Crossed modules and internal categories of Lie-Rinehart algebras

By

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Abstract

We give a definition of internal crossed module in a protomodular, Barr-exact category $\mathscr{C}$ with finite coproducts, and we show that the category of internal crossed modules in $\mathscr{C}$ is equivalent to the category of internal categories in $\mathscr{C}$. A category that is protomodular, Barr-exact, has finite coproducts, and is also pointed is, equivalently, a semi-abelian category. Our definition of internal crossed module is a generalisation of a definition of crossed module in a semi-abelian category due to Janelidze. Similarly, our theorem stating the equivalence of the categories of internal crossed modules and internal categories in $\mathscr{C}$ is a generalisation of a corresponding theorem of Janelidze's. We show that the category **LR** of Lie-Rinehart algebras is protomodular, Barr-exact, and has finite coproducts, showing that our new definition of internal crossed module applies to the category of Lie-Rinehart algebras, and thus that the categories of internal crossed modules in **LR** and internal categories in **LR** are equivalent. We then compare our definition of internal crossed module with the existing definition of a crossed module of Lie-Rinehart algebras.

Contents

Chapter. 1

Introduction

Semi-abelian categories

The notion of an abelian category abstracts from and generalises the category **Ab** of abelian groups [12]. The category **Grp** of groups is not abelian; however, **Grp** shares many of the properties of abelian categories. This motivates the definition of a semi-abelian category, which abstracts from and generalises the category **Grp**. A category $\mathscr{C}$ is *semi-abelian* if $\mathscr{C}$ is pointed, protomodular, Barr-exact, and has finite coproducts [17]. For other examples of semi-abelian categories, all abelian categories are semi-abelian, as are the categories of non-unitial rings, Lie algebras and many other non-abelian categories of algebras.

Internal categories and crossed modules in Grp

Now consider an internal category

$$\begin{tikzcd} C_2 \arrow[r, bend left, "p"] \arrow[r, "m"] \arrow[r, bend right, "q"'] & C_1 \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] & C_0 \arrow[l, "i"'] \end{tikzcd} \tag{1.1}$$

in **Grp**, the category of groups. Let

$$\operatorname{Ker} d = \{f \in C_1 : d(f) = 1_{C_0}\}.$$

Then for all $k \in \operatorname{Ker} d$ and all $x \in C_0$, we have

$$1_x \cdot k \cdot 1_x^{-1} \in \operatorname{Ker} d,$$

and the resulting

$$\phi : C_0 \to Aut(\operatorname{Ker} d) : x \mapsto (k \mapsto {}^x k = 1_x \cdot k \cdot 1_x^{-1}).$$

is a group homomorphism; that is, an action of C_0 on $\operatorname{Ker} d$. We can also define a homomorphism

$$\partial : \operatorname{Ker} d \to C_0 : k \mapsto c(k).$$

These two homomorphisms

$$\begin{aligned} \phi &: C_0 \to Aut(\operatorname{Ker} d), : x \mapsto (k \mapsto {}^x k) \\ \partial &: \operatorname{Ker} d \to C_0 : k \mapsto c(k) \end{aligned} \tag{1.2}$$

together satisfy, for all $x \in C_0$, and for all $k, k' \in \operatorname{Ker} d$,

1. $\partial({}^x k) = x \cdot \partial(k) \cdot x^{-1}$ (Equivariance)

2. ${}^{\partial(k)}k' = k \cdot k' \cdot k^{-1}$ (Peiffer condition)

Any such pair of homomorphisms (an action of a group G on a group H and a homomorphism $H \to G$) that satisfy these conditions is called a *crossed module* of groups.

We have seen that given the internal category (1.1), we can produce the crossed module (1.2). Moreover, we can recover C_1 from the other data. For any $f \in C_1$,

$$f = f \cdot (1_x^{-1} \cdot 1_x) = (f \cdot 1_x^{-1}) \cdot 1_x = k \cdot i(x),$$

with $k \in \operatorname{Ker} d$, and $x \in C_0$, and this representation is unique. So for any $f = k \cdot i(x), g = k' \cdot i(x') \in C_1$,

$$f \cdot g = k \cdot 1_x \cdot k' \cdot (1_x^{-1} \cdot 1_x) \cdot 1'_x = k \cdot (1_x \cdot k' \cdot 1_x^{-1}) \cdot 1_x \cdot 1'_x = k \cdot {}^x k' \cdot i(xx').$$

This looks like the semi-direct product

$$\operatorname{Ker} d \rtimes C_0 = \{(k, x) : k \in \operatorname{Ker} d, x \in C_0\},$$

with group product

$$(k, x) \cdot (k', x') = (k^x \cdot k', x \cdot x').$$

Indeed,

$$C_1 \to \operatorname{Ker} d \rtimes C_0 : k \cdot i(x) \mapsto (k, x)$$

is an isomorphism of groups.

We can also recover the composition of the internal category. It turns out that the composition morphism m of an internal category of groups is determined by the group product of C_1:

$$g \circ f = m(f, g) = g \cdot id(g)^{-1} \cdot f.$$

Writing $f = k \cdot i(x),\ g = k' \cdot i(x')$, we have

$$m(k \cdot i(x), k' \cdot i(x')) = k' \cdot k \cdot i(x).$$

Thus we can define a composition homomorphism m' for composable pairs in $(K \rtimes C_0) \times (K \rtimes C_0)$ by

$$m'((k, x), (k', x')) = (k' \cdot k, x).$$

For more details, see [19], [8].

Crossed modules

Crossed modules of groups were first defined by Whitehead in the 1940s in algebraic topology, see [22]. The equivalence between the categories of crossed modules and internal categories in **Grp** was known by Verdier and Duskin in the 1960s; a proof of the equivalence was first published by Brown and Spencer in 1976 [8]. Since then, various authors have found it expedient to introduce a notion of a crossed module for other structures, for example, Lie algebras [18]. These crossed modules have been defined by analogy, in a more or less ad hoc manner. In the paper [16], Janelidze gives a unifying definition of crossed module in semi-abelian categories such that the equivalence of crossed modules and internal categories holds. Remarkably, the old ad hoc definitions agree with the results of applying the general semi-abelian definition in concrete cases.

Lie-Rinehart algebras

Given a smooth manifold M, there is a commutative algebra $A = C^\infty(M)$ of infinitely differentiable functions $M \to \mathbb{R}$. There is also a Lie $\mathbb{R}$-algebra L of all the vector fields on M, with Lie bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$. The algebra A is an L-module, and the Lie algebra L is an A-module. Furthermore A and L act on each other in a "compatible" way. This is a motivating example of a *Lie-Rinehart algebra*, first defined in [15], see Definition 3.1.2.

Crossed modules and internal categories of Lie-Rinehart algebras

Crossed modules of Lie-Rinehart algebras were defined by Casas et al in [11]. The category **LR** of Lie-Rinehart algebras is not semi-abelian, because it is not pointed. Therefore Janelidze's (semi-abelian) definition of crossed module does not apply to the category of Lie-Rinehart algebras. It turns out, however, that the category **LR** is protomodular, Barr-exact, and has finite coproducts. In this thesis, we give a new definition of an internal crossed module in a protomodular, Barr-exact category $\mathscr{C}$ with finite coproducts, showing that the categories of internal crossed modules and internal categories in $\mathscr{C}$ are equivalent, and thus extending Janelidze's (semi-abelian) result. We then show that the category of Lie-Rinehart algebras is protomodular, Barr-exact, and has finite coproducts, thus showing that the categories of internal crossed modules and internal categories in **LR** are equivalent. Finally, we compare internal crossed modules in **LR** to the classical definition of a crossed module of Lie-Rinehart algebras, with a view towards establishing the equivalence between the categories of (classical) crossed modules and internal categories in **LR**.

In chapter 2 of this thesis, we will give a definition of internal crossed module for any protomodular, Barr-exact category $\mathscr{C}$ that has finite coproducts, and show that this definition ensures that the category of these internal crossed modules is equivalent to the category of internal categories in $\mathscr{C}$. In chapter 3, we will turn our attention to Lie-Rinehart algebras. There we will introduce Lie-Rinehart algebras, and the existing definition of a crossed module of Lie-Rinehart algebras, before showing that the category of Lie-Rinehart algebras is protomodular, Barr-exact, and has finite coproducts. Finally, we compare internal crossed modules of Lie-Rinehart algebras with the existing definition of a crossed module of Lie-Rinehart algebras.

Prerequisites

This thesis is not quite self contained. We collect here some definitions and references that may be useful to the reader.

A category $\mathscr{C}$ is called *regular* if $\mathscr{C}$ is finitely complete and has image factorisations that are stable under pullback. Equivalently, a category $\mathscr{C}$ is regular if (1) it is finitely complete; (2) every kernel pair has a coequaliser; and (3) regular epimorphisms (coequalisers) are stable under pullback. A regular category $\mathscr{C}$ is called *Barr-exact* if every equivalence relation in $\mathscr{C}$ is a kernel pair. We use basic results concerning regular categories, Barr-exact categories, regular epimorphisms, strong epimorphisms, and related topics. A good reference is the appendix of [5].

We also assume the reader knows the definitions of the categories of internal categories (see [19]) and reflexive graphs (see [5]).

Notation

We will use the following notational conventions.

If

$$A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$$

is a product, and $f : X \to A$ and $g : X \to B$ are morphisms, then we write the induced morphism into the product as $\begin{pmatrix} f \\ g \end{pmatrix}$:

$$\begin{tikzcd} & X \arrow[dl, bend right, "f"'] \arrow[d, "\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)"] \arrow[dr, bend left, "g"] & \\ A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B. \end{tikzcd}$$

Similarly, a morphism into a pullback is written $\binom{f}{g}$, as in

$$\begin{tikzcd}
X \arrow[dr, "\binom{f}{g}"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d, "r"] \\
& A \arrow[r, "q"'] & C.
\end{tikzcd}$$

If

$$A \xrightarrow{i_A} A + B \xleftarrow{i_B} B.$$

is a coproduct, and if $f : A \to Y$ and $g : B \to Y$ are morphisms, then we write the induced morphism out of the coproduct as (f, g):

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[dr, bend right, "f"'] & A + B \arrow[d, "{(f,\, g)}"] & B \arrow[l, "i_B"'] \arrow[dl, bend left, "g"] \\
& Y &
\end{tikzcd} .$$

As in the example just given, we will usually write coproduct injections from X to $X + Y$ as

$$X \xrightarrow{i_X} X + Y.$$

Chapter. 2

Crossed modules in protomodular, Barr-exact categories with finite coporoducts

In the paper [16], Janelidze defines a general notion of crossed module in any semi-abelian category, and shows that for any semi abelian category $\mathscr{C}$, the category of crossed modules in $\mathscr{C}$ is equivalent to the category of internal categories in $\mathscr{C}$. In this chapter, we show that this result can be extended to hold more generally. In particular, pointedness is not required, provided that the category $\mathscr{C}$ is Barr-exact, protomodular, and has finite coproducts.

Recall the case of groups that we discussed in the introduction. Notice that the split epimorphism

$$
\begin{tikzcd}
C_1 \arrow[r, shift left, "d"] & C_0 \arrow[l, shift left, "i"]
\end{tikzcd}
$$

is enough to determine the action of C_0 on $\operatorname{Ker} d$. This action, in turn, is enough to recover the original split epimorphism as the projection of $C_0 \rtimes \operatorname{Ker} d$ onto C_0. If we consider the extra structure given by the morphism c, we have a reflexive graph, and giving this reflexive graph is the same as giving the pre-crossed module $\partial : \operatorname{Ker} d \to C_0$. Finally, if we consider the extra structure of the composition morphism m, we have an internal category; but to give this composition m is the same as making the pre-crossed module ∂ into a crossed module.

We will use a similar strategy in this section. First we will search for conditions on a category $\mathscr{C}$ under which we have an equivalence between split epimorphisms in $\mathscr{C}$ and (a suitable general definition of) internal actions in $\mathscr{C}$. Then we will show that giving an extra morphism to build up to a reflexive graph is the same as building up to (a suitable definition of) an internal pre-crossed module. Finally, we will show that giving the reflexive graph a composition morphism, and thus making it into an internal category, is the same as giving (a suitable definition of) an internal crossed module. It should be noted that at times this chapter borrows significantly from Janelidze's approach to the semi-abelian case in [16], adapting his methods and strategy to fit our non-pointed setting.

2.1 Split epimorphisms and "actions"

Split epimorphisms

Definition 2.1.1. Let $\mathscr{C}$ be a category. We let $\mathbf{SplEp}(\mathscr{C})$ denote the *category of split epimorphisms of* $\mathscr{C}$. The objects of $\mathbf{SplEp}(\mathscr{C})$ are split epimorphisms

$$
\begin{tikzcd}
E \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"]
\end{tikzcd}
$$

of $\mathscr{C}$. More precisely, an object of $\mathbf{SplEp}(\mathscr{C})$ is a quadruple (E, B, p, s), where p is a split epimorphism and s is a section of p, that is, $ps = 1_B$. We will sometimes write (p, s) instead of (E, B, p, s). A morphism $(E, B, p, s) \to (E', B', p', s')$ of $\mathbf{SplEp}(\mathscr{C})$ is a pair (f, g) of morphisms of $\mathscr{C}$ such that $p' f = gp$ and $fs = s' g$. In other words, a morphism is a pair (f, g) of morphisms of $\mathscr{C}$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\
E' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

commutes in both directions.

Definition 2.1.2. Let $\mathscr{C}$ be a category. For each object $B \in \mathscr{C}$, let $\mathbf{SplEp}_B(\mathscr{C})$ denote the category of split epimorphisms of $\mathscr{C}$ over B. The objects of $\mathbf{SplEp}_B(\mathscr{C})$ are the objects of $\mathbf{SplEp}(\mathscr{C})$ such that the codomain of the split epimorphism is B. The morphisms of $\mathbf{SplEp}_B(\mathscr{C})$ are the morphisms of $\mathbf{SplEp}(\mathscr{C})$ that have 1_B in the second component.

An adjunction

Let $\mathscr{C}$ be a category with pullbacks and an initial object 0. We begin with the following definition.

Definition 2.1.3 (kernel,Ker)**.** Let $p : E \to B$ be a morphism in $\mathscr{C}$. We can take the pullback of p along the unique morphism $!_B$ from the initial object to B:

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[r, "k_p"] \arrow[d, "\kappa_p"'] & E \arrow[d, "p"] \\
0 \arrow[r, "!_B"'] & B.
\end{tikzcd}$$

We define the object $\operatorname{Ker} p$, along with the morphism $\kappa_p : K \to 0$ to be the *kernel* of f. Of course, the injection $k_p : \operatorname{Ker} p \to E$ is also part of the definition of the kernel. By the universal property of the pullback, this process determines a functor $\operatorname{Ker} : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0$ (see the diagram below), which we refer to as the *kernel functor*.

$$\begin{tikzcd}
& \operatorname{Ker} p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, dashed, "\operatorname{Ker}(h,\, g)"] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p"'] \arrow[dd, "h"'] & & B \arrow[dd, "g"] & \\
& \operatorname{Ker} p' \arrow[rr, "\kappa_{p'}"] \arrow[dl, "k_{p'}"'] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' &
\end{tikzcd}$$

Remark. We observe that if 0 is also a terminal object, then $\mathscr{C}/0 \cong \mathscr{C}$ and our definition agrees with the usual definition of a kernel in a pointed category.

