

Then these arrange into the following progression, via the preorder on modalities from Def. 1.70:

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} \\
\vee && \vee \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} \\
&& \vee && \vee \\
&& \Re & \dashv & \Im & \dashv & \& \\
&&&& \vee && \vee \\
&&&& \int & \dashv & \flat & \dashv & \sharp \\
&&&&&& \vee && \vee \\
&&&&&& \emptyset & \dashv & \ast
\end{array}
$$

where we are displaying, for completeness, also the adjoint modalities at the bottom $\emptyset \dashv *$

and the *top* $\mathrm{id} \dashv \mathrm{id}$ *(Example 1.71).*

Proof. By Prop. 5.13, it just remains to show that for all objects $X \in \mathbf{H}$

1. $\Im X$ is an Rh-modal object, hence $\mathrm{Rh}\, \Im X \simeq X$,
2. $\Re X$ is a bosonic object, hence $\overset{\rightsquigarrow}{\Re X} \simeq \Re X$.

The proof is directly analogous to that of Prop. 5.13, now using the decompositions from Lemma 5.15:

$$\begin{aligned} \mathrm{Rh}\, \Im &= \mathrm{Disc}_{\mathrm{sup}} \underbrace{\Pi_{\mathrm{sup}} \;\; \mathrm{Disc}_{\mathrm{sup}}}_{\simeq \mathrm{id}} \mathrm{Disc}_{\mathrm{inf}}\, \Pi_{\mathrm{inf}} \Pi_{\mathrm{sup}} \\ &\simeq \mathrm{Disc}_{\mathrm{sup}} \mathrm{Disc}_{\mathrm{inf}}\, \Pi_{\mathrm{inf}} \Pi_{\mathrm{sup}} \\ &= \Im \end{aligned}$$

and

$$\begin{aligned} \rightsquigarrow \Re &= \iota_{\mathrm{sup}} \underbrace{\Pi_{\mathrm{sup}} \;\; \iota_{\mathrm{sup}}}_{\simeq \mathrm{id}} \iota_{\mathrm{inf}} \Pi_{\mathrm{inf}} \Pi_{\mathrm{sub}} \\ &\simeq \iota_{\mathrm{sup}} \iota_{\mathrm{inf}} \Pi_{\mathrm{inf}} \Pi_{\mathrm{sub}} \\ &\simeq \Re \end{aligned}$$

■

(...)

6. Basic notions of homotopy theory

Traditionally, mathematics and physics have been founded on set theory, whose concept of *sets* is that of “bags of distinguishable points”.

But fundamental physics is governed by the *gauge principle*. This says that given any two “things”, such as two field histories x and y, it is in general wrong to ask whether they are equal or not, instead one has to ask where there is a *gauge transformation*

$$x \xrightarrow{\gamma} y$$

between them. In mathematics this is called a *homotopy*.

This principle applies also to gauge transformations/homotopies themselves, and thus leads to *gauge-of-gauge transformations* or *homotopies of homotopies*

$$
\begin{tikzcd}
x \arrow[rr, bend left=50, "\gamma_1"] \arrow[rr, bend right=50, "\gamma_2"'] \arrow[rr, phantom, "\Downarrow \kappa"] & & y
\end{tikzcd}
$$

and so on to ever *higher gauge transformations* or *higher homotopies*:

$$
\begin{tikzcd}
x \arrow[rr, bend left=50, "\gamma_1"] \arrow[rr, bend right=50, "\gamma_2"'] \arrow[rr, phantom, "\kappa_1 \Rrightarrow \kappa_2"] & & y
\end{tikzcd}
$$

This shows that what x an y here are elements of is not really a set in the sense of set theory. Instead, such a collection of elements with higher gauge transformations/higher homotopies between them is called a *homotopy type*.

Hence the theory of homotopy types – *homotopy theory* – is much like set theory, but with the concept of gauge transformation/homotopy built right into its foundations. Homotopy theory is gauged mathematics.

A classical model for homotopy types are simply topological spaces: Their points represent the elements, the continuous paths between points represent the gauge transformations, and continuous deformations of paths represent higher gauge transformations. A central result of homotopy theory is the proof of the *homotopy hypothesis*, which says that under this identification homotopy types are equivalent to topological spaces viewed, in turn, up to "weak homotopy equivalence".

In the special case of a homotopy type with a single element x, the gauge transformations necessarily go from x to itself and hence form a *group of symmetries* of x.

$$
\begin{tikzcd}
x \arrow[loop above, "g \in G"]
\end{tikzcd}
$$

This way homotopy theory subsumes group theory.

If there are higher order gauge-of-gauge transformations/homotopies of homotopies between these symmetry group-elements, then one speaks of *2-groups*, *3-groups*, ... *n-groups*, and eventually of *∞-groups*. When homotopy types are represented by topological spaces, then ∞-groups are represented by topological groups.

This way homotopy theory subsumes parts of topological group theory.

Since, generally, there is more than one element in a homotopy type, these are like "groups with several elements", and as such they are called *groupoids* (Def. 1.10).

If there are higher order gauge-of-gauge transformations/homotopies of homotopies between the transformations in such a groupoid, one speaks of *2-groupoids*, *3-groupoids*, ... *n-groupoids*, and eventually of *∞-groupoids*. The plain sets are recovered as the special case of 0-groupoids.

Due to the higher orders n appearing here, mathematical structures based not on sets but on homotopy types are also called *higher structures*.

Hence homotopy types are equivalently *∞-groupoids*. This perspective makes explicit that homotopy types are the unification of plain sets with the concept of gauge-symmetry groups.

An efficient way of handling ∞-groupoids is in their explicit guise as *Kan complexes* (Def. below); these are the non-abelian generalization of the *chain complexes* used in homological algebra. Indeed, *chain homotopy* is a special case of the general concept of homotopy, and hence homological algebra forms but a special abelian corner within homotopy theory. Conversely, homotopy theory may be understood as the non-abelian generalization of homological algebra.

Hence, in a self-reflective manner, there are many different but *equivalent* incarnations of homotopy theory. Below we discuss in turn:

- *Topological homotopy theory*
  ∞-groupoids modeled by topological spaces. This is the classical model of homotopy theory familiar from traditional point-set topology, such as covering space-theory.
- *Simplicial homotopy theory*.
  ∞-groupoids modeled on simplicial sets, whose fibrant objects are the Kan complexes. This simplicial homotopy theory is Quillen equivalent to topological homotopy theory (the "homotopy hypothesis"), which makes explicit that homotopy theory is not really about topological spaces, but about the ∞-groupoids that these represent.

Ideally, abstract homotopy theory would simply be a complete replacement of set theory, obtained by *removing* the assumption of strict equality, relaxing it to gauge equivalence/homotopy. As such, abstract homotopy theory would be part and parcel of the foundations of mathematics themselves, not requiring any further discussion. This ideal perspective is the promise of *homotopy type theory* and may become full practical reality in the next decades.

Until then, abstract homotopy theory has to be formulated on top of the traditional foundations of mathematics provided by set theory, much like one may have to run a Linux emulator on a Windows machine, if one does happen to be stuck with the latter.

A very convenient and powerful such emulator for homotopy theory within set theory is *model category theory*, originally due to Quillen 67 and highly developed since. This we introduce here.

The idea is to consider ordinary categories (Def. 1.1) but with the understanding that some of their morphisms

$$X \xrightarrow{f} Y$$

should be *homotopy equivalences* (Def.), namely similar to isomorphisms (Def. 1.9), but not necessarily satisfying the two equations defining an actual isomorphism

$$f^{-1} \circ f \;=\; \mathrm{id}_X \qquad f \circ f^{-1} \;=\; \mathrm{id}_Y$$

but intended to satisfy this only with equality relaxed to gauge transformation/homotopy:

$$f^{-1} \circ f \overset{\text{gauge}}{\Longrightarrow} \mathrm{id}_X \qquad f \circ f^{-1} \overset{\text{gauge}}{\Longrightarrow} \mathrm{id}_Y \,. \tag{92}$$

Such *would-be homotopy equivalences* are called *weak equivalences* (Def. 1.75 below).

In principle, this information already defines a homotopy theory by a construction called *simplicial localization*, which turns weak equivalences into actual homotopy equivalences in a suitable way.

However, without further tools this construction is unwieldy. The extra structure of a *model category* (Def. 6.1 below) on top of a category with weak equivalences provides a set of tools.

The idea here is to abstract (in Def. 6.20 below) from the evident concepts in topological homotopy theory of *left homotopy* (Def.) and *right homotopy* (Def.) between continuous functions: These are provided by continuous functions out of a cylinder space $\mathrm{Cyl}(X) = X \times [0,1]$ or into a path space $\mathrm{Path}(X) = X^{[0,1]}$, respectively, where in both cases the interval space $[0,1]$ serves to parameterize the relevant gauge transformation/

homotopy.

Now a little reflection shows (this was the seminal insight of Quillen 67) that what really matters in this construction of homotopies is that the path space factors the diagonal morphism from a space X to its Cartesian product as

$$\mathrm{diag}_X \;:\; X \underset{\text{weak equiv.}}{\overset{\text{cofibration}}{\longrightarrow}} \mathrm{Path}(X) \overset{\text{fibration}}{\longrightarrow} X \times X$$

while the cylinder serves to factor the codiagonal morphism as

$$\mathrm{codiag}_X \;:\; X \sqcup X \overset{\text{cofibration}}{\longrightarrow} \mathrm{Cyl}(X) \underset{\text{weak equiv}}{\overset{\text{fibration}}{\longrightarrow}} X$$

where in both cases "fibration" means something like *well behaved surjection*, while "cofibration" means something like *satisfying the lifting property (Def. 6.2 below) against fibrations that are also weak equivalences*.

Such factorizations subject to lifting properties is what the definition of *model category* axiomatizes, in some generality. That this indeed provides a good toolbox for handling homotopy equivalences is shown by the *Whitehead theorem in model categories* (Lemma 6.25 below), which exhibits all weak equivalences as actual homotopy equivalences after passage to "good representatives" of objects (fibrant/cofibrant resolutions, Def. 6.26 below). Accordingly, the first theorem of model category theory (Quillen 67, I.1 theorem 1, reproduced as Theorem 6.29 below), provides a tractable expression for the hom-sets modulo homotopy equivalence of the underlying category with weak equivalences in terms of actual morphisms out of cofibrant resolutions into fibrant resolutions (Lemma 6.35 below).

This is then generally how model category-theory serves as a model for homotopy theory: All homotopy-theoretic constructions, such as that of long homotopy fiber sequences (Prop. below), are reflected via constructions of ordinary category theory but applied to suitably resolved objects.

Literature (Dwyer-Spalinski 95)

Definition 6.1. (model category)

A ***model category*** is

1. a category $\mathcal{C}$ (Def. 1.1) with all limits and colimits (Def. 3.1);

2. three sub-classes $W, \mathrm{Fib}, \mathrm{Cof} \subset \mathrm{Mor}(\mathcal{C})$ of its class of morphisms;

such that

1. the class W makes $\mathcal{C}$ into a ***category with weak equivalences***, def. 1.75;
2. The pairs $(W \cap \mathrm{Cof}\,,\ \mathrm{Fib})$ and $(\mathrm{Cap}\,,\ W \cap \mathrm{Fib})$ are both weak factorization systems, def. 6.3.

One says:

- elements in W are *weak equivalences*,
- elements in Cof are *cofibrations*,
- elements in Fib are *fibrations*,
- elements in $W \cap$ Cof are *acyclic cofibrations*,
- elements in $W \cap$ Fib are *acyclic fibrations*.

The form of def. 6.1 is due to (Joyal, def. E.1.2). It implies various other conditions that (Quillen 67) demands explicitly, see prop. 6.8 and prop. 6.12 below.

We now dicuss the concept of weak factorization systems (Def. 6.3 below) appearing in def. 6.1.

Factorization systems

***Definition 6.2*. (lift and extension)**

Let $\mathcal{C}$ be any category. Given a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {}^{\mathllap{p}}\downarrow \\ B \end{array}$$

then an *extension* of the morphism f along the morphism p is a completion to a commuting diagram of the form

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {}^{\mathllap{p}}\downarrow & \nearrow_{\tilde f} \\ B \end{array} \,.$$

Dually, given a diagram of the form

$$\begin{array}{ccc} && A \\ && \downarrow^{p} \\ X & \overset{f}{\longrightarrow} & Y \end{array}$$

then a ***lift*** of f through p is a completion to a commuting diagram of the form

$$\begin{array}{ccc} && A \\ & {}^{\tilde f}\nearrow & \downarrow^{p} \\ X & \overset{f}{\longrightarrow} & Y \end{array}.$$

Combining these cases: given a commuting square

$$\begin{array}{ccc} X_1 & \overset{f_1}{\longrightarrow} & Y_1 \\ {}^{p_l}\downarrow && \downarrow^{p_r} \\ X_2 & \overset{f_1}{\longrightarrow} & Y_2 \end{array}$$

then a ***lifting*** in the diagram is a completion to a commuting diagram of the form

$$\begin{array}{ccc} X_1 & \overset{f_1}{\longrightarrow} & Y_1 \\ {}^{p_l}\downarrow & \nearrow & \downarrow^{p_r} \\ X_2 & \overset{f_1}{\longrightarrow} & Y_2 \end{array}.$$

Given a sub-class of morphisms $K \subset \mathrm{Mor}(\mathcal{C})$, then

- a morphism p_r as above is said to have the ***right lifting property against*** K or to be a K**-*injective morphism*** if in all square diagrams with p_r on the right and any $p_l \in K$ on the left a lift exists.

dually:

- a morphism p_l is said to have the ***left lifting property against*** K or to be a K**-*projective morphism*** if in all square diagrams with p_l on the left and any $p_r \in K$ on the left a lift exists.

Definition 6.3. (weak factorization systems)

A ***weak factorization system*** (WFS) on a category $\mathcal{C}$ is a pair $(\mathrm{Proj}, \mathrm{Inj})$ of classes of morphisms of $\mathcal{C}$ such that

1. Every morphism $f : X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in Proj followed by one in Inj

$$f \;:\; X \overset{\in \mathrm{Proj}}{\longrightarrow} Z \overset{\in \mathrm{Inj}}{\longrightarrow} Y \;.$$

2. The classes are closed under having the lifting property, def. 6.2, against each other:
   1. Proj is precisely the class of morphisms having the left lifting property against every morphisms in Inj;
   2. Inj is precisely the class of morphisms having the right lifting property against every morphisms in Proj.

***Definition 6.4*. (*functorial factorization*)**

For $\mathcal{C}$ a category, a ***functorial factorization*** of the morphisms in $\mathcal{C}$ is a functor

$$\mathrm{fact} \;:\; \mathcal{C}^{\Delta[1]} \longrightarrow \mathcal{C}^{\Delta[2]}$$

which is a section of the composition functor $d_1 \;:\; \mathcal{C}^{\Delta[2]} \to \mathcal{C}^{\Delta[1]}$.

Remark 6.5. In def. 6.4 we are using the following standard notation, see at *simplex category* and at *nerve of a category*:

Write $[1] = \{0 \to 1\}$ and $[2] = \{0 \to 1 \to 2\}$ for the ordinal numbers, regarded as posets and hence as categories. The arrow category $\mathrm{Arr}(\mathcal{C})$ is equivalently the functor category $\mathcal{C}^{\Delta[1]} \coloneqq \mathrm{Funct}(\Delta[1], \mathcal{C})$, while $\mathcal{C}^{\Delta[2]} \coloneqq \mathrm{Funct}(\Delta[2], \mathcal{C})$ has as objects pairs of composable morphisms in $\mathcal{C}$. There are three injective functors $\delta_i : [1] \to [2]$, where δ_i omits the index i in its image. By precomposition, this induces functors $d_i : \mathcal{C}^{\Delta[2]} \longrightarrow \mathcal{C}^{\Delta[1]}$. Here

- d_1 sends a pair of composable morphisms to their composition;
- d_2 sends a pair of composable morphisms to the first morphisms;
- d_0 sends a pair of composable morphisms to the second morphisms.

Definition 6.6. A weak factorization system, def. 6.3, is called a ***functorial weak factorization system*** if the factorization of morphisms may be chosen to be a functorial factorization fact, def. 6.4, i.e. such that $d_2 \circ \mathrm{fact}$ lands in Proj and $d_0 \circ \mathrm{fact}$ in Inj.

Remark 6.7. Not all weak factorization systems are functorial, def. 6.6, although most (including those produced by the small object argument (prop. 6.15 below), with due care) are.

Proposition 6.8*. Let $\mathcal{C}$ be a category and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a class of morphisms. Write $K\,\mathrm{Proj}$ and $K\,\mathrm{Inj}$, respectively, for the sub-classes of K-projective morphisms and of K-injective morphisms, def. 6.2. Then:*

1. *Both classes contain the class of isomorphism of $\mathcal{C}$.*
2. *Both classes are closed under composition in $\mathcal{C}$.*
   $K\,\mathrm{Proj}$ *is also closed under transfinite composition.*
3. *Both classes are closed under forming retracts in the arrow category $\mathcal{C}^{\Delta[1]}$ (see remark 6.10).*
4. $K\,\mathrm{Proj}$ *is closed under forming pushouts of morphisms in $\mathcal{C}$ ("cobase change").*
   $K\,\mathrm{Inj}$ *is closed under forming pullback of morphisms in $\mathcal{C}$ ("base change").*
5. $K\,\mathrm{Proj}$ *is closed under forming coproducts in $\mathcal{C}^{\Delta[1]}$.*
   $K\,\mathrm{Inj}$ *is closed under forming products in $\mathcal{C}^{\Delta[1]}$.*

Proof. We go through each item in turn.

containing isomorphisms

Given a commuting square

$$
\begin{array}{ccc}
A & \overset{f}{\to} & X \\
{}^{\mathllap{i}}_{\mathllap{\in \mathrm{Iso}}}\downarrow && \downarrow^{\mathrlap{p}} \\
B & \underset{g}{\longrightarrow} & Y
\end{array}
$$

with the left morphism an isomorphism, then a lift is given by using the inverse of this isomorphism $\overset{f\circ i^{-1}}{\nearrow}$. Hence in particular there is a lift when $p \in K$ and so $i \in K\,\mathrm{Proj}$. The other case is formally dual.

closure under composition

Given a commuting square of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow && \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
{}^{\mathllap{i}}_{\mathllap{\in K}}\downarrow && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in K\,\mathrm{Inj}}} \\
B & \longrightarrow & Y
\end{array}
$$

consider its pasting decomposition as

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \searrow & \downarrow^{p_1}_{\in K\,\mathrm{Inj}} \\ {}^{i}_{\in K}\downarrow & & \downarrow^{p_2}_{\in K\,\mathrm{Inj}} \\ B & \longrightarrow & Y \end{array}$$

Now the bottom commuting square has a lift, by assumption. This yields another pasting decomposition

$$\begin{array}{ccc} A & \longrightarrow & X \\ {}^{i}_{\in K}\downarrow & & \downarrow^{p_1}_{\in K\,\mathrm{Inj}} \\ \downarrow & \nearrow & \downarrow^{p_2}_{\in K\,\mathrm{Inj}} \\ B & \longrightarrow & Y \end{array}$$

and now the top commuting square has a lift by assumption. This is now equivalently a lift in the total diagram, showing that $p_1 \circ p_1$ has the right lifting property against K and is hence in $K\,\mathrm{Inj}$. The case of composing two morphisms in $K\,\mathrm{Proj}$ is formally dual. From this the closure of $K\,\mathrm{Proj}$ under transfinite composition follows since the latter is given by colimits of sequential composition and successive lifts against the underlying sequence as above constitutes a cocone, whence the extension of the lift to the colimit follows by its universal property.

