Homotopy and geometric perspectives on string topology

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In these lecture notes I will try to summarize some recent advances in the new area of study known as *string topology.* This subject was initiated in the beautiful paper of Chas and Sullivan [3], and has attracted the attention of many mathematicians over the last few years. In its most basic form, string topology is the study of differential and algebraic topological properties of paths and loops in a manifold.

Throughout this note M^n will denote a closed, n-dimensional, oriented manifold. LM will denote the free loop space,

$$LM = Map(S^1, M).$$

For $D_1, D_2 \subset M$ closed submanifolds, $\mathcal{P}_M(D_1, D_2))$ will denote the space of paths in M that start at D_1 and end at D_2,

$$\mathcal{P}_M(D_1, D_2)) = \{\gamma : [0,1] \to M,\, \gamma(0) \in D_1,\, \gamma(1) \in D_2\}.$$

The paths and loops we consider will always be assumed to be piecewise smooth. Such spaces of paths and loops are well known to be infinite dimensional manifolds, and roughly speaking, string topology is the study of the intersection theory in these manifolds.

Recall that for closed, oriented manifolds, there is an intersection pairing,

$$H_r(M) \times H_s(M) \to H_{r+s-n}(M)$$

which is defined to be Poincare dual to the cup product,

$$H^{n-r}(M) \times H^{n-s}(M) \xrightarrow{\cup} H^{2n-r-s}(M).$$

The geometric significance of this pairing is that if the homology classes are represented by submanifolds, P^r and Q^s with transverse intersection, then the image of the intersection pairing is represented by the geometric intersection, $P \cap Q$.

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The remarkable result of Chas and Sullivan says that even without Poincare duality, there is an intersection type product

$$\mu : H_p(LM) \times H_q(LM) \longrightarrow H_{p+q-n}(LM)$$

that is compatible with both the intersection product on $H_*(M)$ via the map $ev : LM \to M$ $(\gamma \to \gamma(0))$, and with the Pontrjagin product in $H_*(\Omega M)$.

The construction of this pairing involves consideration of the diagram,

$$LM \stackrel{\gamma}{\longleftarrow} Map(\mathbf{8}, M) \stackrel{e}{\to} LM \times LM. \tag{1}$$

Here $Map(\mathbf{8}, M)$ is the mapping space from the figure 8 to M, which can be viewed as the subspace of $LM \times LM$ consisting of those pairs of loops that agree at the basepoint. $\gamma : Map(\mathbf{8}, M) \to LM$ is the map on mapping spaces induced by the pinch map $S^1 \to S^1 \vee S^1$.

Chas and Sullivan constructed this pairing by studying intersections of chains in loop spaces. A more homotopy theoretic viewpoint was taken by Cohen and Jones in [5] who viewed $e : Map(\mathbf{8}, M) \to LM \times LM$ as an embedding, and showed there is a tubular neighborhood homeomorphic to a normal given by the pullback bundle, $ev^*(TM)$, where $ev : LM \to M$ is the evaluation map mentioned above. They then constructed a Pontrjagin-Thom collapse map whose target is the Thom space of the normal bundle, $\tau_e : LM \times LM \to Map(\mathbf{8}, M)^{ev^*(TM)}$. Computing τ_e in homology and applying the Thom isomorphism defines an "umkehr map",

$$e_! : H_*(LM \times LM) \to H_{*-n}(Map(\mathbf{8}, M)).$$

The Chas-Sullivan loop product is defined to be the composition

$$\mu_* = \gamma_* \circ e_! : H_*(LM \times LM) \to H_{*-n}(Map(\mathbf{8}, M)) \to H_{*-n}(LM).$$

Notice that the umkehr map $e_!$ can be defined for a generalized homology theory h_* whenever one has a Thom isomorphism of the tangent bundle, TM, which is to say a generalized homology theory h_* for which the representing spectrum is a ring spectrum, and which supports an orientation of M.

By twisting the Pontrjagin-Thom construction by the virtual bundle $-TM$, one obtains a map of spectra,

$$\tau_e : LM^{-TM} \wedge LM^{-TM} \to Map(\mathbf{8}, M)^{ev^*(-TM)},$$

where LM^{-TM} is the Thom spectrum of the pullback of the virtual bundle $ev^*(-TM)$. Now we can compose, to obtain a multiplication

$$LM^{-TM} \wedge LM^{-TM} \xrightarrow{\tau_e} Map(\mathbf{8}, M)^{ev_0^*(-TM)} \xrightarrow{\gamma} LM^{-TM}.$$

The following was proved by Cohen and Jones in [5].

Theorem 1. *Let M be a closed manifold, then LM^{-TM} is a ring spectrum. If M is orientable the ring structure on LM^{-TM} induces the Chas-Sullivan loop product on $H_*(LM)$ by applying homology and the Thom isomorphism.*

The ring structure on the spectrum LM^{-TM} was also observed by Dwyer and Miller using different methods.

In [4], Cohen and Godin generalized the loop product in the following way. Observe that the figure 8 is homotopy equivalent to the pair of pants surface P, which we think of as a genus 0 cobordism between two circles and one circle.

