FREE GROUPS AND AMALGAMATED PRODUCT

A REPORT

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by

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CERTIFICATE

This is to certify that **Abhay Pratap Singh Chandel**, BS-MS (Mathematics), has worked on the project report entitled **'Free Groups and amalgamated product'** under my supervision and guidance. The content of the project report has not been submitted elsewhere by him/her for the award of any academic or professional degree.

April 2013 **Prof. Kashyap Rajeevsarathy**
IISER Bhopal

Committee Member	Signature	Date

DECLARATION

I hereby declare that this project report is my own work and due acknowledgement has been made wherever the work described is based on the findings of other investigators. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

April 2013 **Abhay Pratap Singh Chandel**
IISER Bhopal

Dedicated to my father

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I would like to dedicate my work to my father. It is his love and inspirations through all these years which kept me enthusiastic and helped me do this project.

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Abhay Pratap Singh Chandel

ABSTRACT

In this project we will study free groups and basic algebraic topology to establish that subgroup of a free group is free with the help of basic graph theory. Then in the second part of project we studied covering space of graph theory. Notice that some notations of graph theory in the second part differ from the first part where we used basic graph theory. Moreover, we studied amalgamated products in detail which helped us in establishing our main result that $\mathrm{SL}(2, \mathbb{Z})$ is the amalgamated product of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ with $\mathbb{Z}_2$ amalgamated that is

$$\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \ .$$

In the first part of the project we tried to establish that subgroup of a free group is free. In order to show this result, we shall introduce CW-pairs and briefly discuss how the homotopy extension property can lead to homotopy equivalences. We then use this concept and the contractibility of tress, to establish that a graph is homotopically equivalent to a wedge of circles and consequently, its fundamental group is free. Finally, using covering space theory, we shall establish the required result.

Then in the second part we studied amalgamated products and their structure for which we rigorously studied direct limits of family of groups. Also, as one of result we establish that every element in amalgamated product can be uniquely composed via homomorphism maps. Further we studied graph of groups and with deep understanding of trees we established some very interesting results *e.g.* tree of even finite diameter, has a vertex, which is invariant under all automorphisms. Then with concluding the project we establish our main result

$$\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \ .$$

LIST OF SYMBOLS OR ABBREVIATIONS

$G = (V, E)$	Graph with vertex V and edge E
T	Tree
$F(X)$	Free group on set X
$\langle R \rangle$	Subgroup generated by R, $R \subset G$
$\langle\langle R \rangle\rangle$	Normal subgroup generated by $R \subset G$
e_α^n	α no. of $n - cells$
$\partial(X)$	boundary of some set X
$f\|_A$	f restricted to A
HEP	Homotopy Extension Property
$f \simeq g$	f and g are homotopic.
e_{a_0}	constant map to a_0
X/A	X quotient A
i_A	identity map on A
$\tau : A \to X$	inclusion map.
$A * B$	Free product of A and B
$\mathbb{Z}$	Group of all Integers
$\mathbb{R}$	Real Number Field
θ_g	orbit of g under a quotient map.
$o(y)\ and\ t(y)$	terminal vertices of edge y of a graph.
$\pi_1(X)$	fundamental group of space X

CONTENTS

1. GRAPHS AND FREE GROUPS

1.1 Introduction

As the first part of the project we establish that subgroup of a free group is free. For this we study some terminologies and definitions in Graph theory. Definition of graph will change in our next chapter, it will be according to sirre's theory [5]. Then we study Free group theory in which we understand universal property for free groups and give a presentation for a group (see [1]). Moreover, we will discuss Cell complex theory and see how to construct a cell complex $n-cells$ and using this we see that a graph is a 1 dimensional cell complex. After that we move to covering space theory, in particular for graphs and as an application of it we establish that subgroup of a free group is free (See [4] and [2]). Since, every group can be realized as the fundamental group of some space, thus in this section we see that how algebraic topology is connected to free group theory which gives us the motivation to study this subject.

1.2 Graph Theory

Definition 1.1. A *graph* is an ordered pair $G = (V, E)$, where V is a set of vertices together with a set E of edges whose element are two element subset of V.

Definition 1.2. A graph $G' = (V', E')$ is *subgraph* of G if $V' \subset V$ and $E' \subset E$.

Definition 1.3. A *Path* in a graph is a sequence of edges $e_1e_2...e_n$ such that it connects a sequence of vertices. A path has a start vertex and an end

vertex which are called as *terminal vertices* of path and the other vertices in path are called *internal vertices*.
A *cycle* is a path whose start and end vertices that is terminal vertices are same.

Definition 1.4. A Graph $G = (V, E)$ is *connected* if there is a path between any two vertices in V.

Definition 1.5. A *simple graph* is a connected graph which contains no cycles. In a simple graph the edges of the graph form a set (rather than a multiset) and each edge is a distinct pair of vertices.

Definition 1.6. A tree is a undirected simple graph i.e. it is a connected and has no cycles.

Definition 1.7. A tree T in X is maximal if there is no tree in X such that it properly contains T.

Theorem 1.8. *Let X be a connected graph. A tree T in X is maximal if and only if it contains all the vertices of X.*

Proof. Suppose $X = (V, E)$ be a graph.
Let T be a tree that contains all vertices of X. Suppose that Y is a subgraph of X such that $T \subset Y$ (proper) that is edges set and vertex set of T are properly contain in edges set and vertex set of Y. We will show T contains a cycle; it follows that T is maximal.
Let e be a edge in $Y \setminus T$. If e has end points a, b, then $a, b \in T \subset Y$. Since T is connected.
Therefore, $\exists$ a path $P = e'_0.e'_1e'_2...e'_n$ in T from a to b.
Thus, passing this path to edge e, we get a cycle $P \cup e$ in Y which implies Y is not a tree. Thus T is maximal.

Now suppose that tree T is maximal. We intend to establish that T contains all vertices of X. Suppose not, let $x_0 \in X \setminus T$. Since X is connected hence there exists a path from x_0 to a vertex v in T. Assume $P = x_0.x_1...x_n$ be a path in X connecting x_0 and $x_n = v$ Let i be the smallest index such that $x_i \in T$.

Let e_i be the edge between x_{i-1} and x_i. Then $T \cup e$ is a tree. Clearly $T \subset T \cup e$, which contradicts the maximality of T. Thus T contains all the vertices of X. $\square$

1.3 Free group

Definition 1.9. A group G is called $free$ if there exists a subset X of G such that every element of G can be written uniquely as the product of finitely many elements from X and $X^{-1} = \{x^{-1} | x \in X\}$. X is called the basis set of G

Example 1.10. The group of integers $(\mathbb{Z}, +)$ is free on the set $X\{= 1\}$. Moreover, free group on two letters $F(a, b)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$

Definition 1.11. Let X be any set and let $X^{-1} = \{x^{-1} | x \in X\}$, then group generated by $X \cup X^{-1}$ is said to be free group on X, denoted as $F(X)$, every element of $F(X)$ is called a *word* (see [3]). A word $w \in F(X)$ is said to be reduced if there exist no twistors(that is subwords) of the form xx^{-1} for all $x \in X$ in w.

Example 1.12. Let $X = \{x, y\}$, $X^{-1} = \{x^{-1}, y^{-1}\}$. Let w be a word in $F(X)$ such that

$$w = x^2yx^{-3}x^2y^{-2}x^5y^2 \ .$$

It is not reduced word since it has two twistors $x^{-1}x$. Reduced word w will look like

$$w = x^2yx^{-1}y^{-2}x^5y^2 \ .$$

Universal property of free groups

Theorem 1.13. *Let G be any group and let $f : X \to G$ be a map. Then there exists a unique group homomorphism $\phi : F(X) \to G$ such that $f = \phi \circ \tau$, where $\tau : X \hookrightarrow F(X)$ is a inclusion map and the following Figure 1.1 commutes. That is we have that homomorphisms $\phi : F(X) \to G$ are in one to one correspondence with the functions $f : X \to G$.*

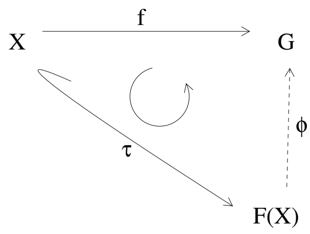

Fig. 1.1:

Proof. It is a well known fact and proof can be refer from [1]. □

Definition 1.14. Let G be a group and R be a subset of G, then $\langle R\rangle$ is the group generated by R and $\langle\langle R\rangle\rangle$ is the *normal* group generated by R and is defined as:

$$\langle R\rangle := \{r_1^{l_1}........r_k^{l_k} | k \in \mathbb{N} \cup \{0\}, r_1, ..., r_k \in R\}.$$

$$\langle R\rangle := \{m_1, ...m_k | k \in \mathbb{N} \cup \{0\}, m_i = \text{ monomonials}\}.$$

$$\langle\langle R\rangle\rangle := \{m_1^{g_1}.........m_k^{g_k} | k \in \mathbb{N} \cup \{0\}, m_i = \text{ monomonials}, g_i \in G\} ,$$

where $m_i^{g_i} = g_i^{-1} m_i g_i$.
Also $\langle R\rangle$ is the smallest subgroup of G containing G and $\langle\langle R\rangle\rangle$ is the smallest normal subgroup of G containing R.

Presentation

Theorem 1.15. *Any group G can be presented as $G = \langle X|R\rangle$, where X = generating set and R = relation set.*
$G = \langle X|R\rangle$ is defined as:

$$G = \langle X|R\rangle = \frac{F(X)}{\langle\langle R\rangle\rangle} ,$$

where $R \subset G$ and $F(X)$ is free group generated by X and $\langle\langle R\rangle\rangle$ is the normal subgroup generated by R.

Proof. Let G be group generated by set $X \subset G$ then by universal property for the inclusion map $i : X \hookrightarrow G$, there exists a surjective homomorphism $\phi : F(X) \to G$, then by the isomorphism theorem we have that

$$G \cong \frac{F(X)}{N} \ ,$$

where $N = ker(\phi) = \{x \in F(X) | \phi(x) = id_G\}$, id_G is the identity element of G. N is normal subgroup of $F(X)$, therefore $N = \langle\langle R \rangle\rangle$ for some $R \subset F(X)$ (see [1]), we call this R as relation set, and thus we have a presentation for a group $G = \langle X | R \rangle$ □

Example 1.16. Consider a cyclic group of order 15 generated by a, then its presentation is $G = \{a | a^{15} = 1\}$.

1.4 Cell Complexes

Cell Complex were first introduce by *J.H.C. Whitehead*. Cell complexes are made up of basic blocks called the $n - cells$. An $n - cell$ is homeomorphic to a closed ball in $\mathbb{R}^n$. For example a $1 - cell$ is a point, a $2 - cell$ is a closed disk in $\mathbb{R}_{\nvDash}$ etc. Given below are the $n - cell$ upto $n = 3$ and their boundary.

