

# Overview of adiabatic quantum computation

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# Adiabatic optimization

*Quantum adiabatic optimization* is a class of procedures for solving optimization problems using a quantum computer.

Basic strategy:

- Design a Hamiltonian whose ground state encodes the solution of an optimization problem.
- Prepare the known ground state of a simple Hamiltonian.
- Interpolate slowly.

Proposed in the context of quantum computation by Farhi, Goldstone, Gutmann, and Sipser (2000). Related ideas suggested by Kadowaki and Nishimori (1998), Brooke, Bitko, Rosenbaum, and Aeppli (1999), and others.

# Outline

- Quantum computation and Hamiltonian dynamics
- Simulated vs. actual dynamics
- The adiabatic theorem
- Adiabatic optimization and spectral gaps
- Examples
- Robustness of adiabatic quantum computation
- Universal quantum computation
- Open problems
  - Stoquastic Hamiltonians
  - Fault tolerance

# Hamiltonian dynamics

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

In the circuit model, we say a unitary operation can be implemented efficiently if it can be realized (approximately) by a short sequence of one- and two-qubit gates.

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- Hamiltonians we can efficiently simulate using quantum circuits (all of the above, plus sparse Hamiltonians, etc.)

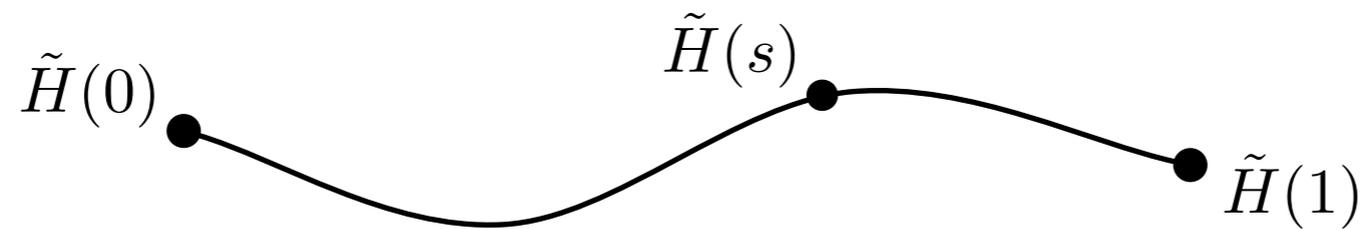
# Simulated vs. actual dynamics

Two perspectives on adiabatic algorithms:

- Simulate the Hamiltonian on a universal quantum computer
  - Can assume fault tolerance
  - Main question: how fast does the noiseless system solve the problem?
- Directly build a system with the desired Hamiltonian
  - Easier than/different from building a universal computer
  - Probably no easier to understand performance
  - Must consider behavior in the presence of noise

# The adiabatic theorem

Let  $\tilde{H}(s)$  be a smoothly varying Hamiltonian for  $s \in [0, 1]$



$$\tilde{H}(s) = \sum_{j=0}^{D-1} E_j(s) |E_j(s)\rangle \langle E_j(s)|$$

where  $E_0(s) < E_1(s) \leq E_2(s) \leq \dots \leq E_{D-1}(s)$

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For large  $T$ ,  $|\psi(T)\rangle \approx |E_0(1)\rangle$ . But how large must it be?

# Approximately adiabatic evolution

The total run time required for adiabaticity depends on the spectrum of the Hamiltonian.

Gap:  $\Delta(s) = E_1(s) - E_0(s)$ ,  $\Delta = \min_{s \in [0,1]} \Delta(s)$

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$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \left\| \left[ \dot{\tilde{H}}(s) \right]^2 \right\|$$

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$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \|\dot{\tilde{H}}(s)\|^2$$

**Theorem.** [Teufel 2003 + perturbation theory; cf. Jansen et al. 2007]

$$T \geq \frac{4}{\epsilon} \left[ \frac{\|\dot{\tilde{H}}(0)\|}{\Delta(0)^2} + \frac{\|\dot{\tilde{H}}(1)\|}{\Delta(1)^2} + \int_0^1 ds \left( 10 \frac{\|\dot{\tilde{H}}\|^2}{\Delta^3} + \frac{\|\ddot{\tilde{H}}\|}{\Delta} \right) \right]$$

implies  $\| |\psi(T)\rangle - |E_0(1)\rangle \| \leq \epsilon$

# Satisfiability problems

- Given  $h: \{0,1\}^n \rightarrow \{0,1,2,\dots\}$ , is there a value of  $z \in \{0,1\}^n$  such that  $h(z)=0$ ?
- Alternatively, what  $z$  minimizes  $h(z)$ ?
- **Example: 3SAT.**  $(z_1 \vee z_2 \vee \bar{z}_3) \wedge \dots \wedge (\bar{z}_{17} \vee z_{37} \vee \bar{z}_{42})$

$$h(z) = \sum_c h_c(z)$$

$$\text{where } h_c(z) = \begin{cases} 0 & \text{clause } c \text{ satisfied by } z \\ 1 & \text{otherwise} \end{cases}$$

# Adiabatic optimization

- Define a *problem Hamiltonian* whose ground state encodes the solution:

$$H_P = \sum_{z \in \{0,1\}^n} h(z) |z\rangle \langle z|$$

- Define a beginning Hamiltonian whose ground state is easy to create, for example

$$H_B = - \sum_{j=1}^n \sigma_x^{(j)}$$

- Choose  $\tilde{H}(s)$  to interpolate from  $H_B$  to  $H_P$ , for example

$$\tilde{H}(s) = (1 - s)H_B + s H_P$$

- Choose total run time  $T$  so the evolution is nearly adiabatic

[Farhi et al. 2000]

# Please mind the gap

Recall rough estimate:

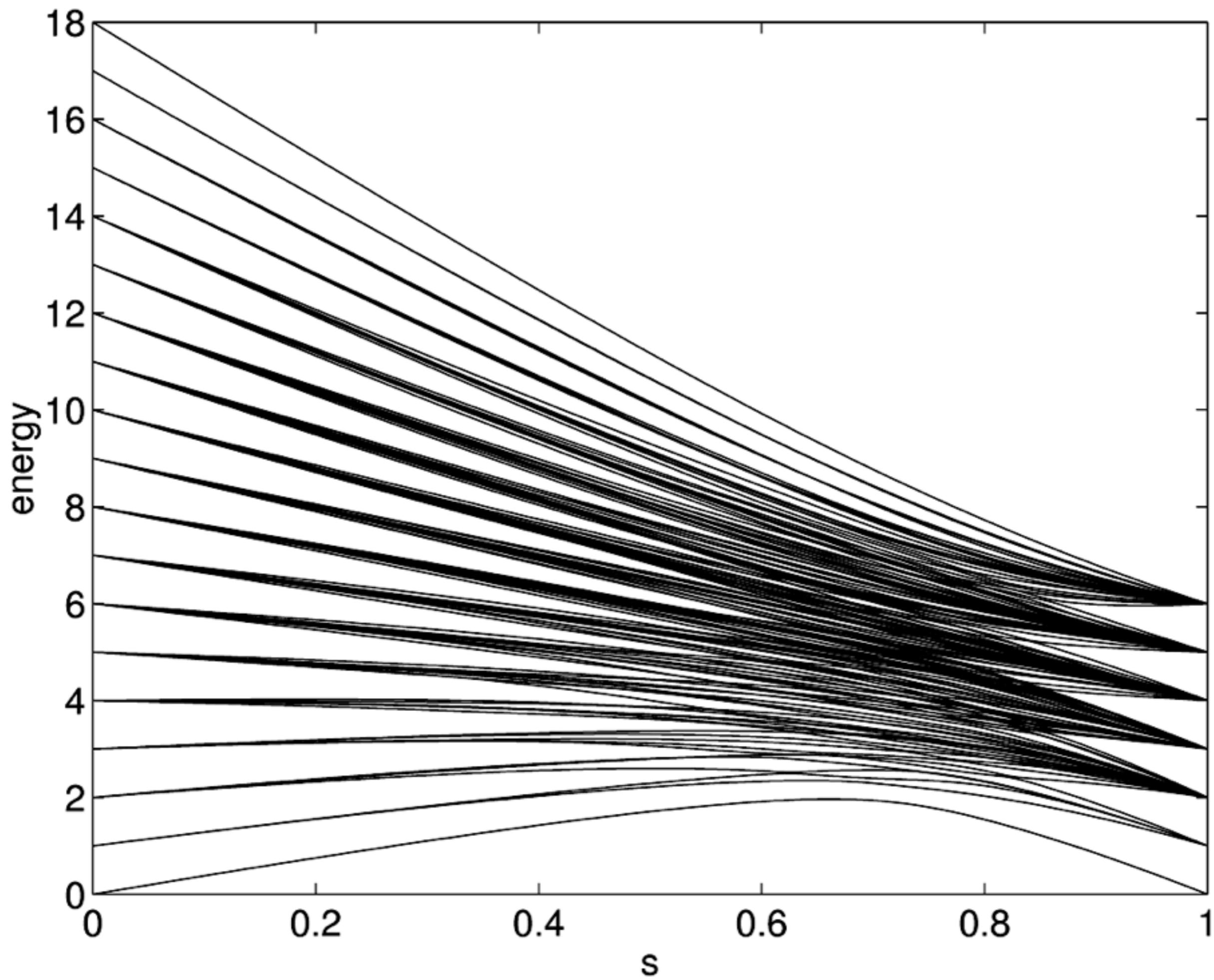
$$T \gg \frac{\Gamma^2}{\Delta^2}, \quad \Gamma^2 = \max_{s \in [0,1]} \left\| \left[ \dot{\tilde{H}}(s) \right]^2 \right\|$$

For  $\tilde{H}(s) = (1-s)H_B + sH_P$ ,

$$\begin{aligned} \|\dot{\tilde{H}}\| &= \|H_P - H_B\| \\ &\leq \|H_B\| + \|H_P\| \end{aligned}$$

Crucial question: How big is  $\Delta$ ?