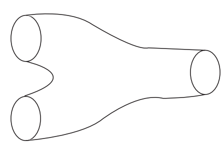

Figure 1: Pair of pants P

Furthermore, diagram 1 is homotopic to the diagram of mapping spaces,

$$LM \xleftarrow{\rho_{out}} Map(P, M) \xrightarrow{\rho_{in}} (LM)^2$$

where ρ_{in} and ρ_{out} are restriction maps to the "incoming" and "outgoing" boundary components of the surface P. So the loop product can be viewed as a composition,

$$\begin{aligned} \mu = \mu_P = (\rho_{out})_* \circ (\rho_{in})_! : (H_*(LM))^{\otimes 2} &\to H_*(Map(P, M)) \\ &\to H_*(LM) \end{aligned}$$

where using the figure 8 to replace the surface P can be viewed as a technical device that allows one to define the umkehr map $(\rho_{in})_!$.

In general if one considers a surface of genus g, viewed as a cobordism from p incoming circles to q outgoing circles, $\Sigma_{g,p+q}$, one gets a similar diagram

$$(LM)^q \xleftarrow{\rho_{out}} Map(\Sigma_{g,p+q}, M) \xrightarrow{\rho_{in}} (LM)^p.$$

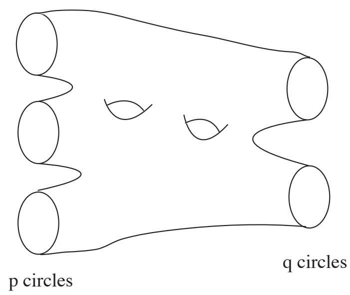

Figure 2: $\Sigma_{g,p+q}$

In [4] Cohen and Godin used the theory of "fat" or "ribbon" graphs to represent surfaces as developed by Harer, Penner, and Strebel [11], [12], [14], in order to define Pontrjagin-Thom maps,

$$\tau_{\Sigma_{g,p+q}} : (LM)^p \to Map(\Sigma_{g,p+q}, M)^{\nu(\Sigma_{g,p+q})}$$

where $\nu(\Sigma_{g,p+q})$ is the appropriately defined normal bundle of ρ_{in}. By applying (perhaps generalized) homology and the Thom isomorphism, they defined the umkehr map,

$$(\rho_{in})_! : H_*((LM)^p) \to H_{*+\chi(\Sigma_{g,p+q})\cdot n}(Map(\Sigma_{g,p+q}, M)),$$

where $\chi(\Sigma_{g,p+q}) = 2 - 2g - p - q$ is the Euler characteristic. Cohen and Godin then defined the string topology operation to be the composition,

$$\mu_{\Sigma_{g,p+q}} = \rho_{out} \circ (\rho_{in})_! : H_*((LM)^p) \to H_{*+\chi(\Sigma_{g,p+q})\cdot n}(Map(\Sigma_{g,p+q}, M)) \to H_{*+\chi(\Sigma_{g,p+q})\cdot n}((LM)^q).$$

They proved that these operations respect gluing of surfaces,

$$\mu_{\Sigma_1 \# \Sigma_2} = \mu_{\Sigma_2} \circ \mu_{\Sigma_1}$$

where $\Sigma_1 \# \Sigma_2$ is the glued surface as in figure 3 below.

The coherence of these operations are summarized in the following theorem.

Theorem 2. *(Cohen-Godin [4]) Let h_* be any multiplicative generalized homology theory that supports an orientation of M. Then the assignment*

$$\Sigma_{g,p+q} \longrightarrow \mu_{\Sigma_{g,p+q}} : h_*((LM)^p) \to h_*((LM)^q)$$

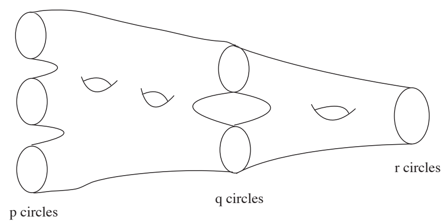

Figure 3: $\Sigma_1 \# \Sigma_2$

is a positive boundary topological quantum field theory. "Positive boundary" refers to the fact that the number of outgoing boundary components, q, must be positive.

A theory with *open strings* was initiated by Sullivan [15] and developed further by A. Ramirez [13] and by Harrelson [10]. In this setting one has a collection of submanifolds, $D_i \subset M$, referred to as "D-branes". This theory studies intersections in the path spaces $P_M(D_i, D_j)$.

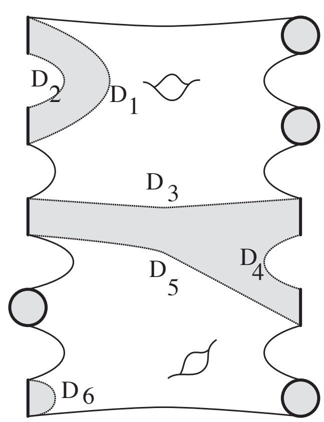

Figure 4: open-closed cobordism

A theory with D-branes involves "open-closed cobordisms" which are cobordisms between compact one dimensional manifolds whose boundary is partitioned into 3 parts:

1. Incoming circles and intervals

2. Outgoing circles and intervals

3. The rest is the "free boundary" which is itself a cobordism between the boundary of the incoming and boundary of the outgoing intervals. Each connected component of the "free boundary" is labelled by a D-brane. See figure 4.

