2. Chern connections and Chern curvatures[1]

Let V be a complex vector space with $\dim_{\mathbb{C}} V = n$. A hermitian metric h on V is

$$h : V \times V \longrightarrow \mathbb{C}$$

such that

$$h(av, bu) = a\bar{b}h(v, u)$$
$$h(a_1v_1 + a_2v_2, u) = a_1h(v_1, u) + a_2h(v_2, u)$$
$$h(v, u) = \overline{h(u, v)}$$
$$h(u, u) > 0, u \neq 0$$

where $v, v_1, v_2, u \in V$ and $a, b, a_1, a_2 \in \mathbb{C}$. If we fix a basis $\{e_i\}$ of V, and set

$$h_{i\bar{j}} = h(e_i, e_j)$$

then

$$h = h_{i\bar{j}}e_i^* \otimes \overline{e}_j^* \in V^* \otimes \overline{V}^*$$

where $e_i^* \in V^*$ is the dual of e_i and $\overline{e}_i^* \in \overline{V}^*$ is the conjugate dual of e_i, i.e.

$$\overline{e}_i^*(\sum a_je_j) = \overline{a}_i$$

It is obvious that $(h_{i\bar{j}})$ is a hermitian positive matrix.

Definition 0.1. A complex vector bundle E is said to be *hermitian* if there is a positive definite hermitian tensor h on E.

Let $\varphi : E|_U \longrightarrow U \times \mathbb{C}^r$ be a trivilization and $e = (e_1, \cdots, e_r)$ be the corresponding frame. The hermitian metric h is represented by a positive hermitian matrix $(h_{i\bar{j}}) \in \Gamma(\Omega, End\mathbb{C}^r)$ such that

$$\langle e_i(x), e_j(x)\rangle = h_{i\bar{j}}(x), \quad x \in U$$

Then hermitian metric on the chart (U, φ) could be written as

$$h = \sum h_{i\bar{j}}e_i^* \otimes \overline{e}_j^*$$

For example, there are two charts (U, φ) and (V, ψ). We set

$$g = \psi \circ \varphi^{-1} : (U \cap V) \times \mathbb{C}^r \longrightarrow (U \cap V) \times \mathbb{C}^r$$

and g is represented by matrix (g_{ij}). On $U \cap V$, we have

$$e_i(x) = \varphi^{-1}(x, \varepsilon_i) = \psi^{-1} \circ \psi \circ \varphi^{-1}(x, \varepsilon_i) = \psi^{-1}(x, \sum g_{ij}\varepsilon_j) = \sum_j g_{ij}\psi^{-1}(x, \varepsilon_j) = \sum_j g_{ij}\tilde{e}_j(x)$$

For the metric

$$h_{i\bar{j}} = \langle e_i(x), e_j(x)\rangle = \langle g_{ik}\tilde{e}_k(x), g_{jl}\tilde{e}_l(x)\rangle = \sum_{k,l} g_{ik}\tilde{h}_{k\bar{l}}\overline{g}_{jl}$$

that is

$$h = g \cdot \tilde{h} \cdot g^*$$

[1]2008.04.30 If there are some errors, please contact to: geal2004@cms.zju.edu.cn

Example 0.2 (Fubini-Study metric on holomorphic tangent bundle $T^{1,0}\mathbb{P}^n$). On the trivialization of $T\mathbb{P}^n$, for example $U_0 = \{Z_0 \neq 0\}$, and basis of the fiber $e_i = \frac{\partial}{\partial z_i}$, we could set

$$h_{i\bar{j}}(z) = \langle e_i(z), e_j(z)\rangle = \frac{\partial^2 \log(1 + \sum_{i=1}^n |z_i^2|)}{\partial z_i \partial z_j}$$

Or

$$h(z) = \sum h_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

When E is a hermitian vector bundle, there is a natural sequilinear map

$$A^p(M, E) \times A^q(M, E) \longrightarrow A^{p+q}(M, \mathbb{C}), \quad (s, t) \longrightarrow \{s, t\}$$