Remark. We also observe that for any morphism $x : X \to 0$, the unique morphism $!_X : 0 \to X$ is a section of x, thus $\mathscr{C}/0 = \mathbf{SplEp}_0(\mathscr{C})$. Indeed, more generally, pulling back a split epimorphism $p : E \to B$ with section s along an arbitrary arrow $f : A \to B$ gives a split epimorphism, with the section given by the unique morphism u induced by the cone $(1_A, sf)$.

We have already defined the kernel functor Ker : $\mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0$. There is also the codomain functor

$$\mathrm{Cod} : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C},$$

which maps

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

to

$$g : B \to B'.$$

So, for any category $\mathscr{C}$ with pullbacks and an initial object 0, we can define a functor

$$U : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

by

$$U = \left(\begin{smallmatrix} \mathrm{Ker} \\ \mathrm{Cod} \end{smallmatrix}\right),$$

defined on objects by sending

$$\begin{tikzcd} E \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

to the pair $(\mathrm{Ker}\, p \xrightarrow{\kappa_p} 0, B)$.

If $\mathscr{C}$ has finite coproducts, then given a pair $(X \xrightarrow{x} 0, A)$ in $\mathscr{C}/0 \times \mathscr{C}$, we can form the coproducts

$$A \xrightarrow{i_A} A + X \xleftarrow{i_X} X$$

and

$$A \xrightarrow{1_A} A = A + 0 \xleftarrow{!_A} 0$$

We define $F(X \xrightarrow{x} 0, A)$ to be the split epimorphism

$$\begin{tikzcd} A + X \arrow[r, shift left, "1_A + x"] & A. \arrow[l, shift left, "i_A"] \end{tikzcd}$$

We denote the kernel of the morphism $1_A + x$ by

$$\begin{tikzcd} A\flat X \arrow[r, "k_{A,x}"] \arrow[d, "\kappa_{A,x}"'] & A + X \arrow[d, "1_A + x"] \\ 0 \arrow[r, "!_A"'] & A. \end{tikzcd}$$

Now, since $A\flat X \xrightarrow{\kappa_{A,x}} 0$ is the kernel of $1_A + x$, we have

$$UF(X \xrightarrow{x} 0, A) = (A\flat X \xrightarrow{\kappa_{A,x}} 0, A).$$

We define the morphism

$$\eta_{A,x} : (X \xrightarrow{x} 0, A) \to (A\flat X \xrightarrow{\kappa_{A,x}} 0, A)$$

by $1_A : A \to A$ on the $\mathscr{C}$ component, and on the $\mathscr{C}/0$ component by

$$\eta_x : X \to A\flat X,$$

where η_x is the unique morphism making the following diagram commute

$$\begin{tikzcd}
X \arrow[dr, dashed, "\eta_x"] \arrow[drr, bend left, "i_X"] \arrow[ddr, bend right, "x"'] & & \\
& A\flat X \arrow[r, "k_{A,x}"] \arrow[d, "\kappa_{A,x}"'] & A+X \arrow[d, "1_A+x"] \\
& 0 \arrow[r, "!_A"'] & A.
\end{tikzcd}$$

Let $\mathscr{C}$ be a category with pullbacks and finite coproducts, and let

$$\begin{tikzcd} E \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

be a split epimorphism of $\mathscr{C}$. Given any morphism

$$(f,g) : (X \xrightarrow{x} 0, A) \to U(E,B,p,s) = (\operatorname{Ker} p \xrightarrow{\kappa_p} 0, B)$$

of $\mathscr{C}/0 \times \mathscr{C}$, the morphism of split epimorphisms

$$\begin{tikzcd}
A+X \arrow[r, "(sg,\ k_p f)"] \arrow[d, shift left, "1_A+x"] & E \arrow[d, shift right, "p"'] \\
A \arrow[u, shift left, "i_A"] \arrow[r, "g"'] & B, \arrow[u, shift right, "s"']
\end{tikzcd} \tag{2.1}$$

is the unique morphism $\overline{(f,g)}$ such that $U\overline{(f,g)}\eta_{A,x} = (f,g)$. This shows that, if $\mathscr{C}$ is a category with pullbacks and finite coproducts, there is an adjunction

$$F \dashv U : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}.$$

From (2.1), it follows that the counit ϵ has components $\epsilon_{(E,B,p,s)} = (s, k_p) : B + \operatorname{Ker} p \to E$.

Internal actions

From now on, we assume that $\mathscr{C}$ is a category with pullbacks and finite coproducts. We have seen that, for such categories $\mathscr{C}$, we have the adjunction

$$\begin{tikzcd}
\mathbf{SplEp}(\mathscr{C}) \arrow[d, shift left, "U"] \\
\mathscr{C}/0 \times \mathscr{C}, \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

with unit η, with components

$$\eta_{A,x} = (\eta_X, 1_A) : (X \xrightarrow{x} 0, A) \to (A\flat X \xrightarrow{\kappa_{A,x}} 0, A),$$

and counit ϵ, with components

$$\epsilon_{p,s} = ((s, k_p), 1_B) : (B + \operatorname{Ker} p, B, 1_B + \kappa_p, i_B) \to (E, B, p, s).$$

This adjunction determines a monad

$$(\flat = UF : \mathscr{C}/0 \times \mathscr{C} \to \mathscr{C}/0 \times \mathscr{C}, \eta, U\epsilon_F),$$

and hence the Eilenberg-Moore category $(\mathscr{C}/0 \times \mathscr{C})^\flat$ of $\flat$-algebras, where the objects satisfy the axioms for algebras for a monad [19].

We will gradually introduce assumptions on $\mathscr{C}$ that force the functor U to be monadic, and thus give an equivalence between the categories $\mathbf{SplEp}(\mathscr{C})$ and $(\mathscr{C}/0 \times \mathscr{C})^\flat$. We will see that for categories $\mathscr{C}$ satisfying these assumptions, the category $(\mathscr{C}/0 \times \mathscr{C})^\flat$ plays an analogous role in the general theory that we develop here to the role played by the category of group actions in establishing the equivalence between internal categories and crossed modules of groups that we mentioned in the introduction. This is the reason for the choice of terminology in the following definition.

Definition 2.1.4 (Internal action)**.** We define an *internal action* in $\mathscr{C}$ to be an object of $(\mathscr{C}/0 \times \mathscr{C})^\flat$. We call $\mathbf{Act}(\mathscr{C}) = (\mathscr{C}/0 \times \mathscr{C})^\flat$ the category of actions internal to $\mathscr{C}$.

Remark. Note that there is a key difference between the internal actions just defined and a group action. In the case of groups, we have an object of **Grp** acting on another object of **Grp**; in our definition an object of $\mathscr{C}$ acts on an object of $\mathscr{C}/0$.

Explicitly, these generalised "actions" consist of objects $X, A \in \mathscr{C}$, along with morphisms $X \xrightarrow{x} 0$ and $A\flat X \xrightarrow{\xi} X$ such that the diagrams

$$\begin{tikzcd} A\flat X \arrow[r, "\xi"] \arrow[dr, "\kappa_{A,x}"'] & X \arrow[d, "x"] \\ & 0, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & A\flat X \arrow[d, "\xi"] \\ & X, \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A\flat(A\flat X) \arrow[r, "U\epsilon_{F(A,x)}"] \arrow[d, "1_A\flat\xi"'] & A\flat X \arrow[d, "\xi"] \\ A\flat X \arrow[r, "\xi"'] & X \end{tikzcd}$$

commute, where $A\flat(A\flat X)$ is the kernel of $1_A + \kappa_{A,x} : A + A\flat x \to A$, and $1_A\flat\xi$ is the morphism $\operatorname{Ker}(1_A + \xi) = \operatorname{Ker} F(\xi, 1_A)$. These data define an internal action

$$(\xi, 1_A) : (A\flat X \xrightarrow{\kappa_{A,x}} 0, A) \to (X \xrightarrow{x} 0, A).$$

We will usually write actions as triples $(X \xrightarrow{x} 0, \xi, A)$.

A morphism of actions

$$(X \xrightarrow{x} 0, \xi, A) \to (X' \xrightarrow{x'} 0, \xi', A)$$

is a pair (α, β), with

$$\alpha : (X \xrightarrow{x} 0) \to (X' \xrightarrow{x'} 0)$$

a morphism of $\mathscr{C}/0$ and

$$\beta : A \to A'$$

a morphism of $\mathscr{C}$ such that

$$\begin{tikzcd} A\flat X \arrow[r, "\beta\flat\alpha"] \arrow[d, "\xi"'] & A'\flat X' \arrow[d, "\xi'"] \\ X \arrow[r, "\alpha"'] & X' \end{tikzcd}$$

commutes, where $\beta\flat\alpha = \operatorname{Ker} F(\alpha, \beta)$.

Remark. Observe that for any pair $(X \xrightarrow{x} 0, A) \in \mathscr{C}/0 \times \mathscr{C}$, we have

$$\flat(X \xrightarrow{x} 0, A) = UF(X \xrightarrow{x} 0, A) = U(A + X, A, 1_A + x, i_A) = (A\flat X \xrightarrow{k_{A,x}} 0, A,),$$

$$\eta_{A,x} = (\eta_x, 1_A),$$

and

$$U\epsilon_{F(A,x)} = U\epsilon_{(1_A+x,\ i_A)} = U((i_A, k_{A,x}), 1_A).$$

It follows that $\flat, \eta$, and $U\epsilon_F$ do not affect anything in the second component. Indeed, for any fixed A, there is another related monad

$$A\flat : \mathscr{C}/0 \to \mathscr{C}/0 : (X \xrightarrow{x} 0) \mapsto (A\flat x \xrightarrow{\kappa_{A,x}} 0).$$

The comparison functor

Provided that $\mathscr{C}$ is a category with pullbacks and finite coproducts, we have our adjunction (F, U, η, ϵ), the induced monad $(\flat, \eta, U\epsilon_F)$, and the corresponding Eilenberg-Moore adjunction $(F^\flat, U^\flat, \eta^\flat, \epsilon^\flat)$ [19]. Let K denote the comparison functor

$$K : \mathbf{SplEp}(\mathscr{C}) \to (\mathscr{C}/0 \times \mathscr{C})^\flat,$$

defined on objects by

$$(E, B, p, s) \mapsto UFU(E, B, p, s) \xrightarrow{U\epsilon_{(E,B,p,s)}} U(E, B, p, s),$$

and on morphisms by

$$(f, g) \mapsto U(f, g).$$

More explicitly, this sends a split epimorphism (E, B, p, s) to the object $(\operatorname{Ker} p \xrightarrow{\kappa_p} 0, B)$ equipped with the action

$$U\epsilon_{(E,B,p,s)} = U((s, k_p), 1_B) = (\operatorname{Ker}((s, k_p), 1_B), 1_B),$$

which is a morphism of $\mathscr{C}/0 \times \mathscr{C}$. The first component, $\operatorname{Ker}((s, k_p), 1_B)$, is defined by the following commutative diagram

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[dr, "\eta_{\operatorname{Ker} p}"] \arrow[drr, bend left, "i_{\operatorname{Ker} p}"] \arrow[ddr, bend right, "1_{\operatorname{Ker} p}"'] \arrow[dddr, bend right=40, "\kappa_p"'] & & \\
& B\flat \operatorname{Ker} p \arrow[r, "k_{B,\kappa_p}"] \arrow[d, "\operatorname{Ker}((s,\ k_p),\ 1_B)"] \arrow[dd, bend right, "\kappa_{B,\kappa_p}"'] & B + \operatorname{Ker} p \arrow[d, "(s,\ k_p)"'] \arrow[dd, bend left, "1_B+\kappa_p"] \\
& \operatorname{Ker} p \arrow[r, "k_p"'] \arrow[d, "\kappa_p"] & E \arrow[d, "p"'] \\
& 0 \arrow[r, "!_B"'] & B,
\end{tikzcd}$$

where $\operatorname{Ker}((s, k_p), 1_B)$ is the unique morphism induced by the cone $(\kappa_{B,\kappa_p}, (s, k_p).k_{B,\kappa_p})$. Also, we see that $1_{\operatorname{Ker} p}$ is the unique morphism such that both

$$\kappa_p.1_{\operatorname{Ker} p} = \kappa_p$$

and

$$k_p.1_{\operatorname{Ker} p} = (s, k_p).i_{\operatorname{Ker} p}.$$

But

$$\kappa_p.\operatorname{Ker}((s, k_p), 1_B).\eta_{\operatorname{Ker} p} = \kappa_{B,\kappa_p}.\eta_{\operatorname{Ker} p} = \kappa_p,$$

and

$$k_p.\operatorname{Ker}((s, k_p), 1_B).\eta_{\operatorname{Ker} p} = (s, k_p).k_{B,\kappa_p}.\eta_{\operatorname{Ker} p} = k_p.$$

So $\operatorname{Ker}((s, k_p), 1_B).\eta_{\operatorname{Ker} p} = 1_{\operatorname{Ker} p}$.

To summarise $U\epsilon_{(E,B,p,s)} = U\epsilon_{p,s}$ is given by

$$\begin{tikzcd}
B\flat \operatorname{Ker} p \arrow[rr, "\operatorname{Ker}((s,\ k_p),\ 1_B)"] \arrow[dr, "\kappa_{B,\kappa_p}"'] & & \operatorname{Ker} p \arrow[dl, "\kappa_p"] \\
& 0 &
\end{tikzcd}$$

in the first component and

$$B \xrightarrow{1_B} B$$

in the second component.

Monadicity

We recall that when $\mathscr{C}$ is a category with pullbacks and finite coproducts, we have our adjunction (F, U, η, ϵ), the induced monad $(\flat, \eta, U\epsilon_F)$, and the corresponding Eilenberg-Moore adjunction $(F^\flat, U^\flat, \eta^\flat, \epsilon^\flat)$. The comparison functor K satisfies $U^\flat K = U$ and $KF = F^\flat$:

$$\begin{tikzcd}
\mathbf{SplEp}(\mathscr{C}) \arrow[rr, "K"] \arrow[d, shift right, "U"'] & & (\mathscr{C}/0 \times \mathscr{C})^\flat \arrow[dll, shift left, "U^\flat"] \\
\mathscr{C} \arrow[u, shift right, "F"'] \arrow[urr, shift left, "F^\flat"] & &
\end{tikzcd}$$

We want to find conditions under which K is an equivalence of categories.

We begin by recalling the following standard definition.

Definition 2.1.5. Let $U : \mathscr{D} \to \mathscr{E}$ be a functor. We say that U is *monadic* if U has a left adjoint F and the comparison functor $K : \mathscr{D} \to \mathscr{E}^{UF}$ is an equivalence of categories.

