T-duality: from circles to spheres and beyond

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Introduction

T-duality

- 87 Buscher rules.
- 90's with Hull, Roček, Verlinde and others.
- 96 SYZ: "Mirror symmetry is T-duality"
- Bouwknegt–Evslin–Mathai (2003): **Topological T-duality**
  - Exchange of curvature and H-flux,
  - Existence ("the T-dual bundle")
  - Twisted cohomology and K-theory.

T-duality

- Bunke–Schick (2005)
  - T-duals are not unique.
- Cavalcanti–Gualtieri (2005)
  - Generalized geometry

$$(TE \oplus T^*E)/S^1 \to (T\hat{E} \oplus T^*\hat{E})/S^1$$

$$X + f\partial_\psi + \xi + g\psi \mapsto X + g\partial_{\hat{\psi}} + \xi + f\hat{\psi}.$$

  - T-duality happens to a manifold. Structures come later.

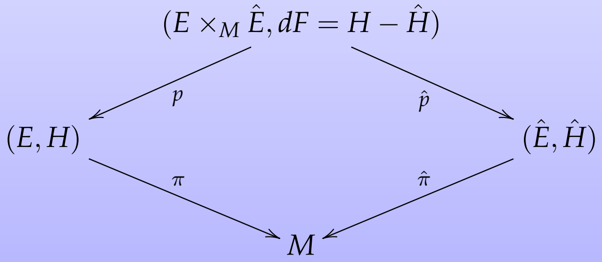
T-duality for torus bundles

T-duality — definition

Definition

Two principal torus bundles (E, H), $(\hat{E}, \hat{H}) \to M$ are *T-dual* if there is a 2-form $F \in E \times_M \hat{E}$ such that

- $dF = p^*H - \hat{p}^*\hat{H}$ and
- $F\colon \mathfrak{t} \times \hat{\mathfrak{t}} \to \mathbb{R}$ is nondegenerate.



T-duality

$$
\begin{array}{ccccc}
 & & (E \times_M \hat{E}, dF = p^*H - \hat{p}^*\hat{H}) & & \\
 & \swarrow^{p} & & \searrow^{\hat{p}} & \\
(E, H) & & & & (\hat{E}, \hat{H}) \\
 & \searrow_{\pi} & & \swarrow_{\hat{\pi}} & \\
 & & M & &
\end{array}
$$

Good to notice:

$$0 = \hat{p}_*([p^*H - \hat{p}^*\hat{H}]) = \hat{p}_*p^*[H] = \hat{\pi}^*\pi_*[H] \Rightarrow \pi_*[H] = \lambda[\hat{\varepsilon}].$$

T-duality

Example (T-duality for S^1-bundles — BEM 03)

Take connections ψ and $\hat{\psi}$ on E and $\hat{E}$.

$$\begin{cases} d\psi &= \varepsilon \\ H &= h + \psi\hat{\varepsilon} \end{cases} \rightsquigarrow \begin{cases} d\hat{\psi} &= \hat{\varepsilon} \\ \hat{H} &= h + \varepsilon\hat{\psi} \end{cases}$$

Then

- $H - \hat{H} = -d(\psi\hat{\psi}) = dF$
- $\pi_* p_* F = 1$

T-duality — Examples

Example (Typical example)

$(S^1 \times S^2, H)$, where $H = \psi \wedge \sigma$ is a unitary volume form and $(S^3, 0)$ are T-dual.

$$\begin{cases} d\psi & = \varepsilon \\ H & = h + \psi\hat{\varepsilon} \end{cases} \quad \rightsquigarrow \quad \begin{cases} d\hat{\psi} & = \hat{\varepsilon} \\ \hat{H} & = h + \varepsilon\hat{\psi} \end{cases}$$

$$F = -\hat{\psi}\psi.$$

T-duality — Examples

Example (Typical example)

$(S^1 \times S^2, H)$, where $H = \psi \wedge \sigma$ is a unitary volume form and $(S^3, 0)$ are T-dual.

$$\begin{cases} d\psi & = \varepsilon = 0 \\ H & = h + \psi\hat{\varepsilon} = 0 + \psi \wedge \sigma \end{cases} \rightsquigarrow \begin{cases} d\hat{\psi} & = \hat{\varepsilon} = \sigma \\ \hat{H} & = h + \varepsilon\hat{\psi} = 0 + 0 \end{cases}$$

$$F = -\hat{\psi}\psi.$$

T-duality — Examples

Example (Special example)

