The Stone-Čech Compactification

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December 15, 2009

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1 Motivation and History

One of the most elementary and functional notions of finiteness in analysis, algebra and topology is the notion of compactness. Yet mathematicians work with many non-compact topological spaces, which have applications that require some properties of compact spaces. In a sense, mathematicians wished to distill out some properties of "large" spaces by looking at compact spaces that resemble these "large" spaces. As such, it is in this vein that the concept of compactifying "large" spaces led to the piecewise development of the Stone-Čech compactification, βX for a topological space X. The Stone-Čech compactification is considered to be one of the more important universal properties in topology due to its importance in applications that range from Ramsey Theory[1] and Topological Dynamical Systems to computing the dual space of $l^\infty(\mathbb{R})$, the space of bounded sequences of real numbers. As is often the case in mathematics, the historical evolution in the understanding and analysis of these applications contributed to the formal construction of the Stone-Čech compactification.

Before looking at the history of the Stone-Čech compactification, there is one patent definition to be made:

Definition. (Munkres) A *compactification* of a topological space X is a compact Hausdorff space Y containing X as a subspace such that $\overline{X} = Y$. Two compactifications Y_1 and Y_2 of X are said to be *equivalent* if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for every $x \in X$.

In order to move past the naïve intuition behind the historical notion of finiteness, the mathematical community required the leadership and direction of mathematicians such as Bernhard Riemann, John von Neumann and Marshall Stone. Riemann (1826-1866) provided the first example of a compactification using the intuitively infinite and non-compact topological space $\mathbb{C}$ with the construction of the Riemann Sphere in 1858-1859 (Laugwitz, 2008). The Riemann Sphere is what is known as the *one-point compactification* of the complex plane $\mathbb{C}$ — it adds one point to represent ∞ at the North Pole and azimuthally compresses the plane to the surface of the 2-sphere S^2. More precisely we can map elements the sphere $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\} \backslash \{(0, 0, 1)\}$ to $\zeta \in \mathbb{C}$ using the map,

$$\zeta = \frac{x + iy}{1 - z} = \cot\left(\frac{\phi}{2}\right) e^{i\theta}$$

where ϕ is the radial angle on the sphere and θ is the azimuthal angle (Stein and Shakarchi, 2003). One rather important application of the compactness of the Riemann Sphere is the *uniformization theorem*, which says that any simply connected Riemann Surface is conformally equivalent[2] to $\mathbb{D}^2$, the open unit disk, $\mathbb{C}$ or the Riemann Sphere. In a sense, we look at the one-point compactification as a *minimal* compactification (Munkres, 2008); in other words, the one-point compactification represents the smallest compact space Y for which one can embed a non-compact space X. This simple and elegant example of compactification intrigued mathematicians and physicists, because it allowed for the reduction of seemingly impossible complex analytic problems to problems on S^2. Subsequently, mathematicians were interested in generalizing the notion of compactification to any topological space. As has been the case with many evolutionary leaps in mathematics, the generalization of this idea required the discovery of novel structures in algebra and analysis. John von Neumann's (1903-1957) study of rings of bounded, continuous functions for his axiomatic formulation of quantum mechanics inspired the notion of a *maximal* compactification for a space (Murray, 1988). The natural motivation for this *maximal* compactification lies in physics. In quantum mechanics, each function in this ring represents the potential energy of a particle in the phase space[3] X and a trivial, realistic condition is that no particle has unbounded potential energy. Yet one simultaneously demands that any quantum mechanical model takes into account as many possible representations

[1] Ramsey Theory is the study of how easily a system (topological space, for example) can become ordered.

[2] Two topological spaces X and Y are said to be conformally equivalent is there exists a conformal map (recall: angle-preserving, bijective, biholomorphic map) between them

[3] The phase space of a physical system is the topological space in which all possible states can be represented. For example, in mechanical systems a phase space consists of position and momentum vectors in each direction/with respect to each degree of freedom.

of potential energy as possible, as the Schrödinger Equation prescribes that the discrete state that a particle is in depends intrinsically on the particle's potential energy. Therefore, if one discovers that X is a viable phase space for a quantum mechanical model then he or she would like to construct maximal phase space Y such that X can be embedded in Y and such that all functions on Y are bounded and continuous (Werner, 1983). This means that we are not only looking for a maximal compactification for our phase space X, but we are also looking to be able to continuously extend continuous functions f on X to continuous functions on Y. von Neumann intuitively realized this and provided the analytic foundations for th what is now known as the $\mathcal{C}^*$*-algebra* of a topological space X, $\mathcal{C}^*(X)$, which converts the ring of all bounded, continuous functions on X to a Banach Algebra.[4]

As Munkres puts it, a natural question that arises when one attempts to find a maximal compactification is,

> *Problem Statement: If Y is a compactification of X, under what conditions can a continuous real-valued function f defined on X be extended continuously to Y?*

In 1937, Marshall H. Stone and Edmund Čech independently found similar answers to this question using results about $\mathcal{C}^*$-algebras and it is this result that is named the Stone-Čech compactification (Walker, 1974). By the 1940s and 1950s, the mathematicians such as H. Wallman and P. Samuel began to realize that one can construct the Stone-Čech compactification without using rings of continuous functions; instead, they opted to use a collection of subsets of a set S called a *filter* on S. In a way, one can look at a filter as a "fine" analogue of a topology as it doesn't preserve unions (the formal definition will be provided in the section 3). Currently the Stone-Čech compactification of the natural numbers, $\beta\mathbb{N}$, is a major object of study in set-theoretic topology as there are many pathologies hidden within the structure of $\beta\mathbb{N}$.