closure under retracts

Let j be the retract of an $i \in K\,\mathrm{Proj}$, i.e. let there be a commuting diagram of the form.

$$\begin{array}{ccccccc} \mathrm{id}_A \colon & A & \longrightarrow & C & \longrightarrow & A \\ & \downarrow^{j} & & \downarrow^{i}_{\in K\,\mathrm{Proj}} & & \downarrow^{j} \\ \mathrm{id}_B \colon & B & \longrightarrow & D & \longrightarrow & B \end{array}.$$

Then for

$$\begin{array}{ccc} A & \longrightarrow & X \\ {}^{j}\downarrow & & \downarrow^{f}_{\in K} \\ B & \longrightarrow & Y \end{array}$$

a commuting square, it is equivalent to its pasting composite with that retract diagram

$$
\begin{array}{ccccccc}
A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\
\downarrow^{j} && \downarrow^{i}_{\in K\,\mathrm{Proj}} & & \downarrow^{j} && \downarrow^{f}_{\in K} \\
B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y
\end{array}
$$

Here the pasting composite of the two squares on the right has a lift, by assumption:

$$
\begin{array}{ccccccc}
A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\
\downarrow^{j} && \downarrow^{i} & & \nearrow && \downarrow^{f}_{\in K} \\
B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y
\end{array}
$$

By composition, this is also a lift in the total outer rectangle, hence in the original square. Hence j has the left lifting property against all $p \in K$ and hence is in $K\,\mathrm{Proj}$. The other case is formally dual.

closure under pushout and pullback

Let $p \in K\,\mathrm{Inj}$ and and let

$$
\begin{array}{ccc}
Z \times_f X & \longrightarrow & X \\
{}^{f^*p}\downarrow && \downarrow^{p} \\
Z & \overset{f}{\longrightarrow} & Y
\end{array}
$$

be a pullback diagram in $\mathcal{C}$. We need to show that $f^* p$ has the right lifting property with respect to all $i \in K$. So let

$$
\begin{array}{ccc}
A & \longrightarrow & Z \times_f X \\
{}^{i}_{\in K}\downarrow && \downarrow^{f^*p} \\
B & \overset{g}{\longrightarrow} & Z
\end{array}
$$

be a commuting square. We need to construct a diagonal lift of that square. To that end, first consider the pasting composite with the pullback square from above to obtain the commuting diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & Z \times_f X & \longrightarrow & X \\
{}^{i}\downarrow && \downarrow^{f^*p} && \downarrow^{p} \\
B & \overset{g}{\longrightarrow} & Z & \overset{f}{\longrightarrow} & Y
\end{array}
.
$$

By the right lifting property of p, there is a diagonal lift of the total outer diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow^{i} & {}^{\widehat{(f g)}}\nearrow & \downarrow^{p} \\ B & \stackrel{f g}{\longrightarrow} & Y \end{array}.$$

By the universal property of the pullback this gives rise to the lift $\hat g$ in

$$\begin{array}{ccccc} && Z \times_f X & \longrightarrow & X \\ & {}^{\hat g}\nearrow & \downarrow^{f^* p} && \downarrow^{p} \\ B & \stackrel{g}{\longrightarrow} & Z & \stackrel{f}{\longrightarrow} & Y \end{array}.$$

In order for $\hat g$ to qualify as the intended lift of the total diagram, it remains to show that

$$\begin{array}{ccc} A & \longrightarrow & Z \times_f X \\ \downarrow^{i} & {}^{\hat g}\nearrow \\ B \end{array}$$

commutes. To do so we notice that we obtain two cones with tip A:

- one is given by the morphisms
  1. $A \to Z \times_f X \to X$
  2. $A \stackrel{i}{\to} B \stackrel{g}{\to} Z$

  with universal morphism into the pullback being

  - $A \to Z \times_f X$
- the other by
  1. $A \stackrel{i}{\to} B \stackrel{\hat g}{\to} Z \times_f X \to X$
  2. $A \stackrel{i}{\to} B \stackrel{g}{\to} Z$.

  with universal morphism into the pullback being

  - $A \stackrel{i}{\to} B \stackrel{\hat g}{\to} Z \times_f X$.

The commutativity of the diagrams that we have established so far shows that the first and second morphisms here equal each other, respectively. By the fact that the universal morphism into a pullback diagram is *unique* this implies the required identity of morphisms.

The other case is formally dual.

closure under (co-)products

Let $\{(A_s \overset{i_s}{\to} B_s) \in K\,\mathrm{Proj}\}_{s \in S}$ be a set of elements of $K\,\mathrm{Proj}$. Since colimits in the presheaf category $\mathcal{C}^{\Delta[1]}$ are computed componentwise, their coproduct in this arrow category is the universal morphism out of the coproduct of objects $\coprod_{s \in S} A_s$ induced via its universal property by the set of morphisms i_s:

$$\underset{s \in S}{\sqcup} A_s \overset{(i_s)_{s \in S}}{\longrightarrow} \underset{s \in S}{\sqcup} B_s \,.$$

Now let

$$\begin{array}{ccc} \underset{s \in S}{\sqcup} A_s & \longrightarrow & X \\ {}^{\mathllap{(i_s)_{s \in S}}}\downarrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ \underset{s \in S}{\sqcup} B_s & \longrightarrow & Y \end{array}$$

be a commuting square. This is in particular a cocone under the coproduct of objects, hence by the universal property of the coproduct, this is equivalent to a set of commuting diagrams

$$\left\{ \array{ A_s &\longrightarrow& X \\ {}^{\mathllap{i_s}}_{\mathllap{\in K\,\mathrm{Proj}}}\downarrow && \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B_s &\longrightarrow& Y } \right\}_{s \in S} \,.$$

By assumption, each of these has a lift ℓ_s. The collection of these lifts

$$\left\{ \array{ A_s &\longrightarrow& X \\ {}^{\mathllap{i_s}}_{\mathllap{\in \mathrm{Proj}}}\downarrow &{}^{\ell_s}\nearrow& \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ B_s &\longrightarrow& Y } \right\}_{s \in S}$$

is now itself a compatible cocone, and so once more by the universal property of the coproduct, this is equivalent to a lift $(\ell_s)_{s \in S}$ in the original square

$$\begin{array}{ccc} \underset{s \in S}{\sqcup} A_s & \longrightarrow & X \\ {}^{\mathllap{(i_s)_{s \in S}}}\downarrow & {}^{(\ell_s)_{s \in S}}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in K}} \\ \underset{s \in S}{\sqcup} B_s & \longrightarrow & Y \end{array}$$

This shows that the coproduct of the i_s has the left lifting property against all $f \in K$ and is hence in $K\,\mathrm{Proj}$. The other case is formally dual. ▮

An immediate consequence of prop. 6.8 is this:

Corollary 6.9. *Let $\mathcal{C}$ be a category with all small colimits, and let $K \subset \mathrm{Mor}(\mathcal{C})$ be a sub-class of its morphisms. Then every K-injective morphism, def. 6.2, has the right lifting property, def. 6.2, against all K-relative cell complexes, def. and their retracts, remark 6.10.*

Remark 6.10. By a retract of a morphism $X \xrightarrow{f} Y$ in some category $\mathcal{C}$ we mean a retract of f as an object in the arrow category $\mathcal{C}^{\Delta[1]}$, hence a morphism $A \xrightarrow{g} B$ such that in $\mathcal{C}^{\Delta[1]}$ there is a factorization of the identity on g through f

$$\mathrm{id}_g \;:\; g \longrightarrow f \longrightarrow g \;.$$

This means equivalently that in $\mathcal{C}$ there is a commuting diagram of the form

$$\begin{array}{cccccc} \mathrm{id}_A \colon & A & \longrightarrow & X & \longrightarrow & A \\ & \downarrow^{g} && \downarrow^{f} && \downarrow^{g} \\ \mathrm{id}_B \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array} \,.$$

Lemma 6.11. *In every category C the class of isomorphisms is preserved under retracts in the sense of remark 6.10.*

Proof. For

$$\begin{array}{cccccc} \mathrm{id}_A \colon & A & \longrightarrow & X & \longrightarrow & A \\ & \downarrow^{g} && \downarrow^{f} && \downarrow^{g} \\ \mathrm{id}_B \colon & B & \longrightarrow & Y & \longrightarrow & B \end{array} \,.$$

a retract diagram and $X \overset{f}{\to} Y$ an isomorphism, the inverse to $A \overset{g}{\to} B$ is given by the composite

$$\begin{array}{ccccc} && X & \longrightarrow & A \\ && \uparrow^{f^{-1}} && \\ B & \longrightarrow & Y && \end{array} \,.$$

■

More generally:

Proposition 6.12. *Given a model category in the sense of def. 6.1, then its class of weak equivalences is closed under forming retracts (in the arrow category, see remark 6.10).*

(Joyal, prop. E.1.3)

Proof. Let

$$
\begin{array}{ccccccc}
\mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\
& {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{w}} && \downarrow^{\mathrlap{f}} \\
\mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B
\end{array}
$$

be a [commuting diagram](https://ncatlab.org/nlab/show/commuting+diagram) in the given model category, with $w \in W$ a weak equivalence. We need to show that then also $f \in W$.

First consider the case that $f \in \mathrm{Fib}$.

In this case, factor w as a cofibration followed by an acyclic fibration. Since $w \in W$ and by [two-out-of-three](https://ncatlab.org/nlab/show/two-out-of-three) (def. 1.75) this is even a factorization through an acyclic cofibration followed by an acyclic fibration. Hence we obtain a commuting diagram of the following form:

$$
\begin{array}{ccccccc}
\mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\
& {}^{\mathllap{\mathrm{id}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\mathrm{id}}} \\
\mathrm{id} \colon & A' & \overset{s}{\longrightarrow} & X' & \overset{t}{\longrightarrow} & A' \,, \\
& {}^{\mathllap{f}}_{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W \cap \mathrm{Fib}}} && \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib}}} \\
\mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B
\end{array}
$$

where s is uniquely defined and where t is any lift of the top middle vertical acyclic cofibration against f. This now exhibits f as a retract of an acyclic fibration. These are closed under retract by prop. 6.8.

Now consider the general case. Factor f as an acyclic cofibration followed by a fibration and form the [pushout](https://ncatlab.org/nlab/show/pushout) in the top left square of the following diagram

$$
\begin{array}{ccccccc}
\mathrm{id} \colon & A & \longrightarrow & X & \longrightarrow & A \\
& {}^{\mathllap{\in W \cap \mathrm{Cof}}}\downarrow & (\mathrm{po}) & \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} && \downarrow^{\mathrlap{\in W \cap \mathrm{Cof}}} \\
\mathrm{id} \colon & A' & \longrightarrow & X' & \longrightarrow & A' \,, \\
& {}^{\mathllap{\in \mathrm{Fib}}}\downarrow && \downarrow^{\mathrlap{\in W}} && \downarrow^{\mathrlap{\in \mathrm{Fib}}} \\
\mathrm{id} \colon & B & \longrightarrow & Y & \longrightarrow & B
\end{array}
$$

where the other three squares are induced by the [universal property](https://ncatlab.org/nlab/show/universal+property) of the pushout, as is the identification of the middle horizontal composite as the identity on A'. Since acyclic cofibrations are closed under forming pushouts by prop. 6.8, the top middle vertical morphism is now an acyclic fibration, and hence by assumption and by two-out-of-three so is the middle bottom vertical morphism.

Thus the previous case now gives that the bottom left vertical morphism is a weak equivalence, and hence the total left vertical composite is. ▮

***Lemma 6.13*. (*retract argument*)**

Consider a composite morphism

$$f : X \xrightarrow{i} A \xrightarrow{p} Y \,.$$

1. *If f has the left lifting property against p, then f is a retract of i.*
2. *If f has the right lifting property against i, then f is a retract of p.*

***Proof*.** We discuss the first statement, the second is formally dual.

Write the factorization of f as a commuting square of the form

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By the assumed lifting property of f against p there exists a diagonal filler g making a commuting diagram of the form

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & A \\ {}^{\mathllap{f}}\downarrow & {}^{g}\nearrow & \downarrow^{\mathrlap{p}} \\ Y & = & Y \end{array} .$$

By rearranging this diagram a little, it is equivalent to

$$\begin{array}{ccccc} & X & = & X \\ & {}^{\mathllap{f}}\downarrow && {}^{\mathllap{i}}\downarrow \\ \mathrm{id}_Y : & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y \end{array} .$$

Completing this to the right, this yields a diagram exhibiting the required retract according to remark 6.10:

$$
\begin{array}{ccccccc}
\mathrm{id}_X : & X & = & X & = & X \\
& {}^{f}\downarrow && {}^{i}\downarrow && {}^{f}\downarrow \\
\mathrm{id}_Y : & Y & \underset{g}{\longrightarrow} & A & \underset{p}{\longrightarrow} & Y
\end{array}
.
$$

▮

Small object argument

Given a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms in some category $\mathcal{C}$, a natural question is how to factor any given morphism $f : X \longrightarrow Y$ through a relative C-cell complex, def. , followed by a C-injective morphism, def.

$$
f \;:\; X \overset{\in C\,\mathrm{cell}}{\longrightarrow} \hat{X} \overset{\in C\,\mathrm{inj}}{\longrightarrow} Y \,.
$$

A first approximation to such a factorization turns out to be given simply by forming $\hat{X} = X_1$ by attaching ***all*** possible C-cells to X. Namely let

$$
(C / f) \coloneqq \left\{ \begin{array}{ccc} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{c \in C}\downarrow && \downarrow^{f} \\ \mathrm{cod}(c) & \longrightarrow & Y \end{array} \right\}
$$

be the set of ***all*** ways to find a C-cell attachment in f, and consider the pushout $\hat{X}$ of the coproduct of morphisms in C over all these:

$$
\begin{array}{ccc}
\coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\
{}^{\coprod_{c \in (C/f)} c}\downarrow & (\mathrm{po}) & \downarrow \\
\coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & X_1
\end{array}
.
$$

This gets already close to producing the intended factorization:

First of all the resulting map $X \to X_1$ is a C-relative cell complex, by construction.

Second, by the fact that the coproduct is over all commuting squres to f, the morphism f itself makes a commuting diagram

$$
\begin{array}{ccc}
\coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\
{}^{\coprod_{c \in (C/f)} c}\downarrow && \downarrow^{f} \\
\coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y
\end{array}
$$

and hence the universal property of the colimit means that f is indeed factored through that

C-cell complex X_1; we may suggestively arrange that factorizing diagram like so:

$$\begin{array}{ccc} \coprod_{c \in (C/f)} \mathrm{dom}(c) & \longrightarrow & X \\ {}^{\mathllap{\mathrm{id}}}\downarrow && \downarrow \\ \coprod_{c \in (C/f)} \mathrm{dom}(c) && X_1 \\ {}^{\mathllap{\coprod_{c \in (C/f)} c}}\downarrow & \nearrow & \downarrow \\ \coprod_{c \in (C/f)} \mathrm{cod}(c) & \longrightarrow & Y \end{array}.$$

This shows that, finally, the colimiting co-cone map – the one that now appears diagonally – ***almost*** exhibits the desired right lifting of $X_1 \to Y$ against the $c \in C$. The failure of that to hold on the nose is only the fact that a horizontal map in the middle of the above diagram is missing: the diagonal map obtained above lifts not all commuting diagrams of $c \in C$ into f, but only those where the top morphism $\mathrm{dom}(c) \to X_1$ factors through $X \to X_1$.

The idea of the small object argument now is to fix this only remaining problem by iterating the construction: next factor $X_1 \to Y$ in the same way into

$$X_1 \longrightarrow X_2 \longrightarrow Y$$

and so forth. Since relative C-cell complexes are closed under composition, at stage n the resulting $X \longrightarrow X_n$ is still a C-cell complex, getting bigger and bigger. But accordingly, the failure of the accompanying $X_n \longrightarrow Y$ to be a C-injective morphism becomes smaller and smaller, for it now lifts against all diagrams where $\mathrm{dom}(c) \longrightarrow X_n$ factors through $X_{n-1} \longrightarrow X_n$, which intuitively is less and less of a condition as the X_{n-1} grow larger and larger.

The concept of *small object* is just what makes this intuition precise and finishes the small object argument. For the present purpose we just need the following simple version:

Definition 6.14. For $\mathcal{C}$ a category and $C \subset \mathrm{Mor}(\mathcal{C})$ a sub-set of its morphisms, say that these have *small domains* if there is an ordinal α (def.) such that for every $c \in C$ and for every C-relative cell complex given by a transfinite composition (def.)

$$f \;:\; X \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \longrightarrow \hat{X}$$

every morphism $\mathrm{dom}(c) \longrightarrow \hat{X}$ factors through a stage $X_\beta \to \hat{X}$ of order $\beta < \alpha$:

$$\begin{array}{ccc} && X_\beta \\ & \nearrow & \downarrow \\ \mathrm{dom}(c) & \longrightarrow & \hat{X} \end{array}.$$

The above discussion proves the following:

***Proposition 6.15**. (small object argument)*

Let $\mathcal{C}$ be a locally small category with all small colimits. If a set $C \subset \mathrm{Mor}(\mathcal{C})$ of morphisms has all small domains in the sense of def. 6.14, then every morphism $f : X \longrightarrow$ in $\mathcal{C}$ factors through a C-relative cell complex, def. , followed by a C-injective morphism, def.

$$f \;:\; X \xrightarrow{\in C\,\mathrm{cell}} \hat{X} \xrightarrow{\in C\,\mathrm{inj}} Y \;.$$

(Quillen 67, II.3 lemma)

Homotopy

We discuss how the concept of homotopy is abstractly realized in model categories, def. 6.1.

Definition 6.16. Let $\mathcal{C}$ be a model category, def. 6.1, and $X \in \mathcal{C}$ an object.

- A ***path space object*** $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\Delta_X \;:\; X \to X \times X$ as

$$\Delta_X \;:\; X \underset{\in W}{\overset{i}{\longrightarrow}} \mathrm{Path}(X) \underset{\in \mathrm{Fib}}{\overset{(p_0, p_1)}{\longrightarrow}} X \times X \;.$$

where $X \to \mathrm{Path}(X)$ is a weak equivalence and $\mathrm{Path}(X) \to X \times X$ is a fibration.

- A ***cylinder object*** $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal (or “fold map”) $\nabla_X \;:\; X \sqcup X \to X$ as

$$\nabla_X \;:\; X \sqcup X \underset{\in \mathrm{Cof}}{\overset{(i_0, i_1)}{\longrightarrow}} \mathrm{Cyl}(X) \underset{\in W}{\overset{p}{\longrightarrow}} X \;.$$

where $\mathrm{Cyl}(X) \to X$ is a weak equivalence. and $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration.

Remark 6.17. For every object $X \in \mathcal{C}$ in a model category, a cylinder object and a path space object according to def. 6.16 exist: the factorization axioms guarantee that there exists

1. a factorization of the codiagonal as

$$\nabla_X \;:\; X \sqcup X \xrightarrow{\in \mathrm{Cof}} \mathrm{Cyl}(X) \xrightarrow{\in W \cap \mathrm{Fib}} X$$

2. a factorization of the diagonal as

$$\Delta_X \;:\; X \xrightarrow{\in W \cap \mathrm{Cof}} \mathrm{Path}(X) \xrightarrow{\in \mathrm{Fib}} X \times X \;.$$

The cylinder and path space objects obtained this way are actually better than required by def. 6.16: in addition to $\mathrm{Cyl}(X) \to X$ being just a weak equivalence, for these this is actually an acyclic fibration, and dually in addition to $X \to \mathrm{Path}(X)$ being a weak equivalence, for these it is actually an acyclic cofibrations.

Some authors call cylinder/path-space objects with this extra property "very good" cylinder/path-space objects, respectively.