(S^3, H) with H a unitary volume form is **self T-dual**.

$$\begin{cases} d\psi &= \sigma \\ H &= \psi\sigma \end{cases} \quad \rightsquigarrow \quad \begin{cases} d\hat{\psi} &= \sigma \\ \hat{H} &= \sigma\hat{\psi} \end{cases}$$

$$F = -\hat{\psi}\psi.$$

T-duality

Example (Torus bundles)

For higher dimensional tori

$$dF = p^*H - \hat{p}^*\hat{H} \rightsquigarrow F = F_{ij}\psi_i\hat{\psi}_j + p^*B_E + \hat{p}^*B_{\hat{E}},$$

with

$$(F_{ij})_{ij} \text{ nondegenerate} \qquad \& \qquad dF_{ij} = 0$$

H and $\hat{H}$ have 'only one leg' on the fibers:

$$H = \psi_1\hat{\varepsilon}_1 + \cdots + \psi_k\hat{\varepsilon}_k + h$$

$$F = -\sum \hat{\psi}_i\psi_i.$$

T-duality

Example (Torus bundles – Existence)

For a T^k-bundle $(E, H) \to M$ and

$$\begin{cases} d\psi_i &= \varepsilon_i \\ H &= \sum \psi_i \hat{\varepsilon}_i + h. \end{cases} \quad \rightsquigarrow \quad \begin{cases} d\hat{\psi}_i &= \hat{\varepsilon}_i \\ \hat{H} &= \sum \varepsilon_i \hat{\psi}_i + h. \end{cases}$$

$$F = -\sum \hat{\psi}_i \psi_i.$$

T-duality

Example (Torus bundles – Non uniqueness BS05)

For a T^2-bundle $(E, H) \to M$,

$$\begin{cases} d\psi_i &= \varepsilon_i \\ H &= 0 \end{cases} \rightsquigarrow \begin{cases} d\hat{\psi}_i &= 0 \\ \hat{H} &= \sum \varepsilon_i \hat{\psi}_i + h. \end{cases}$$

$$F = -\sum \hat{\psi}_i \psi_i.$$

T-duality

Example (Torus bundles – Non uniqueness BS05)

For a T^2-bundle $(E, H) \to M$,

$$\begin{cases} d\psi_i & = \varepsilon_i \\ H & = 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} d\hat{\psi}_i & = \varepsilon_i \\ \hat{H} & = 0. \end{cases}$$

$$F = (\psi_1 - \hat{\psi}_1)(\psi_2 - \hat{\psi}_2).$$

T-duality isomorphisms

T-duality isomorphism I

Theorem (BEM03, BHM03...)

$$\tau_F\colon (\Omega^\bullet(E), d^H) \to (\Omega^\bullet(\hat{E}), d^{\hat{H}}), \qquad \tau_F(\varphi) = \hat{p}_* e^F p^* \varphi$$

induces an isomorphism in twisted cohomology.

Proof.

The condition $dF = p^*H - \hat{p}^*\hat{H}$ implies it is a map of complexes:

$$\begin{aligned}\tau_F(d^H\varphi) &= \hat{p}_* e^F p^* d^H \varphi = \hat{p}_* d^{p^*H - dF}(e^F p^* \varphi)\\ &= \hat{p}_* d^{\hat{p}^*\hat{H}}(e^F p^*\varphi) = d^{\hat{H}} \hat{p}_* e^F p^* \varphi\\ &= d^{\hat{H}} \tau_F(\varphi).\end{aligned}$$

T-duality isomorphism I

Proof.

Decomposing $F = p^*F_E + \lambda\hat{\psi}\wedge\psi + \hat{p}^*F_{\hat{E}}$, we see that up to the action of B-fields, τ_F is equivalent to $\tau_{\lambda\hat{\psi}\wedge\psi}$.
For $\tau_{\lambda\hat{\psi}\wedge\psi}$ we perform a direct computation:

$$\begin{aligned}\tau_{\lambda\hat{\psi}\wedge\psi}(\varphi_0+\psi\varphi_1) &= \hat{p}_* e^{\lambda\hat{\psi}\wedge\psi}\varphi_0+\psi\varphi_1\\ &= \hat{p}_*\varphi_0+\psi\varphi_1+\lambda\hat{\psi}\wedge\psi\varphi_0\\ &= \varphi_1-\lambda\hat{\psi}\varphi_0.\end{aligned}$$

which is an isomorphism. □

Second T-duality Isomorphism
Courant Algebroids

A bracket on $TE \oplus T^*E$

- $X + \xi$ acts on forms via interior and exterior product:

$$(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi$$

- The graded commutator is again a section of $TE \oplus T^*E$:

$$\{\{X + \xi, d\}, Y + \eta\} = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi =: [\![X + \xi, Y + \eta]\!]$$

$$\{\{X+\xi, d^H\}, Y+\eta\} = [X, Y]+\mathcal{L}_X \eta-\iota_Y d\xi-\iota_X \iota_Y H =: [\![X+\xi, Y+\eta]\!]_H$$