2 Initial Construction of the Stone-Čech Compactification

Remark. The development of the Stone-Čech Compactification in this section is due to Walker, *The Stone-Čech Compactification.*

Let X be a topological space. We desire a compact Hausdorff Space, βX, such that all functions on X continuously extend to βX and such that X is a subspace of βX. Formally, we would like an *embedding* of X in βX that continuously extends bounded, continuous functions on X. We define an embedding as follows:

Definition 1. (Adapted from Munkres) Suppose that $f : X \to Y$ is an injective continuous map between topological spaces X, Y. Let $Z = f(X)$, which is a subspace of Y. Moreover the map $f' : X \to Z$, defined as the restriction of f to $B = \{x \in X : f(x) \in Y\}$ is bijective. If f' is a homeomorphism of X with Z then $f : X \to Y$ is said to be an *embedding* of X in Y

The main tool we will use to arrive at this space is that of the $\mathcal{C}^*$*-algebra.* Before constructing the $\mathcal{C}^*$*-algebra*, one first needs to look at the ring of bounded, continuous functions on X, $C^*(X)$. Formally this is defined as:

Definition 2. (Walker) The ring a all real-valued continuous functions defined on a space X will be denoted $C(X)$ and the subring of all bounded members of $C(X)$ will be denoted $C^*(X)$. One says that a subspace S of X is *C-embedded in X if every function $f \in C(S)$ extends to a function $g \in C(X)$. Similarly,* one says that a subspace S of X is *C^*-embedded in X if every function $f \in C^*(S)$ extends to a function $g \in C^*(X)$.*

In order to have compactification that satisfies the problem statement, we would like to discern what, if any, topological constraints are placed upon our compactified space and under what conditions will $C^*(S)$ continuously extend to $C^*(X)$ for $S \subseteq X$, a subspace. An important theorem that shows that at a minimum one requires a form of "topological separation" is *Urysohn's Extension Theorem.*

[4] A Banach Algebra is an associative algebra (i.e. Similar to a module except that you can "multiply" module elements) such that $||xy|| \leq ||x||\,||y||$

2.1 Urysohn's Extension Theorem

Before arriving at the statement of the theorem a notion of "separation" for two subsets $A, B \subseteq X$ must be defined.

Definition 3. Two subsets A and B of a space X are *completely separated in* X if there exists a continuous mapping $f \in C(X)$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Subsequently we have the following theorem of interest, due to Pavel Urysohn (1898-1924):

Theorem. *(Urysohn) A subspace S of a space X is C^*-embedded in X if and only if any two completely separated sets in S are completely separated in X*

Proof. The proof of this theorem can be broken down as follows:

The forward implication ($\Rightarrow$), "If a subspace $S \subset X$ is C^*-embedded in X then any two completely separated sets in S are completely separated in X," follows directly from the definition of a C^*-embedded subspace. If there is a function f such that for $A, B \subset S$, $f(a) = 0, f(b) = 1$ for all $a \in A, b \in B$, then as S is C^*-embedded, f has an extension g on X such that $g|_S = f$. This means that g separates A, B in X

To prove the reverse implication ($\Leftarrow$), "If any two completely separated sets in a subspace S are completely separated in X then S is C^*-embedded in X," we attempt to directly construct an extension for a function $f \in C^*(S)$. This construction is done as follows:

First choose a function $f_1 \in C^*(S)$; as f_1 is bounded we know that $|f_1| \leq M$ for some $M \in \mathbb{R}$. Next, choose a bound $r_{n,M}$ ($r_{n,M}$ depends on n and M) such that for some desired sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subseteq C^*(S)$, $|f_n| \leq 3r_{n,M}$ and $r_{n,M} \downarrow 0$ as $n \nearrow \infty$. To show this we proceeding inductively, we assume that $|f_n| \leq 3r_{n,M}$.

We then fabricate two completely separated sets A_n and B_n that consist of $s \in S$ such that $f_n(s) \leq -r_{n,M}$ and $f_n(s) \geq r_{n,M}$, repsectively. Recall that by hypothesis A_n, B_n are also completely separated in X, so that there exists $g_n \in C^*(X)$ such that $g_n(a) = -r_{n,M}, g_n(b) = r_{n,M}$ and $|g_n| \leq r_{n,M}$ for all $a \in A_n, b \in B_n$. If we let

$$f_{n+1} = (f_n - g_n)\,|_S$$

then we see that our choice of g_n gives us the bound $|f_{n+1}| \leq 3r_{n,M}$.

Finally, we can use the Weierstrass M-test to see that the sequence $\{a_n : a_n = \sum_{i=1}^{n} g_i\}$ convergence uniformly to a continuous function $f = \sum_{i=1}^{\infty} g_i$. By construction we have a telescoping series, that is:

$$\sum_{i=1}^{n} g_i|_S = \sum_{i=1}^{n}(f_i - f_{i+1}) = f_1 - f_{n+1}$$

Recall that $r_{n,M}$ was chosen such that it vanishes as n goes to infinity; subsequently we must have $f_{n+1} \downarrow 0$ as $n \nearrow \infty$ so that f is the continuous extension of f_1 to X. As the choice of f_1 was arbitrary, we have shown that S is C^*-embedded in X. Finally, note that the choice of the constant 3 in the inequality $f_i \leq 3r_i$ was done for convenience. □

Urysohn's Extension Theorem gives us a crucial requirement for our desired compactifcation — namely, we need our space X to have some collection of subsets that are completely separated. As it turns out, we would like the closed subspaces of X to be completely separated. Andrey Tychonoff (1906-1993) made this choice in 1930 since he was looking for a way to preserve compactification[5] for subspaces of X. As closed subspaces of compact spaces are compact, the closed subspaces of X provide a good hint of where to start in the compactification process.

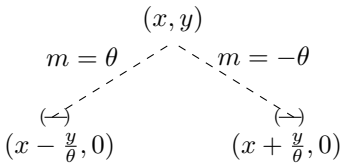
[5]He was looking for this specific classification of topological spaces while he was working on what is now known as Tychonoff's Theorem, which says that the product of an arbitrary collection of compact subspaces is compact

2.2 Completely Regular Spaces

At first glance, one might assume that we simply require that X be Hausdorff in order for the closed sets X to be pairwise completely separated. However, as it turns out that there exist Hausdorff spaces for which this notion fails. An example of this from *Counterexamples in Topology* is the *Irrational Slope Topology*, which is defined as follows:

> Let $X = \{(x, y) | y \geq 0,\ x, y \in \mathbb{Q}\}$ and fix $\theta \in \mathbb{R}\backslash\mathbb{Q}$. The irrational slope topology τ on X is the topology generated by ϵ-neighborhoods of the form,
>
> $$N_\epsilon(x, y) = \{(x, y)\} \cup B_\epsilon(x + \frac{y}{\theta}) \cup B_\epsilon(x - \frac{y}{\theta})$$
>
> where $B_\epsilon(\zeta) = \{r \in \mathbb{Q} : |r - \zeta| < \epsilon\}$ where r is in $\mathbb{Q}$ on the x-axis.