In local coordinate, if $s = \sum \sigma^i \otimes e_i$ and $t = \sum \tau^j \otimes e_j$,

$$\{s, t\} = \sum_{i,j} \sigma^i \wedge \bar{\tau}^j \langle e_i, e_j\rangle = \sum_{i,j} h_{i\bar{j}} \sigma^i \wedge \bar{\tau}^j$$

Definition 0.3. A connection ∇ on E is *compatible with the hermitian structure* of E, or a metric connection if for any $s \in A^p(M, E)$ and $t \in A^q(M, E)$ we have

$$d\{s, t\} = \{\nabla s, t\} + (-1)^p \{s, \nabla t\}$$

Proposition 0.4. *If ∇ is a metric connection, then*

$$\Theta^* = -\Theta$$

with respect to any trivialization.

Proof. It is obvious that

$$0 = d^2\{s, t\} = \{\nabla^2 s, t\} + \{s, \nabla^2 t\} \Longrightarrow \Theta^* = -\Theta$$

□

In the local holomorphic coordinates $\{z^i\}$ of M, we have

$$\frac{\partial}{\partial z^i}\{s, t\} = \{\nabla_{\frac{\partial}{\partial z^i}} s, t\} + (-1)^p \{s, \nabla_{\frac{\partial}{\partial \bar{z}^i}} t\}$$

If (E, ∇, h) is a Hermitian complex vector bundle with a metric connection, there is a decomposition $\nabla = \nabla' + \nabla''$, where

$$\nabla' : A^{p,q}(M, E) \longrightarrow A^{p+1,q}(M, E), \quad \nabla'' : A^{p,q}(M, E) \longrightarrow A^{p,q+1}(M, E)$$

under the decomposition

$$A^l(M, E) = \bigoplus_{p+q=l} A^{p,q}(M, E),$$

and

$$\nabla'(f \wedge s) = \partial f \wedge s + (-1)^{\deg f} f \wedge \nabla' s,$$
$$\nabla''(f \wedge s) = \bar{\partial} f \wedge s + (-1)^{\deg f} f \wedge \nabla'' s.$$

In the local holomorphic coordinates $\{z^i\}_{i=1,\cdots,n}$ on M, and local holomorphic frame $\{e_\alpha\}_{\alpha=1,\cdots,r}$, we set

$$\nabla e_\alpha = (\Gamma^\beta_{\alpha i} dz^i + \Gamma^\beta_{\alpha \bar{l}} d\bar{z}^l) \otimes e_\beta$$

then

$$\nabla' e_\alpha = \Gamma^\beta_{\alpha i} dz^i \otimes e_\beta, \quad \nabla'' e_\alpha = \Gamma^\beta_{\alpha \bar{l}} d\bar{z}^l \otimes e_\beta$$

The formal adjoint operators are denoted by

$$\delta' : A^{p+1,q}(M,E) \longrightarrow A^{p,q}(M,E), \quad \delta'' : A^{p,q+1}(M,E) \longrightarrow A^{p,q}(M,E)$$

with respect to the inner product induced by h. The Laplacian operators are defined by

$$\Delta' = \nabla'\delta' + \delta'\nabla'; \quad \Delta'' = \nabla''\delta'' + \delta''\nabla''; \; \Delta = \nabla\delta + \delta\nabla$$

For the metric connection ∇ on the Hermitian complex vector bundle (E,h), we have the decomposition of the curvature

$$\Theta = \Theta^{2,0} + \Theta^{1,1} + \Theta^{0,2}$$

where

$$\Theta^{2,0} = \nabla'^2 \in \Gamma(M, \Lambda^{2,0}T^*M \otimes E^* \otimes E)$$
$$\Theta^{1,1} = \nabla'\nabla'' + \nabla''\nabla' \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$$
$$\Theta^{2,0} = \nabla''^2 \in \Gamma(M, \Lambda^{0,2}T^*M \otimes E^* \otimes E)$$