Thus we are looking for conditions under which

$$U : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

is guaranteed to be monadic. There are several "monadicity" theorems that give sufficient conditions for a functor to be monadic, for example, see [2]. We will use the "Crude Monadicity Theorem". Before stating it, we need the following standard definition.

Definition 2.1.6 (Reflexive pair)**.** Let $\mathscr{E}$ be a category. A *reflexive pair* of $\mathscr{E}$ is a parallel pair

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

of morphisms of $\mathscr{E}$ with a common section $s : B \to A$. That is, $fs = 1_B = gs$.

Theorem 2.1.7 (Crude Monadicity Theorem (Beck))**.** *A functor $U : \mathscr{D} \to \mathscr{E}$ is monadic if*

(i) *U has a left adjoint;*

(ii) *$\mathscr{D}$ has, and U preserves coequalisers of reflexive pairs;*

(iii) *U reflects isomorphisms.*

Proof. See [2]. □

Basically, $\mathscr{D}$ having reflexive coequalisers allows us to construct a left adjoint L to K. Having U preserve reflexive coequalisers makes the unit of the adjunction $\eta : 1 \implies KL$ an isomorphism. If, in addition, U reflects isomorphisms, then the counit $\epsilon : LK \to 1_{\mathscr{D}}$ is an isomorphism.

We will be especially interested in having an explicit construction for the left adjoint L. As such, we now consider the part of the proof of the Crude Monadicity Theorem involving the construction of the left adjoint L. Let $(F \dashv U : \mathscr{D} \to \mathscr{E}, \eta, \epsilon)$ be an adjunction, and let $(T = UF : \mathscr{E} \to \mathscr{E}, \mu = U\epsilon_F, \eta)$ be the induced monad. Let $(X, x) \in \mathscr{E}^T$. Then since $Fx.F\eta_X = F(x.\eta_X) = 1_{FX}$, and $\epsilon_{FX}.F\eta_X = 1_{FX}$, the pair

$$FUFX \underset{\epsilon_{FX}}{\overset{Fx}{\rightrightarrows}} FX$$

is a reflexive pair. By hypothesis, there is a coequaliser

$$FUFX \underset{\epsilon_{FX}}{\overset{Fx}{\rightrightarrows}} FX \xrightarrow{l(X,\,x)} L(X, x).$$

This defines the left adjoint L.

It will be useful later on to know that under suitable assumptions each component of the counit of the adjunction

$$\begin{tikzcd} \mathbf{SplEp}(\mathscr{C}) \arrow[d, shift left, "U"] \\ \mathscr{C}/0 \times \mathscr{C}, \arrow[u, shift left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

is a coequaliser. First we need the following lemma.

Lemma 2.1.8. *Let $(T : \mathscr{E} \to \mathscr{E}, \eta, \mu)$ be a monad. Let X be an object of $\mathscr{E}$, and let $TX \xrightarrow{x} X$ be an algebra for T. Then*

$$T^2X \underset{\mu_X}{\overset{Tx}{\rightrightarrows}} TX \xrightarrow{x} X \tag{2.2}$$

is a coequaliser.

Proof. The algebra for a monad axioms ensure that (2.2) is a cofork. Let

$$T^2X \underset{\mu_X}{\overset{Tx}{\rightrightarrows}} TX \xrightarrow{y} Y \tag{2.3}$$

be a cofork. For any morphism $f : X \to Y$ such that $y = fx$, we have

$$f = f.x.\eta_X = f.x.\eta_X = y.\eta_X,$$

so that there can be at most one such f. On the other hand, existence follows from the naturality of η, and the monad axioms:

$$y.\eta_X.x = y.Tx.\eta_{TX} = y.\mu_X.\eta_{TX} = y.$$

□

Now we can prove the following.

Proposition 2.1.9. *Let $(F \dashv U : \mathscr{D} \to \mathscr{E}, \eta, \epsilon)$ be an adjunction. If U is monadic, then, for every $Y \in \mathscr{D}$, the component $\epsilon_Y : FUY \to Y$ of the counit ϵ of the adjunction is a coequaliser.*

Proof. Suppose that U is monadic. Let $(T = UF : \mathscr{E} \to \mathscr{E}, \mu = U\epsilon_F, \eta)$ be the induced monad. Let $Y \in \mathscr{D}$. Note that $(UY, U\epsilon_Y)$ is an algebra for the monad T, and that

$$FUFUY \underset{FU\epsilon_Y}{\overset{\epsilon_{FUY}}{\rightrightarrows}} FUY$$

is a reflexive pair in $\mathscr{D}$. Since U is monadic, there is a coequaliser

$$FUFUY \underset{FU\epsilon_Y}{\overset{\epsilon_{FUY}}{\rightrightarrows}} FUY \xrightarrow{c} C$$

in $\mathscr{D}$. By the naturality of ϵ, the diagram

$$\begin{tikzcd} FUFUY \arrow[r, "\epsilon_{FUY}"] \arrow[d, "FU\epsilon_Y"'] & FUY \arrow[d, "\epsilon_Y"] \\ FUY \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

commutes, making

$$FUFUY \underset{FU\epsilon_Y}{\overset{\epsilon_{FUY}}{\rightrightarrows}} FUY \xrightarrow{\epsilon_Y} Y$$

a cofork. By the universal property of the coequaliser, there exists a unique $\theta : C \to Y$ such that

$$\begin{tikzcd} FUY \arrow[r, "c"] \arrow[dr, "\epsilon_Y"'] & C \arrow[d, "\theta"] \\ & Y \end{tikzcd}$$

commutes. Applying U to this diagram, we obtain the commutative diagram

$$\begin{tikzcd} UFUY \arrow[r, "Uc"] \arrow[dr, "U\epsilon_Y"'] & UC \arrow[d, "U\theta"] \\ & UY \end{tikzcd} \tag{2.4}$$

Since

$$UFUY \xrightarrow{U\epsilon_Y} UY$$

is an algebra for the monad UF, Lemma 2.1.8 tells us that $U\epsilon_Y$ is also a coequaliser of the (reflexive) pair $(\epsilon_{FUY}, FU\epsilon_Y)$. Since U preserves coequalisers of reflexive pairs, the morphism $U\theta$ is the unique morphism from the coequaliser UC that makes the diagram (2.4) commute. It follows that $U\theta$ is an isomorphism. Since U reflects isomorphisms, the morphism θ is an isomorphism, making

$$FUY \xrightarrow{\epsilon_Y} Y$$

a coequaliser. $\square$

Protomodularity

To recapitulate, so far we have assumed that $\mathscr{C}$ is a category with pullbacks and finite coproducts. We want to find conditions on $\mathscr{C}$ that make the functor

$$U = \left(\begin{smallmatrix}\mathrm{Ker}\\ \mathrm{Cod}\end{smallmatrix}\right) : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

satisfy the conditions of the Crude Monadicity Theorem, and thus make U a monadic functor, meaning that

$$K : \mathbf{SplEp}(\mathscr{C}) \to (\mathscr{C}/0 \times \mathscr{C})^\flat$$

is an equivalence of categories. We already know that U has a left adjoint. We need U to reflect isomorphisms. This section will address this need.

For each $f : A \to B$ in $\mathscr{C}$, there is a functor

$$f^* : \mathbf{SplEp}_B(\mathscr{C}) \to \mathbf{SplEp}_A(\mathscr{C}),$$

defined on objects by the following procedure. If p is a split epimorphism with section s, then take the pullback

$$\begin{tikzcd} Q \arrow[r, "r"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

The cone $(1_A, sf)$ induces a section of q. Functoriality follows from the universal property of the pullback.

We claim that U reflects isomorphisms if $\mathscr{C}$ satisfies the following definition.

Definition 2.1.10 (Protomodular category)**.** Let $\mathscr{C}$ be a category with pullbacks. We say that $\mathscr{C}$ is *protomodular* if for every morphism $f : A \to B$ of $\mathscr{C}$, the functor $f^* : \mathbf{SplEp}_B(\mathscr{C}) \to \mathbf{SplEp}_A(\mathscr{C})$ reflects isomorphisms.

If $\mathscr{C}$ has an initial object and pullbacks, then, for every object $X \in \mathscr{C}$, there is a unique morphism $!_X : 0 \to X$, which gives rise to a functor

$$!_X^* : \mathbf{SplEp}_B(\mathscr{C}) \to \mathbf{SplEp}_0(\mathscr{C}).$$

In fact, there is an equivalent definition of protomodularity in such categories.

Proposition 2.1.11. *Let $\mathscr{C}$ be a category with pullbacks and an initial object* 0. *The following statements are equivalent.*

1. *$\mathscr{C}$ is protomodular*

2. *For every $X \in \mathscr{C}$, the functor*

$$!_X{}^* : \mathbf{SplEp}_X(\mathscr{C}) \to \mathbf{SplEp}_0(\mathscr{C})$$

reflects isomorophisms.

Proof. If $\mathscr{C}$ is protomodular, then for every morphism f of $\mathscr{C}$, the functor f^* reflects isomorphisms; therefore, in particular, this holds for every $!_X : 0 \to X$.

On the other hand, assuming (2), let $f : A \to B$ be a morphism of $\mathscr{C}$. Note that the following diagram commutes (up to isomorphism).

$$\begin{tikzcd}
\mathbf{SplEp}_B(\mathscr{C}) \arrow[r, "f^*"] \arrow[dr, "!_B^*"'] & \mathbf{SplEp}_A(\mathscr{C}) \arrow[d, "!_A^*"] \\
& \mathbf{SplEp}_0(\mathscr{C})
\end{tikzcd}$$

Let $g : (E, B, p, s) \to (E', B, p', s')$ be a morphism of $\mathbf{SplEp}_B(\mathscr{C})$ for which $f^*(g)$ is an isomorphism. Then $!_A^*(f^*(g)) = !_B^*(g)$ is an isomorphism, and, since $!_B^*$ reflects isomorphisms, g is an isomorphism. Therefore f^* reflects isomorphisms, and $\mathscr{C}$ is protomodular. $\square$

Theorem 2.1.12. *If $\mathscr{C}$ is protomodular, then the functor*

$$U = \left(\begin{smallmatrix}\mathrm{Ker} \\ \mathrm{Cod}\end{smallmatrix}\right) : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

reflects isomorphisms.

Proof. Let $\mathscr{C}$ be a protomodular category, and let

$$\begin{tikzcd}
E \arrow[r, shift left, "p"] \arrow[d, "h"'] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\
E' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

be a morphism of split epimorphisms such that $U(h, g) = (\mathrm{Ker}(h, g), g)$ is an isomorphism in $\mathscr{C}/0 \times \mathscr{C}$. Then, in particular, g is an isomorphism in $\mathscr{C}$, and so gp is a split epimorphism with section sg^{-1}. Consider the diagram

$$\begin{tikzcd}
\mathrm{Ker}\, p \arrow[r, "k_p"] \arrow[d, "\kappa_p"] \arrow[dd, bend right=60, "\kappa_p"'] & E \arrow[d, "p"'] \arrow[dd, bend left=60, "gp"] \\
0 \arrow[r, "!_B"'] \arrow[d, "1_0"] & B \arrow[d, "g"'] \\
0 \arrow[r, "!_{B'}"'] & B'
\end{tikzcd}$$

Since the top and bottom squares are both pullbacks, the pullback lemma says that the outer rectangle is a pullback. Therefore $\mathrm{Ker}\, gp = \mathrm{Ker}\, p$; and we can factorise the diagram

$$\begin{tikzcd}
& \mathrm{Ker}\, p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, "\mathrm{Ker}(h,\, g)" near start] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p" near end] \arrow[dd, "h"'] & & B \arrow[dd, "g" near end] & \\
& \mathrm{Ker}\, p' \arrow[rr, "\kappa_{p'}" near start] \arrow[dl, "k_{p'}"'] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' &
\end{tikzcd}$$

as

$$
\begin{tikzcd}
 & \operatorname{Ker} p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, "1_{\operatorname{Ker} p}" near start] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p" near end, crossing over] \arrow[dd, "1_E"'] & & B & \\
 & \operatorname{Ker} p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, "\operatorname{Ker}(h,\, g)" near start] & & 0 \arrow[dl, "!_{B'}"] \arrow[dd, "1_0"] \\
E \arrow[rr, "gp" near end, crossing over] \arrow[dd, "h"'] & & B' \arrow[from=uu, "g" near start, crossing over] & \\
 & \operatorname{Ker} p' \arrow[rr, "\kappa_{p'}"] \arrow[dl, "k_{p'}"'] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' \arrow[from=uu, "1_{B'}" near start, crossing over] & .
\end{tikzcd}
$$

Thus

$$\operatorname{Ker}(h, g) = !^*_{B'}((h, 1_{B'})).$$

Now $\operatorname{Ker}(h, g)$ is assumed to be an isomorphism, so, since $\mathscr{C}$ is protomodular, $(h, 1_{B'})$ is an isomorphism. Thus h is an isomorphism, making (h, g) an isomorphism, and showing that U reflects isomorphisms. □

SplEp($\mathscr{C}$) has coequalisers of reflexive pairs

From now on, we assume that the category $\mathscr{C}$ is protomodular in addition to having pullbacks and finite coproducts. Since the assumption of protomodularity tells us that U reflects isomorphisms, in order to make U monadic it will suffice to find conditions on $\mathscr{C}$ under which **SplEp**($\mathscr{C}$) has coequalisers of reflexive pairs and U preserves these.

First we want to find conditions under which $\mathscr{C}$ has coequalisers of reflexive pairs. Now in a regular category $\mathscr{E}$, given a reflexive pair

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right=40, "u"']
\end{tikzcd}
$$

in $\mathscr{E}$, the morphism

$$X \xrightarrow{\binom{f}{g}} Y \times Y$$

can be factorised as

$$
\begin{tikzcd}
X \arrow[rr, "\binom{f}{g}"] \arrow[dr, two heads, "e"'] & & Y \times Y \\
 & I \arrow[ur, tail, "\binom{f'}{g'}"'] &
\end{tikzcd},
$$

with e a regular epimorphism and

$$I \rightarrowtail^{\binom{f'}{g'}} Y \times Y$$

a monomorphism, and thus a relation on Y. Indeed,

$$\begin{tikzcd} I \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & Y \arrow[l, bend right, "eu"'] \end{tikzcd}$$

is a reflexive relation. Since e is an epimorphism, a morphism $d : Y \to D$ satisfies $df' = dg'$ if and only if it satisfies $df = df'e = dg'e = dg$. Thus to give a coequaliser of the pair (f, g) is to give a coequaliser of the pair (f', g'). It follows that if $\mathscr{C}$ is a regular category, then, to show that $\mathscr{C}$ has coequalisers of reflexive pairs, it is enough to show that $\mathscr{C}$ has coequalisers of reflexive relations.

From now on, we let $\mathscr{C}$ be a regular category in addition to having finite coproducts and being protomodular. If we assume that $\mathscr{C}$ is also Barr-exact, then our task of showing that $\mathscr{C}$ has coequalisers of reflexive pairs reduces even more.