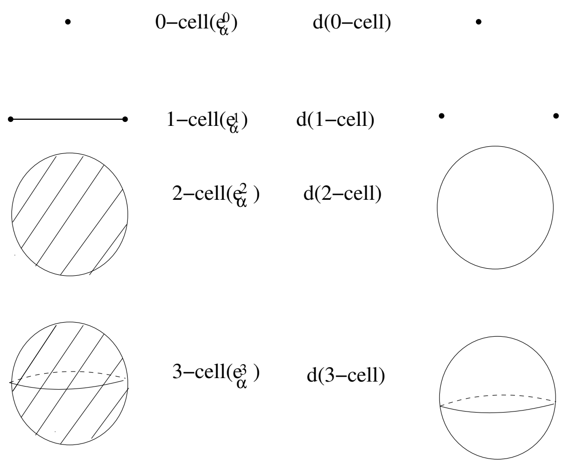

Fig. 1.2:

Construction of Cell Complexes: We start with a discrete set X^0 whose points regarded as $0-cells$.. Then we start attaching the boundary of one cell to the skeleton X^0 via the quotient map $\phi_\alpha : \{x_1, x_2\} \to X_0$, where x_1 and x_2 are boundary of $1-cell$ e^1_α, α runs over J =number of $1-cells$ and we get new skeleton X^1. Inductively, we will get the n-skeleton X^n cell complex from X^{n-1} by attaching n cells via the quotient maps

$$\phi_\alpha : S^{n-1} \longrightarrow X^{n-1} ,$$

where S^{n-1} is the boundary of n cell e^n_α.

$$X^n = X^{n-1} \sqcup \{e^n_\alpha\}_{\alpha \in J} / \sim ,$$

where J is the number of n cells. $\sim$ is a equivalence relation which is defined as
$x = \phi_\alpha(x)$ for each x belongs to the boundary of $n-cell$ e^n_α for every α .
Dimension of X^n we constructed in this fashion is n and is said to be a n dimensional cell complex.

Example 1.17. A very good example of a cell complex is graph. We can construct *graph* using cell with $0-cell$ as the vertices set and edges being the $1-cell$.

Example 1.18. Sphere can be constructed using $n-cells$. In fact sphere is a 2 dimensional cell complex.
Step 1 Take one $0-cell$ e^0_1 and one $1-cell$ e^1_1 and take $\phi : \{x_1, x_2\} \to e^0_1$ to be the constant map, where x_1, x_2 are boundary points of e^1_1. Thus we get

$$X^1 = \frac{e^0_1 \sqcup e^1_1}{\sim}$$

to be a loop based at e^0_1, where $\sim$ is equivalence relation defined as $x = \phi(x)$ for $x \in \partial(e^1_1) = \{x_1, x_2\}$.

Step 2 Now take two $2-cells$ e^1_2 and e^2_2, then $\partial(e^\alpha_2) = \mathbb{S}^1$. Define quotient map $\phi_\alpha : \mathbb{S}^1 \to X^1$ to be the identity map, where $\alpha \in \{1, 2\}$. Thus We get sphere

$$\mathbb{S}^2 = \frac{X_1 \sqcup \{e_1^2, e_2^2\}}{\sim},$$

where $\sim$ is an equivalence relation defined as $x = \phi_\alpha(x)$ for all $x \in \partial(e_2^\alpha)$, $\alpha \in \{1, 2\}$. Thus we have constructed sphere using $0, 1, 2 - cells$ and we see that sphere is 2 dimensional complex.

Definition 1.19. Let f and f' be continous maps from a space X to a space Y. Then f is said to be *homotopic* to f' if there is a continuous map

$$F : X \times I \longrightarrow Y$$

such that $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$ for each x and $I = [0, 1]$. It is sometimes also denoted by F_t. The map F is called a *homotopy* between f and f'. It is denoted by $f \simeq f'$. If $f \simeq f'$ and f' is a constant map then we say that f is *nullhomotopic.*
Note that Homotopy is the continuous deformation of f into f'.

Definition 1.20. Let $f, g : X \longrightarrow Y$ be two continuous functions. Let $A \subseteq X$ and $f|_A \simeq g|_A$(via h). If h can be extended to a homotopy such that $f \simeq g$(via H) s.t. $H|_A = h$, then we say (X, A) has *homotopy extension property*(HEP).

Sufficient condition for HEP : If (X, A) is CW pair and $X \times o \cup A \times I$ is a deformation retract of $X \times I$ hence then (X, A) has the homotopy extension property. (see [2])

Theorem 1.21. *If the pair (X, A) satisfies the HEP and A is contractible then the quotient map $q :\longrightarrow X/A$ is a homotopy equivalence.*

Proof. Since A is contractible there exists a homotopy H_t such that is $i_A \simeq e_{a_0}$(via H_t), where e_{a_0} is constant map taking A to $a_0 \in A$. Consider $i_X : X \longrightarrow X$, then $i_X \simeq e_{a_0}$(via homotopy f_t), where $f_t|A = H_t$. $f_t : X \longrightarrow X$. Since $f_t(A) \subset A$ $\forall t$. $f_0 = id_X$.

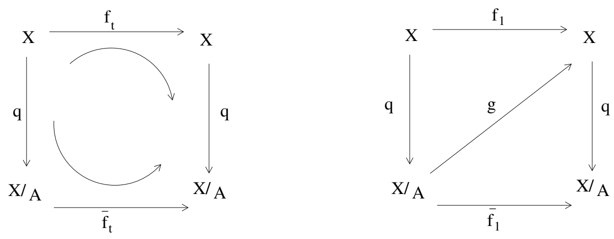

Fig. 1.3:

Therefore, the composition$q \circ f_t : X \longrightarrow X/A$ sends A to a point and we have the composition $X \longrightarrow X/A \longrightarrow X/A$ where $\overline{f_t}(\overline{x}) = \overline{f_t(x)}$ and $x \in X$ such that the above diagram commutes.

$$f_t = \overline{f_t} \circ q \tag{1.1}$$

Now take t=1, $f_1(A) = a_0$(fixed point to which A contracts), hence f_1 induces a map $g : X/A \longrightarrow X$.
It remains to show that $q \circ g = \overline{f_1}$.

$$q \circ g(\overline{x}) = q \circ g \circ q(x) = q \circ f_1(x) = \overline{f_1} \circ q(x) \ (from\ 1.1)$$

we know that which would imply $q \circ g(\overline{x}) = \overline{f_1}(\overline{x})$, and from 1.2 the maps g and q are inverse homotopy equivalences.

$$gq = f_1 \simeq f_0 = i_x \ and \ qg = \overline{f_1} \simeq \overline{f_0} = i_{X/A} \tag{1.2}$$

via $\overline{f_t}$. Hence X and X/A are homotopy equivalences. □

Theorem 1.22. *Every connected graph has a maximal tree.*

Proof. We will prove this using Zorn's lemma. Let T_0 be a tree and $\mathcal{T}$ be collection of all trees in X that contains T_0, strictly order by proper inclusion. To show $\mathcal{T}$ has a maximal element, it is enough to show the following:

If $\mathcal{T}'$ is subcollection of $\mathcal{T}$ which is ordered by proper inclusion, then Y union of the elements of $\mathcal{T}'$ is a tree in X.

Since Y is a union of subgraphs of X, it is a subgraph of X. Also Y is a

union of connected spaces that contain the connected space T_0, therefore Y is also connected.
It is enough to show that Y is a tree. Let $e_1...e_n$ be a cycle in Y. For each i, choose an element T_i of $\mathcal{T}'$ that contains e_i. As $\mathcal{T}'$ is ordered by proper inclusion, one of the trees will contain all the other trees, say T_j, contains all the other trees, which would imply that $e_1...e_n$ is a cycle in T_j because every edge $e_i \in T_j$, and we have a contradiction to hypothesis. □

Theorem 1.23. *Every tree T in graph X is contractible.*

Proof. Refer page 508 of munkres (see[4]) □

Theorem 1.24. *For a connected graph X with a maximal tree T. $\pi_1(X)$ is a free group generated by the elements which are in one to one correspondence with the edges of $X \backslash T$.*

Proof. Let X be a connected graph and T be maximal tree of X. Then by theorem 1.23 T is contractible. Therefore, by the theorem 1.21 we have that quotient map $q : X \to X/T$ is a homotopy equivalence.

Since, T contains all the vertices of X. Therefore, X/T is a cell complex with a single vertex v and an edge set which is in one to one correspondence with the number of edges in $X \backslash T$, call it n which are loops. Since $X \approx X/T$, we have $\pi_1(X) \cong \pi_1(X/T)$. Also we proved in class that fundamental group of wedge of n circles is the free group on n letters. Hence we have $\pi_1(X) = F_n$ where F_n is a free group of $n - letters$. □

Theorem 1.25. *Every covering space of a graph is also a graph with vertices and edges as the lifts of the vertices and edges in the base graph.*

Proof. Let X be a graph and let $P : \widetilde{X} \to X$ be a covering space.
Define vertices of $\widetilde{X}$ as $\widetilde{X_0} = P^{-1}(X_0)$. Graph X is a quotient space obtained by attaching one cell to zero cell via quotient map $X = X_0 \sqcup_\alpha I_\alpha$.
Now Use the path lifting property to define the edges of $\widetilde{X}$ which says that there exists a unique $\tilde{f} : I_\alpha \to \widetilde{X}$. These lifts define the edges of a graph structure of $\widetilde{X}$. □

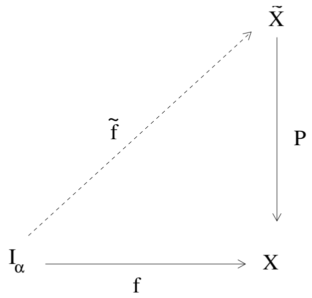

Fig. 1.4:

Theorem 1.26. *Suppose X is path connected, locally path connected and semi locally simply connected. Then for every subgroup $H \leq \pi(X, x_0)$ there exist a covering space $P : X_H \to X$ such that $P_*(\Pi_1(X_H, \widetilde{x_0})) = H$ for suitably chosen base point $\widetilde{x_o} \in X_H$.*

Proof. For a proof, Refer A.Hatcher [2] □

Theorem 1.27. *Every subgroup of a free group is free.*

Proof. Let F be a free group. Let X be a graph with $pi_1(X) \cong F$. Consider G to be a subgroup of F then by theorem 1.26 there exist a covering space $(\widetilde{X}, P : \widetilde{X} \to X)$ with $P_*(\Pi_1(\widetilde{X})) = G$.
Also we have covering space of a graph is also a graph. Hence $\Pi_1(\widetilde{X})$ is also free and since P_* is injective. We have $G \cong \Pi_1(\widetilde{X})$ which is free. □

2. AMALGAMS PRODUCTS

2.1 Introduction

$SL(2,\mathbb{Z})$ also called modular group is a very important class of group of linear fractional transformations of the upper half of the complex plane. One of the significant reasons behind studying $SL(2,\mathbb{Z}) = \langle X, Y\rangle$ is to understand the structure of $GL(2,\mathbb{Z})$ via the epimorphisms $\pi : GL(2,\mathbb{Z}) \to \{\pm 1\}$ whose kernel is $SL(2,\mathbb{Z})$.