- $\geq 1/\text{poly}(n)$ : Efficient quantum algorithm
- $1/\text{exp}(n)$ : Inefficient quantum algorithm



# Unstructured search

Finding a needle in a haystack:  $h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases}$   
(here  $h: \{0, 1, \dots, N-1\} \rightarrow \{0, 1\}$ )

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Query complexity (given black box for  $h$ )

- Classically,  $\Theta(N)$  queries
- Quantumly,  $O(\sqrt{N})$  queries are sufficient to find  $w$   
[Grover 1996]  $(|z\rangle|a\rangle \mapsto |z\rangle|a \oplus h(z)\rangle)$
- This cannot be improved:  $\Omega(\sqrt{N})$  queries are necessary  
[Bennett et al. 1997]

# Example: Adiabatic unstructured search

$$h(z) = \begin{cases} 0 & z = w \\ 1 & z \neq w \end{cases} \Rightarrow H_P = \sum_z h(z) |z\rangle\langle z| = 1 - |w\rangle\langle w|$$

Start in  $|s\rangle = \frac{1}{\sqrt{N}} \sum_z |z\rangle$

$$H_B = 1 - |s\rangle\langle s|$$

$$\begin{aligned} \tilde{H}(s) &= (1 - s)H_B \\ &\quad + s H_P \end{aligned}$$

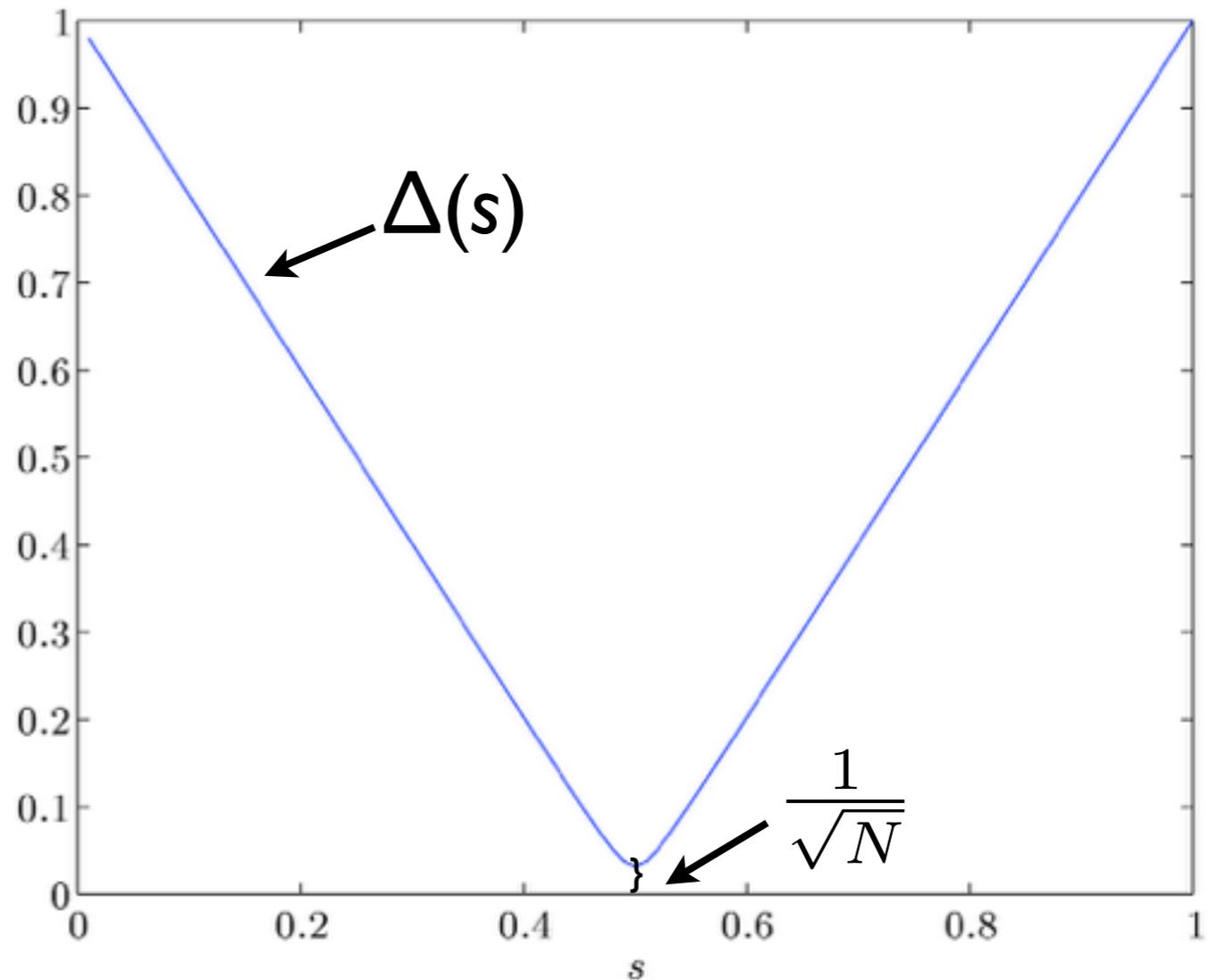
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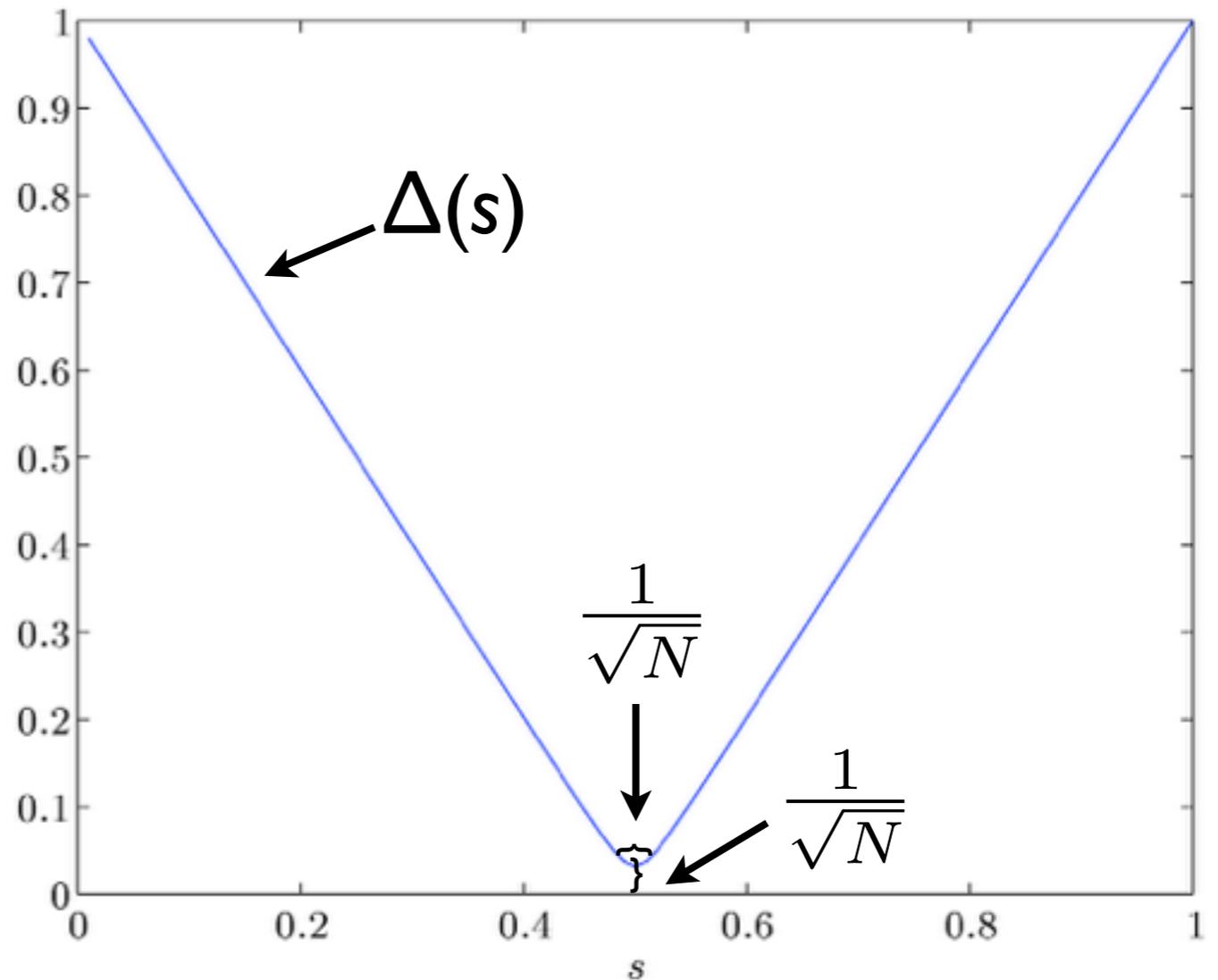
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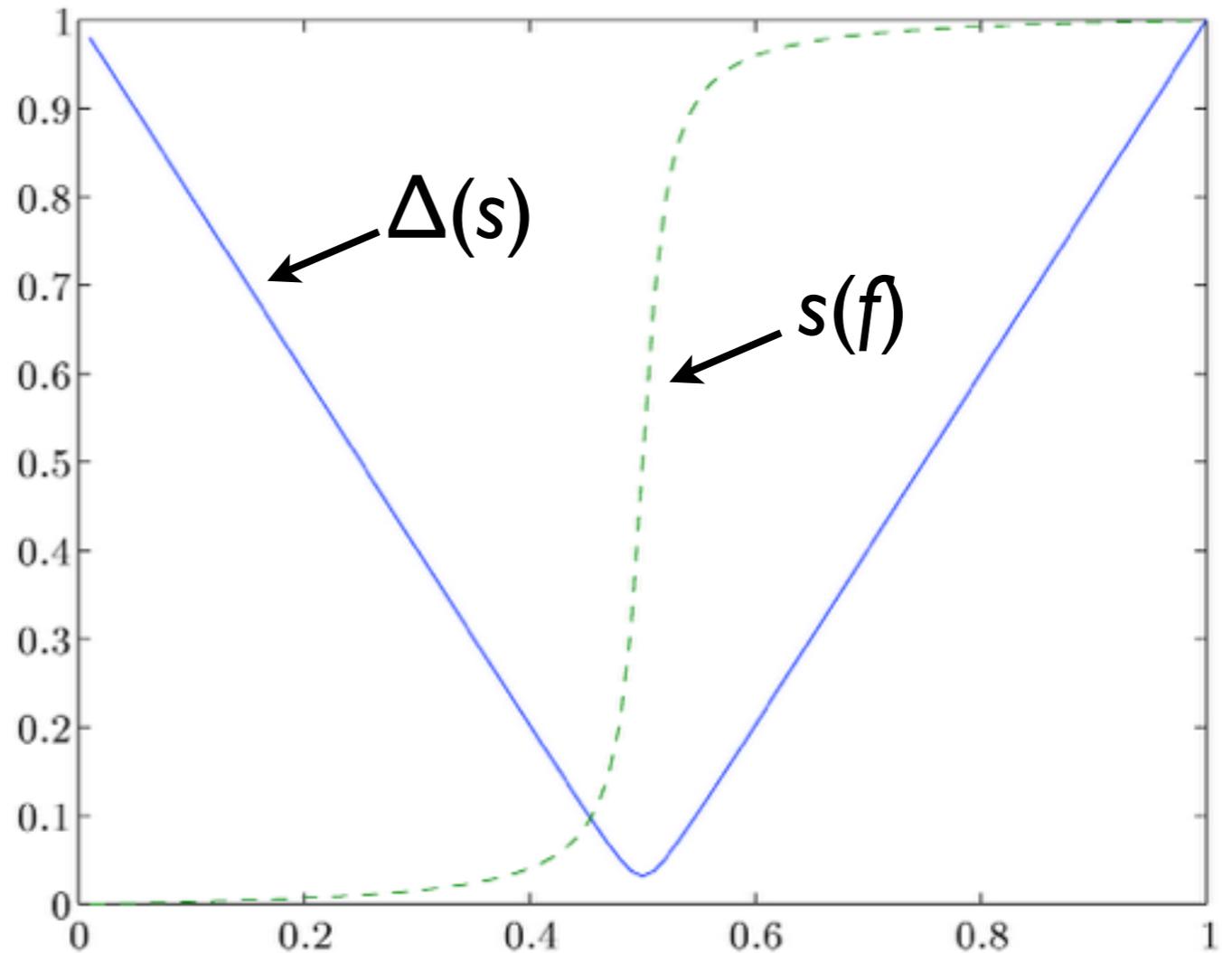
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~~$$\tilde{H}(s) = (1-s)H_B + sH_P$$~~