Proposition 2.1.13. *Let $\mathscr{C}$ be a protomodular, Barr-exact category. Then every reflexive relation in $\mathscr{C}$ is a kernel pair.*

Proof. A Mal'tsev category is defined to be a category in which every reflexive relation is an equivalence relation. Every protomodular category is Mal'tsev: see Proposition 17 of [7]. Since $\mathscr{C}$ is protomodular, every reflexive relation in $\mathscr{C}$ is an equivalence relation. Since $\mathscr{C}$ is Barr-exact, every equivalence relation is a kernel pair. It follows that every reflexive relation is a kernel pair. □

We are now in a position to prove the following proposition.

Proposition 2.1.14. *Let $\mathscr{C}$ be a Barr-exact, protomodular category. Then $\mathscr{C}$ has coequalisers of reflexive pairs.*

Proof. By the discussion at the beginning of section, it suffices to show that $\mathscr{C}$ has coequalisers of reflexive relations. Let

$$\begin{tikzcd} R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right, "u"'] \end{tikzcd}$$

be a reflexive relation in $\mathscr{C}$. Then, by Proposition 2.1.13, there is a morphism $r : Y \to C$ of $\mathscr{C}$ such that (f, g) is the kernel pair of r. Since $\mathscr{C}$ is a regular category, $r = mq$, with $q : Y \to M$ a regular epimorphism, and $m : M \to C$ a monomorphism. Now since m is a monomorphism, and

$$mqf = rf = rg = mqg,$$

we have $qf = qg$ so that

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, two heads, "q"] \\ Y \arrow[r, two heads, "q"'] & M \end{tikzcd}$$

commutes. In fact it is pullback. To see why, let

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "w"'] & Y \arrow[d, two heads, "q"] \\ Y \arrow[r, two heads, "q"'] & M \end{tikzcd}$$

be a commutative square, and consider the solid part of the following diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed, "v"] \arrow[drr, bend left, "z"] \arrow[ddr, bend right, "w"'] & & & \\
& R \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, two heads, "q"] \arrow[ddr, bend left, "r"] & \\
& Y \arrow[r, two heads, "q"'] \arrow[drr, bend right, "r"'] & M \arrow[dr, tail, "m"] & \\
& & & C.
\end{tikzcd}$$

Since $qw = qz$, we must have $rw = mqw = mqz = rz$. So, since (f, g) is the kernel pair of r, there is a unique morphism $v : Z \to R$ such that $fv = w$ and $gv = z$. This makes (f, g) the kernel pair of the regular epimorphism q; therefore, since $\mathscr{C}$ is a regular category, q is a coequaliser of f and g. Thus every reflexive relation in $\mathscr{C}$ has a coequaliser. It follows that every reflexive pair in $\mathscr{C}$ has a coequaliser. $\square$

We have just seen that if $\mathscr{C}$ is Barr-exact and protomodular, then $\mathscr{C}$ has coequalisers of reflexive pairs. We claim that the same is true for **SplEp**($\mathscr{C}$).

Theorem 2.1.15. *Let $\mathscr{C}$ be a Barr-exact, protomodular category. Then* **SplEp**($\mathscr{C}$) *has coequalisers of reflexive pairs.*

Proof. There is a category $\mathscr{R}$ with two objects X and Y, and three non-identity morphisms $\alpha : X \to Y$, $\beta : Y \to X$, and $\beta\alpha : X \to X$, while $\alpha\beta = 1_Y$. Clearly, **SplEp**($\mathscr{C}$) is the functor category $[\mathscr{R}, \mathscr{C}]$. As such, **SplEp**($\mathscr{C}$) has all the colimits (and limits) that $\mathscr{C}$ has, and these are constructed pointwise [21]. By Theorem 2.1.15, $\mathscr{C}$ has coequalisers of reflexive pairs; it follows that **SplEp**($\mathscr{C}$) has coequalisers of reflexive pairs. $\square$

From now on, we assume that $\mathscr{C}$ is protomodular, Barr-exact, and has finite coproducts. Theorem 2.1.15 ensures that **SplEp**($\mathscr{C}$) has coequalisers of reflexive pairs. At this point, it is possible to describe the left adjoint L. From the discussion following the Crude Monadicity Theorem 2.1.7, we know that if $(F \dashv U : \mathscr{D} \to \mathscr{E}, \eta, \epsilon)$ is an adjunction and if $x : UFX \to X$ is an algebra for the induced monad $UF : \mathscr{E} \to \mathscr{E}$, then there is a coequaliser

$$\begin{tikzcd}
FUFX \arrow[r, shift left, "Fx"] \arrow[r, shift right, "\epsilon_{FX}"'] & FX \arrow[r, "l(X,\, x)"] & L(X, x).
\end{tikzcd}$$

Interpreting this for our adjunction $(F \dashv U : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}, \eta, \epsilon)$, we see that for each action $(X \xrightarrow{x} 0, \xi, A)$ in $\mathbf{Act}(\mathscr{C}) = (\mathscr{C}/0 \times \mathscr{C})^\flat$, there is a coequaliser

$$\begin{tikzcd}
A + A\flat X \arrow[r, shift left, "1+\operatorname{Ker}\epsilon_{F(A,x)}"] \arrow[r, shift right, "(i_A,\ \kappa_{A,x})"'] \arrow[d, shift right, "1+\kappa_{A,x}"'] & A + X \arrow[r, "(\alpha,\ \beta)"] \arrow[d, shift right, "1+x"'] & A \rtimes X \arrow[d, shift right, "\delta"'] \\
A \arrow[r, shift left, "1_A"] \arrow[r, shift right, "1_A"'] \arrow[u, shift right, "i_A"'] & A \arrow[r, "1_A"'] \arrow[u, shift right, "i_A"'] & A. \arrow[u, shift right, "\alpha"']
\end{tikzcd} \tag{2.5}$$

We let $L(X \xrightarrow{x} 0, \xi, A)$ be the object

$$\begin{tikzcd}
A \rtimes X \arrow[r, shift left, "\delta"] & A. \arrow[l, shift left, "\alpha"]
\end{tikzcd}$$

of **SplEp**($\mathscr{C}$). Following Bourn and Janelidze [6], who treated the pointed case, we will refer to $A \rtimes X$ as the *semi-direct product* of A and B. Note that this generalised semi-direct product is an abstract definition, and that, although it agrees with classical notions of semi-direct product in concrete cases such as **Grp**, it is not, in general, an algebra structure on the product of underlying sets.

U preserves coequalisers of reflexive pairs

We have seen that if $\mathscr{C}$ is a Barr-exact, protomodular category, then **SplEp**($\mathscr{C}$) has coequalisers of reflexive pairs. We want to show that U preserves these. Since U is a right adjoint functor, it preserves all limits; in particular, it preserves kernel pairs. Clearly, if U preserves coequalisers of reflexive pairs, then it preserves regular epimorphisms; but we will see that, in fact, the converse is also true.

Suppose that U also preserves regular epimorphisms. Given a coequaliser $q : Y \to C$ of a reflexive pair $(f, g) : X \to Y$ in **SplEp**($\mathscr{C}$), we can take the image factorisation

$$\begin{tikzcd} X \arrow[rr, "\binom{f}{g}"] \arrow[dr, two heads, "e"'] & & Y \times Y \\ & I \arrow[ur, tail, "\binom{f'}{g'}"'] & \end{tikzcd},$$

and (f', g') will be the kernel pair of q. Then Uq is a regular epimorphism with kernel pair (Uf', Ug'). Thus it is the coequaliser of (Uf', Ug'). If there is a morphism $h : UY \to Z$ such that $h.Uf = h.Ug$, then, since Ue is a regular epimorphism, and thus an epimorphism, $h.Uf'.Ue = h.Uf = h.Ug = h.Ug'.Ue$ implies that $h.Uf' = h.Ug'$. Hence there is a unique morphism $t : UC \to Z$ such that $h = t.Uq$, making Uq the coequaliser of (Uf, Ug). We see, then, that if U preserves regular epimorphisms, it preserves coequalisers of reflexive pairs. It follows that, to show that U preserves coequalisers of reflexive pairs, we only need to show that U preserves regular epimorphisms.

Since a morphism is a regular epimorphism if and only if it is the coequaliser of its kernel pair, and since both of these are formed pointwise in **SplEp**($\mathscr{C}$), a regular epimorphism of **SplEp**($\mathscr{C}$) is a morphism (f, g) of **SplEp**($\mathscr{C}$) with both f and g regular epimorphisms of $\mathscr{C}$. Let

$$\begin{tikzcd} E \arrow[r, shift left, "p"] \arrow[d, two heads, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, two heads, "g"] \\ E' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"] \end{tikzcd}$$

be a regular epimomorphism of **SplEp**($\mathscr{C}$). Then we have the following commutative diagram

$$\begin{tikzcd} & \operatorname{Ker} p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, "\operatorname{Ker}(f,\, g)" near start] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\ E \arrow[rr, "p" near start, crossing over] \arrow[dd, "f"'] & & B & \\ & \operatorname{Ker} p' \arrow[rr, "\kappa_{p'}" near start] \arrow[dl, "k_{p'}"'] & & 0 \arrow[dl, "!'_B"] \\ E' \arrow[rr, "p'"'] & & B' \arrow[from=uu, crossing over, "g" near end] & \end{tikzcd}$$

where the top and bottom faces of the cube are pullback squares. The following lemma will allow us to factorise it, which will help us to show that U preserves regular epimorphisms.

Lemma 2.1.16. *In any category with pullbacks, a diagram of the form*

$$\begin{tikzcd}
& K \arrow[dl, "k"'] \arrow[rr, "\kappa"] \arrow[dd, "x"'] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p"] \arrow[dd, "f"'] & & B \arrow[dd, "g"] & \\
& K' \arrow[dl, "k'"'] \arrow[rr, "\kappa'"] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' &
\end{tikzcd}$$

can be factorised as

$$\begin{tikzcd}
& K \arrow[dl, "k"'] \arrow[rr, "\kappa"] \arrow[dd, "x"'] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p"] \arrow[dd, "t"'] & & B \arrow[dd, "1_B"] & \\
& K' \arrow[dl, "u"] \arrow[rr, "\kappa'"] \arrow[dd, "1_{K'}"'] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
Q \arrow[rr, "r"] \arrow[dd, "q"'] & & B \arrow[dd, "g"] & \\
& K' \arrow[dl, "k'"'] \arrow[rr, "\kappa'"] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' & .
\end{tikzcd} \tag{2.6}$$

in which all three horizontal squares, the lower front face, and the upper left face are pullbacks.

Proof. We form the pullback

$$\begin{tikzcd}
Q \arrow[r, "r"] \arrow[d, "q"'] & B \arrow[d, "g"] \\
E' \arrow[r, "p'"'] & B'.
\end{tikzcd}$$

Then we have the commutative diagram

$$\begin{tikzcd}
& K' \arrow[rr, "\kappa'"] \arrow[dl, dashed, "u"'] \arrow[dd, "1_{K'}" near start] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
Q \arrow[rr, "r" near end, crossing over] \arrow[dd, "q"'] & & B & \\
& K' \arrow[rr, "\kappa'" near start] \arrow[dl, "k'"'] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B' \arrow[from=uu, crossing over, "g" near start] & ,
\end{tikzcd}$$

where u is the unique morphism induced by the cone $(k', !_B\kappa')$.

The back face is also a pullback since the vertical arrows are identities. Since the right and left squares of the diagram

$$\begin{tikzcd}
K' \arrow[r, "1_{K'}"] \arrow[d, "\kappa'"'] & K' \arrow[r, "k'"] \arrow[d, "\kappa'"] & E' \arrow[d, "p'"] \\
0 \arrow[r, "1_0"'] & 0 \arrow[r, "!_{B'}"'] & B'
\end{tikzcd}$$

are pullbacks, the pullback lemma says that the outer rectangle is a pullback. Then both the outer rectangle and the right hand square of the diagram

$$\begin{tikzcd}
K' \arrow[r, "u"] \arrow[rr, bend left, "k'"] \arrow[d, "\kappa'"'] & Q \arrow[r, "q"] \arrow[d, "r"] & E' \arrow[d, "p'"] \\
0 \arrow[r, "!_B"] \arrow[rr, bend right, "!_{B'}"'] & B \arrow[r, "g"] & B'
\end{tikzcd}$$

are pullbacks. Since the left hand square commutes, the pullback lemma says it is also a pullback. This is the middle horizontal square.

Since (f, g) is a morphism in $\mathbf{SplEp}(\mathscr{C})$, the pair of morphisms (f, p) is a cone over (p', g), and thus there is a unique $t : E \to Q$ such that $rt = p$ and $qt = f$. Now

$$qtk = fk = k'x = qux = qux,$$

and

$$rtk = pk = !_B\kappa = !_B\kappa' x = rux.$$

Since q and r are jointly monic, we have $tk = ux$, and thus the diagram (2.6) in the statement of the lemma commutes.

Consider the diagram

$$\begin{tikzcd}
K \arrow[r, "x"] \arrow[rr, bend left, "\kappa"] \arrow[d, "k"'] & K' \arrow[r, "\kappa'"] \arrow[d, "u"] & 0 \arrow[d, "!_B"] \\
E \arrow[r, "t"] \arrow[rr, bend right, "p"'] & Q \arrow[r, "r"] & B.
\end{tikzcd}$$

The large rectangle and the right square are pullbacks, and the left square commutes; therefore the pullback lemma says that the left square is a pullback. This is the upper left face. □

In fact, the following proposition tells us that the morphism t of the previous lemma is, in fact, a regular epimorphism.

Proposition 2.1.17. *Let $\mathscr{C}$ be a protomodular, Barr-exact category. Then, given a diagram in $\mathscr{C}$ of the form*

$$\begin{tikzcd} E \arrow[r, shift left, "p"] \arrow[d, two heads, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, two heads, "g"] \\ E' \arrow[r, shift left, "p'"] & B', \arrow[l, shift left, "s'"] \end{tikzcd}$$

where (p, s) and (p', s') are split epimorphisms, and f and g are regular epimorphisms, the comparison to the pullback is a regular epimorphism; that is, the unique morphism t making

$$\begin{tikzcd} E \arrow[dr, dashed, "t"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "f"'] & & \\ & P \arrow[r, "r"] \arrow[d, "q"'] & B \arrow[d, two heads, "g"] \\ & E' \arrow[r, "p'"'] & B' \end{tikzcd}$$

commute is a regular morphism.