In this chapter we will establish one known result which was proved by J.Peare Serre that $SL(2,\mathbb{Z})$ is amalgamated product of $\mathbb{Z}_6$ and $\mathbb{Z}_4$ over $\mathbb{Z}_2$ (see [5]). For this we first understand structure of amalgams and then we will rigorously study certain terminologies in graph theory(according to $Serre's$ notation) such as morphism of graphs, graph of groups etc. Notice that $serre's$ definition and terminologies of graph theory differ from what we studied in the previous chapter. Moreover, we will study how a group acts on a graph and with the fact that

$$SL(2,\mathbb{Z}) = \langle X, Y\rangle \ ,$$

where $X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ we will show our main result.

2.2 Direct limits

Let $(G_i)_{i\in I}$ be a family of groups and for each pair (i,j), let $F_{ij} \subseteq \mathrm{Hom}(G_i, G_j)$ where $\mathrm{Hom}(G_i, G_j) = \{\phi : G_i \longrightarrow G_j | \phi \text{ is a group homomorphism}\}$. We are seeking for a group G and a family of homomorphisms $f_i : G_i \to G$ such that

$f_j \circ f = f_i \ \forall \ f \in F_{ij}$ i.e. the following diagram commutes.

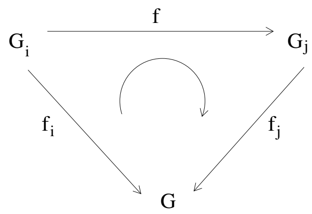

Fig. 2.1:

Then we call G as the direct limit of G_i relative to F_{ij} provided G and the family of homomorphisms $\{f_i\}_{i\in I}$ is universal in the sense that if H is a group and $h_i : G_i \to H$ be another family of homomophism such that $h_j \circ f = h_i \ \forall \ f \in F_{ij}$ then there exist a unique homomorphism $h : G \longrightarrow H$ such that $\triangle_1$ and $\triangle_2$ commutes in figure 2.2 i.e. $h \circ f_i = h_i$ and $h \circ f_j = h_j$

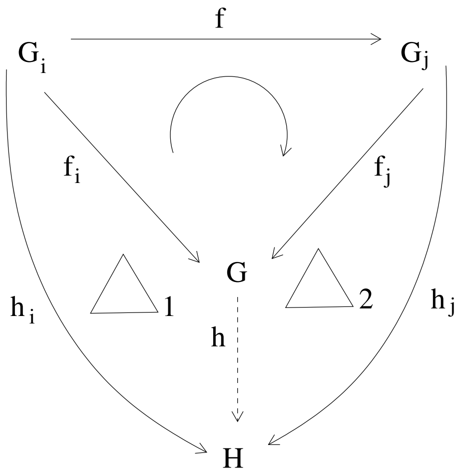

Fig. 2.2:

Theorem 2.1. *The pair consisting of G and the family $(f_i)_{i\in I}$ exists and is unique upto isomorphism.*

Proof. Uniqueness: Let G be direct limit of G_i and a f_i be a family of homomorphism $f_i : G_i \longrightarrow G$ s.t. $f_j \circ f = f_i \ \forall \ f \in Fij$.

Now let H be another group which is also direct limit of G_i and $h_i : G_i \to H$ be a family of homomorphism s.t. $h_j \circ f = h_i$, for all $f \in F_{ij}$ then by universal property of G there exist a unique homomorphism $\phi : G \to H$ such that $h_i = \phi \circ f_i$.

Similarly there exists a unique homomorphism $\psi : H \to G$ whence we have a homomorphism $\psi \circ \phi : G \to G$. But since G is direct limit and hence possess universal property which says there exist a unique homomorphism from G to G. One such is identity homomorphism therefore we have $\psi \circ \phi = \mathrm{id}_G$ which implies that ϕ is injective.

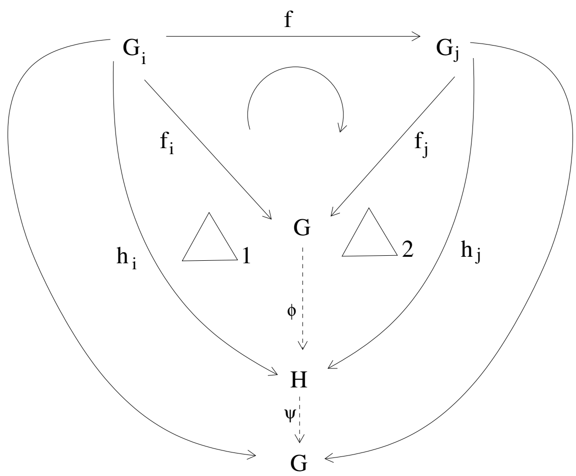

Fig. 2.3:

Similarly if we interchange the role of G and H in above then we will get $\phi \circ \psi = \mathrm{id}_H$ which implies ϕ is surjective and hence ϕ is isomorphism and we have $G \cong H$

Existence: We will define a group G and a family of homomorphism from G_i to G and show that it is actually the direct limit of G_i. Define G as follows $G = \langle \bigsqcup G_i | R \rangle$, where R is the relation set which is given as

1.$xyz^{-1} = 1$ in G if $x, y, z \in G_i$ for some $i \in I$ and $xy = z$ in G_i.

2.$xy^{-1} = 1$ in G where $x \in G_i$, $y \in G_j$ and $y = f(x)$ for some $f \in F_{ij}$.

Consider the inclusion map $\tau_i : G_i \to G$ as a family of homomorphism. Check $\tau_j \circ f = \tau_i \; \forall \; f \in Fij$. Let $x \in G_i$ $\tau_j \circ f(x) = \tau_j(y) = y$ and $\tau_i(x) = x$. But from second relation we have $xy^{-1} = 1$ in G whence $x = y$. Thus we have

$\tau_j \circ f = \tau_i$ as x was arbitrary.
To show G is direct limit it suffices to show that G possess universal property. let H be a group and $h_i : G_i \to H$ such that $h_j \circ f = h_i$ be a family of homomorphism. Define $\phi : G \to H$ as $\phi(g) = h_i(g)$ if $g \in G_i$, then clearly $h_i = \phi \circ \tau_i$, whence we have universal property. Thus G we defined is direct limit of family of groups G_i.

□

2.3 Structure of amalgams

Definition 2.2. Consider $(G_i)_{i \in I}$ be a family of groups and A be another group. For every i we have injection $f_i' : A \to G_i$. Then $f_i'(A) \leq G_i$ and identify $A \equiv f_i'(A)$. Then We denote $*_A G_i$ as the direct limit of the family (A, G_i) w.r.t. these homomorphism and call it the sum of the G_i with A amalgamated.

Example 2.3. Let $A = \{1\}$, then corresponding group is denoted by $*G_i$ which is the free product of the G_i.
For every i Write G_i as presentation $G_i = \langle X_i | R_i \rangle$. Then we claim that $K = \langle \cup X_i | \cup R_i \rangle$

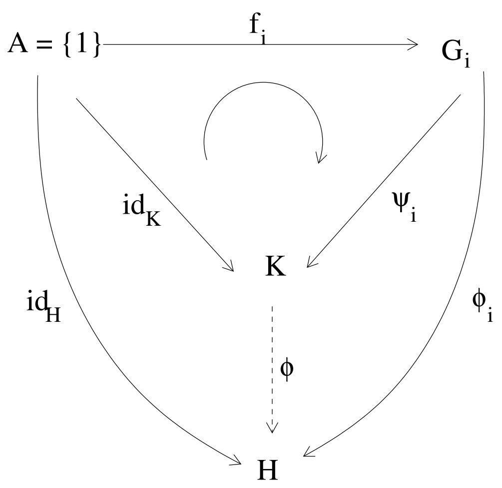

Fig. 2.4:

Let H be a group and $\phi : G_i \to H$ is a homomorphism s.t. $id_H = \phi_i \circ f_i \ \forall$. We want to contruct a homomorphism $\phi : K \to H$, where $K = \frac{F(\cup X_i)}{\ll \cup R_i \gg}$. Define $\phi : K \to H$ as $\phi(x_i) = \phi_i(x_i)$ if $x_i \in X_i$. We want to show that ϕ is well define. Let $r \in \cup R_i \implies r \in R_i$ for some i. $\phi(r) = \phi_i(r) = 1$. Since $\phi_i' s$ are well defined homomorphism. Hence ϕ is a well defined homomorphism(By first isomorphism theorem). Clearly, $\phi \circ \psi_i = \phi_i$. Thus the universal property is satisfied and $*G_i = K$.

We now define notion of *reduced word* in G. Let G_i be a family of group and A be another group. Let $f_i' : A \to G_i$ is injective homomorphism. Identify $A \equiv f_i'(A)$. $G_i/f_i'(A) = \{g_1 f_i'(A) = f_i'(A), g_2 f_i'(A)........., g_n f_i'(A)\}$. Let $S_i := \{$all right coset representatives of $G_a i/f_i'(A)\}$. $1 \in S_i \ \forall \ i$.

Define a map $\theta_i : A \times S_i \to G_i$ as $(a, s) \mapsto f_i'(a)s$. It is easy to check that θ_i is a bijection. injective is clear as $f_i' s$ are injective. All we need to show is that it is surjective. Let $g \in G_i = \bigsqcup_{s \in S_i} f_i'(A).s$. Thus $g \in f_i'(A)s$ for some $s \in S_i \Rightarrow g = f_i'(a)s$. Hence $(a, s) \mapsto f_i'(a)s$, whence θ_i is surjective hence bijective.

Definition 2.4. Let $\mathbf{i} = (i_1, i_2, ..., i_n) \in I^n$ be a sequence where $n \geq 0$ such that

$$i_m \neq \imath_{m+1} \tag{2.1}$$

A reduced word of type $\mathbf{i}$ is any family $m = (a; s_1, s_2, ..., s_n)$ where $a \in A, s_1 \in S_{i_1}, ..., s_n \in S_{i_n}$ and $s_j \neq 1 \ \forall \ j$.

Theorem 2.5. *For every $g \in G$, there is a sequence $\mathbf{i}$ satisfying 2.1 and a reduced word $m = (a; s_1, s_2, ..., s_n)$ of type $\mathbf{i}$ such that*

$$g = f(a)f_{i_1}(s_1), f(a)f_{i_2}(s_2), ..., f_{i_n}(s_n)$$

Furthermore, $\mathbf{i}$ and m are unique.

Remark 2.6. f and f_i are injective maps.