One may also consider dropping a condition in def. 6.16: what mainly matters is the weak equivalence, hence some authors take cylinder/path-space objects to be defined as in def. 6.16 but without the condition that $X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration and without the condition that $\mathrm{Path}(X) \to X$ is a fibration. Such authors would then refer to the concept in def. 6.16 as "good" cylinder/path-space objects.

The terminology in def. 6.16 follows the original (Quillen 67, I.1 def. 4). With the induced concept of left/right homotopy below in def. 6.20, this admits a quick derivation of the key facts in the following, as we spell out below.

Lemma 6.18. *Let $\mathcal{C}$ be a model category. If $X \in \mathcal{C}$ is cofibrant, then for every cylinder object* $\mathrm{Cyl}(X)$ *of X, def. 6.16, not only is $(i_0, i_1) : X \sqcup X \to X$ a cofibration, but each*

$$i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$$

is an acyclic cofibration separately.

Dually, if $X \in \mathcal{C}$ is fibrant, then for every path space object $\mathrm{Path}(X)$ *of X, def. 6.16, not only is* $(p_0, p_1) : \mathrm{Path}(X) \to X \times X$ *a cofibration, but each*

$$p_0, p_1 : \mathrm{Path}(X) \longrightarrow X$$

is an acyclic fibration separately.

Proof. We discuss the case of the path space object. The other case is formally dual.

First, that the component maps are weak equivalences follows generally: by definition they have a right inverse $\mathrm{Path}(X) \to X$ and so this follows by two-out-of-three (def. 1.75).

But if X is fibrant, then also the two projection maps out of the product $X \times X \to X$ are fibrations, because they are both pullbacks of the fibration $X \to *$

$$\begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & (\mathrm{pb}) & \downarrow \\ X & \longrightarrow & * \end{array} .$$

hence $p_i : \mathrm{Path}(X) \to X \times X \to X$ is the composite of two fibrations, and hence itself a fibration, by prop. 6.8. ▮

Path space objects are very non-unique as objects up to isomorphism:

Example 6.19. If $X \in \mathcal{C}$ is a fibrant object in a model category, def. 6.1, and for $\mathrm{Path}_1(X)$ and $\mathrm{Path}_2(X)$ two path space objects for X, def. 6.16, then the fiber product $\mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is another path space object for X: the pullback square

$$\begin{array}{ccc} X & \overset{\Delta_X}{\longrightarrow} & X \times X \\ \downarrow && \downarrow \\ \mathrm{Path}_1(X) \underset{X}{\times} \mathrm{Path}_2(X) & \longrightarrow & \mathrm{Path}_1(X) \times \mathrm{Path}_2(X) \\ {}^{\in \mathrm{Fib}}\downarrow & (pb) & \downarrow^{\in \mathrm{Fib}} \\ X \times X \times X & \overset{(\mathrm{id}, \Delta_X, \mathrm{id})}{\longrightarrow} & X \times X \times X \times X \\ \downarrow^{(\mathrm{pr}_1, \mathrm{pr}_3)}_{\in \mathrm{Fib}} && \downarrow^{(p_1, p_4)} \\ X \times X & = & X \times X \end{array}$$

gives that the induced projection is again a fibration. Moreover, using lemma 6.18 and two-out-of-three (def. 1.75) gives that $X \to \mathrm{Path}_1(X) \times_X \mathrm{Path}_2(X)$ is a weak equivalence.

For the case of the canonical topological path space objects of def , with $\mathrm{Path}_1(X) = \mathrm{Path}_2(X) = X^I = X^{[0,1]}$ then this new path space object is $X^{I \vee I} = X^{[0,2]}$, the mapping space out of the standard interval of length 2 instead of length 1.

Definition 6.20. ***(abstract left homotopy and abstract right homotopy***

Let $f, g : X \longrightarrow Y$ be two parallel morphisms in a model category.

- A ***left homotopy*** $\eta : f \Rightarrow_L g$ is a morphism $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ from a cylinder object of X, def. 6.16, such that it makes this diagram commute:

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Cyl}(X) & \longleftarrow & X \\ & {}_{f}\searrow & \downarrow^{\eta} & \swarrow_{g} & \\ && Y && \end{array} .$$

- A ***right homotopy*** $\eta : f \Rightarrow_R g$ is a morphism $\eta : X \to \mathrm{Path}(Y)$ to some path space object of X, def. 6.16, such that this diagram commutes:

$$
\begin{array}{ccccc}
 & & X & & \\
 & {}^{f}\swarrow & \downarrow^{\eta} & \searrow^{g} & \\
Y & \longleftarrow & \mathrm{Path}(Y) & \longrightarrow & Y
\end{array}
\,.
$$

Lemma 6.21. *Let $f, g : X \to Y$ be two parallel morphisms in a model category.*

1. *Let X be cofibrant. If there is a left homotopy $f \Rightarrow_L g$ then there is also a right homotopy $f \Rightarrow_R g$ (def. 6.20) with respect to any chosen path space object.*
2. *Let X be fibrant. If there is a right homotopy $f \Rightarrow_R g$ then there is also a left homotopy $f \Rightarrow_L g$ with respect to any chosen cylinder object.*

In particular if X is cofibrant and Y is fibrant, then by going back and forth it follows that every left homotopy is exhibited by every cylinder object, and every right homotopy is exhibited by every path space object.

Proof. We discuss the first case, the second is formally dual. Let $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ be the given left homotopy. Lemma 6.18 implies that we have a lift h in the following commuting diagram

$$
\begin{array}{ccc}
X & \overset{i \circ f}{\longrightarrow} & \mathrm{Path}(Y) \\
{}^{i_0}_{\in W \cap \mathrm{Cof}}\downarrow & {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\
\mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y
\end{array}
$$

where on the right we have the chosen path space object. Now the composite $\tilde{\eta} \coloneqq h \circ i_1$ is a right homotopy as required:

$$
\begin{array}{ccccc}
 & & & & \mathrm{Path}(Y) \\
 & & & {}^{h}\nearrow & \downarrow^{p_0, p_1}_{\in \mathrm{Fib}} \\
X & \overset{i_1}{\longrightarrow} & \mathrm{Cyl}(X) & \underset{(f \circ p, \eta)}{\longrightarrow} & Y \times Y
\end{array}
\,.
$$

■

Proposition 6.22. *For X a cofibrant object in a model category and Y a fibrant object, then the relations of left homotopy $f \Rightarrow_L g$ and of right homotopy $f \Rightarrow_R g$ (def. 6.20) on the hom set $\mathrm{Hom}(X, Y)$ coincide and are both equivalence relations.*

Proof. That both relations coincide under the (co-)fibrancy assumption follows directly from lemma 6.21.

The symmetry and reflexivity of the relation is obvious.

That right homotopy (hence also left homotopy) with domain X is a transitive relation follows from using example 6.19 to compose path space objects. ▮

The homotopy category

We discuss the construction that takes a model category, def. 6.1, and then universally forces all its weak equivalences into actual isomorphisms.

***Definition 6.23*. (*homotopy category of a model category*)**

Let $\mathcal{C}$ be a model category, def. 6.1. Write $\mathrm{Ho}(\mathcal{C})$ for the category whose

- objects are those objects of $\mathcal{C}$ which are both fibrant and cofibrant;
- morphisms are the homotopy classes of morphisms of $\mathcal{C}$, hence the equivalence classes of morphism under the equivalence relation of prop. 6.22;

and whose composition operation is given on representatives by composition in $\mathcal{C}$.

This is, up to equivalence of categories, the ***homotopy category of the model category*** $\mathcal{C}$.

Proposition 6.24. *Def. 6.23 is well defined, in that composition of morphisms between fibrant-cofibrant objects in $\mathcal{C}$ indeed passes to homotopy classes.*

Proof. Fix any morphism $X \overset{f}{\to} Y$ between fibrant-cofibrant objects. Then for precomposition

$$(-) \circ [f] \;:\; \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(Y, Z) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}(X,Z))}$$

to be well defined, we need that with $(g \sim h) \;:\; Y \to Z$ also $(f g \sim f h) \;:\; X \to Z$. But by prop 6.22 we may take the homotopy $\sim$ to be exhibited by a right homotopy $\eta : Y \to \mathrm{Path}(Z)$, for which case the statement is evident from this diagram:

$$\begin{array}{ccccc} && && Z \\ && {}^{g}\nearrow && \uparrow^{p_1} \\ X & \overset{f}{\longrightarrow} & Y & \overset{\eta}{\longrightarrow} & \mathrm{Path}(Z) \\ && {}_{h}\searrow && \downarrow_{p_0} \\ && && Z \end{array} \,.$$

For postcomposition we may choose to exhibit homotopy by left homotopy and argue dually. ▮

We now spell out that def. 6.23 indeed satisfies the universal property that defines the localization of a category with weak equivalences at its weak equivalences.

***Lemma 6.25**. (**Whitehead theorem** in **model categories**)*

Let $\mathcal{C}$ be a model category. A weak equivalence between two objects which are both fibrant and cofibrant is a homotopy equivalence (92).

***Proof**.* By the factorization axioms in the model category $\mathcal{C}$ and by two-out-of-three (def. 1.75), every weak equivalence $f : X \longrightarrow Y$ factors through an object Z as an acyclic cofibration followed by an acyclic fibration. In particular it follows that with X and Y both fibrant and cofibrant, so is Z, and hence it is sufficient to prove that acyclic (co-)fibrations between such objects are homotopy equivalences.

So let $f : X \longrightarrow Y$ be an acyclic fibration between fibrant-cofibrant objects, the case of acyclic cofibrations is formally dual. Then in fact it has a genuine right inverse given by a lift f^{-1} in the diagram

$$\begin{array}{ccc} \emptyset & \to & X \\ {}^{\mathllap{\in \mathrm{cof}}}\downarrow & {}^{f^{-1}}\nearrow & \downarrow^{\mathrlap{f}}_{\mathrlap{\in \mathrm{Fib} \cap W}} \\ X & = & X \end{array}$$

To see that f^{-1} is also a left inverse up to left homotopy, let $\mathrm{Cyl}(X)$ be any cylinder object on X (def. 6.16), hence a factorization of the codiagonal on X as a cofibration followed by a an acyclic fibration

$$X \sqcup X \overset{\iota_X}{\longrightarrow} \mathrm{Cyl}(X) \overset{p}{\longrightarrow} X$$

and consider the commuting square

$$\begin{array}{ccc} X \sqcup X & \overset{(f^{-1}\circ f, \mathrm{id})}{\longrightarrow} & X \\ {}^{\mathllap{\in \mathrm{Cof}}}{}^{\iota_X}\downarrow & & \downarrow^{\mathrlap{f}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ \mathrm{Cyl}(X) & \underset{f \circ p}{\longrightarrow} & Y \end{array}$$

which commutes due to f^{-1} being a genuine right inverse of f. By construction, this commuting square now admits a lift η, and that constitutes a left homotopy $\eta : f^{-1} \circ f \Rightarrow_L \mathrm{id}$. ▮

***Definition 6.26**. (**fibrant resolution** and **cofibrant resolution**)*

Given a model category $\mathcal{C}$, consider a *choice* for each object $X \in \mathcal{C}$ of

1. a factorization

$$\emptyset \underset{\in \mathrm{Cof}}{\overset{i_X}{\longrightarrow}} QX \underset{\in W \cap \mathrm{Fib}}{\overset{p_X}{\longrightarrow}} X$$

of the initial morphism (Def. 1.5), such that when X is already cofibrant then $p_X = \mathrm{id}_X$;

2. a factorization

$$X \underset{\in W \cap \mathrm{Cof}}{\overset{j_X}{\longrightarrow}} PX \underset{\in \mathrm{Fib}}{\overset{q_X}{\longrightarrow}} *$$

of the terminal morphism (Def. 1.5), such that when X is already fibrant then $j_X = \mathrm{id}_X$.

Write then

$$\gamma_{P,Q} \;:\; \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for the functor to the homotopy category, def. 6.23, which sends an object X to the object PQX and sends a morphism $f : X \longrightarrow Y$ to the homotopy class of the result of first lifting in

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ {}^{\mathllap{i_X}}\downarrow & {}^{Qf}\nearrow & \downarrow^{\mathrlap{p_Y}} \\ QX & \underset{f \circ p_X}{\longrightarrow} & Y \end{array}$$

and then lifting (here: extending) in

$$\begin{array}{ccc} QX & \overset{j_{QY} \circ Qf}{\longrightarrow} & PQY \\ {}^{\mathllap{j_{QX}}}\downarrow & {}^{PQf}\nearrow & \downarrow^{\mathrlap{q_{QY}}} \\ PQX & \longrightarrow & * \end{array}$$

Lemma 6.27. *The construction in def. 6.26 is indeed well defined.*

Proof. First of all, the object PQX is indeed both fibrant and cofibrant (as well as related by a zig-zag of weak equivalences to X):

$$\begin{array}{ccccc} \emptyset \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow & \searrow^{\mathrlap{\in \mathrm{Cof}}} \\ QX & \underset{\in W \cap \mathrm{Cof}}{\longrightarrow} & PQX & \underset{\in \mathrm{Fib}}{\longrightarrow} & * \\ {}^{\mathllap{\in W}}\downarrow \\ X \end{array}.$$

Now to see that the image on morphisms is well defined. First observe that any two choices $(Qf)_i$ of the first lift in the definition are left homotopic to each other, exhibited by lifting in

$$\begin{array}{ccc} QX \sqcup QX & \overset{((Qf)_1,(Qf)_2)}{\longrightarrow} & QY \\ {}^{\in Cof}\downarrow && \downarrow^{p_Y}_{\in W \cap Fib} \\ \mathrm{Cyl}(QX) & \underset{f \circ p_X \circ \sigma_{QX}}{\longrightarrow} & Y \end{array}$$

Hence also the composites $j_{QY} \circ (Q_f)_i$ are left homotopic to each other, and since their domain is cofibrant, then by lemma 6.21 they are also right homotopic by a right homotopy κ. This implies finally, by lifting in

$$\begin{array}{ccc} QX & \overset{\kappa}{\longrightarrow} & \mathrm{Path}(PQY) \\ {}^{\in W \cap Cof}\downarrow && \downarrow^{\in Fib} \\ PQX & \underset{(R(Qf)_1,P(Qf)_2)}{\longrightarrow} & PQY \times PQY \end{array}$$

that also $P(Qf)_1$ and $P(Qf)_2$ are right homotopic, hence that indeed PQf represents a well-defined homotopy class.

Finally to see that the assignment is indeed functorial, observe that the commutativity of the lifting diagrams for Qf and PQf imply that also the following diagram commutes

$$\begin{array}{ccccc} X & \overset{p_X}{\longleftarrow} & QX & \overset{j_{QX}}{\longrightarrow} & PQX \\ {}^{f}\downarrow && \downarrow^{Qf} && \downarrow^{PQf} \\ Y & \underset{p_y}{\longleftarrow} & QY & \underset{j_{QY}}{\longrightarrow} & PQY \end{array}.$$

Now from the pasting composite

$$\begin{array}{ccccc} X & \overset{p_X}{\longleftarrow} & QX & \overset{j_{QX}}{\longrightarrow} & PQX \\ {}^{f}\downarrow && \downarrow^{Qf} && \downarrow^{PQf} \\ Y & \underset{p_Y}{\longleftarrow} & QY & \underset{j_{QY}}{\longrightarrow} & PQY \\ {}^{g}\downarrow && \downarrow^{Qg} && \downarrow^{PQg} \\ Z & \underset{p_Z}{\longleftarrow} & QZ & \underset{j_{QZ}}{\longrightarrow} & PQZ \end{array}$$

one sees that $(PQg) \circ (PQf)$ is a lift of $g \circ f$ and hence the same argument as above gives that it is homotopic to the chosen $PQ(g \circ f)$. ▮

For the following, recall the concept of natural isomorphism between functors: for $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ two functors, then a *natural transformation* $\eta : F \Rightarrow G$ is for each object $c \in \mathrm{Obj}(\mathcal{C})$ a morphism $\eta_c : F(c) \longrightarrow G(c)$ in $\mathcal{D}$, such that for each morphism $f : c_1 \to c_2$ in $\mathcal{C}$ the following is a commuting square:

$$\begin{array}{ccc} F(c_1) & \overset{\eta_{c_1}}{\longrightarrow} & G(c_1) \\ {}^{\mathllap{F(f)}}\downarrow && \downarrow^{\mathrlap{G(f)}} \\ F(c_2) & \underset{\eta_{c_2}}{\longrightarrow} & G(c_2) \end{array}.$$

Such η is called a *natural isomorphism* if its η_c are isomorphisms for all objects c.

***Definition 6.28*. (localization of a category category with weak equivalences)**

For $\mathcal{C}$ a category with weak equivalences, its ***localization* at the weak equivalences** is, if it exists,

1. a category denoted $\mathcal{C}[W^{-1}]$
2. a functor

$$\gamma \;:\; \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

such that

1. γ sends weak equivalences to isomorphisms;
2. γ is universal with this property, in that:
   for $F : \mathcal{C} \longrightarrow D$ any functor out of $\mathcal{C}$ into any category D, such that F takes weak equivalences to isomorphisms, it factors through γ up to a natural isomorphism ρ

$$\begin{array}{ccccc} \mathcal{C} && \overset{F}{\longrightarrow} && D \\ & {}_{\mathllap{\gamma}}\searrow & \Downarrow^{\rho} & \nearrow_{\mathrlap{\tilde F}} \\ && \mathrm{Ho}(\mathcal{C}) \end{array}$$

   and this factorization is unique up to unique isomorphism, in that for $(\tilde F_1, \rho_1)$ and $(\tilde F_2, \rho_2)$ two such factorizations, then there is a unique natural isomorphism $\kappa : \tilde F_1 \Rightarrow \tilde F_2$ making the evident diagram of natural isomorphisms commute.

***Theorem 6.29*. (convenient localization of model categories)**

For $\mathcal{C}$ a model category, the functor $\gamma_{P,Q}$ in def. 6.26 (for any choice of P and Q) exhibits $\mathrm{Ho}(\mathcal{C})$ as indeed being the localization of the underlying category with weak equivalences at *its weak equivalences, in the sense of def. 6.28:*

$$
\begin{array}{ccc}
\mathcal{C} & = & \mathcal{C} \\
{}^{\mathllap{\gamma_{P,Q}}}\downarrow && \downarrow^{\mathrlap{\gamma}} \\
\mathrm{Ho}(\mathcal{C}) & \simeq & \mathcal{C}[W^{-1}]
\end{array}
\,.
$$

(Quillen 67, I.1 theorem 1)

Proof. First, to see that that $\gamma_{P,Q}$ indeed takes weak equivalences to isomorphisms: By two-out-of-three (def. 1.75) applied to the commuting diagrams shown in the proof of lemma 6.27, the morphism $P Q f$ is a weak equivalence if f is:

$$
\begin{array}{ccccc}
X & \underoverset{\simeq}{p_X}{\longleftarrow} & Q X & \underoverset{\simeq}{j_{Q X}}{\longrightarrow} & P Q X \\
{}^{\mathllap{f}}\downarrow && \downarrow^{\mathrlap{Q f}} && \downarrow^{\mathrlap{P Q f}} \\
Y & \underoverset{p_y}{\simeq}{\longleftarrow} & Q Y & \underoverset{j_{Q Y}}{\simeq}{\longrightarrow} & P Q Y
\end{array}
$$

With this the "Whitehead theorem for model categories", lemma 6.25, implies that $P Q f$ represents an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

Now let $F \colon \mathcal{C} \longrightarrow D$ be any functor that sends weak equivalences to isomorphisms. We need to show that it factors as

$$
\begin{array}{ccccc}
\mathcal{C} && \overset{F}{\longrightarrow} && D \\
& {}_{\mathllap{\gamma}}\searrow & \Downarrow^{\rho} & \nearrow_{\mathrlap{\tilde F}} \\
&& \mathrm{Ho}(\mathcal{C})
\end{array}
$$

uniquely up to unique natural isomorphism. Now by construction of P and Q in def. 6.26, $\gamma_{P,Q}$ is the identity on the full subcategory of fibrant-cofibrant objects. It follows that if $\tilde F$ exists at all, it must satisfy for all $X \overset{f}{\to} Y$ with X and Y both fibrant and cofibrant that

$$
\tilde F([f]) \simeq F(f)\,,
$$

(hence in particular $\tilde F(\gamma_{P,Q}(f)) = F(P Q f)$).