$$\tilde{H}(s) = [1 - f(s)]H_B + f(s)H_P$$



[Roland, Cerf 2002; van Dam et al. 2001]

# Example: Transverse Ising model

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{1}{2} (1 - \sigma_z^{(j)} \sigma_z^{(j+1)}) \quad \text{“agree”}$$

$$H_B = - \sum_{j=1}^n \sigma_x^{(j)} \quad \text{with ground state} \quad |s\rangle = |+\cdots+\rangle$$
$$\tilde{H}(s) = (1-s)H_B + sH_P \quad = \sum_{z \in \{0,1\}^n} |z\rangle$$

Diagonalize by fermionization (Jordan-Wigner transformation)

Result:  $\Delta \propto \frac{1}{n}$  (at critical point of quantum phase transition)

$$|E_0(s \approx 0)\rangle \approx |+\cdots+\rangle$$

$$|E_0(s \approx 1)\rangle \approx \frac{1}{\sqrt{2}} (|0\cdots 0\rangle + |1\cdots 1\rangle)$$

[Farhi et al. 2000]

# Example: The Fisher problem

$$H_P = \sum_{j \in \mathbb{Z}_n} \frac{J_j}{2} (1 - \sigma_z^{(j)} \sigma_z^{(j+1)}) \quad J_j = 1 \text{ or } 2, \text{ chosen randomly}$$