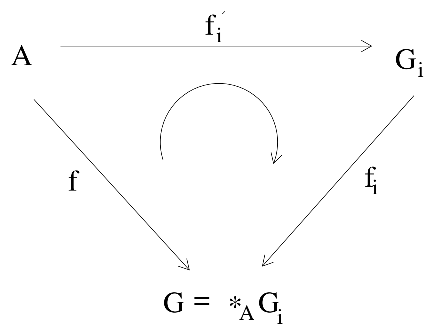

Fig. 2.5:

It is clear by the uniqueness of m. Suppose f is not injective i.e. for some $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. Then for $g = f(a_1) = f(a_2)$ we have two different types $m_1 = (a_1; 0)$ and $m_2 = (a_2; 0)$ with $m_1 \neq m_2$, which contradicts to uniqueness property of m. Hence f is injective. Similarly we can show that $f_i's$ are also injective.

Proof. **Step 1**
Let $X_{\mathbf{i}} = \{m = (a; s_1, s_2, ..., s_n)|m\ is\ set\ of\ reduced\ words\ of\ type\ \mathbf{i}\}$ and $X = X_{\mathbf{i}}$. we want to establish $X \cong G$.
Let $Y_i = \{(1; s_1, s_2, ..., s_n)|i_1 \neq i\}$
Define map $P_i : A \times Y_i \to X$

$$(a, (1; s_1, s_2, ...s_n)) \to (a; s_1, ...s_n) \ .$$

and

$$q_i : A \times (S_i - \{1\}) \times Y_i \to X \ .$$

$$(a, s, (a; s_1, s_2, ...s_n)) \to (a; s, s_1, ..., s_n) \ .$$

This yields a bijection of $A \times Y_i \cup A \times (S_i - 1) \times Y_i$ onto X. Clearly P_i and q_i are injective maps.
Let $x \in X = \cup X_{\mathbf{i}} \Rightarrow \ x = (a; s_1, s_2, ...s_n)$.
If $i_1 \neq i$, then $P_i(a, (1; s_1, s_2, ...s_n)) = (a; s_1, ...s_n)$.
If $i_1 = i$ then $s_1 \in S_{i_1} = S_i$ and $i_2 \neq i$ and $q_i(a, s_1, (a; s_2, s_3, ...s_n)) =$

$(a; s_1, ...s_n)$. This implies Surjection. Hence we have the bijection.

Step 2

If $g \in G_i$, then g has type $(i) \Rightarrow g = as, s \in S_i$. Otherwise g has type $\phi \;\Rightarrow g = a \; where \; a \in A \; g \in G_i = \bigsqcup_{s \in S_i} f_i'(A)s \;\Rightarrow g = f_i'(a)s.$
If $s = 1$ then g is of type ϕ with reduced word $m = (a; ....)$.
If $s \neq 1$ then g is of type (i) with reduced word $m = (a; s)$.
Now define $\theta_i : G_i \times Y_i$ as

$$\theta_i(g, y) = q_i((a, s), y) = (a; s, s_1, ..., s_n) \; if \; type \; (i) \; .$$

and $\theta_i(g, y) = P_i(a, y) = (a; s_1, s_2, ..., s_n) \; if \; type \; (\phi)$ where $y = (1; s_1, s_2, ...s_n)$
The way we have constructed θ it is clear that θ_i is bijective. Hence $X \equiv G_i \times Y$ i.e. every element of X can be uniquely identifies as an element of $G_i \times Y$

Step-3

Define an action of G_i on $G_i \times Y_i$ as $g'.(g, y) = (g'g, y)$. Clearly it is a group action as

$$e.(g, y) = (eg, y) = (g, y)$$

and

$$g_1.(g_2.(g, y)) = g_1.(g_2 g, y) = (g_1 g_2 g, y) = g_1 g_2.(g, y) \; .$$

now transfer this action on X, $G_i \times X \rightarrow X$ as

$$(g', (a; s_1, s_2, ..., s_n)) \mapsto (a'a; s, s_1, s_2, ..., s_n) \; if \; g' = a's \; i.e. \; of \; type \; (i)$$

and

$$(g', (a; s_1, s_2, ..., s_n)) \mapsto (a'a; s_1, s_2, ..., s_n) \; if \; g' = a' \; i.e. \; of \; type \; (\phi) \; .$$

Its restiction to A is given by $a'.(a; s_1, s_2, ...s_n) = (a'a; s_1, ..., s_n)$ which is independent of i. With all these we have an action of G on X.

Step-4

If $m = (a; s_1, s_2, ...s_n)$ is a reduced word and $g = f(a)f_{i_1}(s_1)........f_{i_n}(s_n)$ and

$e = (1; .....)$ (*empty sequence*), then $(g.e) = m$. Now denote $\alpha : G \to X$ such that $g \mapsto g.e$ and

$$\beta : X \to G \ as \ (a; s_1, ..., s_n) \mapsto f(a)f_{i_1}(s_1)........f_{i_n}(s_n) \ .$$

We claim that $\alpha \circ \beta = id$. let $(a; s_1, s_2, ...s_n)$,

$\alpha \circ \beta((a; s_1, s_2, ...s_n) = \alpha(f(a)f_{i_1}(s_1)........f_{i_n}(s_n)) = g.e = (a; s_1, ..., s_n \ ,)$

where

$$g = f(a)f_{i_1}(s_1)........f_{i_n}(s_n)$$

thus β is injective and we have uniqueness of the decomposition which is non trivial part of the theorem. $X \equiv \beta(X) \subset G$.

Step-5

We want to show that $\beta(X) = G$, it suffices to show that $G \subset \beta(X)$. we first show that $G_i\beta(X) \subset \beta(X)$. Let $x = gf(a)f_{i_1}(s_1).........f_{i_n}(s_n) \in G_i\beta(X)$ where $(a; s_1, ...s_n) \in X$ and $g \in G_i$. Thus, g is of type (i) i.e. $g = a's$ or type ϕ ie. $g = a'$ then $\beta(a'a; s, s_1, s_2, ...s_n) = x$ or $\beta(a'a; s_1, s_2, ...s_n) = x$. Hence, $G\beta(X) \subset \beta(X)$, but $1 \in \beta(X) \Rightarrow \ G \subset \beta(X)$ and we have $G = \beta(X)$.

□

Remark 2.7. we want to state previous theorem without involving the sets S_i. let $G'_i = G_i \ - \ A$. every $g \in G$ can be written as $f(a)$ for some $a \in A$ or $f_{\mathbf{i}}(\theta_g)$, where $f_{\mathbf{i}} : G'_{\mathbf{i}} \to G$,

$$G'_{\mathbf{i}} = quotient \ of \ A^{n-1} \times (G_{i'_1} \times .... \times G_{i'_n}) \ .$$

under the quotient map

$$A^{n-1} \times (G_{i'_1} \times .... \times G_{i'_n}) \to G_{i'_1} \times .... \times G_{i'_n} \ .$$

such that

$$(a_1, ...a_{n-1}).(g_1, g_2, ..., g_n) = (g_1 a_1^{-1}, a_1 g_2 a_2^{-1}, a_2 g_3 a_3^{-1}......., a_{n-1} g_n) \ .$$

It is well defined group action. Orbits of $g \in G_{i'_1} \times .... \times G_{i'_n}$

$$\theta_g = \{a.g | a \in A^{n-1}\} \ ,$$

where $g = (g_1, g_2, ..., g_n)$.
Then $G'_{\mathbf{i}} = \{\theta_g | g \in G_{i'_1} \times .... \times G_{i'_n}\}$. Define $f_{\mathbf{i}} : G'_{\mathbf{i}} \to G$ as $\theta_g \mapsto f_{i_1}(g_1).........f_{i_n}(g_n)$. It is well defined and bijective map, whence we have the remark.

Example 2.8. Let $A = \{1\}$, $S_i = G_i$ and $G'_i = G_i - \{1\}$. Suppose

$$G_i = (x_i) = \{x_i^r | r \in (Z)\}$$

Then direct limit of $G'_i s \ and \ A$ is the free group $F((x_i)_{i \in I})$.

Let G be a direct limit of A and G_i and $f_i : G_i \to G$ be a family of homomorphism such that $id_G = f_i \circ f'_i$. Define a homomorphism $\phi : F((x_i)_{i \in I}) \to G$ as $\phi(x_i) = f_i(x_i)$. Clearly $\triangle_1$ and $\triangle_2$ commutes. Let $g \in F((x_i)_{i \in I})$ then by previous theorem there exist $\mathbf{i} = (i_1, i_2, ..., i_n)$ such that $i_m \neq i_{m+1}$ and a reduced word $m = (1; x_{i_1}^{r_1}, ..., x_{i_n}^{r_n})$ with $r_i \neq 0 \ for \ all \ i \in \{1, 2, ..., n\}$ such that $g = x_{i_1}^{r_1}.x_{i_2}^{r_2}......x_{i_n}^{r_n}$.

Remark 2.9. The type of an element $g \in G$ is the sequence $\mathbf{i} = (i_1, i_2, ..., i_n)$ such that $i_m \neq i_{m+1}$, g is of type $\mathbf{i}$. Type $\mathbf{i} = \phi$ if and only if $g \in A$.

Definition 2.10. Let $\mathbf{i} = (i_1, i_2, ..., i_n)$ be the type of an element $g \in G$. then n is called the length of g and we denote it by $l(g)$.

Result: $l(g) \leq 1$ if and only if $g \in G_i$ for some i.
Let $l(g) \leq 1$. This would imply that $\mathbf{i} = (i)$ or $\mathbf{i} = (\phi)$. If $l(g) = 0$ then it is clear that $g \in G_i$ so assume $l(g) = 1$ which means that $G'_{\mathbf{i}} = G'_i$. $\phi : G'_i \sqcup A = G_i \to G$ is a bijective map from *remark* 2.7. Hence $g \in G_i$.

Now let $g \in G_i = \bigsqcup_{s \in S_i} f'_i(A)s$ which would imply that $g = f'_i(a)s$ *or* $g = f'_i(a)$ whence clearly g has type (i) or has type ϕ. Hence we have $l(g) \leq 1$.