Proof. First we claim that

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\ E' \arrow[r, "p'"'] & B' \end{tikzcd}$$

is a pushout. Let $bp = af$. If $hp' = a$, then $h = hp's' = as'$, showing uniqueness. On the other hand,

$$as'g = afs = bps = b$$

and

$$as'p'f = as'gp = bp = af.$$

Since f is a regular epimorphism, $as'p' = a$, giving existence. Thus the square

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\ E' \arrow[r, "p'"'] & B' \end{tikzcd}$$

is a pushout, and, since split epimorphisms are regular epimorphisms, all four morphisms are regular epimorphisms. From Proposition 17 of [7], every protomodular category is Mal'tsev. From Theorem 5.7 of [9], given a pushout of strong epimorphisms in a Barr-exact, Mal'tsev category, the comparison to the pullback is a strong epimorphism. But, in a regular category, strong epimorphisms are exactly the same as regular epimorphisms. Therefore, since $\mathscr{C}$ is assumed to be protomodular and Barr-exact, t is a regular epimorphism. □

We can now prove the following.

Theorem 2.1.18. *If $\mathscr{C}$ is protomodular and Barr-exact, then the functor*

$$U = \left(\begin{smallmatrix}\mathrm{Ker}\\ \mathrm{Cod}\end{smallmatrix}\right) : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

preserves coequalisers of reflexive pairs.

Proof. From the discussion at the beginning of this section, we know that it is sufficient to show that U preserves regular epimorphisms. Let

$$\begin{tikzcd}
E \arrow[r, shift left, "p"] \arrow[d, two heads, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, two heads, "g"] \\
E' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"]
\end{tikzcd}$$

be a regular epimorphism in $\mathbf{SplEp}(\mathscr{C})$. We apply the factorisation of Lemma 2.1.16 to the cube

$$\begin{tikzcd}
& \mathrm{Ker}\,p \arrow[rr, "\kappa_p"] \arrow[dl, "k_p"'] \arrow[dd, "\mathrm{Ker}(f,\,g)"' near start] & & 0 \arrow[dl, "!_B"] \arrow[dd, "1_0"] \\
E \arrow[rr, "p" near end] \arrow[dd, "f"'] & & B \arrow[dd, "g" near end] & \\
& \mathrm{Ker}\,p' \arrow[rr, "\kappa_{p'}" near start] \arrow[dl, "k_{p'}"'] & & 0 \arrow[dl, "!'_B"] \\
E' \arrow[rr, "p'"'] & & B'. &
\end{tikzcd}$$

Lemma 2.1.16 tells us that

$$\begin{tikzcd}
\mathrm{Ker}\,p \arrow[r, "k_p"] \arrow[d, "\mathrm{Ker}(f,\,g)"'] & E \arrow[d, "t"] \\
\mathrm{Ker}\,p' \arrow[r, "u"'] & Q
\end{tikzcd}$$

(the upper left face of the resulting prism) is a pullback. Proposition 2.1.17 tells us that the morphism t, being the comparison to the pullback, is a regular epimorphism. But $\mathscr{C}$ is a regular category, so that regular epimorphisms are stable under pullback in $\mathscr{C}$. Hence $\mathrm{Ker}(f, g)$ is a regular epimorphism of $\mathscr{C}$, and $U(f, g) = (\mathrm{Ker}(f, g), g)$ is a regular epimorphism of $\mathbf{SplEp}(\mathscr{C})$. It follows that U preserves regular epimorphisms. We conclude that U preserves coequalisers of reflexive pairs. □

K is an equivalence of categories

Finally, putting all the pieces together, we can prove the key result.

Theorem 2.1.19. *If $\mathscr{C}$ is protomodular, Barr-exact, and has finite coproducts, then the functor*

$$U = \left(\begin{smallmatrix}\mathrm{Ker}\\ \mathrm{Cod}\end{smallmatrix}\right) : \mathbf{SplEp}(\mathscr{C}) \to \mathscr{C}/0 \times \mathscr{C}$$

is monadic.

Proof. This is a direct result of the Crude Monadicity Theorem 2.1.7 together with Theorems 2.1.15, 2.1.12, and 2.1.18. □

In other words, the comparison functor $K : \mathbf{SplEp}(\mathscr{C}) \to (\mathscr{C}/0 \times \mathscr{C})^{\flat} = \mathbf{Act}(\mathscr{C})$ is an equivalence of categories. We recall, from the discussion following Theorem 2.1.15, that the left adjoint L sends

$$(X \xrightarrow{x} 0, A, \xi) \mapsto \quad A \rtimes X \underset{\alpha}{\overset{\delta}{\rightleftarrows}} A.$$

While K sends

$$E \underset{s}{\overset{p}{\rightleftarrows}} B \quad \mapsto (\operatorname{Ker} p \xrightarrow{\kappa_p} 0, B, \operatorname{Ker}((s, k_p), 1_B)).$$

2.2 Reflexive graphs and pre-crossed modules

Having generalised the equivalence between the categories of split epimorphisms in **Grp** and actions in **Grp** so that the equivalence holds for any Barr exact, protomodular category $\mathscr{C}$ with finite coproducts, we will now do the same for the equivalence between the categories of reflexive graphs and pre-crossed modules.

From split epimorphisms to reflexive graphs

Let $\mathscr{C}$ be a protomodular, Barr-exact category with finite coproducts. We let $\mathbf{RefGph}(\mathscr{C})$ denote the category of reflexive graphs in $\mathscr{C}$. Let

$$E \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a split epimorphism in $\mathscr{C}$. From Proposition 2.1.9, we know that

$$FUFU(p, s) \underset{FU\epsilon_{p,s}}{\overset{\epsilon_{FU(p,s)}}{\rightrightarrows}} FU(p, s) \xrightarrow{\epsilon_{p,s}} (p, s)$$

is a coequlaiser. Explicitly, this means that

$$\begin{tikzcd}
B + B\flat \operatorname{Ker} p \arrow[rrr, shift left, "1_B + \kappa_{B,\kappa_p}"] \arrow[d, shift right, swap, "(i_B,\ k_{B,\kappa_p})"] \arrow[d, shift left, "1_B + \operatorname{Ker} \epsilon_{p,s}"] & & & B \arrow[lll, shift left, "i_B"] \arrow[d, shift right, swap, "1_B"] \arrow[d, shift left, "1_B"] \\
B + \operatorname{Ker} p \arrow[rrr, shift left, "1_B + \kappa_p"] \arrow[d, swap, "(s,\ k_p)"] & & & B \arrow[lll, shift left, "i_B"] \arrow[d, "1_B"] \\
E \arrow[rrr, shift left, "p"] & & & B \arrow[lll, shift left, "s"]
\end{tikzcd}$$

is a coequaliser in $\mathbf{SplEp}(\mathscr{C})$. In particular,

$$B + B\flat \operatorname{Ker} p \underset{(i_B,\ k_{B,\kappa_p})}{\overset{1_B + \operatorname{Ker} \epsilon_{p,s}}{\rightrightarrows}} B + \operatorname{Ker} p \xrightarrow{(s,\ k_p)} E$$

is a coequaliser in $\mathscr{C}$. Now to give a reflexive graph in $\mathscr{C}$ is to give a split epimorphism

$$E \underset{s}{\overset{p}{\rightleftarrows}} B$$

along with another morphism $E \xrightarrow{q} B$ such that $qs = 1_B$. By the universal property of the coequaliser, to give a morphism $q : E \to B$ is, equivalently, to give a morphism

$$(f, g) = q(s, k_p) = (qs, qk_p) : B + K \to B$$

that satisfies

$$(f, g)(i_B, k_{B,\kappa_p}) = (f, g)(1_B + \mathrm{Ker}\,\epsilon_{p,s}). \tag{2.7}$$

Equation (2.7) is equivalent to

$$(f, (f, g)k_{B,\kappa_p}) = (f, g\,\mathrm{Ker}\,\epsilon_{p,s}),$$

which holds if and only if

$$(f, g)k_{B,\kappa_p} = g\,\mathrm{Ker}\,\epsilon_{p,s}.$$

For s to be a section of q, we also require that

$$f = (f, g)i_B = (qs, qk_p)i_B = qs = 1_B.$$

Thus a reflexive graph in $\mathscr{C}$ is basically a split epimorphism

$$\begin{tikzcd} E \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

along with a morphism $g : \mathrm{Ker}\,p \to B$ such that

$$g\,\mathrm{Ker}\,\epsilon_{p,s} = (1_B, g)k_{B,\kappa_p}.$$

Internal pre-crossed modules

In light of the equivalence $\mathbf{SplEp}(\mathscr{C}) \simeq \mathbf{Act}(\mathscr{C})$, the discussion above shows that to give a reflexive graph

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[l] \end{tikzcd}$$

is to give an internal action $(X \xrightarrow{x} 0, A, \xi)$ along with a morphism g satisfying

$$g\xi = (1_A, g)k_{A,x}.$$

This motivates the following definition

Definition 2.2.1 (Internal pre-crossed module, $\mathbf{PXMod}(\mathscr{C})$)**.** Let $\mathscr{C}$ be a Barr exact, protomodular category with finite coproducts. An *internal pre-crossed module* in $\mathscr{C}$ is a quadruple

$$(A, X \xrightarrow{x} 0, \xi, g),$$

where

$$(A, X \xrightarrow{x} 0, \xi)$$

is an internal action (an object of $(\mathscr{C}/0 \times \mathscr{C})^\flat$) and $g : X \to A$ is a morphism of $\mathscr{C}$ such that the diagram

$$\begin{tikzcd} A\flat X \arrow[r, "k_{A,x}"] \arrow[d, "\xi"'] & A + X \arrow[d, "{(1_A,\, g)}"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

commutes. A morphism of internal pre-crossed modules is a morphism (r, s) of actions such that $sg' = gr$. This defines the category of internal pre-crossed modules in $\mathscr{C}$, which we denote by $\mathbf{PXMod}(\mathscr{C})$.

Building on the equivalence $\mathbf{SplEp}(\mathscr{C}) \simeq \mathbf{Act}(\mathscr{C})$, we have the following theorem.

Theorem 2.2.2. *Let $\mathscr{C}$ be a Barr exact, protomodular category with finite coproducts. The category of reflexive graphs in $\mathscr{C}$ is equivalent to the category of internal pre-crossed modules of $\mathscr{C}$.*

Proof. The above discussion shows that everything works at the level of objects. A reflexive graph (C_1, C_0, d, i, c) is sent to

$$(\operatorname{Ker} d \xrightarrow{\kappa_d} 0, C_0, \operatorname{Ker} \epsilon_{d,i}, ck_d) = (K(C_1, C_0, d, i), ck_d).$$

Let

$$(t, s) : (K(C_1, C_0, d, i), ck_d) \to (K(C_1', C_0', d', i'), c'k_d')$$

be a morphism of internal pre-crossed modules. Explicitly, $t : \operatorname{Ker} d \to \operatorname{Ker} d'$ and $s : C_0 \to {C_0}'$ are morphisms making

$$\begin{tikzcd} C_0 \flat \operatorname{Ker} d \arrow[r, "s\flat t"] \arrow[d, "\operatorname{Ker} \epsilon_{d,i}"'] & C_0 \flat \operatorname{Ker} d' \arrow[d, "\operatorname{Ker} \epsilon_{d'i'}"] \\ \operatorname{Ker} d \arrow[r, "t"'] & \operatorname{Ker} d' \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{Ker} d \arrow[r, "t"] \arrow[d, "ck_d"'] & \operatorname{Ker} d' \arrow[d, "c'k_{d'}"] \\ C_0 \arrow[r, "s"'] & {C_0}' \end{tikzcd}$$

commute. Since (t, s) is a morphism of actions, there is exactly one morphism

$$(f, h) : (C_1, C_0, d, i) \to (C_1', C_0', d', i')$$

in $\mathbf{SplEp}(\mathscr{C})$ such that $K(f, h) = (\operatorname{Ker}(f, h), h) = (t, s)$. We claim that this morphism $(f, h) = (f, s)$ of split epimorphisms is also a morphism of reflexive graphs. We only need to show that $c'f = sc$. We have, by definition of $t = \operatorname{Ker}(f, s)$, from the facts that (t, s) is a morphism of pre-crossed modules and that (f, s) is a morphism of split epimorphisms,

$$c' f k_d = c' k_{d'} t = sck_d,$$

while

$$c' f i = c' i' s = s = sci.$$

From Proposition 2.1.9, we know that $(i, k_d) = \epsilon_{d,i}$ is a coequaliser, and thus an epimorphism, so the fact that

$$c' f(i, k_d) = (c' f i, c' f i k_d) = (sci, sck_d) = sc(i, k_d)$$

implies that $c' f = sc$ as required. □

2.3 Internal categories and crossed modules

In the last section, we were able to build on the equivalence $\mathbf{SplEp}(\mathscr{C}) \simeq \mathbf{Act}(\mathscr{C})$ and obtain the equivalence between the categories of reflexive graphs and internal pre-crossed modules in $\mathscr{C}$. We will now build on this equivalence to produce an equivalence between internal categories and (a suitable definition of) crossed modules that generalises the equivalence in the category $\mathbf{Grp}$.

From reflexive graphs to internal categories

What is an internal category in a Barr-exact, protomodular category with finite coproducts? Given a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0 \arrow[l, bend right=60, "i"'] \end{tikzcd}$$

in $\mathscr{C}$, we can form the pullback

$$\begin{tikzcd} C_2 \arrow[r, "q"] \arrow[d, "p"'] & C_1 \arrow[d, "d"] \\ C_1 \arrow[r, "c"'] & C_0, \end{tikzcd}$$

giving C_2, the object of composable pairs. The cone $(1_{C_1}, ic)$ induces a $u : C_1 \to C_2$, which is the unique morphism making

$$\begin{tikzcd} C_1 \arrow[dr, dashed, "u"] \arrow[drr, bend left, "ic"] \arrow[ddr, bend right, "1_{C_1}"'] & & \\ & C_2 \arrow[r, "q"] \arrow[d, "p"'] & C_1 \arrow[d, "d"] \\ & C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

commute. Similarly, there is a unique morphism $v : C_1 \to C_2$ making

$$\begin{tikzcd} C_1 \arrow[dr, dashed, "v"] \arrow[drr, bend left, "1_{C_1}"] \arrow[ddr, bend right, "id"'] & & \\ & C_2 \arrow[r, "q"] \arrow[d, "p"'] & C_1 \arrow[d, "d"] \\ & C_1 \arrow[r, "c"'] & C_0 \end{tikzcd}$$

commute. We can also form the iterated pullback

$$\begin{tikzcd} C_3 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[r, "q"] \arrow[d, "p"] & C_1 \arrow[d, "d"] \\ C_2 \arrow[r, "q"'] \arrow[d, "p"'] & C_1 \arrow[r, "c"'] \arrow[d, "d"] & C_0 \\ C_1 \arrow[r, "c"'] & C_1 & , \end{tikzcd}$$

giving C_3, the object of composable triples. Now a reflexive graph

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & C_0 \arrow[l, bend right=60, "i"'] \end{tikzcd}$$

in $\mathscr{C}$ gives rise to an internal category precisely when there is a morphism $m : C_2 \to C_1$ satisfying the following:

(M.1) $mu = 1_{C_1} = mv$ (guaranteeing that the identities for composition function as such);

(M.2) $dm = dp$ and $cm = cq$ (guaranteeing that the domain and codomain of composites are as they should be);

(M.3) and $m\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right) = m\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right)$ (guaranteeing that composition is associative).