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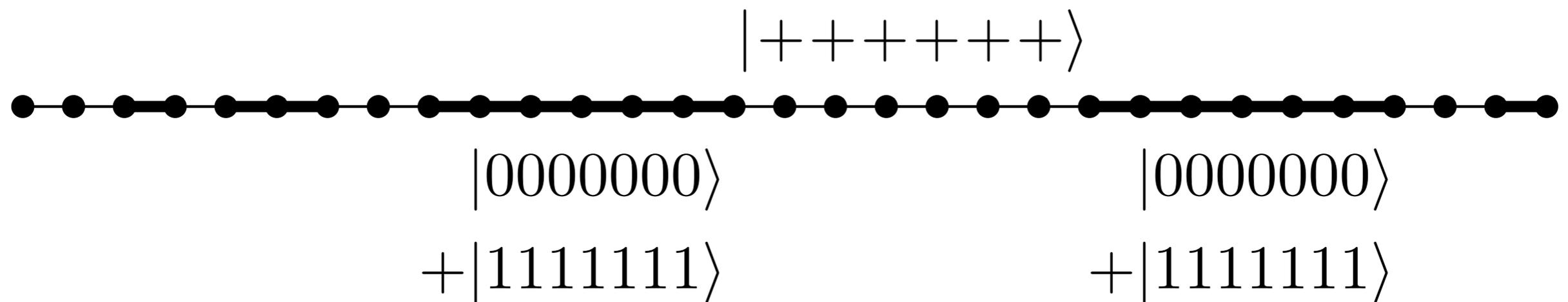
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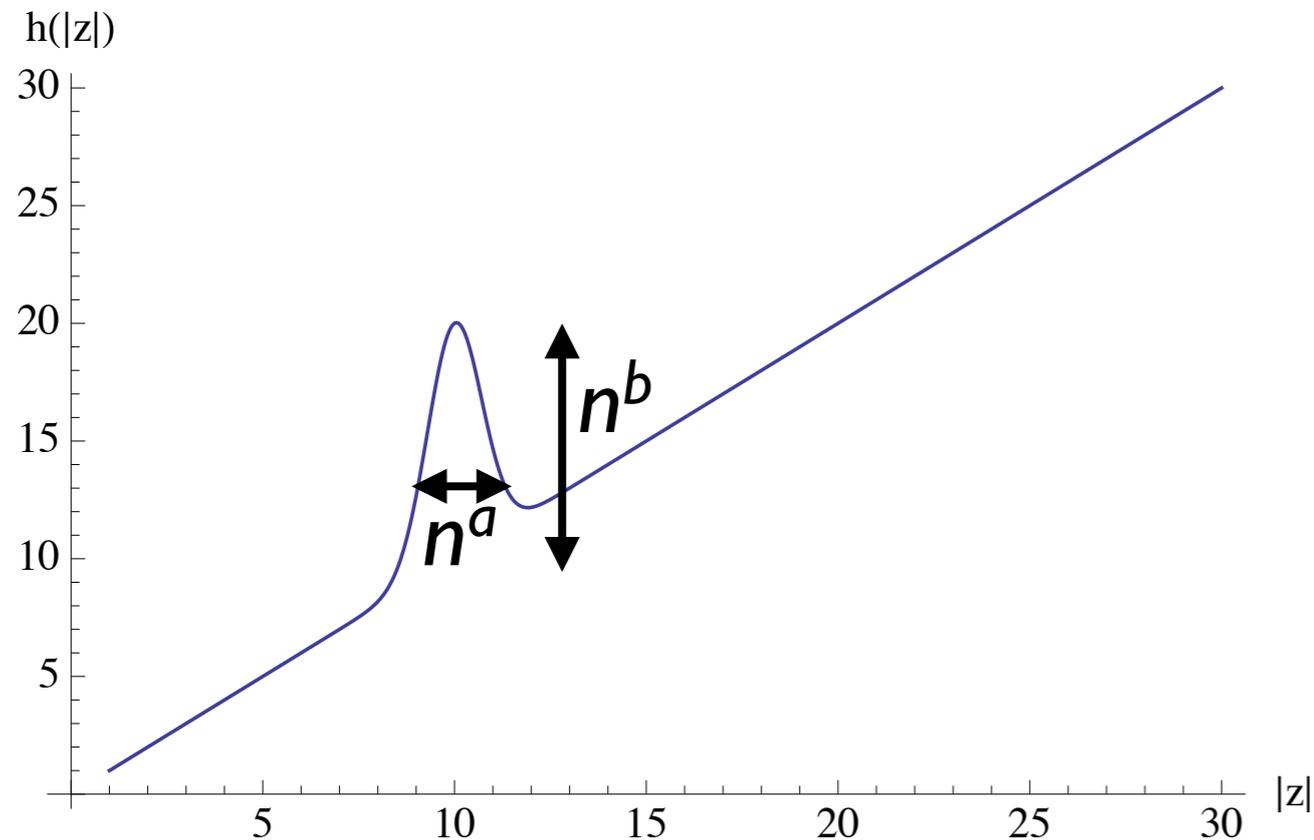
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[Fisher 1992; Reichardt 2004]

# Example: The fat spike

Let  $h(z)$  depend only on the Hamming weight of  $z$ .



- Initial state: binomial distribution in Hamming weight, peaked around  $n/2$
- Efficient tunneling through barrier if  $a+b < 1/2$

[Reichardt 2004]

For more toy examples comparing the performance of adiabatic optimization and simulated annealing, see e.g. Farhi, Goldstone, and Gutmann, [quant-ph/0201031](#) and [quant-ph/0208135](#)

# Random satisfiability problems

Consider random instances of some satisfiability problem (e.g. 3SAT, Exact cover, ...) with a fixed ratio of clauses/bits.

Few clauses: underconstrained. Many solutions, easy to find.

Many clauses: overconstrained. No solutions, easy to find a contradiction.

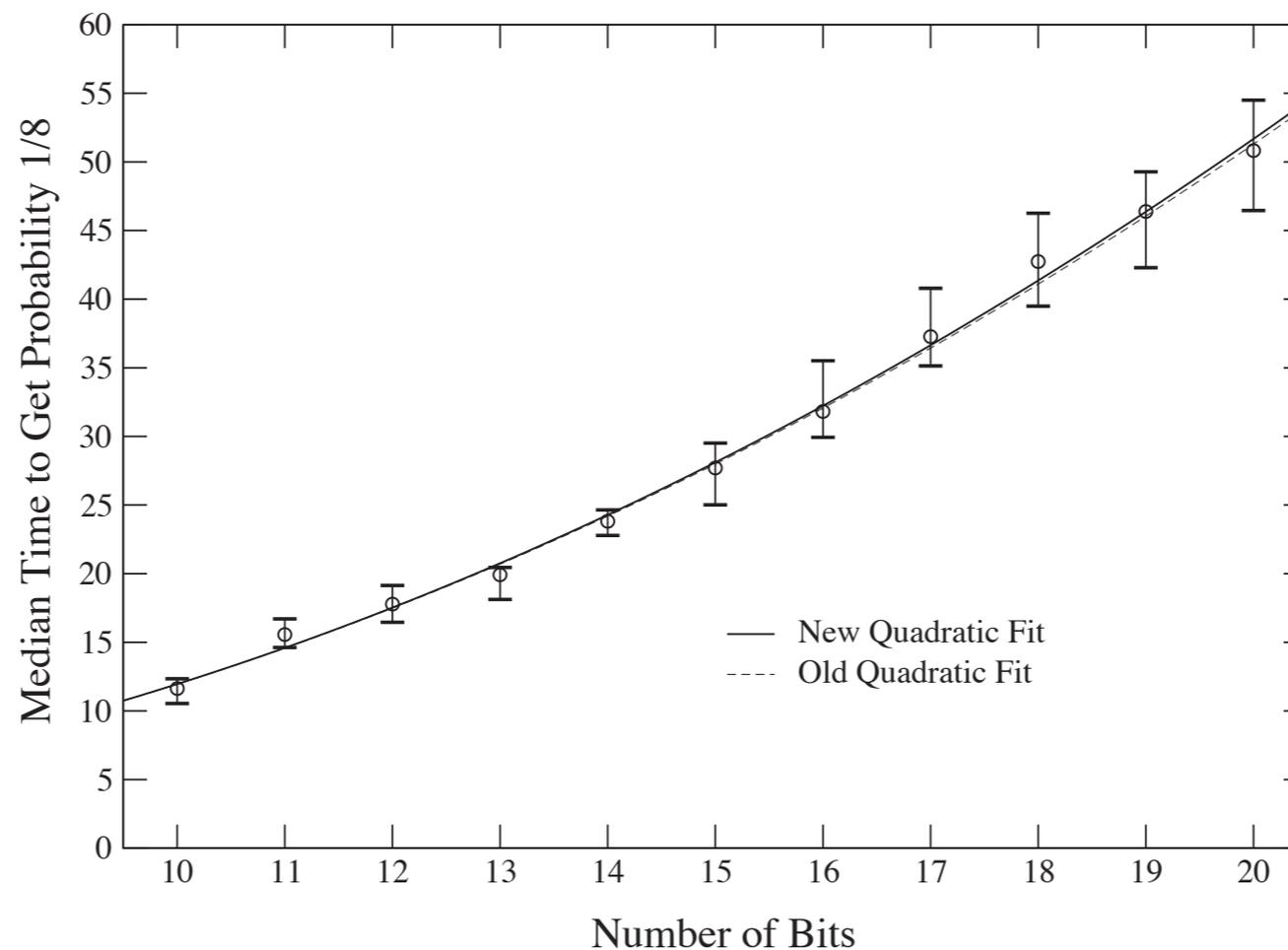
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Simulation results for random exact cover instances with unique satisfying assignments:



[Farhi et al. 2001]