This topology is essentially the point (x, y) plus two intervals (in $\mathbb{Q}$) centered at the irrational points $x \pm \frac{y}{\theta}$; the slope of the lines between (x, y) and $(x \pm \frac{y}{\theta})$ is $\pm\theta$. Graphically, this looks like the following:



As it turns out, this space is Hausdorff because for no two points in X can one point lie on a line with irrational slope θ while the other point lies on a line of irrational slope $-\theta$ (Steen and Seebach, 1995). Therefore any two distinct points of X must project onto distinct pairs of irrational points on the x-axis. Now note that $\overline{N_\epsilon(x, y)}$ contains the union of four strips of slope $\pm\theta$ emanating from $B_\epsilon(x + \frac{y}{\theta}), B_\epsilon(x - \frac{y}{\theta})$ as each point in each ray projects to an irrational number γ such that $d(\gamma, x + \frac{y}{\theta}) \vee d(\gamma, x - \frac{y}{\theta}) < \epsilon$. As a consequence, the closures of every two open sets must intersect. This implies that no two closed neighborhood of two distinct points x, y are disjoint and as such there is no continuous function f such that $f(x) = 0, f(y) = 1$. Although pathological examples such as the irrational slope topology do make life slightly more difficult for topologists, the slightly modified notion of a *completely regular space* (due to Tychonoff) will help characterize the spaces for which closed subspaces are completely separable. Formally, we have:

Definition 4. A space X is *completely regular* if every closed subspace $F \subseteq X$ is completely separated from any point $x \notin F$ and if each point is closed

As it turns out, Tychonoff proved the following theorem in 1931, is almost be equivalent to a simple manifestation of the Stone-Čech Compactification:

Theorem. *(Tychonoff's Theorem for Completely Regular Spaces) The completely regular spaces are precisely those spaces which can be embedded in a product of copies of the closed unit interval* $I = [0, 1]$

To see why this Theorem is extremely important let us recall Urysohn's Lemma:

Lemma. *(Urysohn) X is a normal topological space if and only if whenever A, B are disjoint closed sets in X then there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0, f(b) = 1$ for all $a \in A, b \in B$*

As compact Hausdorff Spaces are normal spaces (Munkres, Theorem 32.3), combining Urysohns Lemma with Tychonoff's Theorem for Completely Regular spaces gives us the following crucial nugget of topological information

> *No larger class of topological spaces can be studied by means of embeddings into compact Hausdorff Spaces*

This means that we need only to look at completely regular spaces when defining our desired compactification. In order to generalize this theorem to any topological space we will need to develop the *Embedding Lemma* and the *Stone-Čech Lemma*.

2.3 Embedding Lemma and the Stone-Čech Lemma

Remark. The technical development of the Embedding Lemma is adapted from Kelley, *General Topology*

While disccusing the construction of βX in the prior subsections, we ignored how βX is explicitly related to the functions on X that are to be extended. As suggested in the statement of Tychonoff's Theorem for Completely Regular spaces, we would like to look at an embedding $X \hookrightarrow [0,1]^J$ for some index set J as $[0,1]^J$ is compact via Tychonoff's Theorem for Products of Compact Spaces. There are some technical requirements that are necessary for a map from a space X to spaces indexed by family $\mathcal{F}$ of functions to be an embedding; these technical requirements are precisely what the embedding lemma will discern. We will deal with functions that preserve the completely separable structure of completely regular spaces, which are described by the following families of functions:

Definition 5. A family $\mathcal{F}$ of functions on X is said to *distinguish points* if for each pair of distinct points x and y, there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. A family $\mathcal{F}$ of functions on X is said to *distinguish points and closed sets* if for each closed set $F \subseteq X$ and each point $x \notin F$, there exists $f \in \mathcal{F}$ such that $f(x) \notin \overline{f[F]}$.

With these definitions in hand, the Embedding Lemma can be stated as follows:

Lemma. *(Embedding) Let $\mathcal{F}$ be a family of mappings such that each member $f \in \mathcal{F}$ maps $X \to Y_f$. Then:*

a. The evaluation mapping $e : X \to \prod_{f\in\mathcal{F}} Y_f$ defined by $\pi_f \circ e(x) = f(x)$, for all $x \in X$, is continuous

b. The mapping e is an open mapping onto $e(X)$ if $\mathcal{F}$ distinguishes points and closed sets

c. The mapping e is one-to-one if and only if $\mathcal{F}$ distinguishes points

d. The mapping e is an embedding if $\mathcal{F}$ distinguished points and $\mathcal{F}$ distinguishes points and closed sets

Proof. a. Let $\pi_g : \prod_{f\in\mathcal{F}} Y_f \to Y_g$ be the projection map to the space Y_g. Then $\pi_g \circ e = g$ so that $\pi_g \circ e$ is continuous. Therefore e must be continuous as g is continuous.

b. Suppose that U is open in X and $x \in U$. Choose $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(X\backslash U)}$. The set $B = \{z \in e(X) | \pi_f(z) \notin \overline{f(X\backslash U)}\}$ is a neighborhood of $e(x)$ as the set is open (it is defined for components not being in the closed set $\overline{f(X\backslash U)}$) and clearly $e(x) \in B$. Moreover, $\pi_f(B) \subset f(U)$ by construction. It is now claimed that $f(U) \subset \pi_f(B)$. This follows trivially from the definition of a family of functions distinguishing points and closed sets. Therefore $f(U) = \pi_f(B)$ and subsequently $f(U)$ is an open subset of $\pi_f \circ e(X)$. Therefore the evaluation map is an open mapping

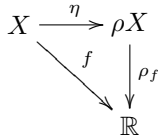
c. The definition of distinguishing points implies injectivity

d. Combining a, b and c, we see that $X \cong e(X)$ as e is a continuous, open, injective, surjective (as a continuous map is always surjective onto its image) map. □

Now that the technicalities of the embedding process have been established, we can prove a result that allows us to embed any topological space into a completely regular space, ρX. Subsequently, Tychonoff's Theorem for Completely Regular spaces provides that there is an embedding that compactifies the completely regular space ρX to the compact space βX. There is also a corollary following Stone-Čech Lemma that describes how continuous maps factor through ρX.