We will show first that given a reflexive graph there is at most one such composition making it into an internal category. Then we will characterise categories internal to $\mathscr{C}$ by giving the necessary and sufficient conditions for the existence of a composition. In order to do this, we want to be able to speak of $\operatorname{Ker} p$ in terms of $\operatorname{Ker} d$. The following lemma shows that we can.

Lemma 2.3.1. *Let $\mathscr{C}$ be a category with pullbacks and an initial object 0. Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"'] & D \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "\kappa"'] & C \arrow[d, "i"] \\ 0 \arrow[r, "!_D"'] & D \end{tikzcd}$$

be pullbacks. Then

$$\begin{tikzcd} K \arrow[r, "\left(\begin{smallmatrix} !_B\kappa \\ k \end{smallmatrix}\right)"] \arrow[d, "\kappa"'] & A \arrow[d, "f"] \\ 0 \arrow[r, "!_B"'] & B \end{tikzcd}$$

is a pullback.

Proof. Consider the diagram

$$\begin{tikzcd} K \arrow[r, dashed, "t"] \arrow[d, "\kappa"'] \arrow[rr, bend left, "k"] & A \arrow[r, "g"] \arrow[d, "f"] & C \arrow[d, "i"] \\ 0 \arrow[r, "!_B"] \arrow[rr, bend right, "!_D"'] & B \arrow[r, "h"] & D, \end{tikzcd}$$

where the large rectangle and the right hand square are pullbacks. Note that, by the universal property of the pullback, $t = \left(\begin{smallmatrix} !_B\kappa \\ k \end{smallmatrix}\right)$ is the unique morphism such that $f\left(\begin{smallmatrix} !_B\kappa \\ k \end{smallmatrix}\right) = !_B\kappa$ and $g\left(\begin{smallmatrix} !_B\kappa \\ k \end{smallmatrix}\right) = k$. Hence the left hand square commutes; therefore the pullback lemma tells us that the left hand square is a pullback. ◻

Uniqueness of the composition

We will now show that given a reflexive graph there is at most one composition m making it into an internal category. Applying Lemma 2.3.1 to the pullbacks

$$
\begin{tikzcd}
\operatorname{Ker} d \arrow[r, "k_d"] \arrow[d, "\kappa_d"'] & C_1 \arrow[d, "d"] \\
0 \arrow[r, "!_{C_0}"'] & C_0,
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C_2 \arrow[r, "q"] \arrow[d, "p"'] & C_1 \arrow[d, "d"] \\
C_1 \arrow[r, "c"'] & C_0,
\end{tikzcd}
$$

shows that the kernel of p is given by

$$
\begin{tikzcd}
\operatorname{Ker} d \arrow[r, "\begin{pmatrix} !_{C_1}\kappa_d \\ k_d \end{pmatrix}"] \arrow[d, "\kappa_d"'] & C_2 \arrow[d, "p"] \\
0 \arrow[r, "!_{C_1}"'] & C_1.
\end{tikzcd}
$$

We let $t = \begin{pmatrix} !_{C_1}\kappa_d \\ k_d \end{pmatrix}$. Since p is a split epimorphism with $pu = 1_{C_1}$, we know from Proposition 2.1.9 that

$$
\begin{tikzcd}[column sep=huge]
C_1 + C_1 \flat \operatorname{Ker} p \arrow[r, shift left, "1_{C_1}+\kappa_{C_1\kappa_p}"] \arrow[d, shift right, "(i_{C_1},\, k_{C_1,\kappa_p})"'] \arrow[d, shift left, "1_{C_1}+\operatorname{Ker}\epsilon_{p,u}"] & C_1 \arrow[l, shift left, "i_{C_1}"] \arrow[d, shift right, "1_{C_1}"'] \arrow[d, shift left, "1_{C_1}"] \\
C_1 + \operatorname{Ker} p \arrow[r, shift left, "1_{C_1}+\kappa_p"] \arrow[d, "(u,\, k_p)"'] & C_1 \arrow[l, shift left, "i_{C_1}"] \arrow[d, "1_{C_1}"] \\
C_2 \arrow[r, shift left, "p"] & C_1 \arrow[l, shift left, "u"]
\end{tikzcd}
$$

is a coequaliser in $\mathbf{SplEp}(\mathscr{C})$. In terms of the morphism $d : C_1 \to C_0$, this can be written as

$$
\begin{tikzcd}[column sep=huge]
C_1 + C_1 \flat \operatorname{Ker} d \arrow[r, shift left, "1_{C_1}+\kappa_{C_1\kappa_d}"] \arrow[d, shift right, "(i_{C_1},\, k_{C_1,\kappa_d})"'] \arrow[d, shift left, "1_{C_1}+\operatorname{Ker}\epsilon_{p,u}"] & C_1 \arrow[l, shift left, "i_{C_1}"] \arrow[d, shift right, "1_{C_1}"'] \arrow[d, shift left, "1_{C_1}"] \\
C_1 + \operatorname{Ker} d \arrow[r, shift left, "1_{C_1}+\kappa_d"] \arrow[d, "(u,\, t)"'] & C_1 \arrow[l, shift left, "i_{C_1}"] \arrow[d, "1_{C_1}"] \\
C_2 \arrow[r, shift left, "p"] & C_1 \arrow[l, shift left, "u"]
\end{tikzcd}
$$

In particular,

$$
\begin{tikzcd}[column sep=huge]
C_1 + C_1 \flat \operatorname{Ker} d \arrow[r, shift left, "1_{C_1}+\operatorname{Ker}\epsilon_{p,u}"] \arrow[r, shift right, "(i_{C_1},\, k_{C_1,\kappa_d})"'] & C_1 + \operatorname{Ker} d \arrow[r, "(u,\, t)"] & C_2
\end{tikzcd}
$$

is a coequaliser in $\mathscr{C}$.

By the universal property of the coequaliser, to give a morphism $m : C_2 \to C_1$ is, equivalently, to give a morphism

$$(a, b) = m(u, t) = (mu, mt) : C_1 + \operatorname{Ker} d \to C_1$$

that satisfies

$$(a, b)(i_{C_1}, k_{C_1,\kappa_d}) = (a, b)(1_{C_1} + \operatorname{Ker} \epsilon_{p,u}). \tag{2.8}$$

That is, to give $m : C_2 \to C_1$ is to give $(a, b) : C_1 + \mathrm{Ker}\, d \to C_1$ such that

$$(a, (a, b)k_{C_1,\kappa_d}) = (a, b\,\mathrm{Ker}\,\epsilon_{p,u})$$

or, equivalently,

$$(a, b)k_{C_1,\kappa_d} = b\,\mathrm{Ker}\,\epsilon_{p,u}.$$

For m to be a composition for an internal category, we also require that $mu = 1_{C_1} = mv$. Thus we require that $(a, b) = m(u, t) = (mu, mt) = (1_{C_1}, mt)$. Now observe that

$$pvk_d = idk_d = i!_{C_0}\kappa_d = !_{C_1}\kappa_d = pt$$

and

$$qvk_d = k_d = qt.$$

Since p and q are jointly monic, we have $t = vk_d$. So, assuming that $mv = 1_{C_1}$, we have

$$b = mt = mvk_d = k_d.$$

This shows that in order to give an internal category, the morphism (a, b) that defines m must satisfy $(a, b) = m(u, t) = (1_{C_1}, k_d)$. Therefore there can be at most one composition m.

Existence of a composition

We have just seen that, for m to be a composition, we need $m(u, t) = (1_{C_1}, k_d)$. The argument at the beginning of section 2.2 shows that, given the split epimorphism

$$\begin{tikzcd} C_2 \arrow[r, shift left, "p"] & C_1, \arrow[l, shift left, "u"] \end{tikzcd}$$

to give $m : C_2 \to C_1$ such that $mu = 1_{C_1}$ is to give $(1_{C_1}, b) = (mu, mt)$ such that

$$k_d\,\mathrm{Ker}\,\epsilon_{p,u} = (1_{C_1}, b)k_{C_1,\kappa_d}.$$

It follows that, for there to be a morphism $m : C_2 \to C_1$, satisfying the conditions (M.1), (M.2), and (M.3), the diagram

$$\begin{tikzcd} C_1 \flat\, \mathrm{Ker}\, d \arrow[r, "k_{C_1,\kappa_d}"] \arrow[d, "\mathrm{Ker}\,\epsilon_{p,u}"'] & C_1 + \mathrm{Ker}\, d \arrow[d, "(1_{C_1},\, k_d)"] \\ \mathrm{Ker}\, d \arrow[r, "k_d"'] & C_1 \end{tikzcd} \tag{2.9}$$

must commute.

In fact, we claim that such an m exists if and only if (2.9) commutes. Suppose that (2.9) commutes. Then the solid part of the following diagram commutes

$$\begin{tikzcd} C_1 + C_1 \flat\, \mathrm{Ker}\, d \arrow[r, shift left, "(i_{C_1},\, k_{C_1,\kappa_d})"] \arrow[r, shift right, "1+\mathrm{Ker}\,\epsilon_{p,u}"'] & C_1 + \mathrm{Ker}\, d \arrow[r, "(u,\, t)"] \arrow[dr, "(1_{C_1},\, k_d)"'] & C_2 \arrow[d, dashed, "m"] \\ & & C_1. \end{tikzcd}$$

The universal property of the coequaliser gives a unique $m : C_2 \to C_1$ satisfying $m(u, t) = (1_{C_1}, k_d)$. Thus $mu = m(u, t)i_{C_1} = (1_{C_1}, k_d)i_{C_1} = 1_{C_1}$. To show that $mv = 1_{C_1}$, note that $pui = i = idi = pvi$ and $qui = ici = i = qvi$. Since p and q are jointly monic, we have $ui = vi$. Then

$$mv(i, k_d) = (mvi, mvk_d) = (mui, mt) = (i, m(u, t)i_{\mathrm{Ker}\, d}) = (i, (1_{C_1}, k_d)i_{\mathrm{Ker}\, d}) = (i, k_d).$$

By Proposition 2.1.9, $(i, k_d) = \epsilon_{d,i}$ is a coequaliser, and thus an epimorphism, so $mv = 1_{C_1}$. This shows that (M.1) holds.

Now we show that (M.2) holds; that is, that $dm = dp$ and $cm = cq$. By Proposition 2.1.9, $(u, t) = \epsilon_{p,u}$ is an epimorphism. Therefore the equality

$$dm(u, t) = d(1_{C_1}, k_d) = (d, dk_d) = (d, !_{C_0}\kappa_d) = (dpu, d!_{C_1}\kappa_d) = (dpu, dpt) = dp(u, t)$$

implies that $dm = dp$. Similarly,

$$cm(u, t) = c(1_{C_1}, k_d) = c(ic, qt) = c(qu, qt) = cq(u, t)$$

implies that $cm = cq$, showing that (M.2) holds.

It remains to show that (M.3) holds; that is, that the composition is associative. We need to show that

$$m\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right) = m\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right).$$

Let $\sigma : C_3 \to C_2$ be the unique morphism making

$$\begin{tikzcd}
C_2 \arrow[dr, dashed, "\sigma"] \arrow[drr, bend left, "uq"] \arrow[ddr, bend right, "1_{C_2}"'] & & \\
& C_3 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "p"] \\
& C_2 \arrow[r, "q"'] & C_1
\end{tikzcd}$$

commute. Then

$$\begin{tikzcd}
C_3 \arrow[r, shift left, "\pi_1"] & C_2 \arrow[l, shift left, "\sigma"]
\end{tikzcd}$$

is a split epimorphism. Let $r : \mathrm{Ker}\, d \to C_3$ be the unique morphism making

$$\begin{tikzcd}
\mathrm{Ker}\, d \arrow[r, dashed, "r"] \arrow[rr, bend left, "t"] \arrow[d, "\kappa_d"'] & C_3 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & C_2 \arrow[d, "p"] \\
0 \arrow[r, "!_{C_2}"'] \arrow[rr, bend right, "!_{C_1}"'] & C_2 \arrow[r, "q"] & C_1
\end{tikzcd}$$

commute. Then, by the pullback lemma, the left hand square is a pullback, and thus a kernel of π_1. Now

$$p\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right)(\sigma, r) = p\pi_1(\sigma, r) = p(\pi_1\sigma, \pi_1 r) = (p, p!_{C_2}\kappa_d) = (p, !_{C_1}\kappa_d) = p(1_{C_2}, t)$$

and

$$q\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right)(\sigma, r) = m\pi_2(\sigma, r) = (muq, mt) = (q, k_d) = (q, qt) = q(1_{C_2}, t).$$

Since p and q are jointly monic, we have

$$\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right)(\sigma, r) = (1_{C_2}, t).$$

Thus

$$m\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right)(\sigma, r) = m(1_{C_2}, t) = (m, k_d). \tag{2.10}$$

On the other hand,

$$p\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right)(\sigma, r) = m\pi_1(\sigma, r) = m(\pi_1\sigma, \pi_1 r) = (m, m!_{C_2}\kappa_d) = (m, !_{C_1}\kappa_d) = p(um, t)$$

and

$$q\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right)(\sigma, r) = q\pi_2(\sigma, r) = (quq, qt) = (icq, qt) = (icm, qt) = q(um, t).$$

Therefore

$$\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right)(\sigma, r) = (um, t);$$

so that

$$m\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right)(\sigma, r) = m(um, t) = (m, k_d). \tag{2.11}$$

From (2.10) and (2.11), and using the fact that $(\sigma, r) = \epsilon_{\pi_1,\sigma}$ is an epimorphism, we have

$$m\left(\begin{smallmatrix} p\pi_1 \\ m\pi_2 \end{smallmatrix}\right) = m\left(\begin{smallmatrix} m\pi_1 \\ q\pi_2 \end{smallmatrix}\right),$$

showing that (M.3) holds.

We have shown that m is a composition if and only if (2.9) commutes. But since t is the kernel of p and $(u, t) = \epsilon_{p,u}$,

$$k_d \operatorname{Ker} \epsilon_{p,u} = qt \operatorname{Ker} \epsilon_{p,u} = q(u, t)\kappa_{C_1,\kappa_d} = (ic, k_d)\kappa_{C_1,\kappa_d}.$$

So a reflexive graph (C_1, C_0, d, c, i) gives rise to an internal category if and only if

$$\begin{tikzcd} C_1 \flat \operatorname{Ker} d \arrow[r, "k_{C_1,\kappa_d}"] \arrow[d, "\kappa_{C_1,\kappa_d}"'] & C_1 + \operatorname{Ker} d \arrow[d, "(1_{C_1},\ k_d)"] \\ C_1 + \operatorname{Ker} d \arrow[r, "(ic,\ k_d)"'] & C_1 \end{tikzcd} \tag{2.12}$$

commutes.