B-field action

- $B \in \Omega^2(E)$ acts on $TE \oplus T^*E$ via

$$e^B(X+\xi) = X - \iota_X B + \xi.$$

- We have

$$e^B((X+\xi)\cdot\varphi) = e^B(X+\xi)\cdot e^B\varphi.$$

Lemma

$$e^B([\![X+\xi, Y+\eta]\!]_H) = [\![e^B(X+\xi), e^B(Y+\eta)]\!]_{H+dB}$$

T-duality isomorphism II

Theorem (— & Gualtieri 05)

$$\mathcal{T}_F \colon (TE \oplus T^*E)/T^k \to (T\hat{E} \oplus T^*\hat{E})/T^k, \qquad \mathcal{T}_F(v) = \hat{p}_* e^F p^* v$$

satisfies

$$\tau_F(v \cdot \varphi) = \mathcal{T}_F(v) \cdot \tau_F(\varphi),$$

hence it is an isomorphism of Courant algebroids.

Proof.

Thing to notice:

- $p^* \colon TE \oplus T^*E \to TE \times_M \hat{E} \oplus T^*E \times_M \hat{E}$ is multivalued.
- $\hat{p}_* \colon TE \times_M \hat{E} \oplus T^*E \times_M \hat{E} \to T\hat{E} \oplus T^*\hat{E}$ is partially defined.

Play these two against each other. □

T-duality isomorphism II

Proof.

$$\begin{aligned}\tau_F([\![v,w]\!]_H\cdot\varphi) &= \tau_F(\{\{v,d^H\},w\}\varphi)\\ &= \{\{\mathcal{T}_F v, d^{\hat{H}}\},\mathcal{T}_F w\}\cdot\tau_F(\varphi)\\ &= [\![\mathcal{T}_F v,\mathcal{T}_F w]\!]_{\hat{H}}\tau_F(\varphi).\end{aligned}$$

□

Example

For T-duals with $F = -\hat{\psi}\psi$, we have

$$\mathcal{T}_F(X + f\partial_\psi + \xi + g\psi) = X + g\partial_{\hat{\psi}} + \xi + f\hat{\psi}$$

What if...

- The group of symmetries is not Abelian?

$\rightsquigarrow$ de la Ossa–Quevedo (92), C. Klimčík and P. Ševera (95).

$\rightsquigarrow$ Mathai *et al* (15, 16), Lint–Sati–Westerland (16), Cavalcanti *et al* (24): Spherical T-duality.

Spherical T-duality

Spherical T-duality — definition

Definition (LSW16,CHU25)

Two oriented sphere bundles $S^{2n-1} \cdots (E, H) \xrightarrow{\pi} M$ and $S^{2k-1} \cdots (\hat{E}, \hat{H}) \xrightarrow{\hat{\pi}} M$ endowed with closed forms of degree $2(n+k)-1$ are *T-dual* if there is a form $F \in \Omega^{2(n+k)-2}(E \times_M \hat{E})$ such that

- $dF = p^*H - \hat{p}^*\hat{H}$ and
- $\pi_* p_* F = \lambda \neq 0$.

If $\lambda = 1$, we say that they are a *unimodular* T-dual pair.