In the local holomorphic coordinates (z^i) of M and coordinates (e_α) of E, we can write

$$\Theta^{2,0} = R^\beta_{ij\alpha} dz^i \wedge dz^j \otimes e^{\alpha*} \otimes e_\beta; \qquad \Theta^{0,2} = R^\beta_{\bar{i}\bar{j}\alpha} d\bar{z}^i \wedge d\bar{z}^j \otimes e^{\alpha*} \otimes e_\beta$$

$$\Theta^{1,1} = R^\beta_{i\bar{j}\alpha} dz^i \wedge d\bar{z}^j \otimes e^{\alpha*} \otimes e_\beta$$

Here and henceforth we sometimes adopt the Einstein convention for summation.

On the holomorphic vector bundle $E \stackrel{\pi}{\longrightarrow} M$, we can define the $\overline{\partial}$-operator:

$$\overline{\partial} : A^p(M,E) \longrightarrow A^{p+1}(M,E)$$

If we take the holomorphic frame $e = (e_1, \cdots, e_n)$ for E over U, then write $s \in A^p(M,E)$ as

$$s = \sum \sigma^i \otimes e_i$$

and set

$$\overline{\partial} s = \overline{\partial}\sigma^i \otimes e_i$$

Proposition 0.5. *The $\overline{\partial}$-operator is well defined.*

Proof. If $e' = (e'_1, \cdots, e'_n)$ is any other holomorphic frame of E over U, with

$$e_i = g_{ij} e'_j$$

If $s = \sigma^i \otimes e_i = \sigma^{i'} \otimes e'_i$, then $\sigma^{i'} = \sum_j g_{ji}\sigma^j$

then we have

$$\overline{\partial} s = \overline{\partial}\sigma^{i'} \otimes e'_i = \overline{\partial}(g_{ji}\sigma^j) \otimes e'_i = g_{ij}\overline{\partial}\sigma^j \otimes e'_i = \overline{\partial}\sigma^j \otimes e_j$$

since g_{ij} is holomorphic. □

Lemma 0.6. *If E is a hermitian holomorphic vector bundle, then there exists a unique connection such that:*

i. ∇ is compatible with the complex structure, *i.e.* $\nabla'' = \overline{\partial}$

ii. ∇ is compatible with the hermitian structure, *i.e.*

$$d\{s,t\} = \{\nabla s, t\} + (-1)^{\deg s}\{s, \nabla t\}$$

for any $s, t \in A^{\bullet}(M, E)$.

Proof. Let $e = (e_1, \cdots, e_r)$ be a holomorphic frame for E under some trivialization, and let $h_{\alpha\overline{\beta}} = \langle e_\alpha, e_\beta \rangle$. If the metric connection ∇ exists, then its connection matrix ω must be type (1, 0), for the $(0,1)$ part of ω for the $\overline{\partial}$ is zero, i.e. $\Gamma^{\beta}_{\alpha\bar{l}} = 0$. Then we have:

$$dh_{\alpha\overline{\beta}} = d\langle e_\alpha,\ e_\beta\rangle = \omega^{\gamma}_{\alpha} h_{\gamma\overline{\beta}} + h_{\alpha\overline{\gamma}}\overline{\omega}^{\gamma}_{\beta}$$

so we have

$$\partial h_{\alpha\overline{\beta}} = \omega^{\gamma}_{\alpha} h_{\gamma\overline{\beta}}, \qquad i.e.\ \ \omega^{\gamma}_{\alpha} = \partial h_{\alpha\overline{\beta}} h^{\gamma\overline{\beta}}$$

$$\overline{\partial} h_{\alpha\overline{\beta}} = h_{\alpha\overline{\gamma}}\overline{\omega}^{\gamma}_{\beta}, \qquad i.e.\ \ \omega^{\gamma}_{\alpha} = \partial h_{\alpha\overline{\beta}} h^{\gamma\overline{\beta}}$$

If we set $\omega = h^{-1} \cdot \partial h$, we know ω satisfies the condition above. Then the connection is determined by ω. □