But by def. 6.23 that already fixes $\tilde F$ on all of $\mathrm{Ho}(\mathcal{C})$, up to unique natural isomorphism. Hence it only remains to check that with this definition of $\tilde F$ there exists any natural isomorphism ρ filling the diagram above.

To that end, apply F to the above commuting diagram to obtain

$$
\begin{array}{ccccc}
F(X) & \underset{\text{iso}}{\overset{F(p_X)}{\longleftarrow}} & F(QX) & \underset{\text{iso}}{\overset{F(j_{QX})}{\longrightarrow}} & F(PQX) \\
{}^{F(f)}\downarrow && \downarrow^{F(Qf)} && \downarrow^{F(PQf)} \\
F(Y) & \underset{F(p_y)}{\overset{\text{iso}}{\longleftarrow}} & F(QY) & \underset{F(j_{QY})}{\overset{\text{iso}}{\longrightarrow}} & F(PQY)
\end{array}
$$

Here now all horizontal morphisms are isomorphisms, by assumption on F. It follows that defining $\rho_X \coloneqq F(j_{QX}) \circ F(p_X)^{-1}$ makes the required natural isomorphism:

$$
\begin{array}{cccccccc}
\rho_X : & F(X) & \underset{\text{iso}}{\overset{F(p_X)^{-1}}{\longrightarrow}} & F(QX) & \underset{\text{iso}}{\overset{F(j_{QX})}{\longrightarrow}} & F(PQX) & = & \tilde F(\gamma_{P,Q}(X)) \\
& {}^{F(f)}\downarrow &&&& \downarrow^{F(PQf)} && \downarrow^{\tilde F(\gamma_{P,Q}(f))} \\
\rho_Y : & F(Y) & \underset{F(p_y)^{-1}}{\overset{\text{iso}}{\longrightarrow}} & F(QY) & \underset{F(j_{QY})}{\overset{\text{iso}}{\longrightarrow}} & F(PQY) & = & \tilde F(\gamma_{P,Q}(X))
\end{array}
.
$$

■

Remark 6.30. Due to theorem 6.29 we may suppress the choices of cofibrant Q and fibrant replacement P in def. 6.26 and just speak of the localization functor

$$\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$$

up to natural isomorphism.

In general, the localization $\mathcal{C}[W^{-1}]$ of a category with weak equivalences $(\mathcal{C}, W)$ (def. 6.28) may invert *more* morphisms than just those in W. However, if the category admits the structure of a model category $(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib})$, then its localization precisely only inverts the weak equivalences:

Proposition 6.31. ***(localization of model categories inverts precisely the weak equivalences)***

Let $\mathcal{C}$ be a model category (def. 6.1) and let $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ be its localization functor (def. 6.26, theorem 6.29). Then a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

(e.g. Goerss-Jardine 96, II, prop 1.14)

While the construction of the homotopy category in def. 6.23 combines the restriction to good (fibrant/cofibrant) objects with the passage to homotopy classes of morphisms, it is often useful to consider intermediate stages:

Definition 6.32. Given a model category $\mathcal{C}$, write

$$\begin{array}{ccccc} & & \mathcal{C}_{fc} & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_c & & & & \mathcal{C}_f \\ & \searrow & & \swarrow & \\ & & \mathcal{C} & & \end{array}$$

for the system of full subcategory inclusions of:

1. the category of fibrant objects $\mathcal{C}_f$
2. the category of cofibrant objects $\mathcal{C}_c$,
3. the category of fibrant-cofibrant objects $\mathcal{C}_{\mathrm{fc}}$,

all regarded a categories with weak equivalences (def. 1.75), via the weak equivalences inherited from $\mathcal{C}$, which we write $(\mathcal{C}_f, W_f)$, $(\mathcal{C}_c, W_c)$ and $(\mathcal{C}_{fc}, W_{fc})$.

***Remark 6.33*. (categories of fibrant objects and cofibration categories)**

Of course the subcategories in def. 6.32 inherit more structure than just that of categories with weak equivalences from $\mathcal{C}$. $\mathcal{C}_f$ and $\mathcal{C}_c$ each inherit "half" of the factorization axioms. One says that $\mathcal{C}_f$ has the structure of a "fibration category" called a "Brown-category of fibrant objects", while $\mathcal{C}_c$ has the structure of a "cofibration category".

We discuss properties of these categories of (co-)fibrant objects below in *Homotopy fiber sequences*.

The proof of theorem 6.29 immediately implies the following:

Corollary 6.34. *For $\mathcal{C}$ a model category, the restriction of the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ from def. 6.26 (using remark 6.30) to any of the sub-categories with weak equivalences of def. 6.32*

$$\begin{array}{ccccc} & & \mathcal{C}_{fc} & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_c & & & & \mathcal{C}_f \\ & \searrow & & \swarrow & \\ & & \mathcal{C} & & \\ & & \downarrow^{\gamma} & & \\ & & \mathrm{Ho}(\mathcal{C}) & & \end{array}$$

exhibits $\mathrm{Ho}(\mathcal{C})$ equivalently as the localization also of these subcategories with weak *equivalences, at their weak equivalences. In particular there are equivalences of categories*

$$\mathrm{Ho}(\mathcal{C}) \simeq \mathcal{C}[W^{-1}] \simeq \mathcal{C}_f[W_f^{-1}] \simeq \mathcal{C}_c[W_c^{-1}] \simeq \mathcal{C}_{fc}[W_{fc}^{-1}] \,.$$

The following says that for computing the hom-sets in the homotopy category, even a mixed variant of the above will do; it is sufficient that the domain is cofibrant and the codomain is fibrant:

***Lemma 6.35*. (hom-sets of homotopy category via mapping cofibrant resolutions into fibrant resolutions)**

For $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, and for P, Q fibrant/cofibrant replacement functors as in def. 6.26, then the morphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(P X, Q Y) = \mathrm{Hom}_{\mathcal{C}}(P X, Q Y)/_\sim \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(j_X, p_Y)} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

(on homotopy classes of morphisms, well defined by prop. 6.22) is a natural bijection.

(Quillen 67, I.1 lemma 7)

***Proof*.** We may factor the morphism in question as the composite

$$\mathrm{Hom}_{\mathcal{C}}(P X, Q Y)/_\sim \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{P X}, p_Y)/_\sim} \mathrm{Hom}_{\mathcal{C}}(P X, Y)/_\sim \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim} \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim \,.$$

This shows that it is sufficient to see that for X cofibrant and Y fibrant, then

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(X, Q Y)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism, and dually that

$$\mathrm{Hom}_{\mathcal{C}}(j_X, \mathrm{id}_Y)/_\sim \;:\; \mathrm{Hom}_{\mathcal{C}}(P X, Y)/_\sim \to \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$$

is an isomorphism. We discuss this for the former; the second is formally dual:

First, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is surjective is the lifting property in

$$\begin{array}{ccc} \emptyset & \longrightarrow & Q Y \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ X & \underset{}{\overset{f}{\longrightarrow}} & Y \end{array}$$

which says that any morphism $f : X \to Y$ comes from a morphism $\hat f : X \to Q Y$ under postcomposition with $Q Y \overset{p_Y}{\to} Y$.

Second, that $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, p_Y)$ is injective is the lifting property in

$$\begin{array}{ccc} X \sqcup X & \overset{(f,g)}{\longrightarrow} & QY \\ {}^{\mathllap{\in \mathrm{Cof}}}\downarrow && \downarrow^{\mathrlap{p_Y}}_{\mathrlap{\in W \cap \mathrm{Fib}}} \\ \mathrm{Cyl}(X) & \underset{\eta}{\longrightarrow} & Y \end{array}$$

which says that if two morphisms $f, g : X \to QY$ become homotopic after postcomposition with $p_Y : QX \to Y$, then they were already homotopic before. ▮

We record the following fact which will be used in part 1.1 (here):

Lemma 6.36. *Let $\mathcal{C}$ be a model category (def. 6.1). Then every commuting square in its homotopy category $\mathrm{Ho}(\mathcal{C})$ (def. 6.23) is, up to isomorphism of squares, in the image of the localization functor $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ of a commuting square in $\mathcal{C}$ (i.e.: not just commuting up to homotopy).*

Proof. Let

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{\mathllap{a}}\downarrow && \downarrow^{\mathrlap{b}} \\ A' & \underset{f'}{\longrightarrow} & B' \end{array} \;\;\;\; \in \mathrm{Ho}(\mathcal{C})$$

be a commuting square in the homotopy category. Writing the same symbols for fibrant-cofibrant objects in $\mathcal{C}$ and for morphisms in $\mathcal{C}$ representing these, then this means that in $\mathcal{C}$ there is a left homotopy of the form

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow} & B \\ {}^{\mathllap{i_1}}\downarrow && \downarrow^{\mathrlap{b}} \\ \mathrm{Cyl}(A) & \underset{\eta}{\longrightarrow} & B' \\ {}^{\mathllap{i_0}}\uparrow && \uparrow^{\mathrlap{f'}} \\ A & \underset{a}{\longrightarrow} & A' \end{array} \,.$$

Consider the factorization of the top square here through the mapping cylinder of f

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & B \\
{}^{\mathllap{i_1}}\downarrow & (\text{po}) & \downarrow^{\mathrlap{\in W}} \\
\mathrm{Cyl}(A) & \longrightarrow & \mathrm{Cyl}(f) \\
{}^{\mathllap{i_0}}\uparrow & {}_{\eta}\searrow & \downarrow \\
A & & B' \\
 & {}_{a}\searrow & \uparrow_{\mathrlap{f'}} \\
 & & A'
\end{array}
$$

This exhibits the composite $A \overset{i_0}{\to} \mathrm{Cyl}(A) \to \mathrm{Cyl}(f)$ as an alternative representative of f in $\mathrm{Ho}(\mathcal{C})$, and $\mathrm{Cyl}(f) \to B'$ as an alternative representative for b, and the commuting square

$$
\begin{array}{ccc}
A & \longrightarrow & \mathrm{Cyl}(f) \\
{}^{\mathllap{a}}\downarrow & & \downarrow \\
A' & \underset{f'}{\longrightarrow} & B'
\end{array}
$$

as an alternative representative of the given commuting square in $\mathrm{Ho}(\mathcal{C})$. ▮

Derived functors

***Definition 6.37*. (*homotopical functor*)**

For $\mathcal{C}$ and $\mathcal{D}$ two categories with weak equivalences, def. 1.75, then a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called a ***homotopical functor*** if it sends weak equivalences to weak equivalences.

***Definition 6.38*. (*derived functor*)**

Given a homotopical functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ (def. 6.37) between categories with weak equivalences whose homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ exist (def. 6.28), then its ("total") *derived functor* is the functor $\mathrm{Ho}(F)$ between these homotopy categories which is induced uniquely, up to unique isomorphism, by their universal property (def. 6.28):

$$
\begin{array}{ccc}
\mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\
{}^{\mathllap{\gamma_{\mathcal{C}}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\mathrlap{\gamma_{\mathcal{D}}}} \\
\mathrm{Ho}(\mathcal{C}) & \underset{\exists\, \mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
\,.
$$

***Remark 6.39*.** While many functors of interest between model categories are not homotopical in the sense of def. 6.37, many become homotopical after restriction to the full subcategories $\mathcal{C}_f$ of fibrant objects or $\mathcal{C}_c$ of cofibrant objects, def. 6.32. By corollary 6.34 this is just as good for the purpose of homotopy theory.

Therefore one considers the following generalization of def. 6.38:

***Definition 6.40*. (left and right derived functors)**

Consider a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ out of a model category $\mathcal{C}$ (def. 6.1) into a category with weak equivalences $\mathcal{D}$ (def. 1.75).

1. If the restriction of F to the full subcategory $\mathcal{C}_f$ of fibrant object becomes a homotopical functor (def. 6.37), then the derived functor of that restriction, according to def. 6.38, is called the *right derived functor* of F and denoted by $\mathbb{R}F$:

$$\begin{array}{ccccccc} & \mathcal{C}_f & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ & {}^{\gamma_{\mathcal{C}_f}}\downarrow & & \swarrow_{\simeq} & & \downarrow^{\gamma_{\mathcal{D}}} & , \\ \mathbb{R}F : & \mathcal{C}_f[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

   where we use corollary 6.34.

2. If the restriction of F to the full subcategory $\mathcal{C}_c$ of cofibrant object becomes a homotopical functor (def. 6.37), then the derived functor of that restriction, according to def. 6.38, is called the *left derived functor* of F and denoted by $\mathbb{L}F$:

$$\begin{array}{ccccccc} & \mathcal{C}_c & \hookrightarrow & \mathcal{C} & \overset{F}{\longrightarrow} & \mathcal{D} \\ & {}^{\gamma_{\mathcal{C}_f}}\downarrow & & \swarrow_{\simeq} & & \downarrow^{\gamma_{\mathcal{D}}} & , \\ \mathbb{L}F : & \mathcal{C}_c[W^{-1}] & \simeq & \mathrm{Ho}(\mathcal{C}) & \underset{\mathrm{Ho}(F)}{\longrightarrow} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

   where again we use corollary 6.34.

The key fact that makes def. 6.40 practically relevant is the following:

***Proposition 6.41*. (Ken Brown's lemma)**

Let $\mathcal{C}$ be a model category with full subcategories $\mathcal{C}_f, \mathcal{C}_c$ of fibrant objects and of cofibrant objects respectively (def. 6.32). Let $\mathcal{D}$ be a category with weak equivalences.

1. *A functor out of the category of fibrant objects*

$$F \;:\; \mathcal{C}_f \longrightarrow \mathcal{D}$$

   is a homotopical functor, def. 6.37, already if it sends acyclic fibrations to weak equivalences.

2. *A functor out of the category of cofibrant objects*

$$F \,:\, \mathcal{C}_c \longrightarrow \mathcal{D}$$

> *is a homotopical functor, def. 6.37, already if it sends acyclic cofibrations to weak equivalences.*

The following proof refers to the factorization lemma, whose full statement and proof we postpone to further below (lemma 6.69).

Proof. We discuss the case of a functor on a category of fibrant objects $\mathcal{C}_f$, def. 6.32. The other case is formally dual.

Let $f : X \longrightarrow Y$ be a weak equivalence in $\mathcal{C}_f$. Choose a path space object $\mathrm{Path}(X)$ (def. 6.16) and consider the diagram

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r, "\in W \cap \mathrm{Fib}"'] \arrow[d, "p_1^* f"', "\in W"] \arrow[dr, phantom, "(\mathrm{pb})"] & X \arrow[d, "f", "\in W"'] \\
\mathrm{Path}(Y) \arrow[r, "p_1", "\in W \cap \mathrm{Fib}"'] \arrow[d, "p_0"', "\in W \cap \mathrm{Fib}"] & Y, \\
Y &
\end{tikzcd}$$

where the square is a pullback and $\mathrm{Path}(f)$ on the top left is our notation for the universal cone object. (Below we discuss this in more detail, it is the *mapping cocone* of f, def. 6.61).

Here:

1. p_i are both acyclic fibrations, by lemma 6.18;
2. $\mathrm{Path}(f) \to X$ is an acyclic fibration because it is the pullback of p_1.
3. $p_1^* f$ is a weak equivalence, because the factorization lemma 6.69 states that the composite vertical morphism factors f through a weak equivalence, hence if f is a weak equivalence, then $p_1^* f$ is by two-out-of-three (def. 1.75).

Now apply the functor F to this diagram and use the assumption that it sends acyclic fibrations to weak equivalences to obtain

$$
\begin{array}{ccc}
F(\mathrm{Path}(f)) & \underset{\in W}{\longrightarrow} & F(X) \\
{}^{F(p_1^\ast f)}\downarrow && \downarrow^{F(f)} \\
F(\mathrm{Path}(Y)) & \underset{\in W}{\overset{F(p_1)}{\longrightarrow}} & F(Y) \\
{}^{F(p_0)}_{\in W}\downarrow && \\
Y &&
\end{array}
\,.
$$

But the factorization lemma 6.69, in addition says that the vertical composite $p_0 \circ p_1^\ast f$ is a fibration, hence an acyclic fibration by the above. Therefore also $F(p_0 \circ p_1^\ast f)$ is a weak equivalence. Now the claim that also $F(f)$ is a weak equivalence follows with applying two-out-of-three (def. 1.75) twice. ▮

Corollary 6.42. *Let $\mathcal{C}, \mathcal{D}$ be model categories and consider $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. Then:*

1. *If F preserves cofibrant objects and acyclic cofibrations between these, then its left derived functor (def. 6.40) $\mathbb{L}F$ exists, fitting into a diagram*

$$
\begin{array}{ccc}
\mathcal{C}_c & \overset{F}{\longrightarrow} & \mathcal{D}_c \\
{}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\
\mathrm{Ho}(\mathcal{C}) & \overset{\mathbb{L}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
$$

2. *If F preserves fibrant objects and acyclic fibrants between these, then its right derived functor (def. 6.40) $\mathbb{R}F$ exists, fitting into a diagram*

$$
\begin{array}{ccc}
\mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D}_f \\
{}^{\gamma_{\mathcal{C}}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\
\mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
\,.
$$

***Proposition 6.43*. (construction of left/right derived functors)**

Let $F \;:\; \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two model categories (def. 6.1).