Lemma. *(Stone-Čech Lemma) For any topological space X, there exists a completely regular space ρX which is a continuous image of X such that any real-valued mapping f from X factors through ρX.*

Proof. Our goal is to arrive at the following universal mapping property:



(1)

To create the above embedding and utilize the technical results provided in the embedding lemma, we need to define an equivalence relation $\sim$ on X such that each $f \in C(X)$ distinguishes points on the quotient space $X/\sim$. We can define the equivalence relation as follows:

$$x \sim y \iff f(x) = f(y), \forall f \in C(X)$$

Let ρX be the set of equivalence classes under this relation and let be the quotient map $\eta : X \to \rho X$ that assigns each element of X to it's equivalence class (i.e. $\eta(x) = [x]$). As each $f \in C(X)$ is constant on $[x]$ for all $x \in X$, we can define $\rho_f : \rho X \to \mathbb{R}$ by:

$$\rho_f(\eta(x)) = f(x)$$

These maps make the diagram commute; now we have to show that ρX is a completely regular space. Intuitively the approach we will take is to minimize the number of closed sets in our topology on ρX in such a way that f is forced to distinguish closed sets and points. Suppose that we provide ρX with the smallest (finest) topology such that each $\rho(f)$ is continuous. Then the closed sets F of ρX are of the form

$$F = \bigcap_{\alpha \in A} \rho_{f_\alpha}^{-1}(F_\alpha) \tag{2}$$

where A is an index set for $C(X)$ and each F_α is closed in $\mathbb{R}$.

Claim 1: With this topology, ρX is Hausdorff

This comes from the fact that points in X that are not separated by some $f \in C(X)$ are identified in ρX.

Claim 2: With this topology, ρX is completely regular

Suppose $F \subseteq \rho X$ is closed and $y \notin F$. By (1) this means that $y \notin \rho_{f_\beta}^{-1}(F_\beta)$ for some $\beta \in A$. Firstly note that $\mathbb{R}$ is a normal space so that $\mathbb{R}$ is completely regular. Therefore, by construction, $\rho_{f_\beta}(y)$ and F_β are completely separated by some real-valued mapping g on $\mathbb{R}$ and $(g \circ \rho_{f_\beta})^{-1}$ completely separates y from F so that ρX is completely regular.

Claim 3: η is continuous

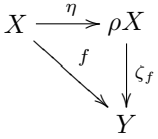
Suppose $F \subseteq \rho X$ is closed. Then we have:

$$\begin{aligned} \eta^{-1}(F) &= \eta^{-1}\left(\bigcap_{\alpha \in A} \rho_{f_\alpha}^{-1}(F_\alpha)\right) \\ &= \bigcap_{\alpha \in A} f_\alpha^{-1}(F_\alpha) \end{aligned}$$

as the diagram commutes. Since each f_α is continuous, $\eta^{-1}(F)$ is the intersection of closed sets and is subsequently closed.

As ρX is Hausdorff it distinguishes points and as ρX is completely regular it distinguishes closed sets and points, we can apply the embedding lemma to show that η is an embedding of X in ρX. This implies that η is a continuous extension of $f \in C(X)$ to ρX as desired. □

Corollary. *If f is a continuous mapping of the space X into a completely regular space Y, then there is a mapping ρ_f of ρX into Y such that the following diagram commutes:*

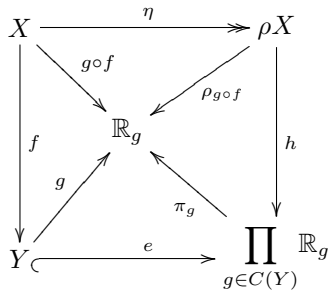


Proof. Let $f : X \to Y$ be a continuous mapping. For each $g \in C(Y)$, let $\mathbb{R}_g$ be a copy of the real line and as in the embedding lemma, let $e : Y \to \prod_{g \in C(Y)} \mathbb{R}_g$ be the evalutation map for the embedding $Y \hookrightarrow \prod_{g \in C(Y)} \mathbb{R}_g$. The commutativity of (2) provides that there is a mapping $\rho_{g \circ f}$ of ρX into $\mathbb{R}_g$ such that $g \circ f = \rho_{g \circ f} \circ \eta$. We want to show that f factors through ρX, that is $\zeta_f \circ \eta = f$; as ρX is a

completely regular space we would ideally want to get this equivalence my looking at an embedding of ρX in $\prod_{g\in C(Y)} \mathbb{R}_g$. We can do this with the map $h : \rho X \to \prod_{g\in C(Y)} \mathbb{R}_g$, defined by:

$$\pi_g \circ h(z) = \rho_{g\circ f}(z)$$

By definition, ρ_f is continuous so h is continuous. Note that the criteria for the use embedding lemma are satisfied; namely, ρX is completely regular and Hausdorff so that h is an embedding. Therefore, we know that $h(\rho X) \subset e(Y)$. As e is an embedding, defining $\zeta_f = e^{-1} \circ h$ gives the following commutative diagram, which completes the proof:



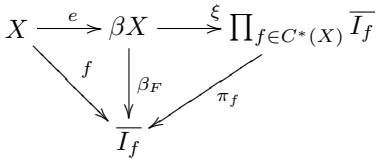
□

2.4 Stone-Čech Compactification: Existence and Uniqueness

We have developed the tools to finally describe the Stone-Čech compactification for a completely regular space and show that it is unique. Note that this is a stronger result than Tychonoff's Theorem for Completely Regular spaces and that Tychonoff's Theorem for Completely Regular spaces follows immediately from the following theorems.