Remark 0.7. Here $h^{-1} = (h^{\alpha\overline{\beta}})$ which is the inverse of h^t, that is

$$\sum h_{\alpha\overline{\beta}} h^{\alpha\overline{\gamma}} = \delta^{\gamma}_{\beta}$$

Now we compute the curvature in the local coordinates. In the local holomorphic coordinates $\{z^i\}_{i=1,\cdots,n}$ on M, and local holomorphic frame $\{e_\alpha\}_{\alpha=1,\cdots,r}$, we have

$$\Theta = \sum_{\alpha,\beta} \Theta^{\beta}_{\alpha} e^{\alpha *} \otimes e_\beta = \sum R^{\beta}_{i\bar{j}\alpha} dz^i \wedge d\overline{z}^j \otimes e^{\alpha *} \otimes e_\beta \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$$

By the relation $\Theta = d\omega - \omega \wedge \omega$, we get

$$\Theta^{\beta}_{\alpha} = d\omega^{\beta}_{\alpha} - \omega^{\gamma}_{\alpha} \wedge \omega^{\beta}_{\gamma}$$

On the other hand

$$\omega^{\beta}_{\alpha} = h^{\beta\overline{\gamma}} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i} dz^i$$

we get

$$d\omega^{\beta}_{\alpha} = -h^{\beta\overline{\gamma}}\left(\frac{\partial^2 h_{\alpha\overline{\gamma}}}{\partial z^i \partial \overline{z}^j} - h^{\delta\overline{\mu}} \frac{\partial h_{\alpha\overline{\mu}}}{\partial z^i} \frac{\partial h_{\delta\overline{\gamma}}}{\partial \overline{z}^j}\right) dz^i \wedge d\overline{z}^j + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} h^{\beta\bar{\lambda}} \frac{\partial h_{\gamma\bar{\lambda}}}{\partial z^j} dz^i \wedge dz^j$$

and

$$\omega^{\gamma}_{\alpha} \wedge \omega^{\beta}_{\gamma} = h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} h^{\beta\bar{\lambda}} \frac{\partial h_{\gamma\bar{\lambda}}}{\partial z^j} dz^i \wedge dz^j$$

then

$$R^{\beta}_{i\bar{j}\alpha} = -h^{\beta\overline{\gamma}}\left(\frac{\partial^2 h_{\alpha\overline{\gamma}}}{\partial z^i \partial \overline{z}^j} - h^{\delta\overline{\mu}} \frac{\partial h_{\alpha\overline{\mu}}}{\partial z^i} \frac{\partial h_{\delta\overline{\gamma}}}{\partial \overline{z}^j}\right).$$

where we use the formula

$$\partial h^{\alpha\overline{\beta}} = -h^{\alpha\overline{\gamma}} \partial h_{\delta\overline{\gamma}} h^{\delta\overline{\gamma}}$$

Here use the formal expression, we have

$$\omega = h^{-1} \cdot \partial h$$

then

$$d\omega = \overline{\partial}(h^{-1}\partial h) + \partial h^{-1} \wedge \partial h$$

For $\omega \wedge \omega = \partial h^{-1} \wedge \partial h$ Then

$$\Theta = \overline{\partial}(h^{-1} \cdot \partial h)$$

If $r = 1$, that is E is a line bundle, then

$$\Theta = \overline{\partial}\partial \log h$$

Now we set

$$R_{i\overline{j}\alpha\overline{\beta}} = R^{\gamma}_{i\overline{j}\alpha} h_{\gamma\overline{\beta}}$$

then

$$R_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i \overline{\partial} z^j} + h^{\delta\overline{\gamma}} \frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i} \frac{h_{\delta\overline{\beta}}}{\overline{\partial} z^j}$$

Theorem 0.8 (Normal frames on holomorphic vector bundles). *For every point x_0 of X and every local holomorphic coordinates $(z_1, \cdots, z_n)$ at x_0, there exists a holomorphic frame $(e_1, \cdots, e_r)$ in a neighborhood Ω of x_0 such that*

$$\langle e_\lambda(z), e_\mu(z) \rangle = \delta_{\lambda\mu} - \sum_{j,k=1}^{n} R_{j\overline{k}\lambda\overline{\mu}} z_j \overline{z}_k + O(|z|^3)$$

where the $(R_{j\overline{k}\lambda\overline{\mu}})$ are the coefficients of the Chern curvature tensor $\Theta^E_{x_0}$ under the above local frame.