1. If F preserves fibrant objects and weak equivalences between fibrant objects, then the total right derived functor $\mathbb{R}F \coloneqq \mathbb{R}(\gamma_{\mathcal{D}} \circ F)$ (def. 6.40) in

$$
\begin{array}{ccc}
\mathcal{C}_f & \overset{F}{\longrightarrow} & \mathcal{D} \\
{}^{\gamma_{\mathcal{C}_f}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\
\mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{R}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a fibrant replacement $P X$ of X and then forming a cofibrant replacement $Q(F(P X))$ of the result:

$$
\mathbb{R}F(X) \simeq Q(F(P X)) \,.
$$

1. If F preserves cofibrant objects and weak equivalences between cofibrant objects, then the total left derived functor $\mathbb{L}F \coloneqq \mathbb{L}(\gamma_{\mathcal{D}} \circ F)$ (def. 6.40) in

$$
\begin{array}{ccc}
\mathcal{C}_c & \overset{F}{\longrightarrow} & \mathcal{D} \\
{}^{\gamma_{\mathcal{C}_c}}\downarrow & \swArrow_{\simeq} & \downarrow^{\gamma_{\mathcal{D}}} \\
\mathrm{Ho}(\mathcal{C}) & \underset{\mathbb{L}F}{\longrightarrow} & \mathrm{Ho}(\mathcal{D})
\end{array}
$$

is given, up to isomorphism, on any object $X \in \mathcal{C} \overset{\gamma_{\mathcal{C}}}{\longrightarrow} \mathrm{Ho}(\mathcal{C})$ by appying F to a cofibrant replacement $Q X$ of X and then forming a fibrant replacement $P(F(Q X))$ of the result:

$$
\mathbb{L}F(X) \simeq P(F(Q X)) \,.
$$

Proof. We discuss the first case, the second is formally dual. By the proof of theorem 6.29 we have

$$
\begin{aligned}
\mathbb{R}F(X) & \simeq \gamma_{\mathcal{D}}(F(\gamma_{\mathcal{C}})) \\
& \simeq \gamma_{\mathcal{D}} F(Q(P(X)))
\end{aligned} \,.
$$

But since F is a homotopical functor on fibrant objects, the cofibrant replacement morphism $F(Q(P(X))) \to F(P(X))$ is a weak equivalence in $\mathcal{D}$, hence becomes an isomorphism under $\gamma_{\mathcal{D}}$. Therefore

$$
\mathbb{R}F(X) \simeq \gamma_{\mathcal{D}}(F(P(X))) \,.
$$

Now since F is assumed to preserve fibrant objects, $F(P(X))$ is fibrant in $\mathcal{D}$, and hence $\gamma_{\mathcal{D}}$ acts on it (only) by cofibrant replacement. ▮

Quillen adjunctions

In practice it turns out to be useful to arrange for the assumptions in corollary 6.42 to be satisfied by pairs of adjoint functors (Def. 1.32). Recall that this is a pair of functors L and R going back and forth between two categories

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot} \mathcal{D}$$

such that there is a natural bijection between hom-sets with L on the left and those with R on the right (10):

$$\phi_{d,c} \;:\; \mathrm{Hom}_{\mathcal{C}}(L(d), c) \underset{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(d, R(c))$$

for all objects $d \in \mathcal{D}$ and $c \in \mathcal{C}$. This being *natural* (Def. 1.23) means that $\phi : \mathrm{Hom}_{\mathcal{D}}(L(-), -) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, R(-))$ is a natural transformation, hence that for all morphisms $g : d_2 \to d_1$ and $f : c_1 \to c_2$ the following is a commuting square:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(L(d_1), c_1) & \underoverset{\simeq}{\phi_{d_1,c_1}}{\longrightarrow} & \mathrm{Hom}_{\mathcal{D}}(d_1, R(c_1)) \\ {}^{\mathllap{L(f)\circ(-)\circ g}}\downarrow && \downarrow^{\mathrlap{g\circ(-)\circ R(g)}} \\ \mathrm{Hom}_{\mathcal{C}}(L(d_2), c_2) & \underoverset{\phi_{d_2,c_2}}{\simeq}{\longrightarrow} & \mathrm{Hom}_{\mathcal{D}}(d_2, R(c_2)) \end{array} \; /$$

We write $(L \dashv R)$ to indicate such an adjunction and call L the *left adjoint* and R the *right adjoint* of the adjoint pair.

The archetypical example of a pair of adjoint functors is that consisting of forming Cartesian products $Y \times (-)$ and forming mapping spaces $(-)^Y$, as in the category of compactly generated topological spaces of def. .

If $f : L(d) \to c$ is any morphism, then the image $\phi_{d,c}(f) : d \to R(c)$ is called its *adjunct*, and conversely. The fact that adjuncts are in bijection is also expressed by the notation

$$\frac{L(c) \overset{f}{\longrightarrow} d}{c \overset{\tilde f}{\longrightarrow} R(d)} \,.$$

For an object $d \in \mathcal{D}$, the adjunct of the identity on $L d$ is called the *adjunction unit* $\eta_d \;:\; d \longrightarrow R L d$.

For an object $c \in \mathcal{C}$, the adjunct of the identity on $R c$ is called the *adjunction counit* $\epsilon_c \;:\; L R c \longrightarrow c$.

Adjunction units and counits turn out to encode the adjuncts of all other morphisms by the formulas

- $\widetilde{(Ld \overset{f}{\to} c)} = (d \overset{\eta}{\to} RLd \overset{Rf}{\to} Rc)$
- $\widetilde{(d \overset{g}{\to} Rc)} = (Ld \overset{Lg}{\to} LRc \overset{\epsilon}{\to} c)$.

***Definition 6.44*. (Quillen adjunction)**

Let $\mathcal{C}, \mathcal{D}$ be model categories. A pair of adjoint functors (Def. 1.32) between them

$$(L \dashv R) \;:\; \mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{} \mathcal{D}$$

is called a *Quillen adjunction*, to be denoted

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

and L, R are called left/right *Quillen functors*, respectively, if the following equivalent conditions are satisfied:

1. L preserves cofibrations and R preserves fibrations;
2. L preserves acyclic cofibrations and R preserves acyclic fibrations;
3. L preserves cofibrations and acyclic cofibrations;
4. R preserves fibrations and acyclic fibrations.

***Proposition 6.45*.** *The conditions in def. 6.44 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

***Proof*.** First observe that

- (i) *A left adjoint L between model categories preserves acyclic cofibrations precisely if its right adjoint R preserves fibrations.*
- (ii) *A left adjoint L between model categories preserves cofibrations precisely if its right adjoint R preserves acyclic fibrations.*

We discuss statement (i), statement (ii) is formally dual. So let $f : A \to B$ be an acyclic cofibration in $\mathcal{D}$ and $g : X \to Y$ a fibration in $\mathcal{C}$. Then for every commuting diagram as on the left of the following, its $(L \dashv R)$-adjunct is a commuting diagram as on the right here:

$$
\begin{array}{ccc}
A & \longrightarrow & R(X) \\
{}^{f}\downarrow && \downarrow^{R(g)} \\
B & \longrightarrow & R(Y)
\end{array}
\quad , \quad
\begin{array}{ccc}
L(A) & \longrightarrow & X \\
{}^{L(f)}\downarrow && \downarrow^{g} \\
L(B) & \longrightarrow & Y
\end{array}
.
$$

If L preserves acyclic cofibrations, then the diagram on the right has a lift, and so the $(L \dashv R)$-adjunct of that lift is a lift of the left diagram. This shows that $R(g)$ has the right lifting property against all acylic cofibrations and hence is a fibration. Conversely, if R preserves fibrations, the same argument run from right to left gives that L preserves acyclic fibrations.

Now by repeatedly applying (i) and (ii), all four conditions in question are seen to be equivalent. ▮

The following is the analog of adjunction unit and adjunction counit (Def. 1.33):

***Definition 6.46*. (derived adjunction unit)**

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories (Def. 6.1), and let

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{Qu}} \mathcal{D}$$

be a Quillen adjunction (Def. 6.44). Then

1. a *derived adjunction unit* at an object $d \in \mathcal{D}$ is a composition of the form

$$Q(d) \overset{\eta_{Q(d)}}{\longrightarrow} R(L(Q(d))) \overset{R(j_{L(Q(d))})}{\longrightarrow} R(P(L(Q(d))))$$

   where

   1. η is the ordinary adjunction unit (Def. 1.33);
   2. $\emptyset \underoverset{\in Cof_{\mathcal{D}}}{i_{Q(d)}}{\longrightarrow} Q(d) \underoverset{\in W_{\mathcal{D}} \cap Fib_{\mathcal{D}}}{p_{Q(d)}}{\longrightarrow} d$ is a cofibrant resolution in $\mathcal{D}$ (Def. 6.26);
   3. $L(Q(d)) \underoverset{\in W_{\mathcal{C}} \cap Cof_{\mathcal{C}}}{j_{L(Q(d))}}{\longrightarrow} P(L(Q(d))) \underoverset{\in Fib_{\mathcal{C}}}{q_{L(Q(d))}}{\longrightarrow} *$ is a fibrant resolution in $\mathcal{C}$ (Def. 6.26);

2. a *derived adjunction counit* at an object $c \in \mathcal{C}$ is a composition of the form

$$L(Q(R(P(c)))) \overset{p_{R(P(c))}}{\longrightarrow} LR(P(c)) \overset{\epsilon_{P(c)}}{\longrightarrow} P(c)$$

   where

   1. ϵ is the ordinary adjunction counit (Def. 1.33);

2. $c \underset{\in W_{\mathcal{C}} \cap \mathrm{Cof}_{\mathcal{C}}}{\overset{j_c}{\longrightarrow}} Pc \underset{\in \mathrm{Fib}_{\mathcal{C}}}{\overset{q_c}{\longrightarrow}} *$ is a fibrant resolution in $\mathcal{C}$ (Def. 6.26);
3. $\emptyset \underset{\in \mathrm{Cof}_{\mathcal{D}}}{\overset{i_{R(P(c))}}{\longrightarrow}} Q(R(P(c))) \underset{\in W_{\mathcal{D}} \cap \mathrm{Fib}_{\mathcal{D}}}{\overset{p_{R(P(c))}}{\longrightarrow}} R(P(c))$ is a cofibrant resolution in $\mathcal{D}$ (Def. 6.26).

We will see that Quillen adjunctions induce ordinary adjoint pairs of derived functors on homotopy categories (Prop. 6.48). For this we first consider the following technical observation:

***Lemma 6.47**. (right Quillen functors preserve path space objects)*

Let $\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{D}$ be a Quillen adjunction, def. 6.44.

1. *For $X \in \mathcal{C}$ a fibrant object and $\mathrm{Path}(X)$ a path space object (def. 6.16), then $R(\mathrm{Path}(X))$ is a path space object for $R(X)$.*
2. *For $X \in \mathcal{C}$ a cofibrant object and $\mathrm{Cyl}(X)$ a cylinder object (def. 6.16), then $L(\mathrm{Cyl}(X))$ is a cylinder object for $L(X)$.*

Proof. Consider the second case, the first is formally dual.

First Observe that $L(Y \sqcup Y) \simeq LY \sqcup LY$ because L is left adjoint and hence preserves colimits, hence in particular coproducts.

Hence

$$L(\mathrm{X} \sqcup X \overset{\in \mathrm{Cof}}{\longrightarrow} \mathrm{Cyl}(X)) = (L(X) \sqcup L(X) \overset{\in \mathrm{Cof}}{\longrightarrow} L(\mathrm{Cyl}(X)))$$

is a cofibration.

Second, with Y cofibrant then also $Y \sqcup \mathrm{Cyl}(Y)$ is a cofibrantion, since $Y \to Y \sqcup Y$ is a cofibration (lemma 6.18). Therefore by Ken Brown's lemma (prop. 6.41) L preserves the weak equivalence $\mathrm{Cyl}(Y) \overset{\in W}{\longrightarrow} Y$. ▮

***Proposition 6.48**. (derived adjunction)*

For $\mathcal{C} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot_{\mathrm{Qu}}}}}} \mathcal{D}$ a Quillen adjunction, def. 6.44, also the corresponding left and right derived functors (Def. 6.40, via cor. 6.42) form a pair of adjoint functors

$$\mathrm{Ho}(\mathcal{C}) \underset{\mathbb{R}R}{\overset{\mathbb{L}L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ho}(\mathcal{D}) \,.$$

Moreover, the adjunction unit and adjunction counit of this derived adjunction are the images of the derived adjunction unit and derived adjunction counit (Def. 6.46) under the localization functors (Theorem 6.29).

(Quillen 67, I.4 theorem 3)

Proof. For the first statement, by def. 6.40 and lemma 6.35 it is sufficient to see that for $X, Y \in \mathcal{C}$ with X cofibrant and Y fibrant, then there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(LX, Y)\,/_\sim \;\simeq\; \mathrm{Hom}_{\mathcal{C}}(X, RY)\,/_\sim \;. \qquad (93)$$

Since by the adjunction isomorphism for $(L \dashv R)$ such a natural bijection exists before passing to homotopy classes $(-)\,/_\sim$, it is sufficient to see that this respects homotopy classes. To that end, use from lemma 6.47 that with $\mathrm{Cyl}(Y)$ a cylinder object for Y, def. 6.16, then $L(\mathrm{Cyl}(Y))$ is a cylinder object for $L(Y)$. This implies that left homotopies

$$(f \Rightarrow_L g) \;:\; LX \longrightarrow Y$$

given by

$$\eta \;:\; \mathrm{Cyl}(LX) = L\,\mathrm{Cyl}(X) \longrightarrow Y$$

are in bijection to left homotopies

$$(\tilde f \Rightarrow_L \tilde g) \;:\; X \longrightarrow RY$$

given by

$$\tilde \eta \;:\; \mathrm{Cyl}(X) \longrightarrow RX \;.$$

This establishes the adjunction. Now regarding the (co-)units: We show this for the adjunction unit, the case of the adjunction counit is formally dual.

First observe that for $d \in \mathcal{D}_c$, then the defining commuting square for the left derived functor from def. 6.40

$$\begin{array}{ccc} \mathcal{D}_c & \overset{L}{\longrightarrow} & \mathcal{C} \\ {}^{\mathllap{\gamma_P}}\downarrow & \swArrow_{\simeq} & \downarrow^{\mathrlap{\gamma_{P,Q}}} \\ \mathrm{Ho}(\mathcal{D}) & \underset{\mathbb{L}L}{\longrightarrow} & \mathrm{Ho}(\mathcal{C}) \end{array}$$

(using fibrant and fibrant/cofibrant replacement functors $\gamma_P, \gamma_{P,Q}$ from def. 6.26 with their universal property from theorem 6.29, corollary 6.34) gives that

$$(\mathbb{L}L)d \simeq PLPd \simeq PLd \quad \in \mathrm{Ho}(\mathcal{C})\,,$$

where the second isomorphism holds because the left Quillen functor L sends the acyclic cofibration $j_d : d \to Pd$ to a weak equivalence.

The adjunction unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ on $Pd \in \mathrm{Ho}(\mathcal{C})$ is the image of the identity under

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}((\mathbb{L}L)Pd, (\mathbb{L}L)Pd) \overset{\simeq}{\to} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(Pd, (\mathbb{R}R)(\mathbb{L}L)Pd)\;.$$

By the above and the proof of prop. 6.48, that adjunction isomorphism is equivalently that of $(L \dashv R)$ under the isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(PLd, PLd) \xrightarrow{\mathrm{Hom}(j_{Ld}, \mathrm{id})} \mathrm{Hom}_{\mathcal{C}}(Ld, PLd)\,/_{\sim}$$

of lemma 6.35. Hence the derived adjunction unit (Def. 6.46) is the $(L \dashv R)$-adjunct of

$$Ld \overset{j_{Ld}}{\longrightarrow} PLd \overset{\mathrm{id}}{\to} PLd\,,$$

which indeed (by the formula for adjuncts, Prop. 1.38) is the derived adjunction unit

$$X \overset{\eta}{\longrightarrow} RLd \overset{R(j_{Ld})}{\longrightarrow} RPLd\;.$$

▮

This suggests to regard passage to homotopy categories and derived functors as itself being a suitable functor from a category of model categories to the category of categories. Due to the role played by the distinction between left Quillen functors and right Quillen functors, this is usefully formulated as a double functor:

***Definition 6.49*. (*double category* of *model categories*)**

The (very large) *double category* of *model categories* $\mathrm{ModCat}_{\mathrm{dbl}}$ is the double category (Def. 1.54) that has

1. as objects: model categories $\mathcal{C}$ (Def. 6.1);
2. as vertical morphisms: left Quillen functors $\mathcal{C} \overset{L}{\longrightarrow} \mathcal{E}$ (Def. 6.44);
3. as horizontal morphisms: right Quillen functors $\mathcal{C} \overset{R}{\longrightarrow} \mathcal{D}$ (Def. 6.44);
4. as 2-morphisms natural transformations between the composites of underlying functors:

$$L_2 \circ R_1 \overset{\phi}{\Rightarrow} R_2 \circ L_1 \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "R_1"] \arrow[d, "L_1"'] \arrow[dr, phantom, "\phi \swArrow"] & \mathcal{D} \arrow[d, "L_2"] \\ \mathcal{C} \arrow[r, "R_2"'] & \mathcal{D} \end{tikzcd}$$

and composition is given by ordinary composition of functors, horizontally and vertically, and by whiskering-composition of natural transformations.

(Shulman 07, Example 4.6)

There is hence a forgetful double functor (Remark 1.55)

$$F \;:\; \mathrm{ModCat}_{\mathrm{dbl}} \longrightarrow \mathrm{Sq}(\mathrm{Cat})$$

to the double category of squares (Example 1.54) in the 2-category of categories (Example 1.49), which forgets the model category-structure and the Quillen functor-property.

The following records the 2-functoriality of sending Quillen adjunctions to adjoint pairs of derived functors (Prop. 6.48):

***Proposition 6.50**. (homotopy double pseudofunctor on the double category of model categories)*

There is a double pseudofunctor (Remark 1.55)

$$\mathrm{Ho}(-) \;:\; \mathrm{ModCat}_{\mathrm{dbl}} \longrightarrow \mathrm{Sq}(\mathrm{Cat})$$

from the double category of model categories (Def. 6.49) to the double category of squares (Example 1.54) in the 2-category Cat (Example 1.49), which sends

1. *a model category $\mathcal{C}$ to its homotopy category of a model category (Def. 6.23);*
2. *a left Quillen functor (Def. 6.44) to its left derived functor (Def. 6.40);*
3. *a right Quillen functor (Def. 6.44) to its right derived functor (Def. 6.40);*
4. *a natural transformation*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "R_1"] \arrow[d, "L_1"'] \arrow[dr, phantom, "\phi \swArrow"] & \mathcal{D} \arrow[d, "L_2"] \\ \mathcal{E} \arrow[r, "R_2"'] & \mathcal{F} \end{tikzcd}$$

to the "derived natural transformation"

$$
\begin{tikzcd}
\mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathbb{R}R_1"] \arrow[d, "\mathbb{L}L_1"'] \arrow[dr, phantom, "\overset{\mathrm{Ho}(\phi)}{\Swarrow}"] & \mathrm{Ho}(\mathcal{D}) \arrow[d, "\mathbb{L}L_2"] \\
\mathrm{Ho}(\mathcal{E}) \arrow[r, "\mathbb{R}R_2"'] & \mathrm{Ho}(\mathcal{F})
\end{tikzcd}
$$

given by the zig-zag

$$
\mathrm{Ho}(\phi) \,:\, L_2 Q R_1 P \longleftarrow L_2 Q R_1 Q P \longrightarrow L_2 R_1 Q P \overset{\phi}{\longrightarrow} R_2 L_1 Q P \longrightarrow R_2 P L1 Q P \longleftarrow R_2 F \tag{94}
$$

where the unlabeled morphisms are induced by fibrant resolution $c \to P c$ and cofibrant resolution $Q c \to c$, respectively (Def. 6.26).

(Shulman 07, Theorem 7.6)

***Lemma 6.51**. (recognizing derived natural isomorphisms)*

For the derived natural transformation $\mathrm{Ho}(\phi)$ in (94) to be invertible in the homotopy category, it is sufficient that for every object $c \in \mathcal{C}$ which is both fibrant and cofibrant the following composite natural transformation

$$
R_2 Q L_1 c \xrightarrow{R_2 p_{L_1 c}} R_2 L_1 c \overset{\phi}{\longrightarrow} L_2 R_1 c \xrightarrow{L_2 j_{R_1 c}} L_2 P R_1 c
$$

(of ϕ with images of fibrant resolution/cofibrant resolution, Def. 6.26) is invertible in the homotopy category, hence that the composite is a weak equivalence (by Prop. 6.31).