From internal pre-crossed modules to internal categories

Now let $(X \xrightarrow{x} 0, A, \xi, g)$ be an internal pre-crossed module. Then

$$\begin{tikzcd} A \rtimes X \arrow[r, bend left, "\delta"] \arrow[r, bend right, "\gamma"'] & A \arrow[l, "\alpha"'] \end{tikzcd}$$

is a reflexive graph, where $((\alpha, \beta), 1_A) : (A + X, A, 1_A + x, i_A) \to (A \rtimes X, A, \delta, \alpha)$ is the coequaliser defined in (2.5), the morphism γ is defined by $\gamma(\alpha, \beta) = (1_A, g)$, and

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "x"'] & A \rtimes X \arrow[d, "\delta"] \\ 0 \arrow[r, "!_A"'] & A \end{tikzcd}$$

is a pullback, and thus a kernel of δ. Making the appropriate substitutions in the diagram (2.12) gives

$$\begin{tikzcd}
(A \rtimes X)\flat X \arrow[r, "k_{(A\rtimes X),x}"] \arrow[d, "k_{(A\rtimes X),x}"'] & (A \rtimes X) + X \arrow[d, "(1_{A\rtimes X},\, \beta)"] \\
(A \rtimes X) + X \arrow[r, "(\alpha\gamma,\, \beta)"'] & A \rtimes X.
\end{tikzcd} \tag{2.13}$$

Now consider the following diagram, which we have borrowed from Janelidze [16].

$$\begin{tikzcd}
(A + X)\flat X \arrow[rrr, "k_{(A+X),x}"] \arrow[ddd, "k_{(A+X),x}"'] \arrow[dr, "(\alpha,\, \beta)\flat 1_X"] & & & (A + X) + X \arrow[dl, "(\alpha,\, \beta)+1_X"'] \arrow[ddd, "(\alpha,\, \beta)(1_A+\nabla_X)"] \\
& (A \rtimes X)\flat X \arrow[r, "k_{(A\rtimes X),x}"] \arrow[d, "k_{(A\rtimes X),x}"'] & (A \rtimes X) + X \arrow[d, "(1_{A\rtimes X},\, \beta)"] & \\
& (A \rtimes X) + X \arrow[r, "(\alpha\gamma,\, \beta)"'] & A \rtimes X \arrow[dr, "1_{A\rtimes X}"] & \\
(A + X) + X \arrow[ur, "(\alpha,\, \beta)+1_X"] \arrow[rrr, "(\alpha,\, \beta)((1_A,\, \gamma\beta)+1_X)"'] & & & A \rtimes X
\end{tikzcd} \tag{2.14}$$

We claim that each of the four quadrialterals surrounding the original square commutes. Since for all morphisms f and h of $\mathscr{C}$ we have $f\flat h = \operatorname{Ker} F(f, h)$, the top quadrilateral and the left quadrilateral both commute by virtue of the definition of the functors Ker and F. The right quadrilateral commutes since

$$(1_{A\rtimes X}, \beta)((\alpha, \beta) + 1_X) = ((\alpha, \beta), \beta) = (\alpha, \beta)(1_A + \nabla_X).$$

The bottom quadrilateral commutes since

$$(\alpha\gamma, \beta)((\alpha, \beta) + 1_X) = (\alpha\gamma(\alpha, \beta), \beta) = (\alpha(1_A, \gamma\beta), \beta) = (\alpha, \beta)((1_A, \gamma\beta) + 1_X).$$

We claim that the inner square commutes if and only if the outer rectangle does. On the one hand, if the inner square commutes, then

$$\begin{aligned}
(\alpha, \beta)(1_A + \nabla_X)k_{(A+X),x} &= (1_{A\rtimes X}, \beta)((\alpha, \beta) + 1_X)k_{(A+X),x} \\
&= (1_{A\rtimes X}, \beta)k_{(A\rtimes X),x}((\alpha, \beta)\flat 1_X) \\
&= (\alpha\gamma, \beta)k_{(A\rtimes X),x}((\alpha, \beta)\flat 1_X) \\
&= (\alpha\gamma, \beta)((\alpha, \beta) + 1_X)k_{(A+X),x} \\
&= (\alpha, \beta)((1_A, \gamma\beta) + 1_X)k_{(A+X),x}.
\end{aligned}$$

On the other hand, from the same chain of equations, we see that if the outer rectangle commutes, then

$$(1_{A\rtimes X}, \beta)k_{(A\rtimes X),x}((\alpha, \beta)\flat 1_X) = (\alpha\gamma, \beta)k_{(A\rtimes X),x}((\alpha, \beta)\flat 1_X).$$

But since (α, β) is a coequaliser, it is a regular epimorphism, and so is $F((\alpha, \beta), 1_X)$. Since U preserves regular epimorphisms, $(\alpha, \beta)\flat 1_X = \operatorname{Ker} F((\alpha, \beta), 1_X)$ is a regular epimorphism. Hence

$$(1_{A\rtimes X}, \beta)k_{(A\rtimes X),x} = (\alpha\gamma, \beta)k_{(A\rtimes X),x}$$

and the inner square commutes. So given an internal pre-crossed module $(X \xrightarrow{x} 0, A, \xi, g)$, we know that the corresponding reflexive graph

$$\begin{tikzcd} A \rtimes X \arrow[r, bend left, "\delta"] \arrow[r, bend right, "\gamma"'] & A \arrow[l, "\alpha"'] \end{tikzcd}$$

is part of an internal category if and only if

$$(\alpha,\ \beta)(1_A + \nabla_X)k_{(A+X),x} = (\alpha,\ \beta)((1_A, g) + 1_X)k_{(A+X),x} \tag{2.15}$$

(recalling that $g = \gamma\beta$); that is, if and only if the diagram

$$\begin{tikzcd} (A+X)\flat X \arrow[r, "k_{(A+X),x}"] \arrow[d, "k_{(A+X),x}"'] & A+X+X \arrow[r, "1_A+\nabla_X"] & A+X \arrow[d, "{(\alpha,\ \beta)}"] \\ A+X+X \arrow[r, "{(1_A,\ g)+1_X}"'] & A+X \arrow[r, "{(\alpha,\ \beta)}"'] & A \rtimes X \end{tikzcd}$$

commutes.

Internal crossed modules

We can “simplify" condition (2.15) further. Notice that the following diagram commutes

$$\begin{tikzcd} (A+X)\flat X \arrow[r, "k_{(A+X),x}"] \arrow[d, "\kappa_{(A+X),x}"'] & A+X+X \arrow[r, "1_A+\nabla_X"] \arrow[d, "1_{A+X}+x"] & A+X \arrow[d, "1_A+x"] \\ 0 \arrow[r, "!_{A+X}"] \arrow[rr, bend right, "!_A"'] & A+X \arrow[r, "1_A+x"] & A, \end{tikzcd}$$

recalling that the left hand square is a (kernel) pullback. The commutativity of the outer rectangle, and the universal property of the pullback ensure that there exists a unique v making

$$\begin{tikzcd} (A+X)\flat X \arrow[dr, dashed, "v"] \arrow[drr, bend left, "(1_A+\nabla_X)k_{(A+X),x}"] \arrow[ddr, bend right, "\kappa_{(A+X),x}"'] & & \\ & A\flat X \arrow[r, "k_{A,x}"] \arrow[d, "\kappa_{A,x}"'] & A+X \arrow[d, "1_A+x"] \\ & 0 \arrow[r, "!_A"'] & A \end{tikzcd} \tag{2.16}$$

commute. Also, since

$$\begin{tikzcd} A + A\flat X \arrow[r, shift left, "1_A+\xi"] \arrow[r, shift right, "{(i_A,\ k_{A,x})}"'] & A+X \arrow[r, "{(\alpha,\ \beta)}"] & A \rtimes X \end{tikzcd}$$

is a coequaliser, we have $(\alpha, \beta)k_{A,x} = \beta\xi$. Thus we have the commutative diagram

$$\begin{tikzcd}
(A+X)\flat X \arrow[r, "v"] \arrow[d, "k_{(A+X),x}"'] \arrow[rr, bend left, "\kappa_{(A+X),x}", to path={-- (\tikztotarget)}] & A\flat X \arrow[r, "\xi"] \arrow[d, "k_{A,x}"] \arrow[ur, "\kappa_{A,x}"] & X \arrow[d, "\beta"] \arrow[u, "x"] \\
(A+X)+X \arrow[r, "(1_A+\nabla_X)"'] & A+X \arrow[r, "(\alpha,\ \beta)"'] & A \rtimes X,
\end{tikzcd}$$

giving

$$(\alpha, \beta)(1_A + \nabla_X)k_{(A+X),x} = \beta\xi v. \tag{2.17}$$

Since $k : \operatorname{Ker} F \implies F$ is a natural transformation, we also have the commutative diagram

$$\begin{tikzcd}
 & & 0 \\
(A+X)\flat X \arrow[r, "(1_A,\ g)\flat 1_X"] \arrow[d, "k_{(A+X),x}"'] \arrow[urr, bend left, "\kappa_{(A+X),x}"] & A\flat X \arrow[r, "\xi"] \arrow[d, "k_{A,x}"] \arrow[ur, "\kappa_{A,x}"] & X \arrow[d, "\beta"] \arrow[u, "x"'] \\
(A+X)+X \arrow[r, "(1_A+g)+1_X"'] & A+X \arrow[r, "(\alpha,\ \beta)"'] & A \rtimes X,
\end{tikzcd}$$

giving

$$(\alpha, \beta)((1_A + g) + 1_X)k_{(A+X),x} = \beta\xi((1_A, g)\flat 1_X). \tag{2.18}$$

Equations (2.15), (2.17), and (2.18) together give

$$\beta\xi v = \beta\xi((1_A, g)\flat 1_X). \tag{2.19}$$

Since (x, β) is a jointly monic pair, and we also have

$$x\xi v = \kappa_{A,x}v = \kappa_{(A+X),x} = \kappa_{A,x}((1_A, g)\flat 1_X) = x\xi((1_A, g)\flat 1_X),$$

we can conclude that the reflexive graph

$$\begin{tikzcd}
A \rtimes X \arrow[r, bend left, "\delta"] \arrow[r, bend right, "\gamma"'] & A, \arrow[l, "\alpha"']
\end{tikzcd}$$

corresponding to the pre-crossed module $(X \xrightarrow{x} 0, A, \xi, g)$, is part of an internal category if and only if the diagram (2.20) below commutes. That is, if and only if

$$\xi v = \xi((1_A, g)\flat 1_X).$$

With this in mind, we give the following defintion.

Definition 2.3.2 (Internal crossed module, $\mathbf{XMod}(\mathscr{C})$)**.** Let $\mathscr{C}$ be a protomodular, Barr-exact category with finite coproducts. An *internal crossed module* of $\mathscr{C}$ is defined to be an internal pre-crossed module

$$(X \xrightarrow{x} 0, A, \xi, g)$$

such that the diagram

$$\begin{tikzcd} (A+X)\flat X \arrow[r, "(1_A,\, g)\flat 1_X"] \arrow[d, "v"'] & A\flat X \arrow[d, "\xi"] \\ A\flat X \arrow[r, "\xi"'] & X \end{tikzcd} \tag{2.20}$$

commutes, where v is defined as in (2.16). A morphism of internal crossed modules is a morphism of internal pre-crossed modules. This determines a category, which we denote by $\mathbf{XMod}(\mathscr{C})$.

Theorem 2.3.3. *Let $\mathscr{C}$ be a protomodular, Barr-exact category with finite coproducts. The category $\mathbf{XMod}(\mathscr{C})$ of internal crossed modules in $\mathscr{C}$ is equivalent to the category of internal categories in $\mathscr{C}$.*

Proof. The above discussion shows that everything works at the level of objects. An internal category (C_1, C_0, d, i, c, m) is sent to the crossed module $(\operatorname{Ker} d \xrightarrow{k_d} 0, C_0, \operatorname{Ker} \epsilon_{d,i}, ck_d) = (K(C_1, C_0, d, i), ck_d)$. Let $(r, s) : (K(C_1, C_0, d, i), ck_d) \to (K(C_1', C_0', d', i'), c'k_d')$ be a morphism of internal crossed modules. Explicitly, $r : \operatorname{Ker} d \to \operatorname{Ker} d'$ and $s : C_0 \to C_0{}'$ are morphisms making the following two diagrams commute.

$$\begin{tikzcd} C_0\flat \operatorname{Ker} d \arrow[r, "s\flat r"] \arrow[d, "\operatorname{Ker}\epsilon_{d,i}"'] & C_0\flat \operatorname{Ker} d' \arrow[d, "\operatorname{Ker}\epsilon_{d',i'}"] \\ \operatorname{Ker} d \arrow[r, "r"'] & \operatorname{Ker} d' \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Ker} d \arrow[r, "r"] \arrow[d, "ck_d"'] & \operatorname{Ker} d' \arrow[d, "c'k_{d'}"] \\ C_0 \arrow[r, "s"'] & C_0{}' \end{tikzcd}$$

Since (r, s) is a morphism of pre-crossed modules, there is exactly one morphism

$$(f, h) : (C_1, C_0, d, i, c) \to (C_1', C_0', d', i', c')$$

of reflexive graphs corresponding to (r, s) under the equivalence $\mathbf{RefGph}(\mathscr{C}) \simeq \mathbf{PXMod}(\mathscr{C})$. Indeed, $(r, s) = K(f, h) = (\operatorname{Ker}(f, h), h)$. We claim that the morphism $(f, h) = (f, s)$ of reflexive graphs is also a morphism of internal categories. We only need to show that

$$\begin{tikzcd} C_2 \arrow[r, "\binom{fp}{fq}"] \arrow[d, "m"'] & C_2' \arrow[d, "m'"] \\ C_1 \arrow[r, "f"'] & C_1' \end{tikzcd}$$

commutes. Consider the diagram

$$\begin{tikzcd} C_1 + \operatorname{Ker} d \arrow[r, "f+\operatorname{Ker}(f,\, s)"] \arrow[d, "(u,\, t)"'] & C_1' + \operatorname{Ker} d' \arrow[d, "(u',\, t')"] \\ C_2 \arrow[r, "\binom{fp}{qp}"] \arrow[d, "m"'] & C_2' \arrow[d, "m'"] \\ C_1 \arrow[r, "f"'] & C_1' \end{tikzcd} \tag{2.21}$$

If the outer rectangle and the top square commute, then

$$fm(u,t) = m'(u',t')(f + \mathrm{Ker}(f,s)) = m' \begin{pmatrix} fp \\ fq \end{pmatrix} (u,\ t).$$

But since $(u,t) = \epsilon_{p,u}$ is a coequaliser by Proposition 2.1.9, we have $fm = m' \begin{pmatrix} fp \\ fq \end{pmatrix}$, as desired. Therefore it suffices to show that both the outer rectangle and the top square commute.