Proof. By linear transformations.

□

Lemma 0.9. *If L is a holomorphic line bundle over a complex manifold M with trivialization functions $\varphi_i : L|_{U_i} \longrightarrow U_i \times \mathbb{C}$ and transition functions $g_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}^*$.*

(1) *If $(\ ,\ )_L$ is a hermitian metric on L and $\mu_j : U_j \longrightarrow \mathbb{R}^+$ is given by $\mu_j(p) = \|\varphi_j^{-1}(p, 1)\|^2$. Then*

$$\mu_j = \mu_i |g_{ij}|^2, \qquad \text{on } U_i \cap U_j$$

(2) *Any collections $\mu_i : U_i \longrightarrow \mathbb{R}^+$ satisfies*

$$\mu_j = \mu_i |g_{ij}|^2, \qquad \text{on } U_i \cap U_j$$

defines a hermitian metric on L.

(3) *The unique Chern connection on L has the curvature*

$$\Theta = -\sqrt{-1}\partial\overline{\partial} \log \mu_i \text{ on } U_i$$

Proof. **(1)** If $p \in U_i \cap U_j$,

$$\begin{aligned} \mu_j(p) &= \|h_j^{-1}(p,1)\|^2 = \|h_i^{-1} \circ h_i \circ h_j^{-1}(p,1)\|^2 \\ &= \|h_i^{-1}(p, g_{ij}(p))\|^2 = |g_{ij}(p)|^2 \|h_i^{-1}(p,1)\|^2 \\ &= \mu_i(p)|g_{ij}(p)|^2 \end{aligned}$$

(2) If $p \in U_i$ and $v_1, v_2 \in L_p$, then $v_i = \varphi_j^{-1}(p, t_i)$ for $i = 1, 2$. We can define a hermitian metric $(\ ,\ )_L$ on L by

$$(v_1, v_2)_L = \mu_i(p) t_1 \bar{t}_2$$

It is a well-defined hermitian metric. In fact, if $p \in U_i \cap U_j$, by the definition of line bundles, there exists t_1', t_2' such that

$$v_i = \varphi_i^{-1}(p, t_i) = \varphi_j^{-1}(p, t_i')$$

for $i = 1, 2$ where

$$t_i = g_{ij}(p) t_i'$$

By definition

$$(v_1, v_2)_L = \mu_j(p) t_1' \bar{t}_2' = \mu_j(p) |g_{ij}(p)|^{-2} t_1 \bar{t}_2 = \mu_i(p) t_1 t_2$$

So the hermitian metric is well-defined on L.

(3) We just need to check that the curvature matrix is globally defined. We have

$$\partial\bar{\partial} \log \mu_j = \partial\bar{\partial} \log \mu_i$$

for g_{ij} is holomorphic. □

Example 0.10 (Universal line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ of $\mathbb{P}^n$). We have a construction for the universal bundle J as follows:

$$J = \{[Z_0, Z_1, \cdots, Z_n], \lambda(Z_0, Z_1, \cdots, Z_n)) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \lambda \in \mathbb{C}\}$$

Proof. Set $L = \{([Z_0, Z_1, \cdots, Z_n], \lambda(Z_0, Z_1, \cdots, Z_n)) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \lambda \in \mathbb{C}\}$. We have