Theorem. *(Stone-Čech compactification, Existence) Every completely regular space X has a Hausdorff compactification βX in which it is C^*-embedded*

Proof. For each $f \in C^*(X)$, let I_f denote the range of f. *As* f is bounded, $\overline{I_f}$ must be compact by Heine-Borel. Now we wish to apply the embedding lemma: Let $\mathcal{F} = C^*(X)$; then X can be embedded in $\prod_{f\in C^*(X)} \overline{I_f}$ by the evaluation map $\pi_f \circ e(x) = f(x)$. Now let $\beta X = \overline{e(X)}$; as $e(X) \subset \prod_{f\in C^*(X)} \overline{I_f}$, there exists a trivial, closed embedding of βX in $\prod_{f\in C^*(X)} \overline{I_f}$, say ξ. Finally, define the map $\beta_f : \beta X \to \overline{I_f}$ by $\beta_f = \pi_f \circ \xi$. This gives us the following commutative diagram:

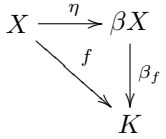


□

From this construction, we see that if X is compact, then $\beta X \cong X$. The maximality of the construction is implicit and comes from the fact that we are embedding X into a compact space indexed by *all* continuous, bounded functions on X. Note that the uniqueness of the Stone-Čech compactification was proved by Marshall Stone, whereas the existence of the extension was proved by Stone and Čech independently.

Theorem. *(Stone-Čech compactification, Uniqueness) Every completely regular space X has a compactification βX such that any mapping of X to a compact space K will extend uniquely to βX. In*

terms of commutative diagrams we have:



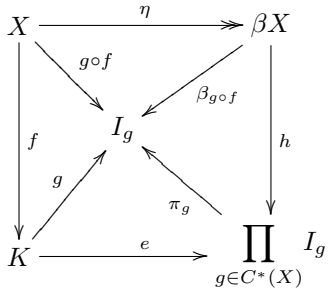
Proof. As in the previous theorem, for each $g \in C^*(K)$, let I_g represent the range of K and let $e : K \to \prod_{g \in C^*(K)} I_g$ be the evaluation map. As $g \circ f(X) \subset I_g$, the Stone-Čech existence theorem provides that there exists an extension $\beta_{g \circ f}$ of $g \circ f$ to βX.

Claim 1: f (uniquely) extends to βX

Suppose we define a mapping $h : \beta X \to \prod_{g \in C^*(K)} I_g$ by

$$\pi_g \circ h(p) = \beta_{g \circ f}(p)$$

for all $p \in \beta X$. Then h is continuous as $\beta_{g \circ f}$ is continuous (by construction) and subsequently this forces the above composition to be continuous in each I_g. This gives us the following commutative diagram



As e is an embedding, by construction, we need to show that $h(\beta X) \subset e(K)$ in order to show that h embeds η in $\prod_{g \in C^*(X)} I_g$. Note that for each $g \in C^*(K)$, $\beta_{g \circ f}(X) \subset g(K)$ so that we have:

$$h(\beta X) = h(\mathrm{cl}_{\beta X}(X)) \subseteq \overline{e(K)} = \mathrm{cl}(K) = K$$

where the last equality holds from the compactness of K. Finally, the uniqueness of the extension h comes directly from the fact that any two extensions of f are equivalent on βX. □

2.5 Final notes on $C^*(X)$ and $\mathcal{C}^*$-algebras

It is interesting to note that we developed the Stone-Čech compactification looking at $C^*(X)$ only as a ring of continuous functions. If we want to look at how products of functions $f, g \in C^*(X)$ behave, we can view $C^*(X)$ as $\mathcal{C}^*(X)$. As it turns out, the Stone-Čech compactification can be constructed for the associative algebra[6] $\mathcal{C}^*(X)$ as well, although the methodology is far less trivial. Whereas von Neumann attempted to look at $\mathcal{C}^*$-algebra from a functional analysis point-of-view, Israel Gelfand (1913-2009) and Mark Naimark (1909-1978) found a way to define $\mathcal{C}^*$-algebras without any reference to functional spaces and/or rings, instead defining a $\mathcal{C}^*$-algebra in an axiomatic fashion just as a ring or a group is often defined. In this more abstract setting, it is possible to describe βX as the $\mathcal{C}^*$-algebra of the dual space, or the space of linear functionals $f : X \to \mathbb{F}$ for a field $\mathbb{F}$, of X. This generalization of the Stone-Čech compactification inspired mathematicians such as Gelfand and Andrey Kolmogorov (1903-1987) to further generalize the construction of the Stone-Čech compactification to a set theoretical definition using filters (Walker, 1974).

[6] In this case, *products* and *norms* of functions are well-defined and we would like to find continuous extensions of products of functions in $\mathcal{C}^*(X)$ under the operator norm, $||f||_{op} = \inf\{c : ||fv|| \leq c||v||, \forall v \in X\}$ where $||\cdot||$ refers to the norm of the space of bounded linear operators (a Banach Space)

3 Alternative constructions of Stone-Čech Compactification using ultrafilters and $\mathcal{C}^*$-Algebras

3.1 Filters and Ultrafilters

As mentioned in the section 2.5, there is an even more general construction of the Stone-Čech compactification that is purely set theoretical and doesn't require dealing with functional spaces. This formulation requires the definition of a *filter*, which one can view from a topological point-of-view as a "locally finer" collection of sets than a topology (Walker, 1974). Formally we have the following definition:

Definition 6. (Walker) A family $\mathcal{K}$ of subsets of a topological space X is called a *ring of sets* if it is closed under finite intersections and unions. A subfamily $\mathcal{F}$ of non-empty members of a ring $\mathcal{K}$ is called a $\mathcal{K}$-*filter* if $\mathcal{F}$ satisfies the following conditions
(1) $\mathcal{F}$ is closed under finite intersections ($\mathcal{F}$ has the finite intersection property)
(2) A member of $\mathcal{K}$ containing a member of $\mathcal{F}$ is in $\mathcal{F}$

Clearly a filter cannot be a topology as it only contains non-empty subsets of X and it at most contains finite unions of some collection of subsets of X. We can think of filters as building blocks for a topology that have the finite intersection property. Note that filters are defined more formally in set-theory for partially ordered sets and lattices; however, this definition is equivalent if we consider the partial ordering of $\mathcal{K}$ under set inclusion, $\subseteq$. As we are looking for a *maximal* compactification for a topological space, it will serve us well to define the maximal filters on a set-theoretical ring:

Definition 7. A $\mathcal{K}$-filter $\mathcal{U}$ is a $\mathcal{K}$-*ultrafilter* if it is not properly contained in any other $\mathcal{K}$-filter. Alternatively we say that a $\mathcal{K}$-filter $\mathcal{U}$ is a $\mathcal{K}$-*ultrafilter* if and only if any member of $\mathcal{K}$ that meets every member of $\mathcal{U}$ is in $\mathcal{U}$.

An ultrafilter is a maximal filter on a set. A crucial result about the Stone-Čech compactification that will not be proven in this paper but is a related generalization of the subsequent section on Boolean algebras is the following Theorem:

Theorem 8. *(Stone and Wallman) Let $\mathcal{K}$ be the ring of all clopen sets of X, define $\omega_X(\mathcal{K})$ to be the set of all $\mathcal{K}$-ultrafilters on X and for $Z \in \mathcal{K}$ let Z^{ω_X} denote the members of $\omega_X(\mathcal{K})$ that contain Z. If X is a discrete topological space, then βX is the set of all ultrafilters on X with the topology defined by the basis,*

$$\mathcal{B} = \{Z^{\omega_X} | Z \in \mathcal{K}\}$$

3.2 The Stone Topology

In order to talk about compact topological spaces, we need to define a topology evolved from filters and ultrafilters. In 1938, Marshall Stone showed that there is a such topology called the *Stone Topology* on a topological space X, such that the ring of clopen sets on X form a compact space Y. As it turns out, X can be embedded in the space Y and as one might expect, Y is homeomorphic to βX. Thes ideas can be formalized using the set-theoretic notion of an ideal. This notion of an ideal stems from the ring theoretic notion of an ideal as a subset of a ring R that is closed under the "multiplication" operation of R when we are dealing with the Boolean Ring (defined in the next section) of a set (Walker, 1974). We can define a set-theoretical ideal for a set X as follows:

Definition 9. Given a set X, an *ideal* on X is a nonempty subset $\mathcal{I} \subseteq \mathcal{P}(X)$, the power set of X, such that
(1). If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$
(2). If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$

In order to make the concept of an ideal useful for an arbitrary (non-empty) set X, we need to define the Boolean Algebra of a power set of X:

Definition 10. A *Boolean Algebra for the power set* of a set X (hereby referred to as "the Boolean Algebra of X") is an algebra on $\mathcal{P}(X)$ such that the following axioms hold for the operations $\cup$ and $\cap$ and any elements $A, B, C \in \mathcal{P}(X)$:

(1) Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

(2) Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$

(3) Absorption: $A \cup (B \cap A) = A$, $A \cap (B \cup A) = A$

(4) Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

(5) Complements: $A \cup (\neg A) = X$, $A \cap (\neg A) = \emptyset$, where $\neg A$ refers to the set of all elements in X that are not in A (i.e. complement)

Note that the Boolean Algebra has a ring structure (set theoretically and algebraically) with $\cup$ being the group operation and $\cap$ being the semi-group operation

In order to arrive at the formal definition Stone Topology on the Boolean Algebra, we need the following set-theoretical definitions:

Definition 11. We say that an ideal $\mathcal{I}$ is *maximal* if there is no other ideal on X that contains $\mathcal{I}$. Let $\mathcal{M}(X)$ denote the colleection of all maximal ideals on X. For some subset $\mathcal{H} \subseteq \mathcal{M}(A)$, we define the *kernel* of $\mathcal{H}$ to be $\ker(\mathcal{H}) = \cap_{h \in \mathcal{H}} h$ and for any ideal $\mathcal{I}$ of X we define the *hull* of $\mathcal{I}$ to be

$$\{M \in \mathcal{M}(X) | \mathcal{I} \subseteq M\}$$

Note that the $\ker(\mathcal{H})$ is not necessarily a maximal ideal.

On a less formal level, the kernel of $\mathcal{H}$ defines a set of "minimal" maximal ideals for some collection of maximal ideals $\mathcal{H}$ while the hull defines the ultrafilters that contain a given ideal $\mathcal{I}$. If we think of the lattice of ideals under the partial order of set inclusion, the hull represents the set of all maximal elements of any chain that contains $\mathcal{I}$. In the construction of the Stone Topology, we are interested in looking at hull of the kernel of any collection of maximal ideals; in essence, we want to find out which maximal ideals "cover" the kernel. Formally we have:

Definition 12. The *Stone Topology* on the set of maximal ideals $\mathcal{M}(X)$ of the Boolean Algebra of a space X is that which defines the closure of any subcollection $\mathcal{H} \subseteq \mathcal{M}(X)$ as the hull of the kernel of $\mathcal{H}$, namely:

$$\overline{\mathcal{H}} = \{M \in \mathcal{M}(A) | \ker(\mathcal{H}) \subseteq M\}$$

In order to characterize the Stone-Čech compactification of X from this perspective, we will need to look at the maximal ideals of the Boolean algebra of $C^*(X)$.