(Shulman 07, Remark 7.2)

***Example 6.52**. (derived functor of left-right Quillen functor)*

Let $\mathcal{C}, \mathcal{D}$ be model categories (Def. 6.1), and let

$$
\mathcal{C} \overset{F}{\longrightarrow} \mathcal{C}
$$

be a functor that is both a left Quillen functor as well as a right Quillen functor (Def. 6.44). This means equivalently that there is a 2-morphism in the double category of model categories (Def. 6.49) of the form

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "F"'] \arrow[dr, phantom, "\mathrm{id}\,\Swarrow"] & \mathcal{D} \arrow[d, "\mathrm{id}"] \\
\mathcal{D} \arrow[r, "\mathrm{id}"'] & \mathcal{D}
\end{tikzcd}
\tag{95}
$$

It follows that the left derived functor $\mathbb{L}F$ and right derived functor $\mathbb{R}F$ of F (Def. 6.40) are naturally isomorphic:

$$\mathrm{Ho}(\mathcal{C}) \xrightarrow{\mathbb{L}F \simeq \mathbb{R}F} \mathrm{Ho}(\mathcal{D}) \;.$$

(Shulman 07, corollary 7.8)

Proof. To see the natural isomorphism $\mathbb{L}F \simeq \mathbb{R}F$: By Prop. 6.50 this is implied once the derived natural transformation $\mathrm{Ho}(\mathrm{id})$ of (95) is a natural isomorphism. By Prop. 6.51 this is the case, in the present situation, if the composition of

$$QFc \overset{p_{Fc}}{\longrightarrow} Fc \overset{j_{Fc}}{\longrightarrow} PFc$$

is a weak equivalence. But this is immediate, since the two factors are weak equivalences, by definition of fibrant/cofibrant resolution (Def. 6.26). ▮

The following is the analog of co-reflective subcategories (Def. 1.60) for model categories:

***Definition 6.53*. (*Quillen reflection*)**

Let $\mathcal{C}$ and $\mathcal{D}$ be model categories (Def. 6.1), and let

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

be a Quillen adjunction between them (Def. 6.44). Then this may be called

1. a *Quillen reflection* if the derived adjunction counit (Def. 6.46) is componentwise a weak equivalence;
2. a *Quillen co-reflection* if the derived adjunction unit (Def. 6.46) is componentwise a weak equivalence.

The main class of examples of Quillen reflections are left Bousfield localizations, discussed as Prop. below.

***Proposition 6.54*. (characterization of *Quillen reflections*)**

Let

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

be a Quillen adjunction (Def. 6.44) and write

$$\mathrm{Ho}(\mathcal{C}) \underoverset{\underset{\mathbb{R}R}{\longrightarrow}}{\overset{\mathbb{L}L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathrm{Ho}(\mathcal{D})$$

for the induced adjoint pair of derived functors on the homotopy categories, from Prop. 6.48.

Then

1. $(L \underset{\mathrm{Qu}}{\dashv} R)$ *is a Quillen reflection (Def. 6.53) precisely if* $(\mathbb{L}L \dashv \mathbb{R}R)$ *is a reflective subcategory-inclusion (Def. 1.60);*
2. $(L \underset{\mathrm{Qu}}{\dashv} R)$ *is a Quillen co-reflection] (Def. 6.53) precisely if* $(\mathbb{L}L \dashv \mathbb{R}R)$ *is a co-reflective subcategory-inclusion (Def. 1.60);*

Proof. By Prop. 6.48 the components of the adjunction unit/counit of $(\mathbb{L}L \dashv \mathbb{R}R)$ are precisely the images under localization of the derived adjunction unit/counit of $(L \underset{\mathrm{Qu}}{\dashv} R)$. Moreover, by Prop. 6.31 the localization functor of a model category inverts precisely the weak equivalences. Hence the adjunction (co-)unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ is an isomorphism if and only if the derived (co-)unit of $(L \underset{\mathrm{Qu}}{\dashv} R)$ is a weak equivalence, respectively.

With this the statement reduces to the characterization of (co-)reflections via invertible units/counits, respectively, from Prop. 1.46. ▮

The following is the analog of adjoint equivalence of categories (Def. 1.56) for model categories:

Definition 6.55. ***(Quillen equivalence)***

For $\mathcal{C}, \mathcal{D}$ two model categories (Def. 6.1), a Quillen adjunction (def. 6.44)

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

is called a *Quillen equivalence*, to be denoted

$$\mathcal{C} \underoverset{\underset{R}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\simeq_{\mathrm{Qu}}} \mathcal{D}\,,$$

if the following equivalent conditions hold:

1. The right derived functor of R (via prop. 6.45, corollary 6.42) is an equivalence of categories

$$\mathbb{R}R : \mathrm{Ho}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{D}) \;.$$

2. The left derived functor of L (via prop. 6.45, corollary 6.42) is an equivalence of categories

$$\mathbb{L}L : \mathrm{Ho}(\mathcal{D}) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{C}) \;.$$

3. For every cofibrant object $d \in \mathcal{D}$, the derived adjunction unit (Def. 6.46)

$$d \overset{\eta_d}{\longrightarrow} R(L(d)) \overset{R(j_{L(d)})}{\longrightarrow} R(P(L(d)))$$

is a weak equivalence;
and for every fibrant object $c \in \mathcal{C}$, the derived adjunction counit (Def. 6.46)

$$L(Q(R(c))) \overset{L(p_{R(c)})}{\longrightarrow} L(R(c)) \overset{\epsilon}{\longrightarrow} c$$

is a weak equivalence.

4. For every cofibrant object $d \in \mathcal{D}$ and every fibrant object $c \in \mathcal{C}$, a morphism $d \longrightarrow R(c)$ is a weak equivalence precisely if its adjunct morphism $L(c) \to d$ is:

$$\frac{d \overset{\in W_{\mathcal{D}}}{\longrightarrow} R(c)}{L(d) \overset{\in W_{\mathcal{C}}}{\longrightarrow} c} \;.$$

Poposition 6.56. *The conditions in def. 6.55 are indeed all equivalent.*

(Quillen 67, I.4, theorem 3)

Proof. That 1) $\Leftrightarrow$ 2) follows from prop. 6.48 (if in an adjoint pair one is an equivalence, then so is the other).

To see the equivalence 1), 2) $\Leftrightarrow$ 3), notice (prop.) that a pair of adjoint functors is an equivalence of categories precisely if both the adjunction unit and the adjunction counit are natural isomorphisms. Hence it is sufficient to see that the derived adjunction unit/derived adjunction counit (Def. 6.46) indeed represent the adjunction (co-)unit of $(\mathbb{L}L \dashv \mathbb{R}R)$ in the homotopy category. But this is the statement of Prop. 6.48.

To see that 4) $\Rightarrow$ 3):

Consider the weak equivalence $LX \overset{j_{LX}}{\longrightarrow} PLX$. Its $(L \dashv R)$-adjunct is

$$X \overset{\eta}{\longrightarrow} RLX \overset{R j_{LX}}{\longrightarrow} RPLX$$

by assumption 4) this is again a weak equivalence, which is the requirement for the derived adjunction unit in 3). Dually for derived adjunction counit.

To see 3) ⇒ 4):

Consider any $f : Ld \to c$ a weak equivalence for cofibrant d, firbant c. Its adjunct $\tilde f$ sits in a commuting diagram

$$\begin{array}{lccccc} \tilde f : & d & \overset{\eta}{\longrightarrow} & RLd & \overset{Rf}{\longrightarrow} & Rc \\ & {}^{=}\downarrow && \downarrow^{R j_{Ld}} && \downarrow^{R j_c} \\ & d & \underset{\in W}{\longrightarrow} & RPLd & \overset{RPf}{\longrightarrow} & RPc \end{array}$$

where Pf is any lift constructed as in def. 6.26.

This exhibits the bottom left morphism as the derived adjunction unit (Def. 6.46), hence a weak equivalence by assumption. But since f was a weak equivalence, so is Pf (by two-out-of-three). Thereby also RPf and Rj_Y, are weak equivalences by Ken Brown's lemma 6.41 and the assumed fibrancy of c. Therefore by two-out-of-three (def. 1.75) also the adjunct $\tilde f$ is a weak equivalence. ▮

***Example 6.57**. (trivial Quillen equivalence)*

Let $\mathcal{C}$ be a model category (Def. 6.1). Then the identity functor on $\mathcal{C}$ constitutes a Quillen equivalence (Def. 6.55) from $\mathcal{C}$ to itself:

$$\mathcal{C} \underoverset{\underset{\mathrm{id}}{\longrightarrow}}{\overset{\mathrm{id}}{\longleftarrow}}{\simeq_{\mathrm{Qu}}} \mathcal{C}$$

***Proof**.* From prop. 6.43 it is clear that in this case the derived functors $\mathbb{L}\,\mathrm{id}$ and $\mathbb{R}\,\mathrm{id}$ both are themselves the identity functor on the homotopy category of a model category, hence in particular are an equivalence of categories. ▮

In certain situations the conditions on a Quillen equivalence simplify. For instance:

***Proposition 6.58**. (recognition of Quillen equivalences)*

If in a Quillen adjunction $\mathcal{C} \underoverset{\underset{R}{\to}}{\overset{L}{\leftarrow}}{\bot} \mathcal{D}$ (def. 6.44) the right adjoint R "creates weak equivalences" (in that a morphism f in $\mathcal{C}$ is a weak equivalence precisely if $U(f)$ is) then $(L \dashv R)$ is a Quillen equivalence (def. 6.55) precisely already if for all cofibrant objects $d \in \mathcal{D}$ the plain adjunction unit

$$d \overset{\eta}{\longrightarrow} R(L(d))$$

is a weak equivalence.

Proof. By prop. 6.56, generally, $(L \dashv R)$ is a Quillen equivalence precisely if

1. for every cofibrant object $d \in \mathcal{D}$, the derived adjunction unit (Def. 6.46)

   $$d \overset{\eta}{\longrightarrow} R(L(d)) \xrightarrow{R(j_{L(d)})} R(P(L(d)))$$

   is a weak equivalence;
2. for every fibrant object $c \in \mathcal{C}$, the derived adjunction counit (Def. 6.46)

   $$L(Q(R(c))) \xrightarrow{L(p_{R(c)})} L(R(c)) \overset{\epsilon}{\longrightarrow} c$$

   is a weak equivalence.

Consider the first condition: Since R preserves the weak equivalence $j_{L(d)}$, then by two-out-of-three (def. 1.75) the composite in the first item is a weak equivalence precisely if η is.

Hence it is now sufficient to show that in this case the second condition above is automatic.

Since R also reflects weak equivalences, the composite in item two is a weak equivalence precisely if its image

$$R(L(Q(R(c)))) \xrightarrow{R(L(p_{R(c)}))} R(L(R(c))) \overset{R(\epsilon)}{\longrightarrow} R(c)$$

under R is.

Moreover, assuming, by the above, that $\eta_{Q(R(c))}$ on the cofibrant object $Q(R(c))$ is a weak equivalence, then by two-out-of-three this composite is a weak equivalence precisely if the further composite with η is

$$Q(R(c)) \xrightarrow{\eta_{Q(R(c))}} R(L(Q(R(c)))) \xrightarrow{R(L(p_{R(c)}))} R(L(R(c))) \overset{R(\epsilon)}{\longrightarrow} R(c) \;.$$

By the formula for adjuncts, this composite is the $(L \dashv R)$-adjunct of the original composite, which is just $p_{R(c)}$

$$\frac{L(Q(R(c))) \xrightarrow{L(p_{R(c)})} L(R(c)) \overset{\epsilon}{\longrightarrow} c}{Q(R(C)) \overset{p_{R(c)}}{\longrightarrow} R(c)} \;.$$

But $p_{R(c)}$ is a weak equivalence by definition of cofibrant replacement. ■

The following is the analog of adjoint triples, adjoint quadruples (Remark 1.34), etc. for model categories:

***Definition 6.59*. (*Quillen adjoint triple*)**

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ be model categories (Def. 6.1), where $\mathcal{C}_1$ and $\mathcal{C}_2$ share the same underlying category $\mathcal{C}$, and such that the identity functor on $\mathcal{C}$ constitutes a Quillen equivalence (Def. 6.55):

$$\mathcal{C}_2 \underoverset{\underset{\mathrm{id}}{\longrightarrow}}{\overset{\mathrm{id}}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{C}_1$$

Then

1. a *Quillen adjoint triple* of the form

$$\mathcal{C}_{1/2} \array{ \overset{L}{\longrightarrow} \\ \bot_{\mathrm{Qu}} \\ \overset{C}{\longleftarrow} \\ \bot_{\mathrm{Qu}} \\ \overset{R}{\longrightarrow} } \mathcal{D}$$

is diagrams in the double category of model categories (Def. 6.49) of the form

$$\begin{tikzcd} & \mathcal{C}_1 \arrow[r, "\mathrm{id}"] \arrow[d, "L"'] \arrow[dr, phantom, "\eta \swArrow"] & \mathcal{C}_2 \arrow[d, "\mathrm{id}"] \\ \mathcal{C}_2 \arrow[r, "R"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\epsilon \swArrow"] & \mathcal{D} \arrow[r, "C"] \arrow[d, "C"'] \arrow[dr, phantom, "\swArrow \mathrm{id}"] & \mathcal{C}_1 \arrow[d, "\mathrm{id}"] \\ \mathcal{C}_2 \arrow[r, "\mathrm{id}"'] & \mathcal{C}_2 \arrow[r, "\mathrm{id}"'] & \mathcal{C}_2 \end{tikzcd}$$

such that η is the unit of an adjunction and ϵ the counit of an adjunction, thus exhibiting Quillen adjunctions

$$\mathcal{C}_1 \underoverset{\underset{C}{\longleftarrow}}{\overset{L}{\longrightarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

$$\mathcal{C}_2 \underoverset{\underset{R}{\longrightarrow}}{\overset{C}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}$$

and such that the derived natural transformation Ho(id) of the bottom right square (94) is invertible (a natural isomorphism);

2. a *Quillen adjoint triple* of the form

$$
\begin{tikzcd}
\mathcal{C}_{1/2} \arrow[r, "C" description] & \mathcal{D} \arrow[l, bend right=40, "L"'] \arrow[l, bend left=40, "R"] \arrow[l, phantom, shift right=3, "\bot_{\mathrm{Qu}}"] \arrow[l, phantom, shift left=3, "\bot_{\mathrm{Qu}}"]
\end{tikzcd}
$$

is diagram in the double category of model categories (Def. 6.49) of the form

$$
\begin{tikzcd}
\mathcal{C}_2 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\mathrm{id}\,\Downarrow"] & \mathcal{C}_1 \arrow[r, "\mathrm{id}"] \arrow[d, "C"] \arrow[dr, phantom, "\epsilon\,\Downarrow"] & \mathcal{C}_1 \arrow[d, "\mathrm{id}"] \\
\mathcal{C}_2 \arrow[r, "C"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\epsilon\,\Downarrow"] & \mathcal{D} \arrow[r, "R"'] \arrow[d, "L"] & \mathcal{C}_1 \\
\mathcal{C}_2 \arrow[r, "\mathrm{id}"'] & \mathcal{C}_2 &
\end{tikzcd}
$$

such that η is the unit of an adjunction and ϵ the counit of an adjunction, thus exhibiting Quillen adjunctions

$$
\mathcal{C}_2 \underoverset{\underset{C}{\longrightarrow}}{\overset{L}{\longleftarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}
$$

$$
\mathcal{C}_1 \underoverset{\underset{R}{\longleftarrow}}{\overset{C}{\longrightarrow}}{\bot_{\mathrm{Qu}}} \mathcal{D}
$$

and such that the derived natural transformation Ho(id) of the top left square square (here) is invertible (a natural isomorphism).

If a Quillen adjoint triple of the first kind overlaps with one of the second kind

$$\mathcal{C}_{1/2} \underoverset{\underset{R_2}{\longleftarrow}}{\overset{L_1}{\longrightarrow}}{\overset{\overset{C_1 = L_2}{\longleftarrow}}{\underset{\underset{R_1 = C_2}{\longrightarrow}}{}}} \mathcal{D}$$

$$\mathcal{C}_{1/2} \begin{array}{c} \xrightarrow{\;L_1\;} \\ \bot_{Qu} \\ \xleftarrow{\;C_1 = L_2\;} \\ \bot_{Qu} \\ \xrightarrow{\;R_1 = C_2\;} \\ \bot_{Qu} \\ \xleftarrow{\;R_2\;} \end{array} \mathcal{D}$$

we speak of a *Quillen adjoint quadruple*, and so forth.

***Proposition 6.60*. (Quillen adjoint triple induces adjoint triple of derived functors on homotopy categories)**

Given a Quillen adjoint triple (Def. 6.59), the induced derived functors (Def. 6.38) on the homotopy categories form an ordinary adjoint triple (Remark 1.34):

$$\mathcal{C}_{1/2} \begin{array}{c} \xrightarrow{\;L\;} \\ \bot_{Qu} \\ \xleftarrow{\;C\;} \\ \bot_{Qu} \\ \xrightarrow{\;R\;} \end{array} \mathcal{D} \qquad \overset{\mathrm{Ho}(-)}{\longmapsto} \qquad \mathrm{Ho}(\mathcal{C}) \begin{array}{c} \xrightarrow{\;\mathbb{L}L\;} \\ \bot \\ \xleftarrow{\;\mathbb{L}C \simeq \mathbb{R}C\;} \\ \bot \\ \xrightarrow{\;\mathbb{R}R\;} \end{array} \mathrm{Ho}(\mathcal{D})$$

$$\mathcal{C}_{1/2} \begin{array}{c} \xrightarrow{\;L\;} \\ \bot_{Qu} \\ \xleftarrow{\;C\;} \\ \bot_{Qu} \\ \xrightarrow{\;R\;} \end{array} \mathcal{D} \qquad \overset{\mathrm{Ho}(-)}{\longmapsto} \qquad \mathrm{Ho}(\mathcal{C}) \begin{array}{c} \xrightarrow{\;\mathbb{L}L\;} \\ \bot \\ \xleftarrow{\;\mathbb{L}C \simeq \mathbb{R}C\;} \\ \bot \\ \xrightarrow{\;\mathbb{R}R\;} \end{array} \mathrm{Ho}(\mathcal{D})$$

***Proof*.** This follows immediately from the fact that passing to homotopy categories of model categories is a double pseudofunctor from the double category of model categories to the double category of squares in Cat (Prop. 6.50). ▮

Mapping cones

In the context of homotopy theory, a pullback diagram, such as in the definition of the fiber in example

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow && \downarrow^{f} \\ * & \longrightarrow & Y \end{array}$$

ought to commute only up to a (left/right) homotopy (def. 6.20) between the outer composite morphisms. Moreover, it should satisfy its universal property up to such homotopies.

Instead of going through the full theory of what this means, we observe that this is plausibly modeled by the following construction, and then we check (below) that this indeed has the relevant abstract homotopy theoretic properties.

Definition 6.61. Let $\mathcal{C}$ be a model category, def. 6.1 with $\mathcal{C}^{*/}$ its model structure on pointed objects, prop. . For $f : X \longrightarrow Y$ a morphism between cofibrant objects (hence a morphism in $(\mathcal{C}^{*/})_c \hookrightarrow \mathcal{C}^{*/}$, def. 6.32), its ***reduced mapping cone*** is the object

$$\mathrm{Cone}(f) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X) \underset{X}{\sqcup} Y$$

in the colimiting diagram

$$\begin{array}{ccccc} && X & \overset{f}{\longrightarrow} & Y \\ && \downarrow^{i_1} && \downarrow^{i} \\ X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) && \\ \downarrow &&& \searrow^{\eta} & \downarrow \\ * & \longrightarrow && \longrightarrow & \mathrm{Cone}(f) \end{array},$$

where $\mathrm{Cyl}(X)$ is a cylinder object for X, def. 6.16.

Dually, for $f : X \longrightarrow Y$ a morphism between fibrant objects (hence a morphism in $(\mathcal{C}^{*})_f \hookrightarrow \mathcal{C}^{*/}$, def. 6.32), its ***mapping cocone*** is the object

$$\mathrm{Path}_*(f) \coloneqq * \underset{Y}{\times} \mathrm{Path}(Y) \underset{Y}{\times} Y$$

in the following limit diagram

$$
\begin{array}{ccccc}
\mathrm{Path}_*(f) & \longrightarrow & & \longrightarrow & X \\
\downarrow & \searrow^{\eta} & & & \downarrow^{f} \\
& & \mathrm{Path}(Y) & \underset{p_1}{\longrightarrow} & Y \\
\downarrow & & \downarrow^{p_0} & & \\
* & \longrightarrow & Y & &
\end{array}
,
$$

where $\mathrm{Path}(Y)$ is a path space object for Y, def. 6.16.

