ISOMORPHISM OF SPLITTING FIELDS

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Using tensor products, we will give a slick proof that any two splitting fields of a polynomial are (non-canonically) isomorphic over the base field.

Theorem 1. Let K be a field and $f(X) \in K[X]$ be nonconstant. Any two splitting fields of f(X) over K are K-isomorphic.

Proof. Let $n = \deg f \ge 1$ and let L_1 and L_2 be splitting fields of f(X) over K, so

$$L_1 = K(\alpha_1, \ldots, \alpha_n), \quad L_2 = K(\beta_1, \ldots, \beta_n),$$

where the α_i 's and β_j 's are full sets of roots of f(X). (Some α_i 's and some β_j 's may be repeated since f(X) might not be separable.) We want to show there is a field isomorphism $L_1 \to L_2$ which fixes the elements of K.

Since L_1 and L_2 are not zero, the ring $L_1 \otimes_K L_2$ is not zero because the tensor product of nonzero vector spaces is not zero. Since L_1/K and L_2/K are algebraic, we can write $L_1 = K[\alpha_1, \ldots, \alpha_n]$ and $L_2 = K[\beta_1, \ldots, \beta_n]$. Thus $L_1 \otimes_K L_2$ is generated as a K-algebra by the 2n elementary tensors $\{\alpha_i \otimes 1, 1 \otimes \beta_j\}$. Pick a maximal ideal \mathfrak{m} in $L_1 \otimes_K L_2$ and consider the composite map

$$L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m},$$

where the first map is $x \mapsto x \otimes 1$ and the second map is the natural reduction. Both are *K*-algebra homomorphisms, so the composite is as well. Since L_1 is a field, the composite map is injective, so we can regard $(L_1 \otimes_K L_2)/\mathfrak{m}$ as a field extension of L_1 . The α_i 's are a full set of roots of f(X) in L_1 , so the only roots of f(X) in $(L_1 \otimes_K L_2)/\mathfrak{m}$ are the $\alpha_i \otimes 1 \mod \mathfrak{m}$. Each $1 \otimes \beta_j \mod \mathfrak{m}$ is a root of f(X), so $1 \otimes \beta_j \equiv \alpha_i \otimes 1 \mod \mathfrak{m}$ for some *i*. Therefore $(L_1 \otimes_K L_2)/\mathfrak{m}$ is generated as a *K*-algebra by all $\alpha_i \otimes 1 \mod \mathfrak{m}$, which proves the above map $L_1 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ is surjective, and hence is a *K*-algebra isomorphism.

We get a K-algebra isomorphism $L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ in a similar way. Composing $L_1 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ with the inverse of $L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ gives us a K-algebra isomorphism from L_1 to L_2 .

Remark 2. Each $\alpha_i \otimes 1$ and $1 \otimes \beta_j$ in $L_1 \otimes_K L_2$ is a solution to f(t) = 0. This typically gives us 2n solutions to f = 0 in $L_1 \otimes_K L_2$ when f(X) is separable,¹ so we should anticipate a collapsing of these roots into each other when we reduce $L_1 \otimes_K L_2$ modulo a maximal ideal and get a field, where f(X) always has at most n roots.

It might at first seem curious that the construction of a K-algebra isomorphism $L_1 \rightarrow L_2$ succeeded using any maximal ideal in $L_1 \otimes_K L_2$. In fact, different maximal ideals provide us with all the different isomorphisms. Let's look at an example before proving the general result.

¹This isn't always true: if $\alpha_i \in K$ then α_i is some β_j and $\alpha_i \otimes 1 = 1 \otimes \alpha_i$.

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Example 3. Two splitting fields for $X^2 - 2$ over \mathbf{Q} are $L_1 = \mathbf{Q}[T]/(T^2 - 2)$ and $L_2 = \mathbf{Q}(\sqrt{2})$ (a subfield of \mathbf{R}). There are two \mathbf{Q} -isomorphisms $L_1 \to L_2$, determined by the identification of T in L_1 with $\pm \sqrt{2}$ in L_2 . The tensor product of L_1 and L_2 over \mathbf{Q} is

$$L_1 \otimes_{\mathbf{Q}} L_2 = \mathbf{Q}[T] / (T^2 - 2) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}) \cong \mathbf{Q}(\sqrt{2})[T] / (T^2 - 2) = \mathbf{Q}(\sqrt{2})[T] / (T - \sqrt{2})(T + \sqrt{2}).$$

Using the Chinese remainder theorem,

$$\mathbf{Q}(\sqrt{2})[T]/(T-\sqrt{2})(T+\sqrt{2}) \cong \mathbf{Q}(\sqrt{2})[T]/(T-\sqrt{2}) \times \mathbf{Q}(\sqrt{2})[T]/(T+\sqrt{2}) \cong \mathbf{Q}(\sqrt{2}) \times \mathbf{Q}(\sqrt{2}),$$

where T on the left corresponds to $(\sqrt{2}, -\sqrt{2})$ on the right. The ring $\mathbf{Q}(\sqrt{2}) \times \mathbf{Q}(\sqrt{2})$ has two maximal ideals, $\{0\} \times \mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{2}) \times \{0\}$. The quotient by each of these maximal ideals is isomorphic to $\mathbf{Q}(\sqrt{2})$, with one sending T to $\sqrt{2}$ and the other sending T to $-\sqrt{2}$.

Theorem 4. With notation as in the proof of Theorem 1, the set of maximal ideals in $L_1 \otimes_K L_2$ is in bijection with the set of K-algebra isomorphisms $L_1 \to L_2$.