Spherical T-duality

Good to notice:

$$0 = \hat{p}_*([p^*H - \hat{p}^*\hat{H}]) = \hat{p}_* p^*[H] = \hat{\pi}^* \pi_*[H] \Rightarrow \pi_*[H] = \lambda[\hat{\varepsilon}].$$

Spherical T-duality

Example

Take global angular forms ψ and $\hat{\psi}$ on E and $\hat{E}$.

$$\begin{cases} d\psi &= \varepsilon \\ H &= h + \psi\hat{\varepsilon} \end{cases} \rightsquigarrow \begin{cases} d\hat{\psi} &= \hat{\varepsilon} \\ \hat{H} &= h + \varepsilon\hat{\psi} \end{cases}$$

Then

- $H - \hat{H} = -d(\psi\hat{\psi}) = dF$
- $\pi_* p_* F = 1$

Spherical T-duality

Example

- $S^3 \cdots (S^7, 0) \to S^4$ is T-dual to $(S^3 \times S^4, H)$, with H a unit volume form.
- $S^3 \cdots (S^7, H) \to S^4$ with H a unit volume form is self T-dual.
- $S^1 \cdots S^{2n+1} \to \mathbb{C}P^n$ with H a volume form is T-dual to $ST\mathbb{C}P^n$, the sphere bundle of $T\mathbb{C}P^n$, with $\hat{H}$ a generator of $H^{2n+1}(ST\mathbb{C}P^n)$.

The Gysin complex

Given $S^{2n-1} \cdots (E, H) \xrightarrow{\pi} M$, pick a *global angular form* $\psi \in \Omega^{2n-1}(E)$:

- $\pi_* \psi = 1$,
- $d\psi = \pi^* \varepsilon, \quad ([\varepsilon] = \text{Euler class of } E)$.

Definition

The *Gysin complex*

$$\Omega_\psi = \wedge^\bullet \langle \psi \rangle \otimes \Omega(M) \subset \Omega(E).$$

Theorem (Gysin 42)

The inclusion $\Omega_\psi \hookrightarrow \Omega(E)$ is a quasi-isomorphism.

Assume H and $\hat{H}$ lie in the corresponding Gysin complexes.

Existence

Theorem

$S^{2n-1} \cdots (E, H) \xrightarrow{\pi} M$ has a T-dual iff $\pi_* H$ is a multiple of an integral class.
It has a unimodular T-dual iff $\pi_* H$ is the Euler class of some sphere bundle.

Isomorphisms

Isomorphism I

Theorem (BEM15, LSW16,CHU25)

If $(E,H)\xrightarrow{\pi} M$ *and* $(\hat{E},\hat{H})\xrightarrow{\hat{\pi}} M$ *are T-dual with* $dF = H - \hat{H}$*, then*

$$\tau_F\colon (\Omega^\bullet(E), d^H) \to (\Omega^\bullet(\hat{E}), d^{\hat{H}}), \quad \tau_F = \hat{p}_* e^F p^*$$

is a quasi-isomorphism.

Proof.

$dF = H - \hat{H} \quad \Rightarrow \quad \tau_F$ is a map of complexes.

The Gysin complex

Lemma

Given $S^{2n-1} \cdots (E,H) \xrightarrow{\pi} M$ and a global angular form $\psi \in \Omega^{2n-1}(E)$, if $H \in \Omega_\psi$, then (Ω_ψ, d^H) is a dga and the inclusion $(\Omega_\psi, d^H) \to (\Omega(E), d^H)$ a quasi-isomorphism.

Lemma

If $(E,H) \xrightarrow{\pi} M$ and $(\hat{E},\hat{H}) \xrightarrow{\hat{\pi}} M$ are T-dual and H and $\hat{H}$ are in the Gysin complexes, then there is $F' \in \Omega_{\psi,\hat{\psi}}$ such that τ_F and $\tau_{F'}$ induce the same map in twisted cohomology.

end of the proof of he first isomorphism.

Proof boils down to the case $F = \lambda\hat{\psi} \wedge \psi$.
In this case τ_F is an *isomorphism of Gysin complexes*:

$$\tau_F(\varphi_0 + \psi\varphi_1) = \varphi_1 - \lambda\hat{\psi}\varphi_0.$$

□

Isomorphism II – Extended Higher Courant Algebroids

Guess 1: Higher Courant algebroid:

$$TE \oplus \wedge^{2(n+k)-3} T^*E$$

No direct analogue over M.

Guess 2:

$$C_\psi =$$

Extended Higher Courant Algebroids

Guess 1: Higher Courant algebroid:

$$TE \oplus \wedge^{2(n+k)-3} T^*E$$

No direct analogue over M.