# Robustness of adiabatic QC

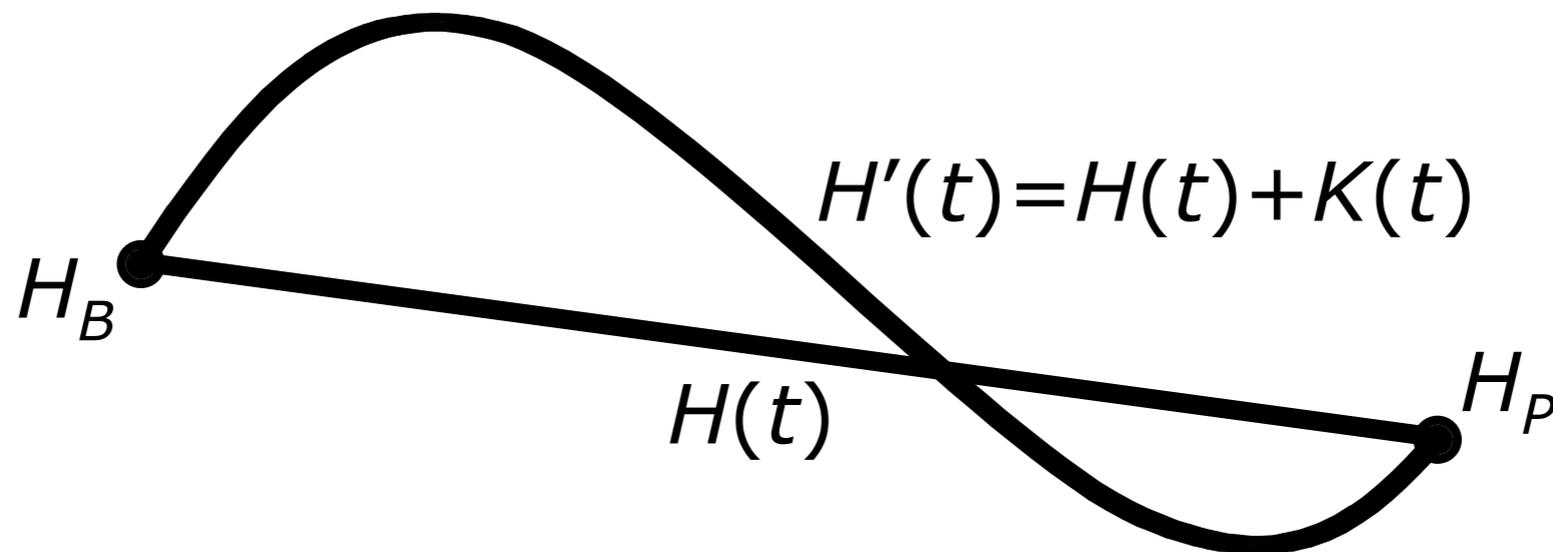
Potential sources of error:

- Unitary control error
- Dephasing in instantaneous eigenstate basis
- Transitions between instantaneous eigenstates: thermal noise

# Unitary control error

Adiabatic algorithm depends on going smoothly from  $H_B$  to  $H_P$ , not on the particular path between them.