In a topological field theory with D-branes, one associates to each boundary circle a vector space V_{S^1}, (in our case $V_{S^1} = H_*(LM)$) and to an interval whose endpoints are labeled by D_i, D_j, one associates a vector space V_{D_i,D_j} (in our case $V_{D_i,D_j} = H_*(P_M(D_i, D_j))$).

To an open-closed cobordism as above, one associates an operation from the tensor product of these vector spaces corresponding to the incoming boundaries to the tensor product of the vector spaces corresponding to the outgoing boundaries. Of course these operations have to respect the relevant gluing of open-closed cobordisms.

By developing a theory of fat graphs that encode the open-closed boundary data, Ramirez was able to prove that there are string topology operations that form a positive boundary, topological quantum field theory with D-branes. [13]

We end these notes by a discussion of three applications of string topology to classifying spaces of groups.

Example 1: *Application to Poincare duality groups.* This is work of H. Abbaspour, R.Cohen, and K. Gruher [2].

For G any discrete group, one has that the loop space of the classifying space satisfies

$$LBG \simeq \coprod_{[g]} BC_g$$

where $[g]$ is the conjugacy class determined by $g \in G$, and $C_g < G$ is the centralizer of g.

When BG is represented by a closed manifold, or more generally, when G is a Poincare duality group, the Chas-Sullivan loop product then defines pairings among the homologies of the centralizer subgroups. In [2] the authors describe this loop product entirely in terms of group homology, thus giving structure to the homology of Poincare-duality groups that had not been previously known.

Example 2 : *Applications to 3-manifolds.* This is work of H. Abbaspour [1]

Let $\iota : H_*M \to H_*(LM)$ be induced by inclusion of constant loops. This is a split injection of rings. Write $H_*(LM) = H_*(M) \oplus A_M$. We say $H_*(LM)$ has nontrivial extended loop products if the composition

$$A_M \otimes A_M \hookrightarrow H_*(LM) \otimes H_*(LM) \xrightarrow{\mu} H_*(LM)$$

is nontrivial.

Let M be a closed, irreducible 3-manifold. In a remarkable piece of work, Abbaspour showed the relationship between having a trivial extended loop product and M being "algebraically hyperbolic".

This means that M is a $K(\pi, 1)$ and its fundamental group has no rank 2 abelian subgroup. (If geometrization conjecture is true, this is equivalent to M admitting a complete hyperbolic metric.)

Example 3: *The string topology of classifying spaces of compact Lie groups.* This is work of K. Gruher [8] and of Gruher and Salvatore [9].

The goal of Gruher's work is to construct string topological invariants of $LBG \simeq EG \times_G G$, where G acts on itself via conjugation. Ultimately, one would like to understand the relationship between this structure and the work of Freed, Hopkins, Teleman [6] on twisted equivariant K-theory, $K_G^\tau(G)$ and the Verlinde algebra.

The first observation in this program was to notice that the key ingredient in the forming of the Chas-Sullivan loop product is that the fibration $ev : LM \to M$ is a fiberwise monoid over a closed oriented manifold. The fiber is ΩM, which has the usual Pontrjagin product.

The following was proved by Gruher and Salvatore:

Lemma 3. *Let $G \to E \to M$ be a fiberwise monoid over a closed manifold M. Then E^{-TM} is a ring spectrum.*

The following construction gives a large supply of examples of such fiberwise monoids over manifolds.

Let $G \to P \to M$ be a principal G bundle over a closed manifold M. We can construct the corresponding adjoint bundle,

$$Ad(P) = P \times_G G \to M.$$

It is an easy observation that $G \to Ad(P) \to M$ is a fiberwise monoid.

Theorem 4. *$Ad(P)^{-TM}$ is a ring spectrum. This ring structure is natural with respect to maps of principal G-bundles.*

Let BG be classifying space of compact Lie groups. It is possible to construct a filtration of BG,

$$M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_i \subset M_{i+1} \hookrightarrow \cdots \hookrightarrow BG$$

where the $M_i's$ are compact, closed manifolds. An example of this is filtering $BU(n)$ by Grassmannians.

Let $G \to P_i \to M_i$ be the restriction of $EG \to BG$. By the above theorem one obtains an inverse system of ring spectra

$$P_1^{-TM_1} \leftarrow P_2^{-TM_2} \leftarrow \cdots \leftarrow P_i^{-TM_i} \leftarrow P_{i+1}^{-TM_{i+1}} \leftarrow \cdots$$

Theorem 5. *The homotopy type of this pro-ring-spectrum is a well defined invariant of BG. It is referred to as the "string topology of BG".*

Work in progress by Gruher: Apply K_* to this pro-ring spectrum, as well as twisted K-theory. Relate the string topology multiplicative structure to the multiplicative structure on $K_G^\tau(G)$ found by Freed, Hopkins, and Teleman , known as the "fusion product" in the Verlinde algebra.

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