Proof. We want to describe a bijection between the sets

 $\{K\text{-algebra isomorphisms } L_1 \to L_2\} \longleftrightarrow \{\text{Maximal ideals in } L_1 \otimes_K L_2\}.$

From K-algebra isomorphism to maximal ideal: Let $L_1 \xrightarrow{\varphi} L_2$ be a K-algebra isomorphism. To construct from φ a maximal ideal in $L_1 \otimes_K L_2$, we will construct a homomorphism from $L_1 \otimes_K L_2$ onto the field L_2 and then take its kernel. The function $L_1 \times L_2 \to L_2$ where $(x, y) \mapsto \varphi(x)y$ is K-bilinear, so there is a K-linear map

$$L_1 \otimes_K L_2 \xrightarrow{f_{\varphi}} L_2$$

where $f_{\varphi}(x \otimes y) = \varphi(x)y$. This is onto since $f_{\varphi}(1 \otimes y) = y$. A computation shows f_{φ} is multiplicative on products of elementary tensors, so f_{φ} is a K-algebra homomorphism. Since f_{φ} is surjective and L_2 is a field, the kernel of f_{φ} is a maximal ideal. Set $M_{\varphi} = \ker f_{\varphi}$.

From maximal ideal to K-algebra isomorphism: Let \mathfrak{m} be a maximal ideal in $L_1 \otimes_K L_2$. We will construct from \mathfrak{m} a K-algebra isomorphism $L_1 \to L_2$. By the proof of Theorem 1, the natural composite maps

$$L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$$
 and $L_2 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$

are K-algebra isomorphisms. Call the first composite map $\psi_{1,\mathfrak{m}}$ and call the second one $\psi_{2,\mathfrak{m}}$. Set $\psi_{\mathfrak{m}} = \psi_{2,\mathfrak{m}}^{-1} \circ \psi_{1,\mathfrak{m}}$, so $\psi_{\mathfrak{m}}$ is a K-algebra isomorphism from L_1 to L_2 .

We will now show $\varphi \rightsquigarrow M_{\varphi}$ and $\mathfrak{m} \rightsquigarrow \psi_{\mathfrak{m}}$ are inverses of each other: $\psi_{M_{\varphi}} = \varphi$ and $M_{\psi_{\mathfrak{m}}} = \mathfrak{m}$.

Starting with φ , that $\psi_{M_{\varphi}} = \varphi$ means $\psi_{1,M_{\varphi}} = \psi_{2,M_{\varphi}} \circ \varphi$ as maps $L_1 \to (L_1 \otimes_K L_2)/M_{\varphi}$. For

 $\psi_{1,M_{\varphi}} \colon L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/M_{\varphi}$

the effect is $x \mapsto x \otimes 1 \mapsto x \otimes 1 \mod M_{\varphi}$. For

$$\psi_{2,M_{\varphi}} \circ \varphi \colon L_1 \to L_2 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/M_{\varphi}$$

the effect is $x \mapsto \varphi(x) \mapsto 1 \otimes \varphi(x) \mapsto 1 \otimes \varphi(x) \mod M_{\varphi}$. Therefore we need to show $x \otimes 1 \equiv 1 \otimes \varphi(x) \mod M_{\varphi}$. Recall that $M_{\varphi} = \ker f_{\varphi}$, so this congruence amounts to saying $f_{\varphi}(x \otimes 1) = f_{\varphi}(1 \otimes \varphi(x))$. From the definition of f_{φ} we have $f_{\varphi}(x \otimes 1) = \varphi(x) \cdot 1 = \varphi(x)$ and $f_{\varphi}(1 \otimes \varphi(x)) = \varphi(1)\varphi(x) = \varphi(x)$.

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Starting with \mathfrak{m} , that $M_{\psi_{\mathfrak{m}}} = \mathfrak{m}$ means ker $f_{\psi_{\mathfrak{m}}} = \mathfrak{m}$. We will show the diagram



commutes. Then since $\psi_{2,\mathfrak{m}}$ is an isomorphism, the kernels of the two maps out of $L_1 \otimes_K L_2$ would be equal, so ker $f_{\psi_{\mathfrak{m}}} = \mathfrak{m}$. To verify commutativity of the diagram, it suffices (by additivity of all the maps) to focus

on elementary tensors $x \otimes y$ in $L_1 \otimes_K L_2$, where we want to check

$$\psi_{2,\mathfrak{m}}(f_{\psi_{\mathfrak{m}}}(x\otimes y))\stackrel{!}{=} x\otimes y \bmod \mathfrak{m}.$$

The left side is

$$\begin{split} \psi_{2,\mathfrak{m}}(f_{\psi_{\mathfrak{m}}}(x\otimes y)) &= \psi_{2,\mathfrak{m}}(\psi_{\mathfrak{m}}(x)y) \\ &= \psi_{2,\mathfrak{m}}(\psi_{\mathfrak{m}}(x))\psi_{2,\mathfrak{m}}(y) \\ &= (\psi_{2,\mathfrak{m}}\circ\psi_{\mathfrak{m}})(x)\psi_{2,\mathfrak{m}}(y) \\ &= \psi_{1,\mathfrak{m}}(x)\psi_{2,\mathfrak{m}}(y) \\ &= (x\otimes 1) \bmod \mathfrak{m} \cdot (1\otimes y) \bmod \mathfrak{m} \\ &= x\otimes y \bmod \mathfrak{m}. \end{split}$$