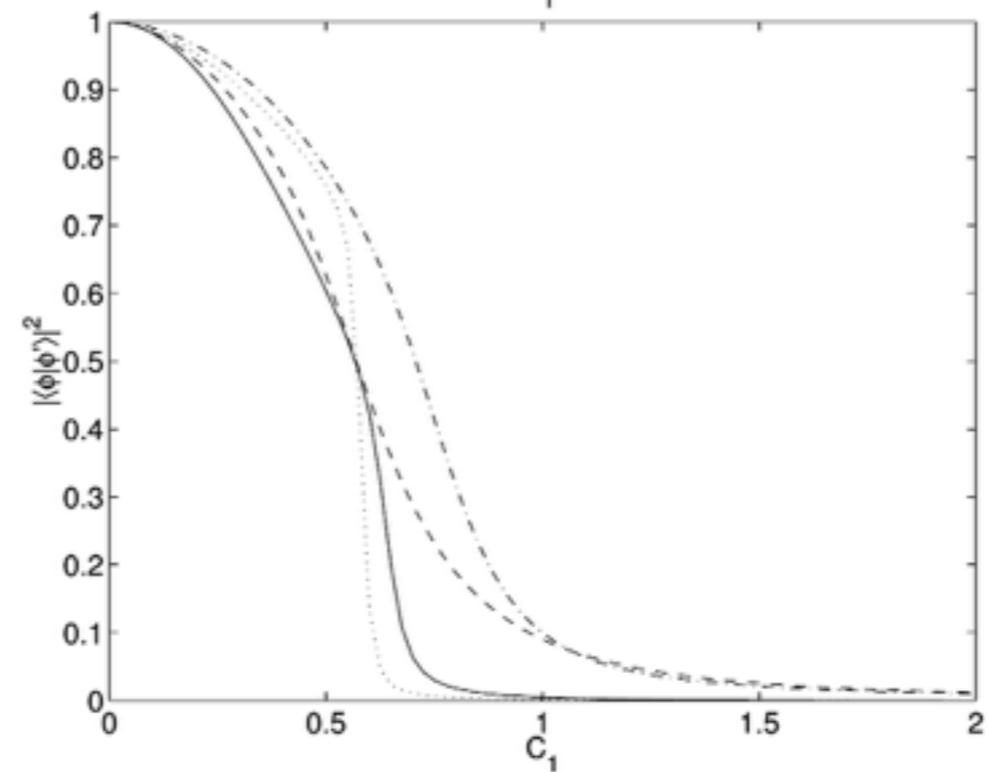
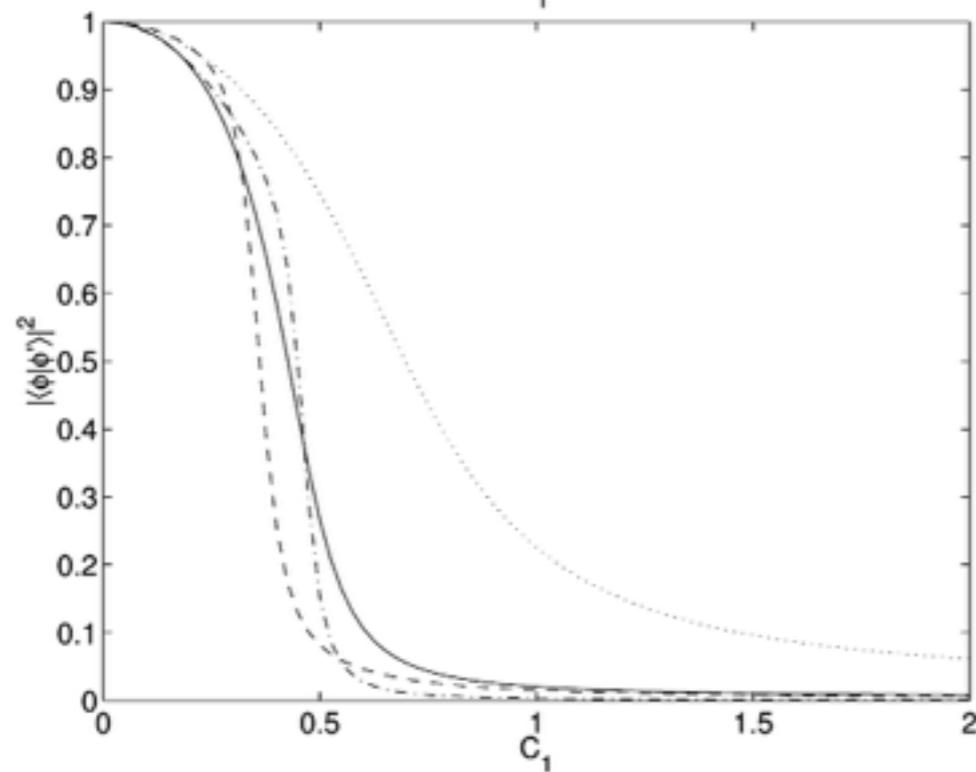
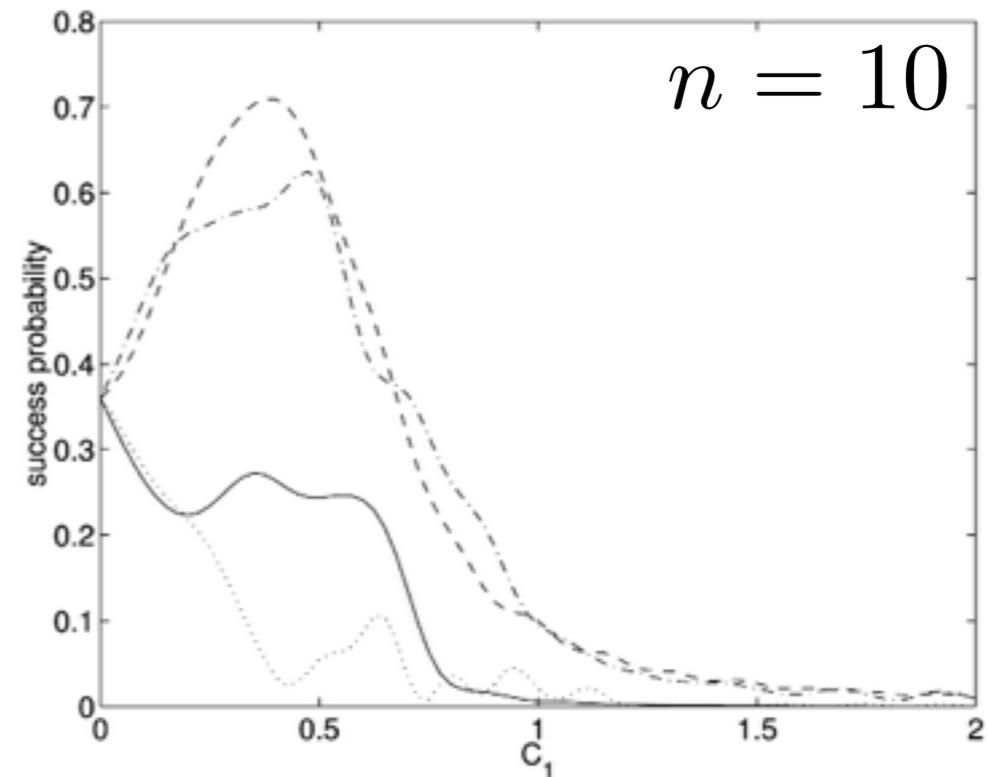
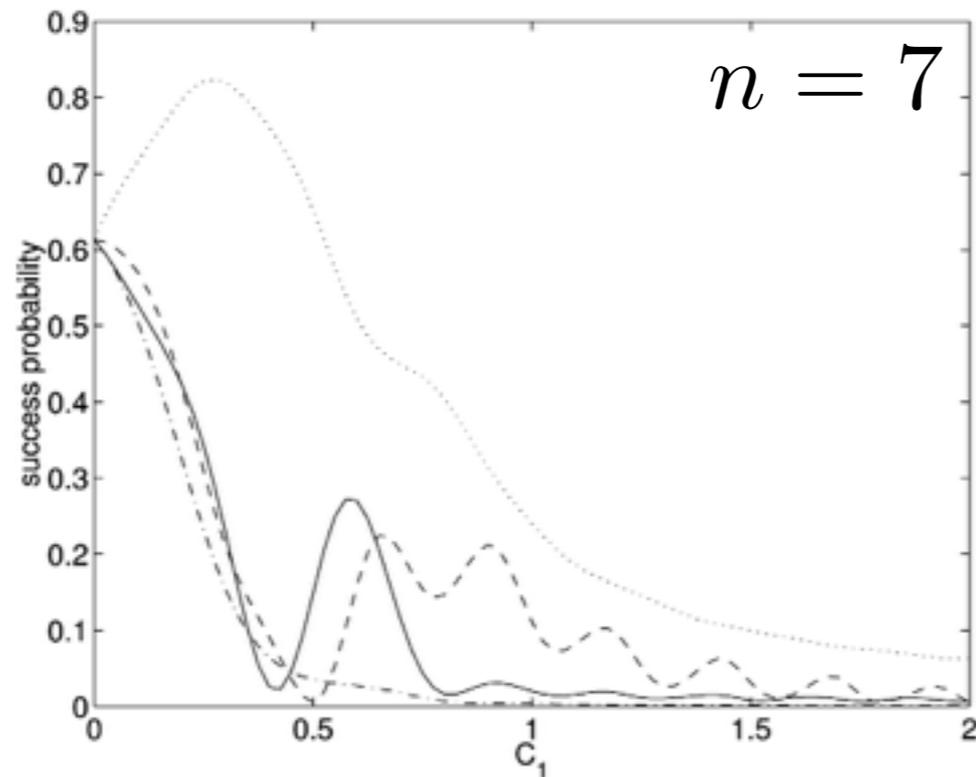
For smooth perturbations, we have no reason to expect the gap will become smaller rather than larger, even if the perturbation is not small (provided it is zero at the beginning and end of the evolution).