Definition 2.11. Let $g \in G$ is of type $\mathbf{i} = (i_1, i_2, ...i_n), n \geq 2$, g is said to be cyclically reduced if $i_1 \neq i_n$

Theorem 2.12. (a) *Every element* g *of* G *is conjugate to a cyclically reduced element, or conjugate to an element of the* G_i.
(b)*Every cyclically reduced element is of infinite order.*

Proof. (a) We will prove this statement by using induction. Suppose $l(g) = 2$ ie. $\mathbf{i} = (i_i, i_2)$ and assume it is not cyclically reduced ie. $i_1 = i_2 = i$. Then $g = f_i(g_1)f_i(g_2)$ is of type $\mathbf{i}$. $g = g_1g_2$ (f'_is *are injective maps*). $g_1, g_2 \in G'_i$. Then g is an element of $G'_i \subset G_i$. Hence the result is clear.
Now assume the result is true for $l(g) < n$. We will try to establish the result for $l(g) = n$. Let $g \in G$ is of type $\mathbf{i} = (i_1, i_2, ...i_n)$ and let g is not cyclically reduced ie. $i_1 = i_n$. Then

$$g = g_1g_2, ...g_n \ s.t. \ g_1 \in G'_{i_1}, ..., g_n \in G'_{i_n} \ .$$

then $g_1^{-1}gg_1 = g_2........g_{n-1}g_ng_1$. $g_ng_1 \in G_{i_1} = G_{i_n}$. This would imply that $g_1^{-1}gg_1$ is of length $n-1$ if $g_ng_1 \notin A$.
and is of length $n-2$ if $g_ng_1 \in A$. By induction hypothesis we have $g_1^{-1}gg_1$ is conjugate to some cyclically reduced element or conjugate to an element of G_i. Then same is true for g. Since
$g_1^{-1}gg_1 = g'^{-1}{}''g'$ where g'' is either a cyclically reduced element or an an element of G'_i.
$\Rightarrow \ g = (g'g_1^{-1})^{-1}g''g'g_1^{-1}$.
(b) Let g be cyclically reduced word of type $\mathbf{i} = (i_1, i_2, ...i_n)$ such that $i_1 \neq i_n$ then g^2 will be of type $2\mathbf{i} = (i_1, i_2, ...i_n, i_1, ...i_n)$ and length $2n$. More generally $g^k(k \geq 1)$ is of length kn and therefore never equal to 1. Hence is of infinite order. □

Remark 2.13. Every element of G of finite order is conjugate to an element of one of the G_i.
It is clear from part (a) and (b) of previous theorem.

2.4 Graphs

Definition 2.14. A graph Γ has a vertex set X and an edge set Y with two maps

$$Y \to X \times X, \qquad y \mapsto (o(y), t(y))$$

and

$$Y \to Y, \qquad y \to \overline{y}$$

satisfying for every $y \in Y$ we have $\bar{\bar{y}} = y, \bar{y} \neq y$ and $o(y) = t(\bar{y})$. Every element of X and Y is called the vertex and respectively edge of Γ. y is called the oriented edge and $\bar{y}$ is called the inverse edge. origin vertex of edge e is defined as $o(y) = t(\bar{y})$ and $t(y) = o(\bar{y})$ is called the terminal vertex of y. These two vertices are refer to as extremities of edge y.

Definition 2.15. Let $a, b \in X$. We say that a, b are adjacent if they are extremities of some edge.

Definition 2.16. (**Morphism of graphs**)
Let A, B be graphs with $(V(A), E(A))$ $(V(B), E(B))$ be the vertex and edge set of A *and* B respectively. Then

$$\phi : A \to B$$

is a morphism provided we have maps

$$\phi_V : V(A) \to V(B)$$

and

$$\phi_E : E(A) \to E(B)$$

such that
(1) $\phi_V(o(e)) = o(\phi_E(e))$ and
(2) $\phi_V(t(e)) = t(\phi_E(e))$ and
(3) $\phi_E(\bar{e}) = \overline{\phi_E(e)}$.

Definition 2.17. Let Γ be a graph with X *and* Y as set of vertices and edges respectively. Then *orientation* of graph Γ is subset Y_+ of Y such that Y is the disjoint union of Y_+ and $\overline{Y_+}$.

Definition 2.18. (**Diagrams**) A graph is represented using a diagram such that every point on the diagram represent the vertex of graph and line joining any two marked points represent edges of the form $y, \bar{y}$.

Example 2.19. Let Γ has three vertices P, Q, R, S and 6 edges $p, q, r, \bar{p}, \bar{q}, \bar{r}$ such that $o(p) = o(r) = o(q) = S$ and $t(p) = P, t(r) = R, t(q) = Q$. This graph will be represented as

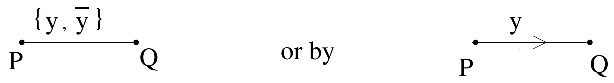

Fig. 2.6:

Example 2.20. Consider the following diagram with vertices P, Q, R and 8 edges $r, s, t, u, \bar{r}, \bar{s}, \bar{t}, \bar{u}$ such that r, s, t, u have the extremities $\{P, P\}$, $\{P, Q\}$, $\{P, Q\}$, $\{Q, R\}$ respectively. Also we have $o(r) = t(r) = P$ but we don't have information whether P is terminal or origin of edges *s and t*.

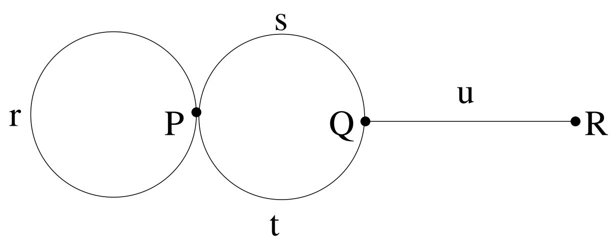

Fig. 2.7:

Definition 2.21. Paths: A $Path_n$ where $n \geq 0$ is an oriented graph which has $n + 1$ vertices given as $1, 2, ..., n$ and n edges denoted as $[i, i + 1]$, $0 \leq i < n$ which gives the orientation as $o([i, i + 1]) = i$ and $t([i, i + 1]) = i + 1$.

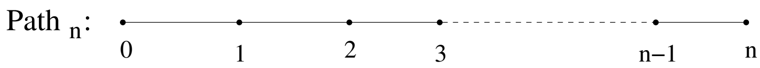

Fig. 2.8:

A Path in a graph Γ is a morphism ϕ of $path_n$ into Γ.

Definition 2.22. A pair of the form $(y_i, y_{i+1}) = (y_i, \overline{y_i})$ in the path is called *backtracking*.

Definition 2.23. A graph Γ is connected if there exist atleast one path between any two vertices of graph Γ. Moreover, the maximal connected subgraphs og graph called the connected component of the graph.

Definition 2.24. Circuit: Consider the oriented graph $circ_n$ where $n \geq 0$ and the set of vertices are from $\mathbb{Z}/n\mathbb{Z}$ and edges $[i, i+1]$ which gives the orientation as $o([i, i+1]) = i$ and $t([i, i+1]) = i+1$.

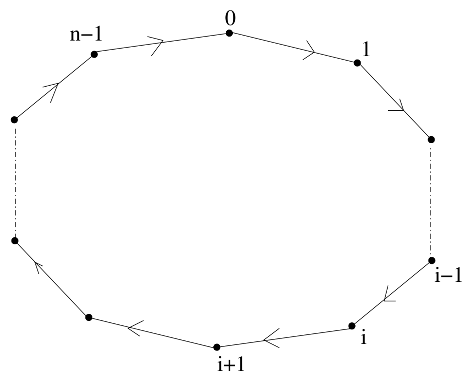

Fig. 2.9:

A circuit in a graph of length n is any subgraph which is isomorphic to $Circ_n$. A circuit of length 1 is called a *loop*.

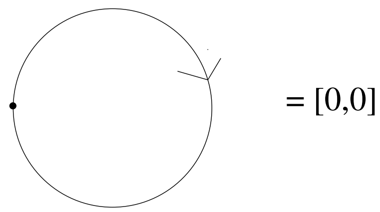

Fig. 2.10:

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Definition 2.25. A graph Γ is said to be combinatorial if it has no circuit of length ≤ 2.
Suppose Γ be a combinatorial graph and let X and Y be the vertex and edge set respectively of Γ, then a set $\{P, Q\}$ of the extremities of some edge $y \in Y$ is called an *geometric edge*. Thus geometric edges actually determines the set of orientated edge $\{y, \bar{y}\}$.

Definition 2.26. Graph of groups: Suppose G be any group and S be any subset of G. Define $\Gamma(G, S)$ to be an oriented graph with G as its set of vertices and $G \times S$ as the set of edges such that the orientation is given by

$$o(g, s) = g \text{ and } t(g, s) = gs \text{ for all edge } (g, s) \in G \times S \ .$$

Example 2.27. Suppose G be a cyclic group of order 15 and S be the generating element of G that is

$$G = \{x | x^{15} = 1\} \text{ and } S = \{x\}$$

. So the vertex set of graph $\Gamma(G, S)$ is (x) and $edge(\Gamma) = ((x) \times \{x\})$ such that $x_{15} = 1$. let (x^t, x), then orientation of edge is given as $o(x^t, x) = x^t$ and $t(x^t, x) = x_{t+1}$ where $0 \leq t \leq 14$. Then graph $\Gamma(G, S)$ will be a circuit of length 15. .

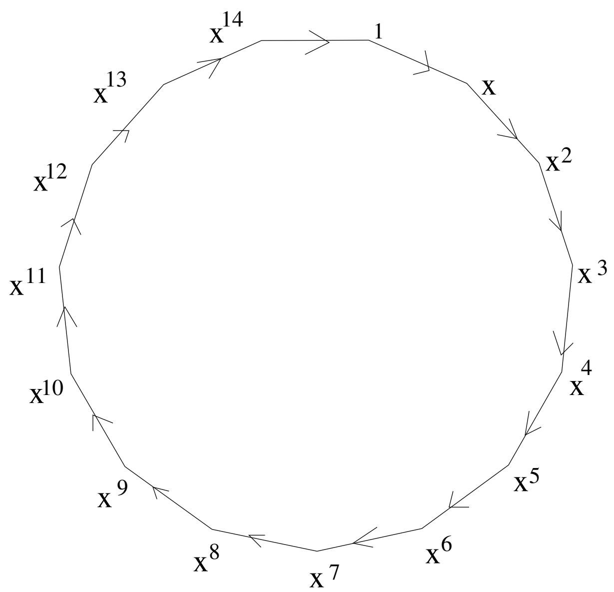

Fig. 2.11:

Example 2.28. Now suppose in the previous example instead of taking $S = \{x\}$ we take $S = \{x^5\}$, let (x^t, x^5) be a edge in $\Gamma(G \times S)$. Then the graph will look like as following because $o(x^t, x^5) = x^t$ and $t(x^t, x^5) = x^{t+5}$. Therefore in the graph there will be 5 circuits of length 3. .

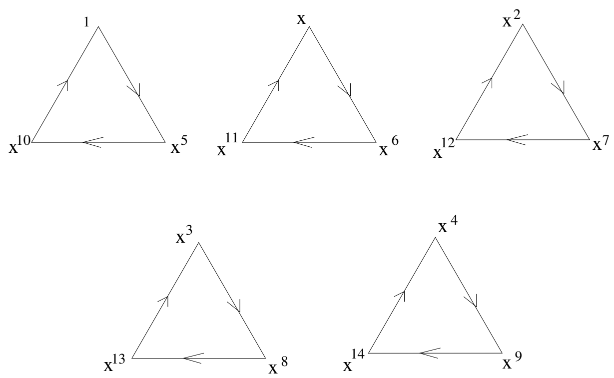

Fig. 2.12:

The graph of $\Gamma(G \times S)$ is connected since there is no path from 1 to x. Infact we have the following result for this particular graph

Remark 2.29. Result: Let $G = \{x|x^{15} = 1$ and $S = \{x^t\}$, then there is a path from 1 to x in $\Gamma(G \times S)$ if and only if $g.c.d.(t, 15) = 1$. More generally there is a path from x^i to x^j in $\Gamma(G \times S)$ if and only if $j - i$ *divides* $g.c.d.(t, 15)$.

