Math 210B. Group cohomology and group extensions

1. Motivation

There is a general principle (best learned through experience with many examples) that when cohomology (of various sorts) is used to classify obstructions to constructions then H^2 classifies isomorphism classes of structures (up to suitable equivalence) and H^1 acts simply transitively on the set of (equivalence classes of) automorphisms of a given structure. Thus, for example, when we have vanishing theorems for H^1 (which occurs in some important situations) then structures being studied do not have "non-trivial" automorphisms. In this handout we make this vague principle precise in the setting of group cohomology.

Let G be a group, and let M be a G-module. Consider exact sequences of groups

$$1 \to M \to E \to G \to 1$$

in which the left action by $G = E/M$ on M induced by E-conjugation on the commutative normal subgroup M is the given G-module structure on M. (For example, we could take E to be the semidirect product $E = M \rtimes G$ with the action $gmg^{-1} = g.m$ (using the evident inclusion $M \hookrightarrow E$ and quotient map $E \twoheadrightarrow G$ modulo M.) An *isomorphism* between two such extensions (with the same G and M) is defined to be a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \simeq \big\downarrow f & & \| & & \\ 1 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

in which $f : E' \to E$ is a group isomorphism restricting to the identity on M and inducing the identity on the common quotient G. Of course, if f is merely assumed to be a group homomorphism respecting the extension structures in this way then it is automatically an isomorphism (by a simple diagram chase).

In this handout, we will see that $\mathrm{H}^2(G, M)$ is naturally identified with the set of isomorphism classes of such extensions of G by M. But we emphasize that just as this particular group cohomology depends very much on the G-module structure on M, it will be essential that we have fixed the G-action induced on M from the extension structures that we consider. For example, if M has *trivial* G-action then $\mathrm{H}^2(G, M)$ is the set of isomorphism classes of central extensions of G by M (i.e., exact sequences as above for which M is in the center of E), but if we modify the G-module structure on M to be nontrivial then $\mathrm{H}^2(G, M)$ completely changes in general and likewise the class of extensions of G by M that we are considering completely changes too. So don't forget that the G-action on M in the exact sequences which we consider has been specified in advance!

2. Interpretation of H^2

To describe the possible exact sequences as above (inducing a given G-action on M!), let us first describe E as a set: we choose a set-theoretic section $s : G \to E$ to the given quotient map $\pi : E \twoheadrightarrow G$, so the M-cosets of E have a unique representative $s(g)$ for varying $g \in G$. We do *not* assume $s(1) = 1$. As a set, we have a disjoint union decomposition

$$E = \coprod_{g \in G} M \cdot s(g) = M \times G,$$

where $M \cdot s(g) = \pi^{-1}(g)$. To describe the group structure on this disjoint union, we note that the subgroup structure on M has been specified (with $1 \in M$ as the identity for the group law on

E) but $s(1)$ may not equal the identity of E, so we cannot expect the element $(0,1) \in M \times G$ to correspond to the identity of the group law.

As far as the composition law on E is concerned, what needs to be defined is $s(g_1)s(g_2)$ for $g_1, g_2 \in G$, since the way that $s(g)$ acts on M by conjugation within E (i.e., $s(g) \cdot m \cdot s(g)^{-1}$ for $g \in G$ and $m \in M$) has been specified in advance. Since π is to be a group homomorphism, we must have $s(g_1)s(g_2) \in \pi^{-1}(g_1g_2)$, which is to say $s(g_1)s(g_2) = c_{g_1,g_2}s(g_1g_2)$ for a unique $c_{g_1,g_2} \in M$. (Note in particular that $s(1) = c_{1,1}$.) Thus, there is a function $c : G \times G \to M$ such that the group law on the set $E = M \times G$ is defined by the rule

$$(m, g)(m', g') = (m + g.m' + c(g, g'), gg').$$

The condition that this be associative says

$$g.c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g') = 0, \tag{2.1}$$

and this relation forces $c(1, g) = c(1, 1)$ and $g.c(1, 1) = c(g, 1)$ for all $g \in G$ (by specializing $g = g' = 1$ and $g' = g'' = 1$). The condition in (2.1) is exactly the 2-cocycle condition: it says $c \in Z^2(G, M)$. Also, if we change s to another section $s' : G \to E$, which is to say we replace s with $s' = f \cdot s$ for a function $f : G \to M$ then c is replaced with $c' = c + \mathrm{d}f$, or in other words $c' - c \in B^2(G, M)$. Hence, the class $[c] \in \mathrm{H}^2(G, M)$ is independent of s and so depends only on the isomorphism class of the given extension structure E of G by M.