3.3 Characterization of $\mathcal{M}(C^*(X))$

In order to motivate the definition of maximal ideals in $\mathcal{M}(C^*(X))$, we now turn to a space with a trivial Stone-Čech compactification, $I = [0, 1]$. In this case, we can show that the maximal ideal of $C^*(I)$ is the set of functions that have a zero at some point $c \in [0, 1]$. In algebra, the notion of a *prime ideal* provides a methodology

The following proposition, which was given (and subsequently proven by the author) in a problem set from Math 4340.

Proposition. *Let R be the ring of all continuous functions from the closed inteval $[0, 1]$ to $\mathbb{R}$ and for each $c \in [0, 1]$ let $M_c = \{f \in R : f(c) = 0\}$ (which is a maximal ideal). Then if M is* any *maximal ideal of R then there is a real number $c \in [0, 1]$ such that $M = M_c$*

Proof. Suppose that $M \neq M_c$ for all $c \in [0, 1]$. For each $f \in M$, define

$$C_f := \{f^{-1}(0)\}$$

Clearly $C_f = \overline{C_f}$ as $\{0\}$ is closed in $\mathbb{R}$ and f is continuous. Therefore $[0, 1] \backslash C_f$ is an open set for all $f \in M$. Now consider $C = \bigcap_{f \in M} C_f$. If S is nonempty, then we would have $M \subseteq M_c$ (as all

functions would have a zero at some c) which would imply that $M = M_c$ as M, M_c are maximal. Thus suppose $C = \emptyset$. Then $[0,1]\backslash C = [0,1]$ and we have:

$$\begin{aligned} [0,1]\backslash S &= [0,1]\backslash \bigcap_{f\in M} C_f \\ &= \bigcup_{f\in M} [0,1]\backslash C_f \\ &= D \end{aligned}$$

Therefore, D is an open cover of $[0,1]$ and by Heine-Borel, there exists a finite subcover. Thus for some $f_1, \cdots, f_n$:

$$[0,1] = ([0,1]\backslash C_{f_1}) \cup \cdots \cup ([0,1]\backslash C_{f_n}) = [0,1]\backslash \bigcap_{1\leq i\leq n} C_{f_i}$$

Now note that we can construct a positive function g in M from these f_i such that $\inf(g) > 0$ (i.e. g has no zeroes), by defining $g = \sum f_i^2$. Since g has no roots and is continuous, $\frac{1}{g} \in R$ and thus $\frac{1}{g} \cdot g = 1 \in M$. However if $1 \in M$ then M is equal to the whole ring R, which contradicts the maximality of M. Therefore M must be equal to M_c for some c. □

To simplify notation, we will define a zero-set of a function as:

Definition 13. A *zero-set* of a function $f \in C(X)$, $Z(f)$ is the set:

$$Z(f) = \{x \in X | f(x) = 0\}$$

Gelfand and Kolmogorov generalized this idea to any $C(X)$ in the following theorem:

Theorem. *(Gelfand and Kolmogorov) The maximal ideals of $C(X)$ are in one-to-one correspondance with the points ofβX and are given by*

$$M^p = \{f \in C(X) | p \in cl_{\beta X} Z(f)\}$$

for $p \in \beta X$

The proof of this theorem requires quite a few technical lemmas about ultrafilters and is thus omitted. However, many of the same ideas contained within the proposition extend to arbitrary completely regular spaces (not just $\mathbb{R}$). As we now have three characterizations of the Stone-Čech Compactification, we can look at some interesting applications of the Stone-Čech Compactification to physics, probability theory and set theory.

4 Atypical Applications of the Stone-Čech Compactification

4.1 Physics

As mentioned in the introductory section on the history of the Stone-Čech Compactification, there are some interesting historical ties between the compactification, $\mathcal{C}^*$-algebras and physics. In physics, there is a clear, intuitive notion of the relative uniformity or homogeneity a phase space. These "intuitive" notions include assumptions along the lines of "the flat surface of a table can be approximated by a compact subspace of $\mathbb{R}^2$" or "the electron is roughly in the shape of a sphere." However, we can make these notions far more precise by defining homogeneity in terms of topological invariants; for example we can say far more precise statements such as "the flat surface of a table is homeomorphic to a compact subspace of $\mathbb{R}^2$." These statements allow for many qualitative changes to take place without constraining the underlying mathematics of a process within a topologically-defined phase space. It is in this regard that the Stone-Čech Compactification becomes very useful in quantum

mechanics and string theory. Within the major models for both of these areas of physics is the assumption that results that take place in a high-dimensional[7] (or possibly infinite-dimensional phase space) can be applied to three-space or Lorentzian space (the four-space of Special Relativity; this space is homeomorphic to $\mathbb{R}^4$ with a specific metric). However, it is certainly possible that some components of our phase space are not compact due to requirements of the problem being solved. Therefore, the problem that physicists face is along the lines of:

> *How can one prevent pathologies due to compactness in non-compact subspaces of a phase space X that are not homeomorphic to $\mathbb{R}^3$ or $\mathbb{R}^4$ from affecting solutions in $\mathbb{R}^3$ or $\mathbb{R}^4$?*

As it turns out, the Stone-Čech Compactification provides a way to compactify these extra dimensions while simultaneously providing geometric details about our results in $\mathbb{R}^3$ or $\mathbb{R}^4$. For example, consider the following theorem due to Isbell (Keesling, 1977):

Theorem. *(Isbell) The Lebesgue Covering Dimension of a normal space X is the supremum over $n \in \mathbb{Z}$ such that there is a map $f : X \to [0,1]^n$ that has the property that $f|_{f^{-1}(\partial[0,1]^n)}$ cannot be extended to all of X*