Guess 2:

$$C_\psi = TM \oplus \wedge^{2n-2} T^*M \otimes \partial_\psi \oplus \psi \otimes \wedge^{2k-2} T^*M \oplus \wedge^{2(n+k)-3} T^*M$$

Isomorphism II

Theorem (CHU24)

Given a T-dual pair, with H, $\hat{H}$ and F in the appropriate Gysin complexes $\mathcal{T}_F = \hat{p}_* e^F p^*$ is an isomorphism of higher extended Courant algebroids.
Further

$$\tau_F(v \cdot \varphi) = \mathcal{T}_F(v) \cdot \tau_F(\varphi).$$

Extended Higher Courant Algebroids

Remark

- *An isomorphism of higher twisted K-theory still holds in this case [LSW16].*
- *The space of self-dual sections C_ψ*

$$C_\psi^{SD} = TM \oplus \psi \otimes \wedge^{2n-2} T^*M \oplus \wedge^{4n-3} T^*M$$

  appear also in the context of B_n-geometry and exceptional generalized geometry ($n = 1, 2$ cases).
- *Self-T-duality equations also appear in the work of Lind, Sati, Schreiber, Westerland.*

Transgressive T-duality

Transgressive Fibrations

Definition

Given a fibration $F \cdots E \xrightarrow{\pi} M$, a cohomology class $A \in H^\bullet(F)$ is *transgressive* if there is a form $\psi \in \Omega^\bullet(E)$ such that

- $[\iota^*\psi] = A$ (where $\iota\colon F \to E$ is the inclusion of a reference fiber) and
- $d\psi = \pi^*\varepsilon$ for some $\varepsilon \in \Omega^\bullet(M)$.

Transgressive Fibrations

Definition

A fibration $F \cdots E \xrightarrow{\pi} M$ is *transgressive* if the cohomology algebra $H^\bullet(F)$ is generated by transgressive elements.
An *odd transgressive fibration* is one whose fibers have the same cohomology algebra of a product of odd spheres.

Transgressive Fibrations

Example

Given a Hermitian vector bundle $V \to M$, any partial frame bundle $\mathrm{U}(n)/\mathrm{U}(k) \cdots E \xrightarrow{\pi} M$ is an odd transgressive fibration over M.

- Line bundles $\rightsquigarrow$ $U(1) = S^1$ $\rightsquigarrow$ principal circle bundles.
- Partial frame with a single vector $\rightsquigarrow$ S^{2n-1}-bundles over M.
- Full frame $\rightsquigarrow$ principal $\mathrm{U}(n)$-bundle.
- In general, the cohomology of $\mathrm{U}(n)/\mathrm{U}(k)$ is generated by one element in each degree from $2k+1$ to $2n-1$ with

$$d\psi_{2j-1} = c_{2j}(V).$$

Transgressive Fibrations

Lemma

If $E \to M$ and $\hat{E} \to M$ are transgressive fibrations, so is $E \times_M \hat{E}$.

Transgressive Fibrations

Definition

Given an odd transgressive fibration $E \xrightarrow{\pi} M$, and a choice of forms $\psi^j_{2i-1} \in \Omega^{2i-1}(E)$ such that

- $\Psi = \{\psi^j_{2i-1} : j = 1, \ldots, n_i\}$ generates the cohomology of the fiber
- $d\psi^j_{2i-1} = \pi^* c^j_{2i}$.

Ψ is a *transgressive generating set* and the complex

$$\Omega^\bullet_\Psi(M) = \wedge^\bullet \langle \psi^j_{2i-1} : i, j\rangle \otimes \Omega^\bullet(M) \subset \Omega^\bullet(E)$$

is the *transgressive subcomplex*.

Transgressive Fibrations

Lemma

The inclusion $(\Omega_\Psi(M), d) \hookrightarrow (\Omega^\bullet(E), d)$ *is a quasi-isomorphism.*

Lemma

Given a transgressive generating set Ψ*, if* $H \in \Omega_\Psi(M)$ *is closed, then the inclusion* $(\Omega_\Psi(M), d^H) \hookrightarrow (\Omega^\bullet(E), d^H)$ *is a quasi-isomorphism.*

T-duality

Definition

Two odd transgressive fibrations $(E, \Psi, H) \xrightarrow{\pi} M$ and $(\hat{E}, \hat{\Psi}, \hat{H}) \xrightarrow{\hat{\pi}} M$ are T-dual if

- $p^*H - \hat{p}^*\hat{H} = dF$
- $\hat{p}_* \circ e^{F_0'} \circ p^* : \wedge^\bullet \langle\Psi\rangle \to \wedge^\bullet\langle\hat{\Psi}\rangle$ is an isomorphism where F_0 is the basic degree 0 component of F.