Theorem 2.30. *Let* $\Gamma(G, S)$ *be the graph defined by the group* G *and a subset* S *of* G. *Then the following are true.*
(a) Γ *is connected if and only if* S *generate* G.
(b) Γ *contain a loop if and only if* 1 *belong to* S.
(c) Γ *is a combinatorial graph if and only if* $S \cap S^{-1} = \phi$.

Proof. (a) Suppose Γ is connected that is between any two vertex of Γ there exist a path connecting them.
Let $g, g' \in G = Ver\ \Gamma$. Consider the path from g *to* g' as
$(g, s_1).(gs_1, s2).......(gs_1s_2....s_{n-1}, s_n)$ such that $g' = gs_1........s_n$. In particular if we take $g = 1$ then clearly S generate G.

Now assume S generate G that is every element of G can be written as combination of elements of S *or* S^{-1}. consider $g, g' \in G = Ver\ \Gamma$. Since S generate G hence we have $g' = g.s_1.....s_n$ and where $s_i \in S \cup S^{-1}$. Clearly then we have a path from g *to* g' as $(g, s_1).(gs_1, s2).......(gs_1s_2....s_{n-1}, s_n)$.

(b) Let Γ contains a loop that is we have an edge (g, s) such that
$o(g, s)$ and $t(g, s) = gs = g$
$\Rightarrow s = 1$, Hence clearly $1 \in S$.
Now suppose $1 \in S$ then clearly we have a loop with edge $(g, 1) \in G \times S$.

(c) Let Γ is combinatorial graph that is it has no circuit of length ≤ 2. Suppose $s \neq 1 \in S \cap S^{-1}$ because if $s = 1$ then by previous part Γ will contain a loop. As $s^{-1} \in S$ consider the edge
(s, s^{-1}), $o(s^{-1}, s) = s^{-1}$ and $t(s_{-1}, s) = 1$ Also consider the edge $(1, s)$, $o(1, s) = 1$ and $t(1, s) = s$. This will give us a circuit of length 2 hence we have a contradiction. Thus $S \cap S^{-1} = \phi$.

Assume $S \cap S^{-1} = \phi$. We want to show that Γ is a combinatorial graph. Suppose it is not combinatorial graph that it has circuit of length 2 or has

a loop. Suppose first that it has circuit of length 2 that is we have g and g' and edges (g, s) and (g', s') such that $o(g, s) = g$ and $t(g, s) = gs = g'$. Also we have $o(g', s') = g'$ and $t(g'.s') = g's' = g$. Thus we have
$g' = g's's \Rightarrow s's = 1 \Rightarrow s = s'^{-1}$. Since $s \in S$ and $s = s'^{-1} \in S^{-1}$ we have $s \in S \cap S^{-1}$. Hence we have a contradiction. □

2.5 Trees

Definition 2.31. A *tree* T is a connected graph which has no circuits.

Example 2.32. Given are some examples of trees .

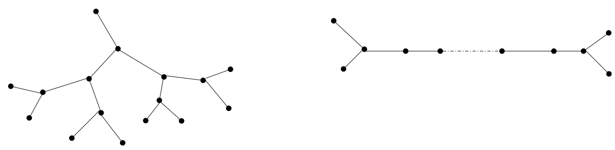

Fig. 2.13:

Definition 2.33. Let T be a tree. A geodesic in T is a path without backtracking.

Theorem 2.34. *Let P and Q be two vertices in a tree T, then there exists a unique geodesic from P to Q and it is an injective path.*

Proof. Existence is trivial since T is connected.

Injectivity: Suppose $c : path_n \to T$ be a geodesic from P to Q such that $P = c(0)$ and $Q = c(n)$ and put $P_i = c(i)$. We want to show that c is injective, it suffices to show that all vertex P_i are different. Then we can assume that c is defined by the sequence of edges $(y_1, y_2, ..., y_n)$ such that $o(y_1) = P$ and $t(y_1) = P_1$, $o(y_{i+1}) = P_i$, $t(y_{i+1}) = P_{i+1}$ and $t(y_n) = Q$.
Suppose if it were not injective meaning for some $i \neq j$ $P_i = P_j$. Then $(y_{i+1}, ..., y_j$ will form a circuit from P_i to P_j which is a contradiction since tree cannot have circuits.

Uniqueness: Assume $P \neq Q$ because if it not then a geodesic of length> 0 from P to Q would define a circuit as it is injective.
Let there are two geodesic from P to Q given as $(x_1, ...x_n)$ and $(y_1, ..., y_m)$. Suppose if $x_n \neq y_m$ then we will have a geodesic from P to P given as $(x_1, ..., x_n, \bar{y_n}, ..., \bar{y_1})$, which is a contradiction hence $x_n = y_m$. Thus by induction geodesics $(x_1, ..., x_{n-1})$ and $y_1, ..., y_{m-1}$ having the same terminal point must coincide. □

Definition 2.35. Subtree generated by a set of vertices: Let T be a tree and X' be subset of vertex set of T. Consider all geodesics in tree T whose extremities are in X'. Take all the vertices and edges of these geodesics which will form a subtree T' containing X'. Such a subtree T' is said to be generated by X'.

Definition 2.36. Let Γ be a graph. Let X and Y be vertex and edge set respectively of Γ and suppose $P \in X$ then $st(P) = \{e \in Y : t(e) = P\}$. Valency of vertex P is defined as $V(P)$ =no. of elements in $st(P)$.

Definition 2.37. A vertex $P \in ver(\Gamma)$ is said to be terminal if valency of that vertex is one that is $V(P) = 1$. If $V(P) = 0$ then we say that P is isolated vertex of Γ.

Theorem 2.38. *Let P be a non- isolated terminal vertex of a graph Γ. Then*
(a) Γ *is connected if and only if* $\Gamma - P$ *is connected.*
(b) *Every circuit of* Γ *is contained in* $\Gamma - P$.
(c) Γ *is a tree if and only if* $\Gamma - p$ *is a tree.*

Proof. Since P is terminal vertex hence it is terminus of a unique edge y. Thus (a) is clear.
Every vertex belonging to circuit have valency two whence we have (b). And we get (c) from (a) and (b). □

Definition 2.39. Let T be a tree and X be the set of vertices. Define a metric on X. Let $x, y \in X$ then metric $l(x, y)$ =no. of edges in shortest path from x to y. Define $Diameter(\Gamma) = sup\{l(x, y)|x, y \in X\}$.

Theorem 2.40. *Let T be a tree of finite diameter n.*
(a) The set $t(T)$ of terminal vertices of T is non empty.
(b) If $n \geq 2$, $VerT - t(T)$ is the vertex set of a subtree of diameter $n-2$.
(c) If $n = 0$ we have $T \cong Path_0$(diagram:○) and if $n = 1$ we have $T \cong Path_1$

Fig. 2.14: $Path_1$

Proof. It suffices to show (b) since (a) follows clearly from (b) and (c) and (c) is trivial.
(b) Let $X' = verT - t(T)$. Let $P, Q \in X'$. Then any point of the geodesics from P to Q is non terminal. This will imply that subtree T' generared by X' has all of its vertices same as of X'. Now suppose that $l(P, Q) = m$.
Now add two more edges from $t(T)$ to both the vertex P and Q then length of geodesic will be $m+2$. But as n is diameter of T we have $m+2 \leq n$. Thus $Diam(T') \leq n-2$. As T is of diameter n there exist a geodesic of length n remove the first and last edge from this geodesic we will get a geodesic of length $n-2$ in T'. Thus we have $DiamT' = n-2$. □

Result:
Let Γ be a graph. If $\alpha \in Aut(\Gamma)$ then for all $P \in V(\Gamma)$ valency(P) = valency$(\alpha(P))$.
Result: If $\alpha \in Aut(\Gamma)$ then $t(\Gamma) = \alpha(t(\Gamma))$, where $t(\Gamma)$ =set of terminal vertices.
It is clear since valency remains unchanged on application of automorphism from previous result that is we have $V(\alpha(t(e))) = V(t(e)) = 1$.

Theorem 2.41. *A tree of even finite diameter (respectively odd diameter) has a vertex (respectively a edge) which remains fixed under all automorphisms.*

Proof. Let n be the diameter of tree T. If $n = 0$ then the result is trivial. Without loss of generality assume that n is even and positive. Let $\alpha \in Aut(T)$ be any arbitrary automorphism then by previous result α just permuting the elements of $t(T)$ so we can remove these many elements from the vertex set. Now use part (b) which says $Ver(T)-t(T)$ is vertex set of subtree of diameter $n-2$. In this way deleting the terminal vertices we will get a graph consisting of a fixed point with diameter zero, that point will remain fixed under all automorphism.

Following the same procedure we will get an edge if n is odd and positive. □

2.6 Trees and amalgams

Definition 2.42. Let G be a group and X be a graph. A **Group** G acts on a **graph** X, denoted by $G \times X \to X$, if G acts on the vertices and edges of X:

(a) $G \times vert(X) \to vert(X)$

(b) $G \times edge(X) \to edge(X)$

and the action commutes with the maps $o, t : edge(X) \to vert(X)$ that is following holds true:

$o(gy) = g(o(y))$

$t(gy) = g(t(y))$

where $g \in G$, $y \in edge(X)$.

Definition 2.43. Let G be a group and X a graph upon which G acts.

(a) An *inversion* is a pair consisting of some $g \in G$ and an edge y of X such that $gy = \bar{y}$ (where $\bar{y}$ is the reverse edge of y).

(b) If no such pair exists we say that G acts without inversion on X. In other words, the action does not map any edge to its reverse edge (and thus we say that G preserves the orientation of X).

If G acts on X without inversion, then we can define the quotient graph denoted as $G\backslash X$ (which we read as : X mod G) in an obvious way:

(a) The vertex set of $G\backslash X$ is the quotient of $vert(X)$ under the action of G: $vert(G\backslash X) = \{Gx : x \in vert(X)\}$.

(b) Similarly, the edge set of $G\backslash X$ is the quotient of $edge(X)$ under the action of G: $edge(G\backslash X) = \{Gy : y \in edge(X)\}$.

Theorem 2.44. *Let X be a connected graph, acted upon without inversion by a group G. Every subtree T' of $G\backslash X$ lifts to a subtree of X.*

Proof. Let T' be a tree in $G\backslash X$. Consider ω to be a set of all those trees T in X which project injectively into T'. This is a directed set under the relation of inclusion. Hence by Zorn's lemma it has a maximal element T_0, let T_0' be its image in T' ie. $G\backslash T_0 \equiv T_0 \subseteq T'$. If T_0 is properly contained in T' then there exist an edge $y' \in T'$ but $y' \notin T_0'$. As T' is connected we can assume that $o(y') \in T_0'$ but $t(y') \notin T_0'$ (because if $t(y') \in T_0'$ then as T_0' is connected there is a geodesic from $o(y')$ to $t(y')$ in T_0', this geodesic will contain in T also as $T_0' \subset T'$, this geodesic from $o(y')$ to $t(y')$ followed by $\bar{y'}$ will give a circuit in T' which is a contradiction since T' is a tree.
Now let y be a lift of y'; then $G.(o(y)) \cap V(T_0) \neq \phi$ (since image of both sets is in T')

Thus we can assume that $o(y) \in T_0$. Let T_1 be a graph which we get by joining the vertex $t(y)$ to $o(y)$ via the edges $y, \bar{y}$ and T_1 is tree from $Theorem$ 2.38(c) such that T' is properly contained in T_1 . Also by the way we have chosen y', T_1 projected injectively into T', which is a contradiction to the maximality of T', hence we have the theorem. □

Definition 2.45. Let G be a group acting on a graph X. A fundamental domain of X mod G is a subgraph T of X such that $T \to G\backslash X$ is a isomorphism.