Note that the last condition, $f|_{f^{-1}(\partial[0,1]^n)}$ cannot be extended to all of X implies that from a physical standpoint, Dirichlet boundary conditions[8] cannot be applied. More specifically, this theorem essentially tells us that if we have a normal subspace Y of our phase space X with a finite Stone-Čech Compactification (i.e. Y can be embedded in $[0,1]^n$ for some $n \in \mathbb{Z}$), then we have a subspace of our phase space of dimension n for which we cannot apply Dirichlet boundary conditions. An example of how important the Dirichlet boundary conditions are in both physics and mathematics, note that the heat equation, $\Delta u = \frac{\partial u}{\partial t}$ requires Dirichlet boundary conditions. More advanced geometric results that are expressed in terms of the cohomology groups of a topological space and in terms of Lie groups can also be shown. These geometric results can often be far more important to physicists as analytic results are often hard to interpret physically. In many cases, physicists use the Stone-Čech Compactification implicitly when they look for irreducible representations of the objects they are looking at. For instance in String Theory when we look at subsets of $\mathcal{C}^*$-algebras that are invariant under the Lagrangian (a way to formulate the idea of "total energy" on manifolds in a coordinate-free manner), we decompose this space into the direct sum or direct product of "irreducible" $\mathcal{C}^*$-algebras that are isomorphic to our original $\mathcal{C}^*$-algebra (**?**). Such a decomposition is done because it is much easier to analyze the "smaller," irreducible $\mathcal{C}^*$-algebras. This is similar to the decomposition of finite abelian groups, in which we find an isomorphism between some direct sum of irreducible finite groups (i.e. $\mathbb{Z}_q$ for $q = p^n$ for any $n \geq 1$, p a prime or $\mathbb{Z}^n$, the free abelian group on n generators) and the finite abelian group we started with. An important theorem in representation theory is the Gelfand-Naimark theorem which (informally) says that every irreducible decomposition of $\mathcal{C}^*$-algebras is isomorphic to some Hilbert Space (a closed subspace of L^2) (Rudin, 1991). There is a result in spectral theory (that is a corollary of the Gelfand-Naimark Theorem), that says that the irreducible-$\mathcal{C}^*$ decomposition of $\mathcal{C}^*(H)$ for a Hilbert Space H is isomorphic to the Stone-Čech Compactification of H (Rudin, 1991). As Hilbert Spaces of operators are the main functional spaces of interest in quantum mechanics and string theory (due to their geometric properties), the compactification that takes place implicitly guarantees that non-compactified dimensions will not impact the expected value of a wavefunction. Therefore the next time a physicist tells you that he has simplified a high-dimensional model by decomposing Hilbert Spaces, you can tell him that he or she owes a little bit of his or her Nobel Prize to M. Stone, I. Gelfand and M. Naimark.

[7] The notion of dimension being used is *Lebesgue Covering Dimension*, as defined in Hatcher's Point-Set Topology Notes

[8] Dirichlet Boundary Conditions are conditions that dictate how a solution to a differential equation behaves on the boundary of the solution space

4.2 Probability Theory

Within probability theory there is an area of study called *large-deviations theory*, which studies how probability measures[9] deal with unusual or tail events (i.e. events with low probability). Probabilists are often interested in whether a sequence of measures on a probability space converges and large-deviations theory attempts to provide and answer to how "fast" probability measures converge pointwise for low-probability events. S. R. Srinivasa Varadhan (1940-) formalized the definition of the *large-deviation property* in 1966 which one could call the central definition in large-deviations theory (Cattiaux and Gamboa, 1999). This property gives conditions for a probability measure to converge exponentially; that is for a sequence of probability measures $\{\mu_n\}$, we know the conditions so that $\log(\mu_n) \to I(x)$, where $I(x)$ is a function that represents the asymptotic "rate of convergence" of the measure). One crucial result in large-deviations theory is Varadhan's Lemma which is stated as follows (Dembo and Zeitouni, 1993):

Lemma. *(Varadhan) Let X be a regular topological space; let $(Z_\epsilon)_{\epsilon \geq 0}$ be a family of random variables taking values in X and let μ_ϵ be the probability measure associated with the random variable Z_ϵ. Suppose that $(\mu_\epsilon)_{\epsilon \geq 0}$ is such that for for every $F \subseteq X$ closed and $G \subseteq X$ open, we have:*

$$\begin{aligned} \limsup_{\delta \downarrow 0} \delta \log \mu_\delta(F) &\leq -\inf_{x \in F} I(x) \\ \liminf_{\delta \downarrow 0} \delta \log \mu_\delta(G) &\geq -\inf_{x \in G} I(x) \end{aligned}$$

for some rate function $I(x)$. Let $\phi : X \to \mathbb{R}$ be any continuous function. Further suppose that for some $\gamma > 1$,

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{E}[\exp(\gamma \varphi(Z_\epsilon/\epsilon)] < +\infty$$

where $\mathbb{E}[\cdot]$ refers to expected value. Then:

$$\lim_{\epsilon \downarrow 0} \epsilon \log \mathbb{E}[\exp(\gamma \varphi(Z_\epsilon/\epsilon)] = \sup_{x \in X} (\varphi(x) - I(x))$$

In informal terms, this lemma gives the asymptotic distribution (i.e. the distribution Φ such that $\mathbb{E}_\epsilon[\varphi] = \lim_{\epsilon \downarrow 0} \frac{\int_X \varphi d\mu_\epsilon}{\int_X \varphi d\Phi} \to 1$ for any $\varphi \in C(X)$) in terms of the rate function $I(x)$. As the rate function can take into account the severity of an event $x \in X$, this allows us to effectively simulate how low-probability events will effect a model. Now how does this relate to the Stone-Čech Compactification?

The Stone-Čech Compactification is an essential condition in the proof of Varadhan's Lemma because it allows for the lemma to hold for any $\varphi \in C(X)$. As X is regular, we can look at the continuous extension of φ to βX, $\tilde{\varphi} \in C(\beta X)$ and apply theorems such as the Riesz Representation Theorem (every measure on X corresponds to a linear functional in X^*) to simplify the proof of this fact. It is crucial that this result hold for all $\varphi \in C(X)$ as convergence in the sense of distributions requires thats $\mathbb{E}_n[\varphi] \to K$ for all $\varphi \in C(X)$. This novel application of the Stone-Čech Compactification is rather useful in probability theory; in fact, Varadhan won the Abel Prize in 2007 for topics including his proof of this lemma.

[9] A *probability measure* is a measure μ on a measure space (X, d, μ) such that $\int_X d\mu = 1$, where X is a metric space with metric d

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