Remark 6.62. When we write homotopies (def. 6.20) as double arrows between morphisms, then the limit diagram in def. 6.61 looks just like the square in the definition of fibers in example , except that it is filled by the right homotopy given by the component map denoted η:

$$
\begin{array}{ccc}
\mathrm{Path}_*(f) & \longrightarrow & X \\
\downarrow & \swArrow_{\eta} & \downarrow^{f} \\
* & \longrightarrow & Y
\end{array}
.
$$

Dually, the colimiting diagram for the mapping cone turns to look just like the square for the cofiber, except that it is filled with a left homotopy

$$
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
\downarrow & \swArrow_{\eta} & \downarrow \\
* & \longrightarrow & \mathrm{Cone}(f)
\end{array}
$$

Proposition 6.63. *The colimit appearing in the definition of the reduced mapping cone in def. 6.61 is equivalent to three consecutive pushouts:*

$$
\begin{array}{ccccc}
& & X & \overset{f}{\longrightarrow} & Y \\
& & \downarrow^{i_1} & (po) & \downarrow^{i} \\
X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) & \longrightarrow & \mathrm{Cyl}(f) \\
\downarrow & (po) & \downarrow & (po) & \downarrow \\
* & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \mathrm{Cone}(f)
\end{array}
.
$$

The two intermediate objects appearing here are called

- *the plain **reduced cone*** $\mathrm{Cone}(X) \coloneqq * \underset{X}{\sqcup} \mathrm{Cyl}(X)$*;*
- *the **reduced mapping cylinder*** $\mathrm{Cyl}(f) \coloneqq \mathrm{Cyl}(X) \underset{X}{\sqcup} Y$.

Dually, the limit appearing in the definition of the mapping cocone in def. 6.61 is equivalent to three consecutive pullbacks:

$$
\begin{array}{ccccc}
\mathrm{Path}_*(f) & \longrightarrow & \mathrm{Path}(f) & \longrightarrow & X \\
\downarrow & (pb) & \downarrow & (pb) & \downarrow^{f} \\
\mathrm{Path}_*(Y) & \longrightarrow & \mathrm{Path}(Y) & \underset{p_1}{\longrightarrow} & Y \\
\downarrow & (pb) & \downarrow^{p_0} \\
* & \longrightarrow & Y
\end{array}
.
$$

The two intermediate objects appearing here are called

- *the **based path space object*** $\mathrm{Path}_*(Y) \coloneqq * \prod_Y \mathrm{Path}(Y)$*;*
- *the **mapping path space** or **mapping co-cylinder*** $\mathrm{Path}(f) \coloneqq X \underset{Y}{\times} \mathrm{Path}(X)$*.*

Definition 6.64. Let $X \in \mathcal{C}^{*/}$ be any pointed object.

1. The mapping cone, def. 6.63, of $X \to *$ is called the ***reduced suspension*** of X, denoted

$$\Sigma X = \mathrm{Cone}(X \to *)\ .$$

Via prop. 6.63 this is equivalently the coproduct of two copies of the cone on X over their base:

$$
\begin{array}{ccccc}
 & & X & \longrightarrow & * \\
 & & \downarrow^{i_1} & (po) & \downarrow \\
X & \overset{i_0}{\longrightarrow} & \mathrm{Cyl}(X) & \longrightarrow & \mathrm{Cone}(X) \\
\downarrow & (po) & \downarrow & (po) & \downarrow \\
* & \longrightarrow & \mathrm{Cone}(X) & \longrightarrow & \Sigma X
\end{array}
.
$$

This is also equivalently the cofiber, example of (i_0, i_1), hence (example) of the wedge sum inclusion:

$$X \vee X = X \sqcup X \overset{(i_0,i_1)}{\longrightarrow} \mathrm{Cyl}(X) \overset{\mathrm{cofib}(i_0,i_1)}{\longrightarrow} \Sigma X\ .$$

2. The mapping cocone, def. 6.63, of $* \to X$ is called the ***loop space object*** of X, denoted

$$\Omega X = \mathrm{Path}_*(* \to X)\ .$$

Via prop. 6.63 this is equivalently

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \mathrm{Path}_*(X) & \longrightarrow & * \\ \downarrow & (\mathrm{pb}) & \downarrow & (\mathrm{pb}) & \downarrow \\ \mathrm{Path}_*(X) & \longrightarrow & \mathrm{Path}(X) & \underset{p_1}{\longrightarrow} & X \\ \downarrow & (\mathrm{pb}) & \downarrow^{p_0} \\ * & \longrightarrow & X \end{array} \,.$$

This is also equivalently the fiber, example of (p_0, p_1):

$$\Omega X \overset{\mathrm{fib}(p_0,p_1)}{\longrightarrow} \mathrm{Path}(X) \overset{(p_0,p_1)}{\longrightarrow} X \times X \,.$$

Proposition 6.65. *In pointed topological spaces* $\mathrm{Top}^{*/}$*,*

- *the reduced suspension objects (def. 6.64) induced from the standard reduced cylinder* $(-)\wedge (I_+)$ *of example are isomorphic to the smash product (def.) with the 1-sphere, for later purposes we choose to smash* ***on the left*** *and write*

$$\mathrm{cofib}(X \vee X \to X \wedge (I_+)) \simeq S^1 \wedge X \,,$$

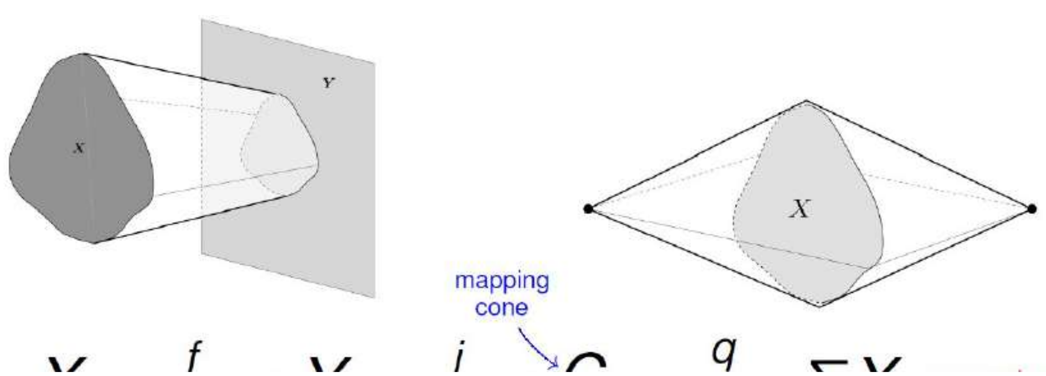
Dually:

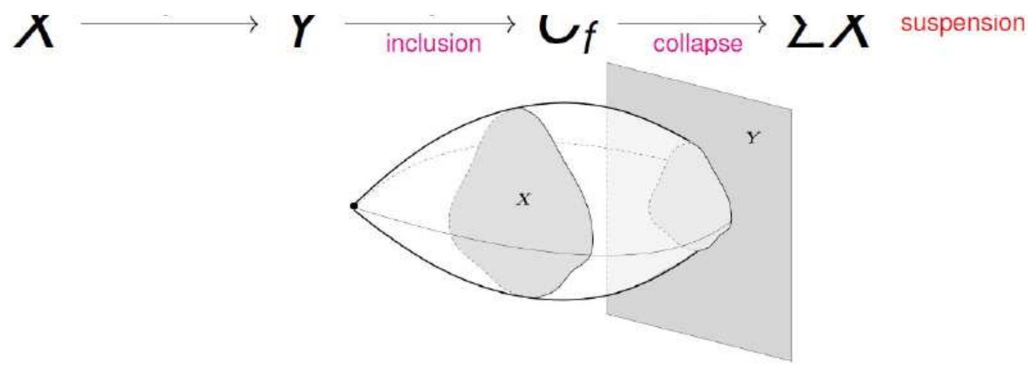
- *the loop space objects (def. 6.64) induced from the standard pointed path space object* $\mathrm{Maps}(I_+, -)_*$ *are isomorphic to the pointed mapping space (example) with the 1-sphere*

$$\mathrm{fib}(\mathrm{Maps}(I_+, X)_* \to X \times X) \simeq \mathrm{Maps}(S^1, X)_* \,.$$

Proof. By immediate inspection: For instance the fiber of $\mathrm{Maps}(I_+,X)_* \longrightarrow X \times X$ is clearly the subspace of the unpointed mapping space X^I on elements that take the endpoints of I to the basepoint of X. ▮

Example 6.66. For $\mathcal{C} =$ Top with $\mathrm{Cyl}(X) = X \times I$ the standard cyclinder object, def. , then by example , the





mapping cone, def. 6.61, of a continuous function $f : X \longrightarrow Y$ is obtained by

1. forming the cylinder over X;
2. attaching to one end of that cylinder the space Y as specified by the map f.
3. shrinking the other end of the cylinder to the point.

Accordingly the suspension of a topological space is the result of shrinking both ends of the cylinder on the object two the point. This is homeomoprhic to attaching two copies of the cone on the space at the base of the cone.

(graphics taken from Muro 2010)

Below in example we find the homotopy-theoretic interpretation of this standard topological mapping cone as a model for the *homotopy cofiber*.

Remark 6.67. The *formula* for the mapping cone in prop. 6.63 (as opposed to that of the mapping co-cone) does not require the presence of the basepoint: for $f : X \longrightarrow Y$ a morphism in $\mathcal{C}$ (as opposed to in $\mathcal{C}^{*/}$) we may still define

$$\mathrm{Cone}'(f) \coloneqq Y \underset{X}{\sqcup} \mathrm{Cone}'(X)\,,$$

where the prime denotes the *unreduced cone*, formed from a cylinder object in $\mathcal{C}$.

Proposition 6.68. *For $f : X \longrightarrow Y$ a morphism in Top, then its unreduced mapping cone, remark 6.67, with respect to the standard cylinder object $X \times I$ def. , is isomorphic to the reduced mapping cone, def. 6.61, of the morphism $f_+ : X_+ \to Y_+$ (with a basepoint adjoined, def.) with respect to the standard reduced cylinder (example):*

$$\mathrm{Cone}'(f) \simeq \mathrm{Cone}(f_+)\ .$$

Proof. By prop. and example , $\mathrm{Cone}(f_+)$ is given by the colimit in Top over the following diagram:

$$
\begin{array}{ccccc}
* & \longrightarrow & X \sqcup * & \stackrel{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\
\downarrow && \downarrow && \downarrow \\
X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\
\downarrow &&&& \downarrow \\
* & \longrightarrow && \longrightarrow & \mathrm{Cone}(f_+)
\end{array}
\,.
$$

We may factor the vertical maps to give

$$
\begin{array}{ccccc}
* & \longrightarrow & X \sqcup * & \stackrel{(f,\mathrm{id})}{\longrightarrow} & Y \sqcup * \\
\downarrow && \downarrow && \downarrow \\
X \sqcup * & \longrightarrow & (X \times I) \sqcup * && \\
\downarrow &&&& \downarrow \\
* \sqcup * & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f)_+ \\
\downarrow &&&& \downarrow \\
* & \longrightarrow && \longrightarrow & \mathrm{Cone}'(f)
\end{array}
\,.
$$

This way the top part of the diagram (using the pasting law to compute the colimit in two stages) is manifestly a cocone under the result of applying $(-)_+$ to the diagram for the unreduced cone. Since $(-)_+$ is itself given by a colimit, it preserves colimits, and hence gives the partial colimit $\mathrm{Cone}'(f)_+$ as shown. The remaining pushout then contracts the remaining copy of the point away. ▮

Example 6.66 makes it clear that every cycle $S^n \to Y$ in Y that happens to be in the image of X can be *continuously* translated in the cylinder-direction, keeping it constant in Y, to the other end of the cylinder, where it shrinks away to the point. This means that every homotopy group of Y, def. , in the image of f vanishes in the mapping cone. Hence in the mapping cone ***the image of X under f in Y is removed up to homotopy***. This makes it intuitively clear how $\mathrm{Cone}(f)$ is a homotopy-version of the cokernel of f. We now discuss this formally.

Lemma 6.69. ***(factorization lemma)***

Let $\mathcal{C}_c$ be a category of cofibrant objects, def. 6.32. Then for every morphism $f : X \longrightarrow Y$ the mapping cylinder-construction in def. 6.63 provides a cofibration resolution of f, in that

1. *the composite morphism $X \stackrel{i_0}{\longrightarrow} \mathrm{Cyl}(X) \stackrel{(i_1)_* f}{\longrightarrow} \mathrm{Cyl}(f)$ is a cofibration;*
2. *f factors through this morphism by a weak equivalence left inverse to an acyclic cofibration*

$$f \,:\, X \underset{\in \mathrm{Cof}}{\overset{(i_1)_* f \circ i_0}{\longrightarrow}} \mathrm{Cyl}(f) \underset{\in W}{\longrightarrow} Y \,,$$

Dually:

Let $\mathcal{C}_f$ be a category of fibrant objects, def. 6.32. Then for every morphism $f : X \longrightarrow Y$ the mapping cocylinder-construction in def. 6.63 provides a fibration resolution of f, in that

1. *the composite morphism $\mathrm{Path}(f) \overset{p_1^* f}{\longrightarrow} \mathrm{Path}(Y) \overset{p_0}{\longrightarrow} Y$ is a fibration;*
2. *f factors through this morphism by a weak equivalence right inverse to an acyclic fibration:*

$$f \,:\, X \underset{\in W}{\longrightarrow} \mathrm{Path}(f) \underset{\in \mathrm{Fib}}{\overset{p_0 \circ p_1^* f}{\longrightarrow}} Y \,,$$

Proof. We discuss the second case. The first case is formally dual.

So consider the mapping cocylinder-construction from prop. 6.63

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{\mathllap{p_1^* f}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{f}} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{\mathllap{\in W \cap \mathrm{Fib}}}\downarrow^{\mathrlap{p_0}} \\ Y \end{array} \,.$$

To see that the vertical composite is indeed a fibration, notice that, by the pasting law, the above pullback diagram may be decomposed as a pasting of two pullback diagram as follows

$$\begin{array}{ccccc} \mathrm{Path}(f) & \underset{\in \mathrm{Fib}}{\overset{(f,\mathrm{id})^*(p_1,p_0)}{\longrightarrow}} & X \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & X \\ \downarrow && \downarrow^{\mathrlap{(f,\mathrm{Id})}} && \downarrow^{\mathrlap{f}} \\ \mathrm{Path}(Y) & \overset{(p_1,p_0) \in \mathrm{Fib}}{\longrightarrow} & Y \times Y & \overset{\mathrm{pr}_1}{\longrightarrow} & Y \\ {}^{\mathllap{p_0}}\downarrow & \swarrow_{\underset{\in \mathrm{Fib}}{\mathrm{pr}_2}} \\ Y \end{array} \,.$$

Both squares are pullback squares. Since pullbacks of fibrations are fibrations by prop. 6.8, the morphism $\mathrm{Path}(f) \to X \times Y$ is a fibration. Similarly, since X is fibrant, also the projection map $X \times Y \to Y$ is a fibration (being the pullback of $X \to *$ along $Y \to *$).

Since the vertical composite is thereby exhibited as the composite of two fibrations

$$\mathrm{Path}(f) \xrightarrow{(f,\mathrm{id})^*(p_1,p_0)} X \times Y \xrightarrow{\mathrm{pr}_2 \circ (f,\mathrm{Id}) = \mathrm{pr}_2} Y\,,$$

it is itself a fibration.

Then to see that there is a weak equivalence as claimed:

The universal property of the pullback $\mathrm{Path}(f)$ induces a right inverse of $\mathrm{Path}(f) \to X$ fitting into this diagram

$$\begin{array}{ccccccc} \mathrm{id}_X : & X & \underset{\in W}{\overset{\exists}{\longrightarrow}} & \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ & {}^{\mathllap{f}}\downarrow & & \downarrow & & \downarrow^{\mathrlap{f}} \\ \mathrm{id}_Y : & Y & \underset{\in W}{\overset{i}{\longrightarrow}} & \mathrm{Path}(Y) & \overset{p_1}{\to} & Y\,' \\ & & {}_{\mathllap{\mathrm{Id}}}\searrow & \downarrow^{\mathrlap{p_0}} \\ & & & Y \end{array}$$

which is a weak equivalence, as indicated, by two-out-of-three (def. 1.75).

This establishes the claim. ▮

Categories of fibrant objects

Below we discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above. Before we do so, we here establish a collection of general facts that hold in categories of fibrant objects and dually in categories of cofibrant objects, def. 6.32.

Literature (Brown 73, section 4).

Lemma 6.70*. Let $f : X \longrightarrow Y$ be a morphism in a category of fibrant objects, def. 6.32. Then given any choice of path space objects $\mathrm{Path}(X)$ and $\mathrm{Path}(Y)$, def. 6.16, there is a replacement of $\mathrm{Path}(X)$ by a path space object $\widetilde{\mathrm{Path}(X)}$ along an acylic fibration, such that $\widetilde{\mathrm{Path}(X)}$ has a morphism ϕ to $\mathrm{Path}(Y)$ which is compatible with the structure maps, in that the following diagram commutes*

$$
\begin{array}{ccccccc}
 & & X & \xrightarrow{f} & Y \\
 & \swarrow & \downarrow & & \downarrow \\
\mathrm{Path}(X) & \underset{\in W \cap \mathrm{Fib}}{\longleftarrow} & \widetilde{\mathrm{Path}(X)} & \xrightarrow{\phi} & \mathrm{Path}(Y) \\
 & {}_{(p_0^X, p_1^X)}\searrow & \downarrow^{(p_0^Y, p_1^Y)} & & \downarrow^{(\tilde p_0^X, \tilde p_1^X)} \\
 & & X \times X & \xrightarrow{(f,f)} & Y \times Y
\end{array}
$$

([Brown 73, section 2, lemma 2])

Proof. Consider the commuting square

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{Path}(Y) \\
\downarrow & & & & \downarrow^{(p_0^Y, p_1^Y)} \\
\mathrm{Path}(X) & \xrightarrow{(p_0^X, p_1^X)} & X \times X & \xrightarrow{(f,f)} & Y \times Y
\end{array}
$$

Then consider its factorization through the pullback of the right morphism along the bottom morphism,

$$
\begin{array}{ccccc}
X & \longrightarrow & (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y) & \longrightarrow & \mathrm{Path}(Y) \\
 & {}_{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{(p_0^Y, p_1^Y)}_{\in \mathrm{Fib}} \\
 & & \mathrm{Path}(X) & \xrightarrow{(f \circ p_0^X, f \circ p_1^X)} & Y \times Y
\end{array}
$$

Finally use the factorization lemma 6.69 to factor the morphism $X \to (f \circ p_0^X, f \circ p_1^X)^* \mathrm{Path}(Y)$ through a weak equivalence followed by a fibration, the object this factors through serves as the desired path space resolution

$$
\begin{array}{ccccc}
X & \xrightarrow{\in W} & \widetilde{\mathrm{Path}(X)} & \longrightarrow & \mathrm{Path}(Y) \\
 & {}_{\in W}\searrow & \downarrow^{\in W \cap \mathrm{Fib}} & & \downarrow^{(p_0^Y, p_1^Y)} \\
 & & \mathrm{Path}(X) & \xrightarrow{(f \circ p_0^X, f \circ p_1^X)} & Y \times Y
\end{array}
$$

▮

Lemma 6.71. *In a category of fibrant objects $\mathcal{C}_f$, def. 6.32, let*

$$\begin{array}{ccccc} A_1 && \stackrel{f}{\longrightarrow} && A_2 \\ & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} \\ && B \end{array}$$

be a morphism over some object B in $\mathcal{C}_f$ and let $u\colon B' \to B$ be any morphism in $\mathcal{C}_f$. Let

$$\begin{array}{ccccc} u^* A_1 && \stackrel{u^* f}{\longrightarrow} && u^* A_2 \\ & {}_{\in \mathrm{Fib}}\searrow && \swarrow_{\in \mathrm{Fib}} \\ && B' \end{array}$$

be the corresponding morphism pulled back along u.