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$$

where $U_i = \{Z_i \neq 0\} \subset \mathbb{P}^n$. Then the trivialiZation functions

$$\varphi_i : L|_{U_i} \longrightarrow U_i \times \mathbb{C} \cong \mathbb{C}^{n+1}$$

where

$$\varphi_i([Z_0, Z_1, \cdots, Z_n], \lambda(Z_0, Z_1, \cdots, Z_n))) = ([Z_0, Z_1, \cdots, Z_n], \lambda Z_i)$$

or equivalently

$$\varphi_i([Z_0, Z_1, \cdots, Z_n], \lambda(Z_0, Z_1, \cdots, Z_n))) = (Z_0/Z_i, Z_1/Z_i, \cdots, \widehat{Z_i/Z_i}, \cdots Z_n/Z_i, \lambda Z_i)$$

It is a biholomorphic map and the inverse

$$\varphi_i^{-1} : U_i \times \mathbb{C} \longrightarrow L|_{U_i}$$

is given by

$$\varphi_i^{-1}([Z_0, Z_1, \cdots, Z_n], \lambda Z_i) = ([Z_0, Z_1, \cdots, Z_n], \lambda(Z_0, Z_1, \cdots, Z_n)))$$

or equivalently

$$\varphi_i^{-1}(Z_1, Z_2, \cdots, Z_n, \lambda) = ([Z_1, Z_2, \cdots, 1, \cdots Z_n], \lambda(Z_1, Z_2, \cdots, 1, \cdots Z_n))$$

So we get

$$\varphi_i \circ \varphi_j^{-1}([Z_0, Z_1, \cdots, Z_n], \lambda Z_j) = ([Z_0, Z_1, \cdots, Z_n], \lambda Z_i)$$

The transition functions are given by

$$g_{ij}([Z_0, Z_1, \cdots, Z_n]) = Z_i/Z_j$$

on $U_i \cap U_j$. □

Example 0.11 (Metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ of $\mathbb{P}^n$). The tautological line bundle L is the dual line bundle of the universal line bundle $J = \mathcal{O}_{\mathbb{P}^n}(-1)$, and we denote it by $L = \mathcal{O}_{\mathbb{P}^n}(1)$. The line bundle L, dual bundle of J, by definition, it represented the transition function

$$\tilde{g}_{ij} = \frac{Z_j}{Z_i}$$

Now we construct a hermitian metric on L. On the chart U_i, we set

$$h_i = \frac{1}{\sum_{k=0}^n \left|\frac{Z_k}{Z_i}\right|^2}$$

Then

$$h_j = h_i \left|\frac{Z_j}{Z_i}\right|^2 = h_i |g_{ij}|^2$$

So we define a metric on the line bundle L. The curvature of this metric is given by

$$\omega = \partial\overline{\partial} \log(\sum_{k=0}^n \left|\frac{Z_k}{Z_i}\right|^2)$$

For example, we choose $i = 0$ and set $z_i = Z_i/Z_0$, $i = 1, \cdots, n$. Then

$$\omega = \partial\overline{\partial} \log(1 + \sum_{i=1}^n |z_i|^2)$$

which is the Fubini-Study metric on $T^{1,0}\mathbb{P}^n$.

Example 0.12 (Curvature of the Fubini-Study metric on $T^{1,0}\mathbb{P}^n$). On the chart U_0

$$h_{i\bar{j}} = \frac{\partial^2 \log(1 + \sum_{i=1}^n |z_i|^2)}{\partial z_i \overline{\partial} z_j}$$

Now we choose the point $P = [1, 0, \cdots, 0] = (0, \cdots, 0) \in U_0$. By Taylor's expansion, we have

$$\log(1 + \sum |z_i|^2) = \sum |z_i|^2 - (\sum |z_i|^2)^2 + O(|z|^6)$$

Then, around the point P,

$$h_{i\bar{j}} = \delta_{ij} - (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) z_k \overline{z}_l + O(|z|^4)$$

that is

$$R_{i\bar{j}k\bar{l}}(P) = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} = h_{i\bar{j}}(P) h_{k\bar{l}}(P) + h_{i\bar{l}}(P) h_{k\bar{j}}(P)$$

In fact, by a linear transformation,

$$R_{i\bar{j}k\bar{l}} = h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}$$

for any $P \in U_0$, and we will prove it later.