Isomorphism I

Theorem (Cavalcanti 25)

Given a pair of T-dual odd transgressive fibrations $(E, H) \xrightarrow{\pi} M$ *and* $(\hat{E}, \hat{H}) \xrightarrow{\hat{\pi}} M$*, the map*

$$\tau_F = \hat{p}_* e^F p^*$$

induces an isomorphism in cohomology.

Proof.

- $p^* H - \hat{p}^* \hat{H} = dF \quad \Rightarrow \quad \tau_F$ is a map of complexes.
- F nondegenerate $\Rightarrow \tau_F$ is a quasi-isomorphism.

□

Further extended even higher Courant algebroids

Definition

Given (M, Ψ) and a closed odd form $H \in \Omega_\Psi(M)$, let

$$E_\Psi := TM \oplus (\mathrm{Clif}(\langle \Psi \rangle \oplus \langle \Psi \rangle^*) \otimes \wedge T^*M)$$

E_Ψ acquires a natural Courant-like bracket

$$[v, w]_H \cdot \varphi := \{\{v, d^H\}, w\}\varphi.$$

This is the *Clifford–Courant algebroid* of associated to (M, Ψ, H).

Isomorphism II

Theorem (Cavalcanti 25)

If (E, Ψ, H) *and* $(\hat{E}, \hat{\Psi}, \hat{H})$ *are transgressive T-duals, then there is a isomorphism* $\mathcal{T}_F$ *of Clifford–Courant algebroids satisfying*

$$\tau_F(v \cdot \varphi) = \mathcal{T}_F(v) \cdot \tau_F(\varphi).$$

Examples

Examples

Example (High turns low, and low turns high)

Frame bundle $\mathrm{U}(n) \cdots (E, H) \to M$.

$$\begin{cases} d\psi_i & = c_i \\ H & = \sum \lambda_i \psi_i \hat{c}_{n-i} + h. \end{cases} \quad \rightsquigarrow \quad \begin{cases} d\hat{\psi}_{n-i} & = \hat{c}_{n-i} \\ \hat{H} & = \sum \lambda_{n-i} c_i \hat{\psi}_{n-i} + h. \end{cases}$$

$$F = -\lambda_i \sum \psi_i \hat{\psi}_{n-i}.$$

Examples

Example (Quadratic relations)

Given two bundles $V^n, \hat{V}^m \to M$ and a quadratic relation with nonzero coefficients

$$0 = [\lambda_1 \hat{c}_m c_k + \lambda_2 \hat{c}_{m-1} c_{k+1} + \cdots + \lambda_{n-k+1} \hat{c}_{m+k-n} c_n]$$

The partial frame bundles

$$U(n; V)/U(k-1; V) \quad U(m; \hat{V})/U(m-(n-k+1); \hat{V})$$

are T-dual

Examples

Example (Inverse bundle)

Let V^{-1} be an additive inverse of $V \to M$, that is $V \oplus V^{-1} = \mathbb{C}^m \times M$. From the product formula

$$\sum_{k=1}[c_k(V)c_{j-k}(\hat{V})] = 0 \text{ for } j > 1$$

All the partial frame bundles of V and $\hat{V}$ are unimodular T-dual for appropriate choices of H.
This is still true after stabilization.

Examples

Example (Low stays low)

If the Chern classes of two vector bundles $V, \hat{V} \to M$ satisfy a collection of relations

$$[c_i \wedge \hat{c}_i] = 0, \quad i = 1, \dots, n.$$

Then there is h_i on the frame bundle of V for which $H_i = \psi \wedge \hat{c}_i + h_i \in \Omega^{4i-1}|(E)$ is closed (similarly for $\hat{H}_i$) Then $(E, H_3 + \cdots + H_{4n-1})$ is T-dual to $(\hat{E}, \hat{H}_3 + \cdots + \hat{H}_{4n-1})$
In this case $F = \sum \hat{\psi}_{2i+1} \wedge \psi_{2i+1}$.

Examples

Example (Nearly the identity)

Let $V, \hat{V} \to M$ be a Hermitian vector bundle of rank $2n+1$. Assume that for each i, there are nonzero constants such that

$$c_i(V) + \lambda_i c_i(\hat{V}) = 0$$

Let $(E, 0), (\hat{E}, 0) \to M$ be their frame bundles and take

$$F = \Pi_i(\psi_i + \lambda_i \hat{\psi}_i)$$

- $dF = 0 = H - \hat{H}$
- τ_F is an isomorphism
- F induces no "duality" of generators.