Theorem 2.46. *Let G be a group acting without inversion on a tree X. A fundamental domain of $G\backslash X$ exists if and only if $G\backslash X$ is a tree.*

Proof. Let T be a fundamental domain of $G\backslash X$ that is T is a subgraph of X such that $T \cong G\backslash X$. As X is connected and non-empty therefore $G\backslash X$ is also connected and non-empty. So, T is a tree as a non empty, connected subgraph of X. Thus $G\backslash X$ is also a tree.

Now suppose $G\backslash X$ is a tree. By previous theorem there is a tree T which is isomorphic to $G\backslash X$ such that $G\backslash X \cong T$. This T is a fundamental domain of $G\backslash X$. Hence it exists. □

Definition 2.47. A graph is called segment if it is isomorphic to $Path_1 =$

Theorem 2.48. *Let G be a group acting without inversion on a graph X, and let T be a segment of X that has edge y(reverse edge$\bar{y}$) with $o(y) = P$ and $t(y) = Q$. Suppose that T is a fundamental domain of X mod G. Let G_P, G_Q and $G_y = G_{\bar{y}}$ be the stabilizers of the vertices and edges of T, then X is a tree if and only if the homomorphism $G_P *_{G_y} G_Q \to G$ induced by the inclusions $G_P \to G$ and $G_Q \to G$ is an isomorphism.*

(Note:Amalgam makes sense because $G_P \cap G_Q = G_y$).

First we will show that $G_y \subset G_P \cap G_Q$, let $g \in G_y$ then $g.y = y$, thus we have $o(g(y)) = g(o(y))$. Hence $o(y) = g.P \Rightarrow P = g.P$.
Similarly we have $t(g(y)) = g(t(y))$. Hence $t(y) = g.Q \Rightarrow Q = g.Q$.

Thus $g \subset G_P \cap G_Q$. Now we want to show that $G_P \cap G_Q \subset G_y$. Let $g \in G_P \cap G_Q$. Then we have $g.P = P$ and $g.Q = Q$. We want to show that $g \in G_y$, but this is clear as there is only one edge on which when g acts we will get the same thing. Thus $G_P \cap G_Q = G_y$.

Proof. It follows from the proof of the next two theorems: □

Theorem 2.49. *X is connected if and only if G is generated by $G_P \cup G_Q$.*

Proof. Let X' be the connected component of X containing T, let

$$G' = \{g \in G | gX' = X'\}$$

and G'' be the subgroup generated bt $G_P \cup G_Q$. We intend to establish that $G'' = G'$.
Let $h \in G_P \cup G_Q$, suppose if $h \in G_P$ then $h.P = P$. Thus, clearly T and hT will have a common vertex P. Then as X' is connected component containing T, we have $h.T \subset X'$.
Claim: $h.X' = X'$.
First we show that $h.X' \subset X'$, let $v \in X'$ but $v \notin V(T) = \{P, Q\}$, we need to show that $h.v \in Vert(X')$. Then as X' is connected there is a path

$e_1, e_2, ..., e_n$ from P to v such that $o(e_1) = P$ and $t(e_n) = v$. If $h \in G_P$ then $h.P = P$.
$h.v = h.t(e_n) = t(h.e_n)$ and $h.P = h.o(e_1) = o(h.e_1)$. Thus we have a path from $P = h.P$ to $h.v$ and since $P \in X'$ with X' is connected we have that $h.v \in Vert(X')$.
Suppose if $h \in G_Q$ that is $h.Q = Q$. Consider the same path as in above from $h.P$ to $h.v$ and attach this path to the edge $h.Q = Q, h.P$. Thus again we have a path from Q to $h.v$ in X'. Hence, $h.v \in Vert(X')$ Now we want to show that if $e \in edge(X')$ then $h.e \in edge(X')$. Since X' is connected there will be a path γ in X' containing e and y. Therefore γ and $h.\gamma$ will have one vertex in common.
This implies that $h.\gamma$ is a path contained in X' since X' is connected component. Thus $h.e \in edge(X')$. And hence we have $h.X' \subset X'$.

Similarly, $h^{-1}.X' \subset X'$. From this we want to establish that $X' \subset h.X'$.
Let $v \in Vert(X')$. As, $h^{-1}.X' \subset X'$
$\Rightarrow h^{-1}.v = v'$ for some $v' \in VErt(X')$.
$\Rightarrow h.h^{-1}.v = h.v'$.
$\Rightarrow v = h.v'$.
$\Rightarrow v \in h.X'$.
Hence we have that $h.X' = X'$. Therefore $h \in G'$ and we established that $G'' \subset G'$.
Now we want to show that $G' \subset G''$. Firstly, we show that $G''T$ and $(G - G'')T$ are the disjoint subgraphs of X whose union is X. We start with showing that vertex set of these two subgraphs has no intersection .
$gv \in G''T$, where $g \in G''$ and $v \in \{P, Q\}$. Suppose if $gv \in (G - G'')T$ that is $gv = g'v'$, where $g' \in (G - G'')$ and $v' \in \{P, Q\}$. If $v = v'$, then we have $gv = g'v$,
$\Rightarrow v = (g^{-1}g').v$, $\Rightarrow g^{-1}g' \in G_P \cup G_Q$,
$\Rightarrow g^{-1}g' \in G''$, which implies $g' \in G''$ which is a contradiction (as by hypothesis $g' \in (G - G''))$.

Now suppose if $v \neq v'$. Without loss of generality assume $v = P$ and $v' = Q$ (proof of other case will be exactly same by just interchanging vertices P,

Q'), then we have $g.P = g'.Q$. This implies $P = (g^{-1}g')Q$, whence we have $G.P = G.Q$, where $G.P, G.Q \in Vert(G\backslash X)$, which is a contradiction since P, Q are vertices of segment T which will go to different class in $Vert(G\backslash X)$. Thus we have that vertex set of $G''T$ and $(G - G'')T$ are disjoint.

Edge set of $G''T$ and $(G - G'')T$ will also be disjoint since if it were not then that is if $e \in Edge(G''T) \cap Edge((G - G'')T)$,then $o(e), t(e) \in Vert(G''T) \cap Vert((G - G'')T)$, which is a contradiction because we showed that vertex set of $G''T$ and $(G - G'')T$ are disjoint.

Now we will show that $X = G''T \cup (G - G'')T$. Let $v \in Vert(X)$. We are also given that T is fundamental domain that is $T \cong G\backslash X$. Therefore

$$G.v = \{g.v | g \in G\} \ .$$

will either equal to

$$G.P = \{g.P | g \in G\} \ .$$

or

$$G.Q = \{g.Q | \in G\} \ .$$

Suppose if $G.v = G.P$, then $v = g.P$ for some $g \in G$, g either belong to G'' or $(G - G'')$ and hence we have

$$g.v \in G''T \ or \ g.v \in (G - G'')T \ .$$

and hence $X \subseteq G''T \cup (G - G'')T$. Thus, we have that the required result that union of subgraphs $G''T$ and $(G - G'')T$ is X.

From this we have that $X' \subset G''T$ since $T = 1_G T \in G''T$ and as $T \in X'$ therefore X' has to be contained in $G''T$ because $G''T$ and $(G - G'')T$ are disjoint subgraphs and X' is connected.

Claim: $G' \subseteq G''$

Let $g \in G'$. This implies that $g.X' = X'$, suppose if $g \notin G''$ that is $g \in G - G''$.
$P \in Vert(T) \subseteq Vert(X')$.
$\Rightarrow g.P \in Vert(g.X') = Vert(X')$.
Also we have that $X' \subset G''T$, thus we have

$gP \in Vert(G''T)$ and by assumption we have $gP \in (G - G'')T$ and we have a contradiction since subgraphs $G''T$ and $(G - G'')T$ are disjoint. Thus we have $G \subseteq G''$.
Combining this with the above $G'' \subseteq G$, we have $G'' = G$

The graph X is connected if and only if $X = X'$ that is if $G = G' = G''$ and we have the required theorem. □

Theorem 2.50. *X contains no circuit if and only if $G_P *_{G_y} G_Q \to G$ is injective.*

Proof. X contains a circuit if and only if there exists a path $c = (w_0, ...w_n)$, where $n \geq 1$ in X without backtracking such that $o(w_0) = t(w_n)$. As $G \backslash X = T$, we have only two class of edges in $G \backslash X$, which are $Gy = \{gy|g \in G\}$ and $G\bar{y} = \{g\bar{y}|g \in G\}$ thus for every edge w_i we have an $h_i \in G$ such that $w_i = h_i y_i$, where $y_i = y$ and $y_i = \bar{y}$. Passing this to the quotient we have that $Gy_i = G\overline{y_{i-1}}$, suppose if $y_i = y$, then there exists a $g \in G$ such that $gy = \overline{y_{i-1}}$, as G acts without inversion on X, we have that $y_{i-1} = \overline{y} = \overline{y_i}$, similarly if $y_i = \bar{y}$, we have $y_{i-1} = \overline{y_i}$. Now let $P_i = o(y_i) = t(y_{i-1})$. Notice that

$$h_i P_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i \ .$$

Thus, $h_{i-1}^{-1} h_i \in G_{P_i}$, hence there exists a $g_i \in G_{P_i}$ such that $h_i = h_{i-1} g_i$.
Also $g_i \notin G_y$, suppose if $g_i \in G_y$,
$\Rightarrow g_i y = y$, using last equation we have that,

$$h_i y = h_{i-1} y \ ,$$

which is a contradiction, since

$$\overline{h_i y_i} \neq h_{i-1} y_{i-1} \ ,$$

where $y_i \in \{y, \bar{y}\}$.
Notice that $o(w_0) = t(w_n)$ is equivalent to writing that $t(y_n) = P_0$, which is

also equivalent to

$$h_0P_0 = h_nP_0 = h_{n-1}g_nP_0 = h_{n-2}g_{n-1}g_nP_0 = ...h_0g_1g_2...g_nP_0 \ ,$$

that is $g_1g_2...g_n \in G_{P_0}$.