[Childs, Farhi, Preskill 2001]

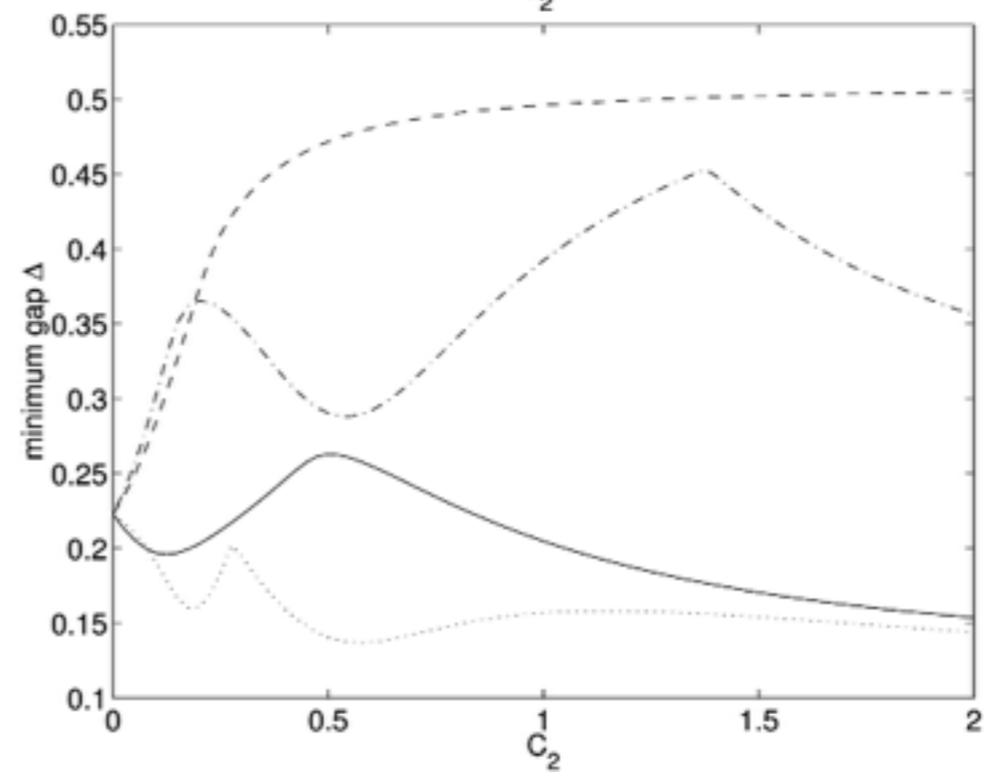
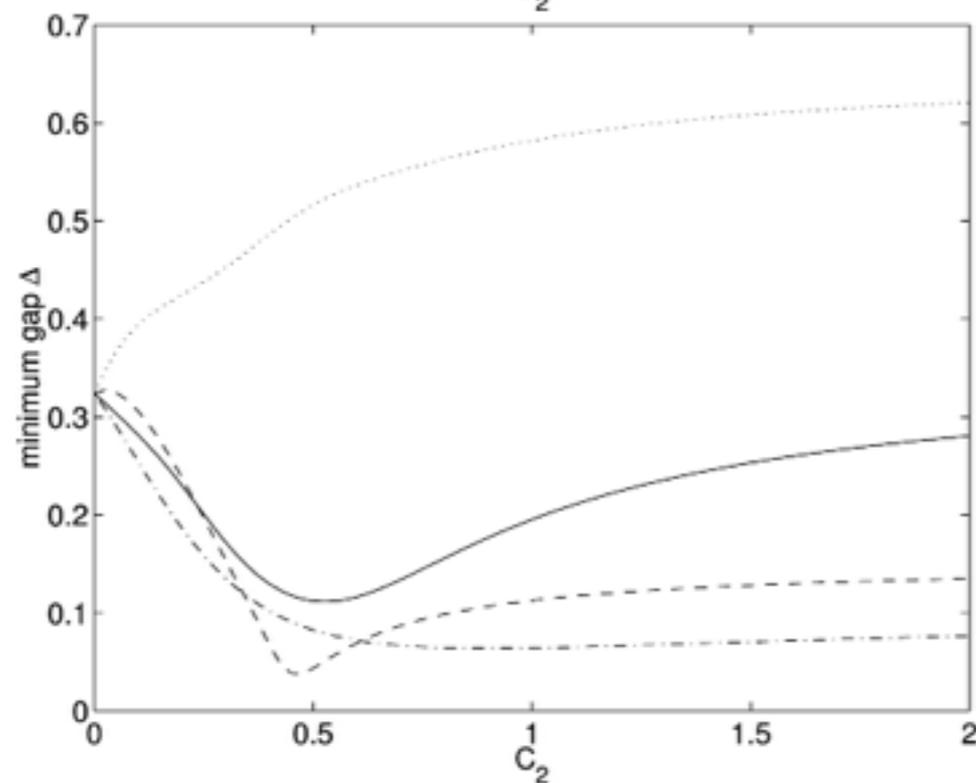
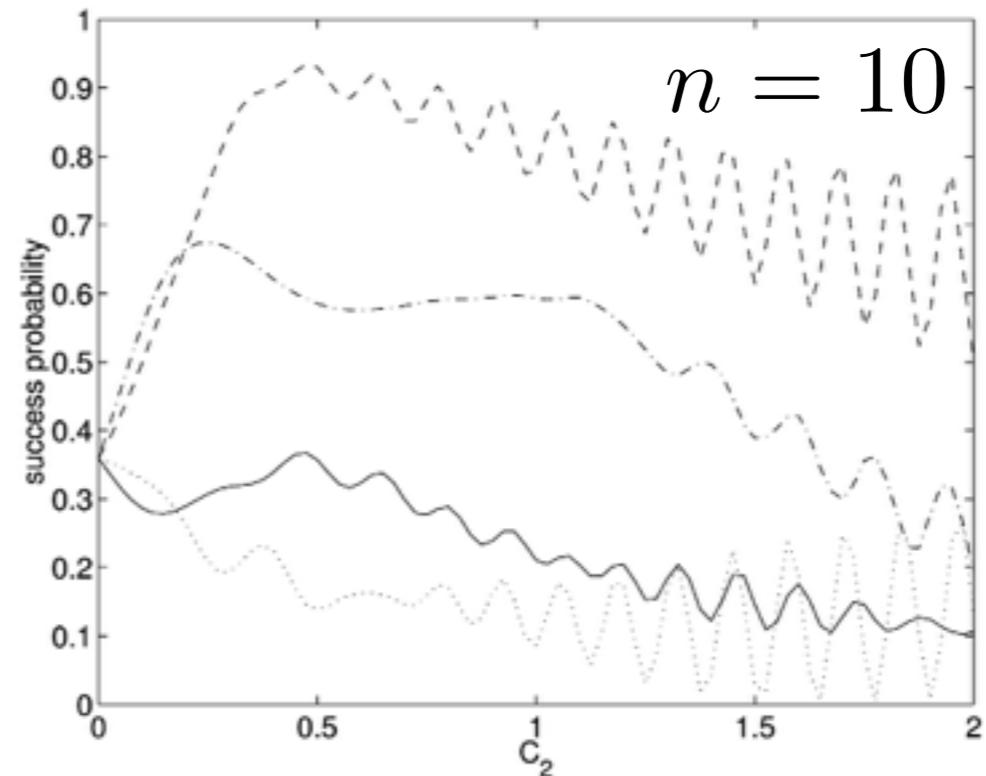
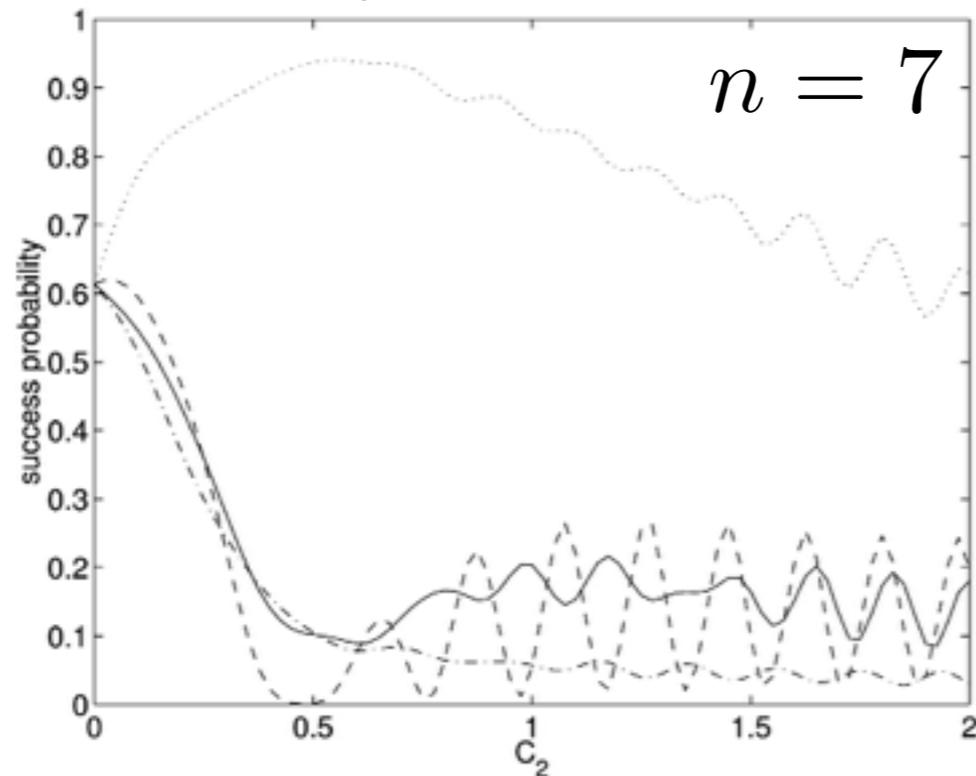
# Error in the final Hamiltonian

