PONTRJAGIN DUALITY FOR FINITE GROUPS

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Motivation: duality for vector spaces

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$T : V \to W, S : W \to U, (S \circ T)^* = T^* \circ S^* : U^* \to W^*$

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(b) G *is isomorphic with* $\widehat{G}$, *in particular* $|G| = |\widehat{G}|$.

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- Assertion holds for all finite abelian groups.

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$\widehat{G}$ is always abelian

Since $\widehat{G}$ is always abelian for nonabelian groups one should not hope to have $G \cong \widehat{\widehat{G}}$.

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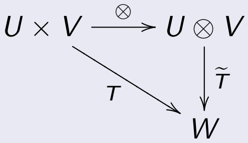

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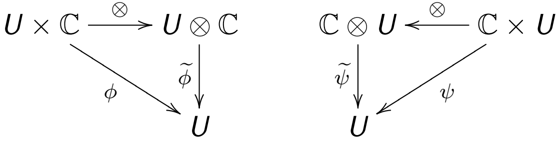

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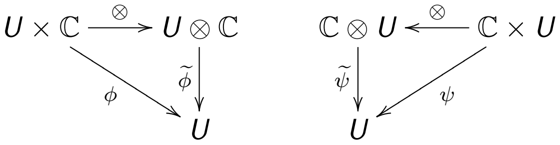

$U \otimes \mathbb{C} \cong U \cong \mathbb{C} \otimes U$

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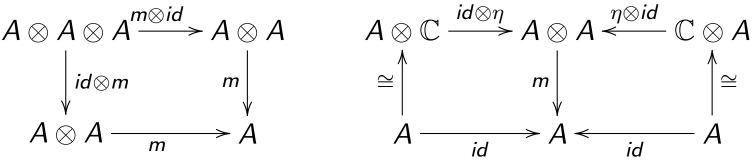

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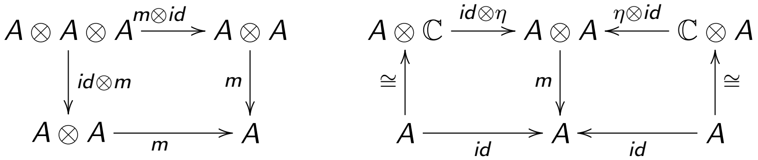

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$(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$, $a \cdot \eta(1) = \eta(1) \cdot a = a$

The algebra $(A \otimes A, m_2, \eta_2)$

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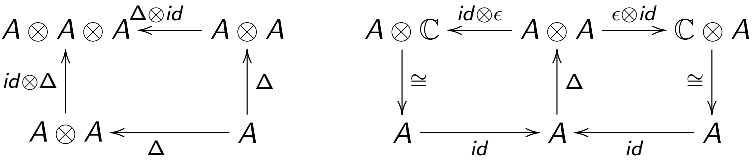

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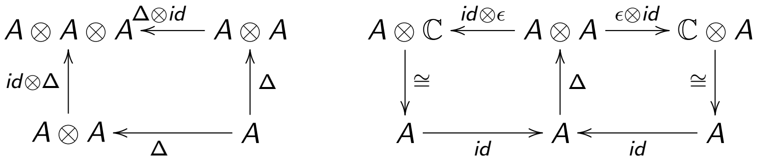

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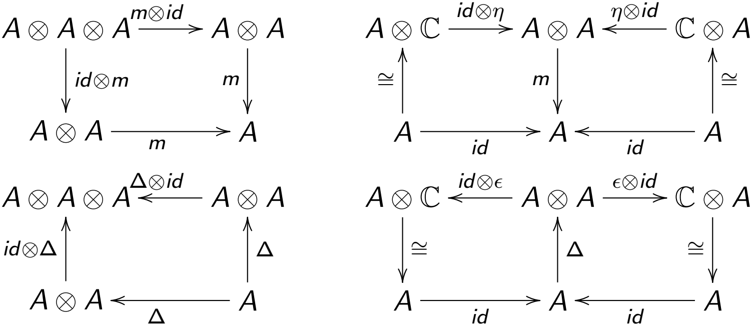

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Hopf algebra

Hopf algebra $(A, m, \eta, \Delta, \epsilon, S)$

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$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \eta \circ \epsilon$

$$
\begin{array}{ccccc}
A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A \\
{\scriptstyle S\otimes id}\downarrow & & \downarrow{\scriptstyle \epsilon\circ\eta} & & \downarrow{\scriptstyle id\otimes S} \\
A \otimes A & \xrightarrow[m]{} & A & \xleftarrow[m]{} & A \otimes A
\end{array}
$$

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$(H, m, \eta, \Delta, \epsilon, S)$ Hopf algebra, H finite dimensional

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$\{\xi_g | g \in G\}$ is the dual basis of $\{\delta_g | g \in G\}$.
$\langle m_G^*(\xi_{g_1} \otimes \xi_{g_2}), \delta_g\rangle = \delta_{g,g_1g_2} = \langle \xi_{g_1} \otimes \xi_{g_2}, \Delta_G(\delta_g)\rangle$

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provided G *is abelian*

Proof: (c) Note $C(G)$ has basis $\{\delta_g | \delta_g : h \mapsto \delta_{g,h}, g \in G\}$
$m_G(\delta_g \otimes \delta_h) = \delta_{g,h}\delta_g, \Delta_G(\delta_g) = \sum_{g=g_1g_2} \delta_{g_1} \otimes \delta_{g_2}$
$\{\xi_g | g \in G\}$ is the dual basis of $\{\delta_g | g \in G\}$.
$\langle m^*_G(\xi_{g_1} \otimes \xi_{g_2}), \delta_g \rangle = \delta_{g,g_1g_2} = \langle \xi_{g_1} \otimes \xi_{g_2}, \Delta_G(\delta_g) \rangle$
$\langle \Delta^*_G(\xi_g), \delta_{g_1} \otimes \delta_{g_2} \rangle = \delta_{g,g_1}\delta_{g,g_2} = \langle \xi_g, m_G(\delta_{g_1} \otimes \delta_{g_2} \rangle$

Pontryagin duality

$$G \longleftrightarrow C(G)$$

Pontryagin duality

$$\begin{array}{ccc} G & \longleftrightarrow & C(G) \\ \downarrow & & \downarrow \\ \widehat{G} & \longleftrightarrow & C(\widehat{G}) \cong C^*(G) \cong (C(G)^* \end{array}$$

Pontryagin duality

$$\begin{array}{ccc}
G & \longleftrightarrow & C(G) \\
\downarrow & & \downarrow \\
\widehat{G} & \longleftrightarrow & C(\widehat{G}) \cong C^*(G) \cong (C(G))^* \\
\downarrow & & \downarrow \\
\widehat{\widehat{G}} \cong G & \longleftrightarrow & C(\widehat{\widehat{G}}) \cong C(\widehat{G})^* \cong (C(G))^{**} \cong C(G)
\end{array}$$

Thank you for your attention !

slides are available at

www.imsc.res.in/∼parthac/talks/Summer-talk-2012.pdf