Then

- *if f is a fibration then also $u^* f$ is a fibration;*
- *if f is a weak equivalence then also $u^* f$ is a weak equivalence.*

(Brown 73, section 4, lemma 1)

Proof. For $f \in \mathrm{Fib}$ the statement follows from the pasting law which says that if in

$$\begin{array}{ccc} B' \times_B A_1 & \longrightarrow & A_1 \\ \downarrow^{u^* f \in \mathrm{Fib}} && \downarrow^{f \in \mathrm{Fib}} \\ B' \times_B A_2 & \longrightarrow & A_2 \\ \downarrow^{\in \mathrm{Fib}} && \downarrow^{\in \mathrm{Fib}} \\ B' & \stackrel{u}{\longrightarrow} & B \end{array}$$

the bottom and the total square are pullback squares, then so is the top square. The same reasoning applies for $f \in W \cap \mathrm{Fib}$.

Now to see the case that $f \in W$:

Consider the full subcategory $(\mathcal{C}_{/B})_f$ of the slice category $\mathcal{C}_{/B}$ (def.) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$
\begin{tikzcd}
(\Delta_X)/B : & X \arrow[r, "\in W"] \arrow[dr, "\in \mathrm{Fib}"'] & \mathrm{Path}_B(X) \arrow[r, "\in \mathrm{Fib}"] \arrow[d] & X \underset{B}{\times} X \arrow[dl, "\in \mathrm{Fib}"] \\
& & B &
\end{tikzcd}
.
$$

With these, the factorization lemma (lemma 6.69) applies in $(\mathcal{C}_{/B})_f$.

(Notice that for this we do need the restriction of $\mathcal{C}_{/B}$ to the fibrations, because this ensures that the projections $p_i : X_1 \times_B X_2 \to X_i$ are still fibrations, which is used in the proof of the factorization lemma (here).)

So now given any

$$
\begin{tikzcd}
X \arrow[rr, "f", "\in W"'] \arrow[dr, "\in \mathrm{Fib}"'] & & Y \arrow[dl, "\in \mathrm{Fib}"] \\
& B &
\end{tikzcd}
$$

apply the factorization lemma in $(\mathcal{C}_{/B})_f$ to factor it as

$$
\begin{tikzcd}
X \arrow[r, "i \in W"] \arrow[dr, "\in \mathrm{Fib}"'] & \mathrm{Path}_B(f) \arrow[r, "\in W \cap \mathrm{Fib}"] \arrow[d] & Y \arrow[dl, "\in \mathrm{Fib}"] \\
& B &
\end{tikzcd}
.
$$

By the previous discussion it is sufficient now to show that the base change of i to B' is still a weak equivalence. But by the factorization lemma in $(\mathcal{C}_{/B})_f$, the morphism i is right inverse to another acyclic fibration over B:

$$
\begin{tikzcd}
\mathrm{id}_X : & X \arrow[r, "i \in W"] \arrow[dr, "\in \mathrm{Fib}"'] & \mathrm{Path}_B(f) \arrow[r, "\in W \cap \mathrm{Fib}"] \arrow[d] & X \arrow[dl, "\in \mathrm{Fib}"] \\
& & B &
\end{tikzcd}
.
$$

(Notice that if we had applied the factorization lemma of Δ_X in $\mathcal{C}_f$ instead of $(\Delta_X)/B$ in $(\mathcal{C}_{/B})$ then the corresponding triangle on the right here would not commute.)

Now we may reason as before: the base change of the top morphism here is exhibited by the following pasting composite of pullbacks:

$$\begin{array}{ccc} B' \underset{B}{\times} X & \longrightarrow & X \\ \downarrow & (pb) & \downarrow \\ B' \underset{B}{\times} \mathrm{Path}_B(f) & \longrightarrow & \mathrm{Path}_B(f) \\ \downarrow & (pb) & \downarrow^{\in W \cap \mathrm{Fib}} \\ B' \underset{B}{\times} X & \longrightarrow & X \\ \downarrow & (pb) & \downarrow \\ B' & \longrightarrow & B \end{array}$$

The acyclic fibration $\mathrm{Path}_B(f)$ is preserved by this pullback, as is the identity $\mathrm{id}_X : X \to \mathrm{Path}_B(X) \to X$. Hence the weak equivalence $X \to \mathrm{Path}_B(X)$ is preserved by two-out-of-three (def. 1.75).

Lemma 6.72. *In a category of fibrant objects, def. 6.32, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

(Brown 73, section 4, lemma 2)

Proof. Let $u : B' \to B$ be a weak equivalence and $p : E \to B$ be a fibration. We want to show that the left vertical morphism in the pullback

$$\begin{array}{ccc} E \times_B B' & \longrightarrow & B' \\ \downarrow^{\Rightarrow \in W} && \downarrow^{\in W} \\ E & \overset{\in \mathrm{Fib}}{\longrightarrow} & B \end{array}$$

is a weak equivalence.

First of all, using the factorization lemma 6.69 we may factor $B' \to B$ as

$$B' \overset{\in W}{\longrightarrow} \mathrm{Path}(u) \overset{\in W \cap F}{\longrightarrow} B$$

with the first morphism a weak equivalence that is a right inverse to an acyclic fibration and the right one an acyclic fibration.

Then the pullback diagram in question may be decomposed into two consecutive pullback diagrams

$$
\begin{array}{ccc}
E \times_B B' & \to & B' \\
\downarrow && \downarrow \\
Q & \overset{\in \mathrm{Fib}}{\longrightarrow} & \mathrm{Path}(u)\,, \\
\downarrow^{\mathrlap{\in W \cap \mathrm{Fib}}} && \downarrow^{\mathrlap{\in W \cap \mathrm{Fib}}} \\
E & \overset{\in \mathrm{Fib}}{\longrightarrow} & B
\end{array}
$$

where the morphisms are indicated as fibrations and acyclic fibrations using the stability of these under arbitrary pullback.

This means that the proof reduces to proving that weak equivalences $u \colon B' \overset{\in W}{\longrightarrow} B$ that are right inverse to some acyclic fibration $v \colon B \overset{\in W \cap F}{\longrightarrow} B'$ map to a weak equivalence under pullback along a fibration.

Given such u with right inverse v, consider the pullback diagram

$$
\begin{array}{cccc}
& E \\
& {}^{\mathllap{(p,\mathrm{id})}}_{\mathllap{\in W}}\downarrow & \searrow^{\mathrlap{\mathrm{id}}} \\
E_1 \coloneqq & B \times_{B'} E & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & E \\
& \downarrow^{\mathrlap{\in \mathrm{Fib}}} && \downarrow^{\mathrlap{p \in \mathrm{Fib}}} \\
& & (\mathrm{pb}) & B \\
& \downarrow && \downarrow^{\mathrlap{v \in W \cap \mathrm{Fib}}} \\
& B & \overset{v \in \mathrm{Fib} \cap W}{\longrightarrow} & B'
\end{array}
$$

Notice that the indicated universal morphism $p \times \mathrm{Id} \colon E \overset{\in W}{\longrightarrow} E_1$ into the pullback is a weak equivalence by two-out-of-three (def. 1.75).

The previous lemma 6.71 says that weak equivalences between fibrations over B are themselves preserved by base extension along $u \colon B' \to B$. In total this yields the following diagram

$$
\array{
u^* E = B' \times_B E &\longrightarrow& E
\\
{}^{\mathllap{u^*(p\times Id)}}_{\mathllap{\in W}}\downarrow && {}^{\mathllap{p \times Id}}_{\mathllap{\in W}}\downarrow & \searrow^{\mathrlap{id}}
\\
u^* E_1 &\longrightarrow& E_1 &\overset{\in W \cap Fib}{\longrightarrow}& E
\\
\downarrow^{\mathrlap{\in Fib}} && \downarrow^{\mathrlap{\in Fib}} && \downarrow^{\mathrlap{p \in Fib}}
\\
&& && B
\\
\downarrow && \downarrow && \downarrow^{\mathrlap{v \in W \cap Fib}}
\\
B' &\overset{u}{\longrightarrow}& B &\overset{v \in W \cap Fib}{\longrightarrow}& B'
}
$$

so that with $p \times Id \colon E \to E_1$ a weak equivalence also $u^*(p \times Id)$ is a weak equivalence, as indicated.

Notice that $u^* E = B' \times_B E \to E$ is the morphism that we want to show is a weak equivalence. By two-out-of-three (def. 1.75) for that it is now sufficient to show that $u^* E_1 \to E_1$ is a weak equivalence.

That finally follows now since, by assumption, the total bottom horizontal morphism is the identity. Hence so is the top horizontal morphism. Therefore $u^* E_1 \to E_1$ is right inverse to a weak equivalence, hence is a weak equivalence. ▮

Lemma 6.73. *Let $(\mathcal{C}^{*/})_f$ be a category of fibrant objects, def. 6.32 in a model structure on pointed objects (prop.). Given any commuting diagram in $\mathcal{C}$ of the form*

$$
\array{
X'_1 &\underoverset{t}{\in W}{\longrightarrow}& X_1 & \underoverset{g}{f}{\rightrightarrows} & X_2
\\
&& \downarrow^{\mathrlap{p_1}}_{\mathrlap{\in Fib}} && \downarrow^{\mathrlap{p_2}}_{\mathrlap{\in Fib}}
\\
&& B &\overset{u}{\longrightarrow}& C
}
$$

(meaning: both squares commute and t equalizes f with g) then the localization functor $\gamma \colon (\mathcal{C}^{*/})_f \to Ho(\mathcal{C}^{*/})$ (def. 6.26, cor 6.34) takes the morphisms $fib(p_1) \rightrightarrows fib(p_2)$ induced by f and g on fibers (example) to the same morphism, in the homotopy category.

(Brown 73, section 4, lemma 4)

Proof. First consider the pullback of p_2 along u: this forms the same kind of diagram but with the bottom morphism an identity. Hence it is sufficient to consider this special case.

Consider the full subcategory $(\mathcal{C}^{*/}_{/B})_f$ of the slice category $\mathcal{C}^{*/}_{/B}$ (def.) on its fibrant objects, i.e. the full subcategory of the slice category on the fibrations

$$\begin{array}{c} X \\ \downarrow^{p}_{\in \mathrm{Fib}} \\ B \end{array}$$

into B. By factorizing for every such fibration the diagonal morphisms into the fiber product $X \underset{B}{\times} X$ through a weak equivalence followed by a fibration, we obtain path space objects $\mathrm{Path}_B(X)$ relative to B:

$$\begin{array}{ccccc} (\Delta_X)\,/\,B\;: & X & \overset{\in W}{\longrightarrow} & \mathrm{Path}_B(X) & \overset{\in \mathrm{Fib}}{\longrightarrow} & X \underset{B}{\times} X \\ & {}_{\in \mathrm{Fib}}\searrow & & \downarrow & & \swarrow_{\in \mathrm{Fib}} \\ & & & B & & \end{array} \,.$$

With these, the factorization lemma (lemma 6.69) applies in $(\mathcal{C}^{*/}_{/B})_f$.

Let then $X \overset{s}{\to} \mathrm{Path}_B(X_2) \xrightarrow{(p_0,p_1)} X_2 \times_B X_2$ be a path space object for X_2 in the slice over B and consider the following commuting square

$$\begin{array}{ccc} X'_1 & \overset{s f t}{\longrightarrow} & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W}\downarrow & & \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} \,.$$

By factoring this through the pullback $(f,g)^*(p_0,p_1)$ and then applying the factorization lemma 6.69 and then two-out-of-three (def. 1.75) to the factoring morphisms, this may be replaced by a commuting square of the same form, where however the left morphism is an acyclic fibration

$$\begin{array}{ccc} X''_1 & \longrightarrow & \mathrm{Path}_B(X_2) \\ {}^{t}_{\in W \cap \mathrm{Fib}}\downarrow & & \downarrow^{(p_0,p_1)}_{\in \mathrm{Fib}} \\ X_1 & \overset{(f,g)}{\longrightarrow} & X_2 \underset{B}{\times} X_2 \end{array} \,.$$

This makes also the morphism $X''_1 \to B$ be a fibration, so that the whole diagram may now be regarded as a diagram in the category of fibrant objects $(\mathcal{C}_{/B})_f$ of the slice category over B.

As such, the top horizontal morphism now exhibits a right homotopy which under localization $\gamma_B : (\mathcal{C}_{/B})_f \longrightarrow \mathrm{Ho}(\mathcal{C}_{/B})$ (def. 6.26) of the slice model structure (prop.) we have

$$\gamma_B(f) = \gamma_B(g) \,.$$

The result then follows by observing that we have a commuting square of functors

$$\begin{array}{ccc} (\mathcal{C}^{*/}_{/B})_f & \overset{\text{fib}}{\longrightarrow} & \mathcal{C}^{*/} \\ \downarrow^{\gamma_B} & \swarrow & \downarrow^{\gamma} \\ \text{Ho}(\mathcal{C}^{*/}_{/B}) & \longrightarrow & \text{Ho}(\mathcal{C}^{*/}) \end{array} \,,$$

because, by lemma 6.71, the top and right composite sends weak equivalences to isomorphisms, and hence the bottom filler exists by theorem 6.29. This implies the claim. ▮

Homotopy fibers

We now discuss the homotopy-theoretic properties of the mapping cone- and mapping cocone-constructions from above.

Literature (Brown 73, section 4).

Remark 6.74. The factorization lemma 6.69 with prop. 6.63 says that the mapping cocone of a morphism f, def. 6.61, is equivalently the plain fiber, example , of a fibrant resolution $\tilde f$ of f:

$$\begin{array}{ccc} \text{Path}_*(f) & \longrightarrow & \text{Path}(f) \\ \downarrow & (\text{pb}) & \downarrow^{\tilde f} \\ * & \longrightarrow & Y \end{array} \,.$$

The following prop. 6.75 says that, up to equivalence, this situation is independent of the specific fibration resolution $\tilde f$ provided by the factorization lemma (hence by the prescription for the mapping cocone), but only depends on it being *some* fibration resolution.

Proposition 6.75. *In the category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 6.32, of a model structure on pointed objects (prop.) consider a morphism of fiber-diagrams, hence a commuting diagram of the form*

$$\begin{array}{ccccc} \text{fib}(p_1) & \longrightarrow & X_1 & \underset{\in \text{Fib}}{\overset{p_1}{\longrightarrow}} & Y_1 \\ \downarrow^{h} & & \downarrow^{g} & & \downarrow^{f} \\ \text{fib}(p_2) & \longrightarrow & X_2 & \underset{\in \text{Fib}}{\overset{p_2}{\longrightarrow}} & Y_2 \end{array} .$$

If f and g weak equivalences, then so is h.

Proof. Factor the diagram in question through the pullback of p_2 along f

$$\begin{tikzcd} \mathrm{fib}(p_1) \arrow[r] \arrow[d, "h"] & X_1 \arrow[d, "\in W"'] \arrow[dr, "p_1"] & \\ \mathrm{fib}(f^* p_2) \arrow[r] \arrow[d, "\simeq"] & f^* X_2 \arrow[r, "f^* p_2", "\in \mathrm{Fib}"'] \arrow[d, "\in W"] & Y_1 \arrow[d, "f", "\in W"'] \\ \mathrm{fib}(p_2) \arrow[r] & X_2 \arrow[r, "p_2", "\in \mathrm{Fib}"'] & Y_2 \end{tikzcd}$$

and observe that

1. $\mathrm{fib}(f^* p_2) = \mathrm{pt}^* f^* p_2 = \mathrm{pt}^* p_2 = \mathrm{fib}(p_2)$;
2. $f^* X_2 \to X_2$ is a weak equivalence by lemma 6.72;
3. $X_1 \to f^* X_2$ is a weak equivalence by assumption and by two-out-of-three (def. 1.75);

Moreover, this diagram exhibits $h : \mathrm{fib}(p_1) \to \mathrm{fib}(f^* p_2) = \mathrm{fib}(p_2)$ as the base change, along $* \to Y_1$, of $X_1 \to f^* X_2$. Therefore the claim now follows with lemma 6.71. ▮

Hence we say:

Definition 6.76. Let $\mathcal{C}$ be a model category and $\mathcal{C}^{*/}$ its model category of pointed objects, prop. . For $f : X \longrightarrow Y$ any morphism in its category of fibrant objects $(\mathcal{C}^{*/})_f$, def. 6.32, then its ***homotopy fiber***

$$\mathrm{hofib}(f) \longrightarrow X$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C}^{*/})$, def. 6.23, which is represented by the fiber, example , of any fibration resolution $\tilde{f}$ of f (hence any fibration $\tilde{f}$ such that f factors through a weak equivalence followed by $\tilde{f}$).

Dually:

For $f : X \longrightarrow Y$ any morphism in its category of cofibrant objects $(\mathcal{C}^{*/})_c$, def. 6.32, then its ***homotopy cofiber***

$$Y \longrightarrow \mathrm{hocofib}(f)$$

is the morphism in the homotopy category $\mathrm{Ho}(\mathcal{C})$, def. 6.23, which is represented by the cofiber, example , of any cofibration resolution of f (hence any cofibration $\tilde{f}$ such that f factors as $\tilde{f}$ followed by a weak equivalence).

Proposition 6.77. *The homotopy fiber in def. 6.76 is indeed well defined, in that for f_1 and f_2 two fibration replacements of any morphisms f in $\mathcal{C}_f$, then their fibers are isomorphic in $\mathrm{Ho}(\mathcal{C}^{*/})$.*

Proof. It is sufficient to exhibit an isomorphism in $\mathrm{Ho}(\mathcal{C}^{*/})$ from the fiber of the fibration replacement given by the factorization lemma 6.69 (for any choice of path space object) to the fiber of any other fibration resolution.

Hence given a morphism $f : Y \longrightarrow X$ and a factorization

$$f \;:\; X \underset{\in W}{\longrightarrow} \hat X \underset{f_1}{\overset{\in \mathrm{Fib}}{\longrightarrow}} Y$$

consider, for any choice $\mathrm{Path}(Y)$ of path space object (def. 6.16), the diagram

$$\begin{array}{ccc} \mathrm{Path}(f) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & X \\ {}^{\mathllap{\in W}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{\in W}} \\ \mathrm{Path}(f_1) & \overset{\in W \cap \mathrm{Fib}}{\longrightarrow} & \hat X \\ {}^{\mathllap{\in \mathrm{Fib}}}\downarrow & (\mathrm{pb}) & \downarrow^{\mathrlap{f_1}}_{\mathrlap{\in \mathrm{Fib}}} \\ \mathrm{Path}(Y) & \underset{\in W \cap \mathrm{Fib}}{\overset{p_1}{\longrightarrow}} & Y \\ {}^{\mathllap{p_0}}_{\mathllap{\in W \cap \mathrm{Fib}}}\downarrow \\ Y \end{array}$$

as in the proof of lemma 6.69. Now by repeatedly using prop. 6.75:

1. the bottom square gives a weak equivalence from the fiber of $\mathrm{Path}(f_1) \to \mathrm{Path}(Y)$ to the fiber of f_1;
2. The square

$$\mathrm{Path}(f_1$$

▮