Conversely, given c satisfying (2.1) and defining a composition law on $E = M \times G$ as indicated above, one checks that $(-c(1, 1), 1)$ is a 2-sided identity element for this composition law and that the maps $M \to E$ defined by $m \mapsto (m - c(1, 1), 1)$ and $E \to G$ defined by $(m, g) \mapsto g$ are compatible with the composition laws. Finally, one checks that $(-g^{-1}.m - c(g^{-1}, g) - c(1, 1), g^{-1})$ is a 2-sided inverse to (m, g) (using that $g.c(g^{-1}, g) - c(1, g) + c(g, 1) - c(g, g^{-1}) = 0$ with $c(1, g) = c(1, 1)$ and $c(g, 1) = g.c(1, 1)$ for all $g \in G$). Thus, we have constructed a group extension of G by M inducing the given G-module structure on M via conjugation on the extension structure. Moreover, if we replace c with any 2-cocycle c' representing the same cohomology class then the new extension structure thereby constructed is isomorphic to the one constructed from c. (Explicitly, if $c' = c + \mathrm{d}f$ and E' denotes the group extension structure on $M \times G$ defined via c' then the asserted isomorphism $E' \simeq E$ as group extensions is $(m, g) \mapsto (m + f(g), g)$.)

The preceding considerations provide a natural bijection between the set $\mathrm{H}^2(G, M)$ and the set of isomorphism classes of group extensions of G by M inducing the given G-module structure on M via conjugation on the group extension. Note that in $\mathrm{H}^2(G, M)$ there is a distinguished element, the origin, and this is represented by the 2-cocycle $c = 0$. Hence, the corresponding group extension is easily seen to be the semidirect product $E = M \rtimes G$ associated to the given G-action on M, where E is given its evident extension structure.

2.1. **Automorphisms and H^1.** Having interpreted degree-2 group cohomology in terms of isomorphism classes of group extensions, we now interpret degree-1 group cohomology in terms of automorphisms of a fixed such group extension. To this end, consider an automorphism of a group extension

$$1 \to M \to E \xrightarrow{\pi} G \to 1,$$

which is to say an automorphism $f : E \simeq E$ respecting the extension structure. We write $\mathrm{Aut}(E)$ to denote the set of such automorphisms (the extension structure on E being understood from context). One trivial example of such an automorphism is conjugation $\gamma_m : E \simeq E$ by some $m \in M$. We say that two automorphisms $f_1, f_2 : E \simeq E$ of this extension structure are *equivalent* if $f_1 = \gamma_m \circ f_2$ for some $m \in M$; we then write $f_1 \sim f_2$, and $[f]$ will denote the equivalence class of f in $\mathrm{Aut}(E)$.

There is a natural action of $\mathrm{H}^1(G, M)$ on $\mathrm{Aut}(E)/\sim$ as follows. If $\xi \in \mathrm{H}^1(G, M)$ and $c : G \to M$ is a 1-cocycle representing the cohomology class ξ, then for any $f \in \mathrm{Aut}(E)$ it is easy to check that $c.f : x \mapsto c(\pi(x)) \cdot f(x)$ is another such automorphism of E as a group extension. (It is a group automorphism of E since it is clearly a group homomorphism from E to E that induces the identity automorphisms on the subgroup M and on the quotient $G = E/M$.). Moreover, if we change the choice of representative c for ξ then this automorphism of E changes up to equivalence as just defined. Hence, we get a well-defined pairing

$$\mathrm{H}^1(G, M) \times (\mathrm{Aut}(E)/\sim) \to \mathrm{Aut}(E)/\sim$$

via $([c], [f]) \mapsto [c.f]$. One checks (on HW10) that this really is an action of the group $\mathrm{H}^1(G, M)$ on the set $\mathrm{Aut}(E)/\sim$.

Rather interestingly, the verification that the equivalence class of $c.f$ only depends on the equivalence class of f (and the cohomology class of c) shows that this action by $\mathrm{H}^1(G, M)$ is simply transitive on $\mathrm{Aut}(E)/\sim$. This is part of HW10. In particular, if $\mathrm{H}^1(G, M) = 0$ then *all* automorphisms of E as a group extension are *necessarily* of the trivial type arising from conjugation by an element of M! This is useful in conjunction with vanishing theorems for degree-1 G-cohomology (of which we shall see a couple of examples in important cases with G a Galois group).