Thus, X contains a circuit if and only if we can find a sequence of vertices of T with $\{P_{i-1}, P_i\} = \{P, Q\}$ for all i and a sequence of elements $g_i \in G_{P_i} - G_y$ $(0 \leq i \leq n)$, such that $g_1g_2...g_n \in G_{P_0}$, thus there exist an element $g_0 \in G_{P_0}$

$$g_0g_1...g_n = 1 \ .$$

Thus we have the map $f_{\mathbf{i}} : G'_{\mathbf{i}} \to G$ is not injective, where $\mathbf{i} = (0, 1, ..., n)$, and $G'_{\mathbf{i}}$ is same as defined in *Remark* 2.7 of *Theorem* 2.5 taking $A = G_y$ and $G'_i = G_{P_i} - G_y$, because

$$\theta_g = G_y^{n-1}.(g_0g_1...g_n) \mapsto \tau_{P_0}(g_0)\tau_{P_1}(g_1)...\tau_{P_n}(g_n) = g_0g_1...g_n = 1 \ ,$$

where $g = g_0g_1...g_n$ and $\tau_{P_i} : G_y \to G_{P_i}$ are inclusion maps, which is a contradiction, since it were a bijective map.
Hence, we have our result.

These two theorems together form the statement that X is a tree if and only if $G_P *_{G_y} G_Q \to G$ is isomorphism. □

Definition 2.51. $SL(2, \mathbb{Z})$ is a group consisting of all matrices of rank two, whose entries are from set of integers and determinant is one.

Theorem 2.52. *We want to establish that* $SL(2, \mathbb{Z}) = \langle X, Y \rangle$*, where*

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Proof. Since, $X, Y \in SL(2,\mathbb{Z})$. Therefore, clearly

$$\langle X, Y\rangle \subseteq SL(2,\mathbb{Z}) \ .$$

Therefore, it suffices to show that $SL(2,\mathbb{Z}) \subseteq \langle X, Y\rangle$.
Let $g \in SL(2,\mathbb{Z})$ such that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. We want to show that

$$gX^{n_1}Y^{m_1}....X^{n_k}Y^{m_k} = \pm I \ .$$

This will imply that $g \in \langle X, Y\rangle$. We want that $|a|$ should be smaller than $|b|$. If not then by applying following operation we can achieve this.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b & a \\ -d & c \end{bmatrix} \qquad (Op\ 1)$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \qquad (Op\ 2) \ .$$

Let $XY = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = S^{-1}$, where $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Suppose $b \neq 0$, we want to produce $g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ from g, such that $b' = 0$.
This can be done by the following operations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b-a \\ c & d-c \end{bmatrix} \qquad (Op\ 3) \ .$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b+a \\ c & d+c \end{bmatrix} \qquad (Op\ 4) \ .$$

If $b = \pm na$, then this could be done by repeated operation of $(Op\ 3)$ and

(Op 4).

If $b \neq \pm na$, then apply (Op 3) and (Op 4) until we obtain $g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, with $|a'| > |b'|$.

Now apply ($Op1$) to get

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b' & a' \\ -d' & c' \end{bmatrix} = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}, such\ that\ |a''| < |b''| \ .$$

Now repeat this procedure to get

$$g''' = \begin{bmatrix} a''' & b''' \\ c''' & d''' \end{bmatrix}, with\ b''' = 0\ .$$

This implies $a'''d''' = 1 \Rightarrow a''' = d''' = \pm 1$. Hence upto multiplying $-I$, we have $g''' = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$.

Let $YX = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, then

$$T = YX.(-I) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\ ,\ T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\ ,$$

then we have $g''' = T^c \in \langle X, Y \rangle$.

Hence in this way we have $A_1, A_2, ..., A_n \in \langle X, Y \rangle$ such that $g(A_1.A_2....A_n) = \pm I$, which would imply that $g \in \langle X, Y \rangle$. Hence we have the required result. □

Theorem 2.53. *Establish that* $SL(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$

Proof. To show that $SL(2, \mathbb{Z})$ is amalgamated product of two groups we need to have

(1) A tree with a segment as a fundamental domain upon which $SL(2, \mathbb{Z})$ acts without inversion.

(2) Compute the stabilizers of the vertices and the edge of the fundamental domain.

$SL(2,\mathbb{Z})$ acts on the upper half plane via Mobius transformation which is given as

$$SL(2,\mathbb{Z}) \times H\mathbb{C} \to H\mathbb{C}$$

such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .z = \frac{az+b}{cz+d} \ ,$$

where $H\mathbb{C}$ =upper half of complex plane. It is a well defined action. Let y be the circular arc consisting of points $z = e^{i\theta}$ for $\frac{\pi}{3} \leq \theta < \frac{\pi}{2}$. Let

$$P = o(y) = e^{\frac{i\pi}{3}} \ and \ Q = t(y) = e^{\frac{i\pi}{2}} = i \ .$$

Let $g \in SL(2,\mathbb{Z}) = \langle X, Y \rangle$, then $g = X^{n_1}Y^{m_1}.....X^{n_k}Y^{m_k}$, where $n_1, ...n_k \in \{0,1\}$ and $m_1, ...m_k \in \{0,1,2\}$. Thus to construct tree we just have to see how does X, X^2, Y, Y^2, Y^3 acts on P and Q, notice that

$$X.Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .i = \frac{-i}{-1} = i$$

and

$$X.P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .e^{i\pi/3} = \frac{1}{-e^{i\pi/3}} = e^{i2\pi/3} \ .$$

Now act X on $X.P = e^{i2\pi/3}$. We get,

$$X.e^{i2\pi/3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .e^{i2\pi/3} = \frac{1}{-e^{i2\pi/3}} = e^{i\pi/3} \ .$$

Moreover,

$$Y.o(y) = Y.P = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .e^{i\pi/3} = \frac{1}{-e^{i\pi/3}+1} = e^{i\pi/3}$$

$$Y.t(y) = Y.Q = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .i = \frac{1}{-e^{i\pi/2}+1} = \frac{1}{-i+1} = \frac{1}{2} + \frac{i}{2} \ .$$

Now again act Y on $Y.Q$ we get,

$$Y.(Y.Q) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .(\frac{1}{2} + \frac{i}{2}) = 1 + i \ .$$

Now we have

$$Y.(1+i) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .(1+i) = i \ .$$

Now act X on $Y.Q$ we have

$$X.(Y.Q) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .(1/2 + i/2) = -1 + i \ .$$

Also

$$Y.(X.P) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .e^{i2\pi/3} = 1/2 + \sqrt{3}i/12 \ .$$

In this way we have constructed a following graph X which is in fact a tree of which T is segment with P and Q as vertices and y as its edge and is fundamental domain of $X\backslash SL(2,\mathbb{Z})$. .

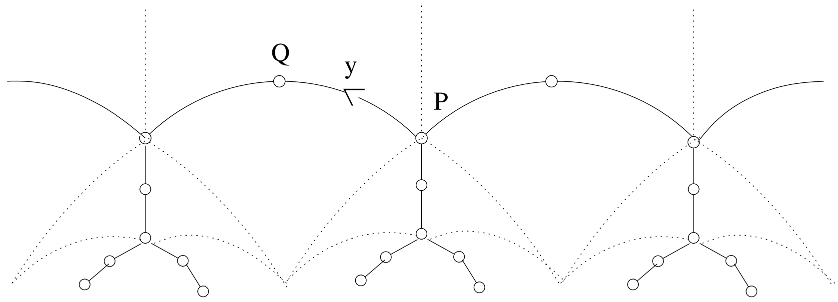

Fig. 2.15: $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$

From Theorem 2.48 $SL(2,\mathbb{Z})$ is isomorphic to amalgam of stabilizers of P and Q over the stabilizer of y that is $SL(2,\mathbb{Z}) \cong G_P *_{G_y} G_Q$. Thus all remain is to compute stabilizers of P, Q and y.

First we compute $G_P = \{g \in SL(2,\mathbb{Z}) | g.P = P\}$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$

$SL(2,\mathbb{Z})$ such that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ and \ g.e^{i\pi/3} = e^{i\pi/3}$$

Thus we have

$$\frac{ae^{i\pi/3} + b}{ce^{i\pi/3} + d} = e^{i\pi/3}$$

$\Rightarrow ae^{i\pi/3} + b = ce^{2i\pi/3} + de^{i\pi/3}$
$\Rightarrow (a-d)(cos(\pi/3) + isin(\pi/3)) + b = c(cos(2\pi/3) + isin(2\pi/3))$
$\Rightarrow (a-d+c)cos(\pi/3) + b + i(a-d-c)sin(\pi/3) = 0$
$\Rightarrow \frac{a-d+c}{2} + b = 0$ and $a = d + c$.

On substituting $a = d + c$ in $a - d + c + 2b = 0$, we have $c = -b$. Now substitute this in $ad - bc = 1$, we get
$ad + b^2 = 1$, only choices for b are $b = 0$ or $b = -1$ or $b = 1$.

Case 1

Suppose $b = 0$, then $ad = 1$, thus $a = 1, d = 1$ or $a = -1, d = -1$, Hence for this case we have two matrices given as

$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ and \ g_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Case 2

Suppose $b = 1$, then $c = -1$ thus $ad = 0$, also we have that $a = d + c$, on solving this we have $a = 0, d = 1$ or $a = -1, d = 0$, Hence for this case we have following two matrices

$$g_3 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \ and \ g_4 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Case 3 Now comes the last case when $b = -1$, thus $c = 1$, thus $ad = 0$, also we have that $a = d + c$, On solving these we have that $a = 0, d = -1$ or

$a = 1, d = 0$, thus for this case we have following two matrices

$$g_5 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \ and \ g_6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} .$$

Hence We have that G_P is cyclic group of order 6 generated by

$$Y = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .$$

which is isomorphic to $\mathbb{Z}_6$, thus we have $G_P \cong \mathbb{Z}_6$.

Now we compute $G_Q = \{g \in SL(2,\mathbb{Z}) | g.Q = Q\}$, Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$ such that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ and \ g.e^{i\pi/2} = g.i = e^{i\pi/2} = i \ ,$$

Thus we have that

$$\frac{ai + b}{ci + d} = i \ ,$$

on solving this equation we have following $(b+c)+(a-d)i = 0$, which implies that $a = d$, $b = -c$, also we have that $ad-bc = 1$, substituting $a = d$, $b = -c$ in this we have $a^2 + b^2 = 1$, Since $a, b, c, d \in \mathbb{Z}$, hence it gives rise to these 4 matrices

$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ and \ g_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} .$$

and

$$g_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ and \ g_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Hence G_Q is a cyclic group of order 4 generated by

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which is isomorphic to $\mathbb{Z}_4$ that is $G_Q \cong \mathbb{Z}_4$.

Now all remain is to compute G_y but its easy since we have $G_P \cap G_Q = G_y$ and hence G_y consist of these two matrices

$$g_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \; and \; g_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is a cyclic group of order 2 generated by

$$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

which is isomorphic to $\mathbb{Z}_2$ that is $G_y \cong \mathbb{Z}_2$.
Hence we have the required result

$$SL(2,\mathbb{Z}) \cong \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4 \ .$$

□

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