$$\tilde{K}_1(s) = C_1 s \sum_{j=1}^n \hat{m}_j \cdot \vec{\sigma}^{(j)}$$



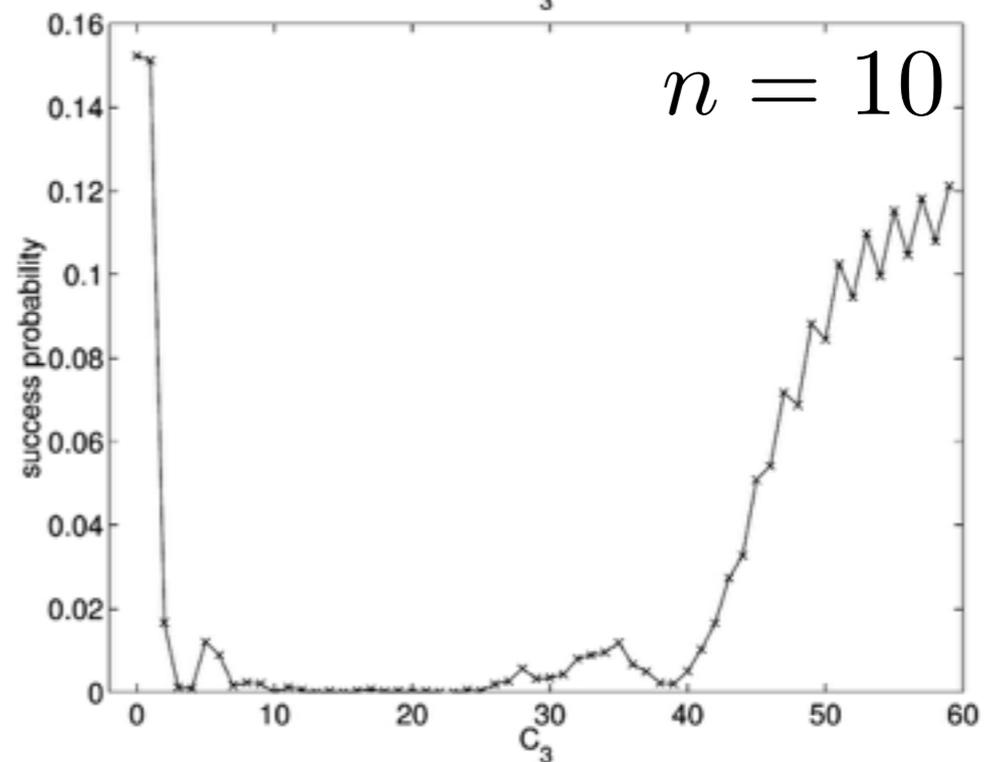
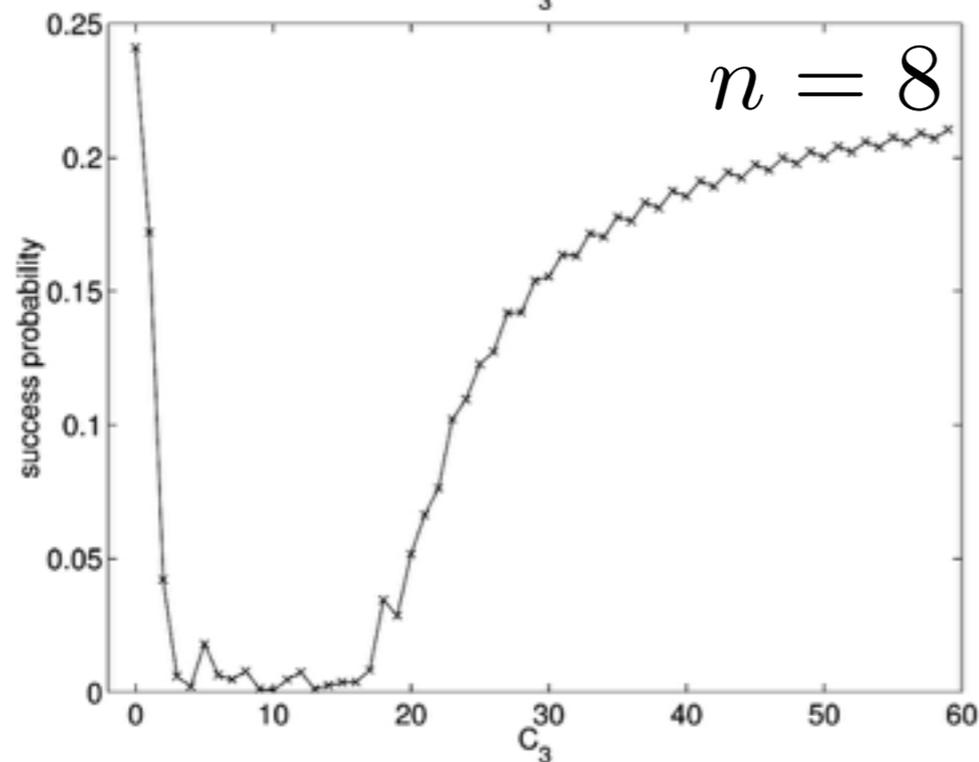
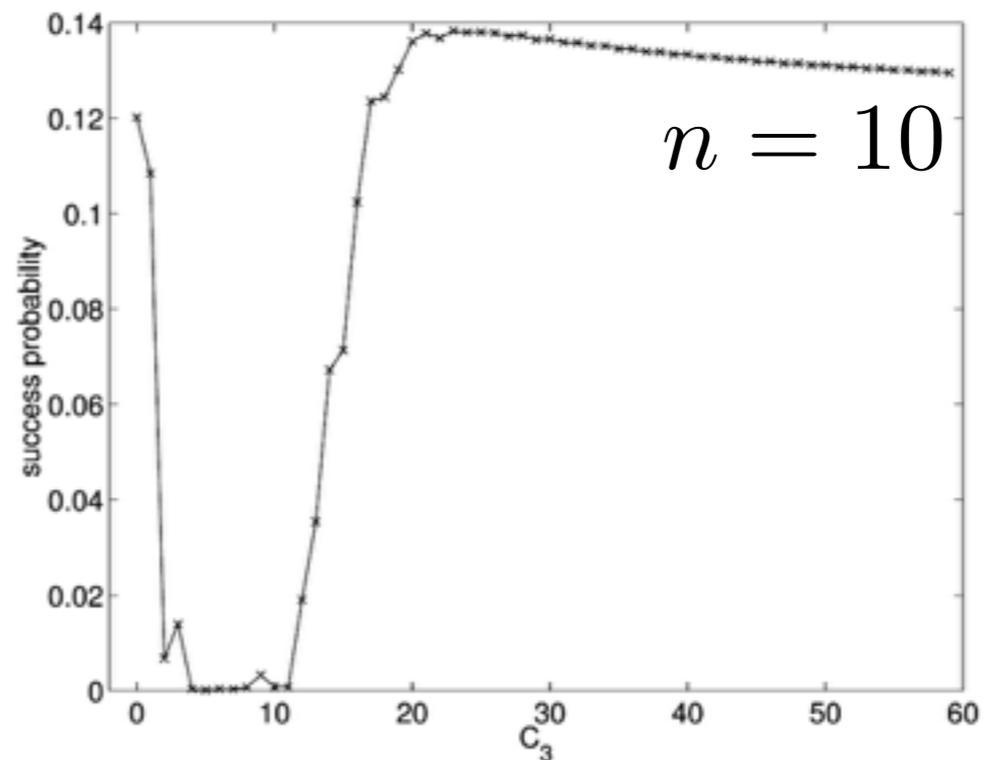
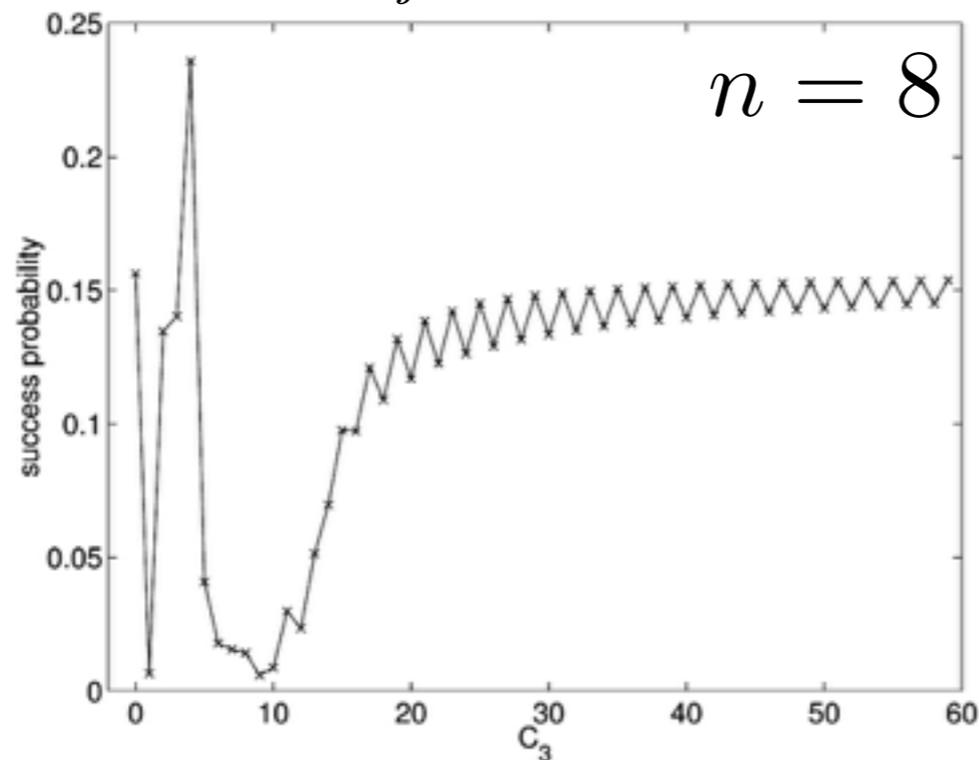
# Error in the interpolation

$$\tilde{K}_2(s) = C_2 \sin(\pi s) \sum_{j=1}^n \hat{m}_j \cdot \vec{\sigma}^{(j)}$$



# High frequency error

$$\tilde{K}_3(s) = \frac{1}{2} \sin(C_3 \pi s) \sum_{j=1}^n \hat{m}_j \cdot \vec{\sigma}^{(j)}$$



# Thermal noise

Efficient adiabatic quantum computation requires that the minimum gap  $\Delta$  is not too small.

Provided  $k_B T \ll \Delta$ , thermal fluctuations are unlikely to drive the system out of the ground state. So a big gap not only allows for adiabaticity, but also provides protection against thermal noise!

Here it is important that  $H$  is the actual Hamiltonian of the quantum computer, not just a simulated Hamiltonian.

Even for tractable problems, typical gaps decrease as  $1/\text{poly}(n)$ , so this may not provide any asymptotic protection in realistic systems.

[Childs, Farhi, Preskill 2001]

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$$H = \sum_{j=1}^k [U_j \otimes |j+1\rangle\langle j| + U_j^\dagger \otimes |j\rangle\langle j+1|]$$

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Basic idea [Aharonov et al. 2004]: Use this as  $-H_P$ .

Final ground state: 
$$\frac{1}{\sqrt{k}} \sum_{j=1}^k U_j U_{j-1} \cdots U_1 |0\rangle \otimes |j\rangle$$

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Add energy penalties to stay in an appropriate subspace.

Note: This is adiabatic, but not adiabatic *optimization*.

# Open questions: Stoquastic Hamiltonians

A Hamiltonian is *stoquastic* if all its off-diagonal matrix elements are non-positive.

Example:  $-(1 - s) \sum_j \sigma_x^{(j)} + sH_P$  with  $H_P$  diagonal

Stoquastic Hamiltonians are potentially less powerful:

- Perron-Frobenius: ground state has all positive amplitudes
- Quantum Monte Carlo methods can be applied (though they're not necessarily efficient)
- Ground energy problem is in AM [Bravyi et al. 2009]

Open questions:

- Is universal adiabatic quantum computation possible with a stoquastic Hamiltonian?
- Exponential speedup using stoquastic adiabatic optimization? (Also open even without the stoquastic restriction.)

# Open questions: Fault tolerance

Can we develop a theory of fault-tolerant adiabatic quantum computing (for inherently adiabatic devices)?

Some preliminary studies show improved performance using quantum error correcting codes (e.g., [Jordan, Farhi, Shor 05], [Lidar et al. 13]), but these fall far short of anything like the threshold theorem for quantum circuits.

Related questions:

- Can we map any adiabatic optimization problem with an inverse polynomial gap to one with a constant gap?
- Can we perform universal adiabatic computation with a constant gap?