Recall that $\mathrm{Ker}(f,s)$ is defined to be the unique morphism making the diagram

$$\begin{tikzcd}
& \mathrm{Ker}\, d \arrow[rr, "\kappa_d"] \arrow[dl, "k_d"'] \arrow[dd, "\mathrm{Ker}(f,\ s)"' near start] & & 0 \arrow[dl, "!_{C_0}"] \arrow[dd, "1_0"] \\
C_1 \arrow[rr, "d" near end] \arrow[dd, "f"'] & & C_0 \arrow[dd, "s" near end] & \\
& \mathrm{Ker}\, d' \arrow[rr, "\kappa_{d'}" near start] \arrow[dl, "k_{d'}"'] & & 0 \arrow[dl, "!_{C_0'}"] \\
C_1' \arrow[rr, "d'"'] & & C_0'. &
\end{tikzcd} \tag{2.22}$$

commute. To see that the outer rectangle of (2.21) commutes, observe that

$$\begin{aligned} m'(u',t')(f + \mathrm{Ker}(f,s)) &= (1_{C_1'}, k_{d'})(f + \mathrm{Ker}(f,s)) \\ &= (f, k_{d'} \mathrm{Ker}(f,s)) = (f, fk_d) = f(1_{C_1}, k_d) = fm(u,t). \end{aligned}$$

Now we show that the top square of (2.21) commutes. On the one hand, we have

$$\begin{pmatrix} fp \\ fq \end{pmatrix} (u,t) = \begin{pmatrix} fp \\ fq \end{pmatrix} \begin{pmatrix} 1_{C_1} & !_{C_1}\kappa_d \\ ic & k_d \end{pmatrix} = \begin{pmatrix} f & f!_{C_1}\kappa_d \\ fic & fk_d \end{pmatrix}.$$

On the other hand, we have

$$(u',t')(f + \mathrm{Ker}(f,s)) = \begin{pmatrix} 1_{C_1'} & !_{C_1'}\kappa_{d'} \\ i'c' & k_{d'} \end{pmatrix} (f + \mathrm{Ker}(f,s)) = \begin{pmatrix} f & !_{C_1'}\kappa_{d'}\,\mathrm{Ker}(f,s) \\ i'c'f & k_{d'}\,\mathrm{Ker}(f,s) \end{pmatrix}.$$

We show that

$$\begin{pmatrix} f & f!_{C_1}\kappa_d \\ fic & fk_d \end{pmatrix} = \begin{pmatrix} f & !_{C_1'}\kappa_{d'}\,\mathrm{Ker}(f,s) \\ i'c'f & k_{d'}\,\mathrm{Ker}(f,s) \end{pmatrix}$$

by equating components. First note that, from (2.22), we have

$$!_{C_1'}\kappa_{d'}\,\mathrm{Ker}(f,s) = !_{C_1'}\kappa_d = f!_{C_1}\kappa_d.$$

Next, since (f,s) is a morphism of reflexive graphs, we have $fi = i's$ and $sc = c'f$. It follows that

$$fic = i'sc = i'c'f.$$

Finally, the equation

$$k_{d'}\,\mathrm{Ker}(f,s) = fk_d$$

is immediate from (2.22). It follows that the top square commutes. Therefore the bottom square commutes, showing that (f,s) is a morphism of internal categories. $\square$

Chapter. 3

Crossed modules in the category of Lie-Rinehart algebras

This chapter consists of two parts: in part 1, we will define Lie-Rinehart algebras and show that, for a given field k and commutative algebra A over k, there is a category of Lie-Rinehart algebras that is protomodular, Barr-exact, and has finite coproducts; in part 2, we will give the classical definitions of action, semi-direct product, and crossed module of Lie-Rinehart algebras before comparing the "classical" crossed modules to our definition of internal crossed modules in the category of Lie-Rinehart algebras.

3.1 The category of Lie-Rinehart algebras

Before defining Lie-Rinehart algebras, we need the following definition.

Definition 3.1.1 (Derivation, $\mathrm{Der}(A)$)**.** Let k be a field, and let A be a commutative algebra over k. A k-*derivation* of A is a k-linear endomorphism $D : A \to A$ such that, for all $a, b \in A$, the "product rule"

$$D(ab) = aD(b) + D(a)b$$

holds. We let $\mathrm{Der}(A)$ denote the set of all k-derivations of A.

The set of linear endomorphisms $\mathrm{End}(A)$ is a ring with composition as an operation. It becomes a Lie k-algebra under the bracket

$$[T, T'] = TT' - T'T.$$

Since $\mathrm{Der}(A)$ is a vector subspace of $\mathrm{End}(A)$, and since for all $D, D' \in \mathrm{Der}(A)$, and for all $a, b \in A$, we have

$$\begin{aligned}
(DD' - D'D)(ab) &= D(aD'(b)) + D(D'(a)b) - (D'(aD(b)) + D'(D(a)b)) \\
&= aD(D'(b)) + D(a)D'(b) + D'(a)D(b) + D(D'(a))b \\
&\quad - (aD'(D(b)) + D'(a)D(b) + D(a)D'(b) + D'(D(a))b)) \\
&= a((D(D'(b)) - D'(D(b))) + (D(D'(a)) - D'(D(a)))b \\
&= (a(DD' - D'D)(b) + (DD' - D'D)(a)b,
\end{aligned}$$

the subspace $\mathrm{Der}(A)$ is a Lie subalgebra under the same bracket. With the obvious scalar multiplication $(a, D) \mapsto ((aD) : A \to A : b \mapsto aD(b))$, $\mathrm{Der}(A)$ is also an A-module. In fact, it will be our first example of a Lie-Rinehart algebra.

Lie-Rinehart algebras

Definition 3.1.2 (Lie-Rinehart algebra, [15])**.** Let k be a field, and let A be a commutative algebra over k. A Lie-Rinehart algebra (over k and A) is a Lie k-algebra L that is simultaneously an A-module together with a map $\alpha : L \to \mathrm{Der}(A)$ that is simultaneously a morphism of Lie k-algebras and a morphism of A-modules such that, for all $x, y \in L$ and for all $a \in A$, we have

$$[x, ay] = a[x, y] + \alpha_x(a)y,$$

where α_x is the image of x under α.

A morphism

$$f : (L, \alpha : L \to \mathrm{Der}(A)) \to (L', \alpha' : L' \to \mathrm{Der}(A))$$

of Lie-Rinehart algebras is defined to be a morphism $f : L \to L'$ of Lie k-algebras that is also a morphism of A-modules such that

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & L' \arrow[dl, "\alpha'"] \\ & \mathrm{Der}(A) & \end{tikzcd}$$

commutes. It follows that for a fixed pair (A, k) there is a category $\mathbf{LR}(A, k)$ of Lie-Rinehart algebras. Here we are considering a fixed A and k, so we will usually omit the (A, k) and simply write $\mathbf{LR}$ for the category of Lie-Rinehart algebras over the given A and k.

Some trivial cases are worth mentioning here. On the one hand,

$$\mathrm{Der}(A) \xrightarrow{1_{\mathrm{Der}(A)}} \mathrm{Der}(A)$$

is clearly a Lie-Rinehart algebra; indeed, it is the terminal object of $\mathbf{LR}$. On the other hand, the zero Lie-Rinehart algebra (with underlying set $\{0\}$) is the initial object, which we denote by

$$0 \xrightarrow{0} \mathrm{Der}(A),$$

or simply by 0. If, in the definition of a Lie k-algebra, we replace the field k with a commutative algebra over a field k, then we get a *Lie A-algebra*. If R is a Lie A-algebra, then $R \xrightarrow{0} \mathrm{Der}(A)$ is a Lie-Rinehart algebra. Conversely, a Lie-Rinehart algebra of the form $R \xrightarrow{0} \mathrm{Der}(A)$ is a Lie A-algebra. We will soon see that $\mathbf{LR}$ has pullbacks, so that the kernel functor Ker, defined in Definition 2.1.3, can be defined on $\mathbf{SplEp}(\mathbf{LR})$. For any split epimorphism $p : L \to L'$ of Lie-Rinehart algebras, the kernel of p, defined as the pullback of p along the unique morphism $!_{L'} : 0 \to L'$, is a Lie A-algebra.

LR is Barr-exact and has finite coproducts

Let k be a field, and let A be a commutative algebra over k. There is a category $\mathscr{A}$ that has Lie k-algebras with an A-module structure as objects (with no compatibility required); the morphisms of $\mathscr{A}$ are the Lie algebra homomorphisms that are also A-module homomorphisms. Since an object of $\mathscr{A}$ is merely a set along with some operations of finite arity satisfying certain equations, the category $\mathscr{A}$ is finitarily monadic over set, and so $\mathscr{A}$ is complete, cocomplete, and Barr-exact, see [1].

Since $\mathrm{Der}(A)$ is an object of $\mathscr{A}$, we can form the slice category $\mathscr{A}/\mathrm{Der}(A)$. We claim that $\mathscr{A}/\mathrm{Der}(A)$ is also complete, cocomplete, and Barr-exact. Indeed, we have the following.

Lemma 3.1.3. *Let k be a field, and let A be a commutative algebra over k. Let $\mathscr{A}$ be the category of Lie k-algebras with an A-module structure defined above. Then the following statements hold.*

1. *The category $\mathscr{A}/\mathrm{Der}(A)$ is complete and cocomplete.*

2. *Colimits and connected limits in $\mathscr{A}/\mathrm{Der}(A)$ are computed as in $\mathscr{A}$.*

3. *The category $\mathscr{A}/\mathrm{Der}(A)$ is Barr-exact*

Proof. For (1) and (2), see the proof of Proposition 2.16.3 of [3].

For (3), it follows from (1) and (2) that $\mathscr{A}/\mathrm{Der}(A)$ has all finite limits and coequalisers; in particular, $\mathscr{A}/\mathrm{Der}(A)$ certainly has coequalisers of kernel pairs. Thus, to show that $\mathscr{A}/\mathrm{Der}(A)$ is regular, it is enough to show that regular epimorphisms are stable under pullback. Let q be the pullback of a regular epimorphism (equivalently a coequaliser) p along some f in $\mathscr{A}/\mathrm{Der}(A)$. Then, since pullbacks in $\mathscr{A}/\mathrm{Der}(A)$ are computed as in $\mathscr{A}$, the morphism q is the pullback of p along f in $\mathscr{A}$. Since coequalisers are computed in $\mathscr{A}/\mathrm{Der}(A)$ as in $\mathscr{A}$, the morphism p is a coequaliser in $\mathscr{A}$. Since $\mathscr{A}$ is regular, q, being the pullback of the regular epimorphism p along f in $\mathscr{A}$, is a regular epimorphism (equivalently a coequaliser) in $\mathscr{A}$. It follows that it is a coequaliser in $\mathscr{A}/\mathrm{Der}(A)$, and thus a regular epimorphism $\mathscr{A}/\mathrm{Der}(A)$.

To see that $\mathscr{A}/\mathrm{Der}(A)$ is Barr-exact, let $\binom{k}{l} : X \to Y \times Y$ be an equivalence relation in $\mathscr{A}/\mathrm{Der}(A)$. Then $\binom{k}{l}$ also an equivalence relation in $\mathscr{A}$. Since $\mathscr{A}$ is exact, $\binom{k}{l}$ is the kernel pair of its coequaliser in $\mathscr{A}$. This coequaliser is also a coequaliser in $\mathscr{A}/\mathrm{Der}(A)$. And since pullbacks are computed in the same way in $\mathscr{A}/\mathrm{Der}(A)$ as in $\mathscr{A}$, same, kernel pairs lift. Hence every equivalence relation in $\mathscr{A}$ $\mathrm{Der}(A)$ is s kernel pair in $\mathscr{A}/\mathrm{Der}(A)$, and $\mathscr{A}$ $\mathrm{Der}(A)$ is Barr-exact. □

Now an object $L \xrightarrow{\alpha} \mathrm{Der}(A) \in \mathscr{A}/\mathrm{Der}(A)$ is an object of **LR** precisely when for all $x, x' \in L$, and for all $a \in A$, the equation

$$[x, ax'] = a[x, x'] + \alpha_x(a)x'$$

holds. This equation determines the category **LR** as a full subcategory of $\mathscr{A}$. Since equationally determined full subcategories of locally presentable categories are always reflective, **LR** is reflective, and thus cocomplete. We claim that **LR** is also closed under limits, subobjects, and images. More precisely.

Proposition 3.1.4. *Let k be a field, and let A be a commutative algebra over k. Let $\mathscr{A}$ be the category of Lie k-algebras with an A-module structure defined above. Then the following statements hold.*

1. *The subcategory* **LR** *of* $\mathscr{A}/\mathrm{Der}(A)$ *is closed under limits. That is, if a diagram in* **LR** *has a limit L in* $\mathscr{A}/\mathrm{Der}(A)$*, the limit L is in* **LR**.

2. *The subcategory* **LR** *of* $\mathscr{A}/\mathrm{Der}(A)$ *is closed under subobjects. That is, if L in* **LR** *has a subobject L' in* $\mathscr{A}/\mathrm{Der}(A)$*, then L' is in* **LR**.

3. *The subcategory* **LR** *of* $\mathscr{A}/\mathrm{Der}(A)$ *is closed under images. That is, if L is in* **LR***, and $p : L \to Y$ is a surjective morphism of* $\mathscr{A}/\mathrm{Der}(A)$*, then Y is in* **LR**.

Proof. (1) and (2) follow directly from the fact that an object of **LR** is an object of $\mathscr{A}/\mathrm{Der}(A)$ such that a certain equation holds. If the equation holds individually for all objects of the diagram, then it also must hold for the limit of the diagram. If the equation holds for all elements of a set, it certainly holds for all elements of the subset.

For (3), let $L \xrightarrow{\alpha} \mathrm{Der}(A) \in$ **LR**, and let

$$\begin{tikzcd} L \arrow[rr, "p"] \arrow[dr, "\alpha"'] & & L' \arrow[dl, "\alpha'"] \\ & \mathrm{Der}(A) & \end{tikzcd}$$

be a surjection in $\mathscr{A}/Der(A)$, Let $x', y' \in L'$. Then, since p is surjective, there exist $x, y \in L$ such that $p(x) = x'$ and $p(y) = y'$. Then

$$\begin{aligned}
[x', ay'] = [p(x), ap(y)] = [p(x), p(ay)] = p[x, ay] &= p(a[x, y] + \alpha_x(a)y) \\
&= a[p(x), p(y)] + (\alpha' p)_x(a)p(y) \\
&= a[p(x), p(y)] + \alpha'_{p(x)}(a)p(y) \\
&= a[x', y'] + \alpha'_{x'}(a)y',
\end{aligned}$$

making L' a Lie-Rinehart algebra. □

We now show that **LR** is Barr-exact.

Proposition 3.1.5. *The category* **LR** *is Barr-exact.*

Proof. Since **LR** is closed under limits and $\mathscr{A}/(\mathrm{Der}(A))$ is complete, **LR** is complete. We have already seen that regular epimorphisms are stable under pullback in $\mathscr{A}/\mathrm{Der}(A)$, but **LR** is closed under limits, hence to show that **LR** is regular, it is enough to show that every kernel pair has a coequaliser. Now a kernel pair $\binom{k}{l} : X \to Y \times Y$ in **LR** has a coequaliser $q : Y \to C$ in $\mathscr{A}/\mathrm{Der}(A)$. The fact that C is a (surjective) image of the Lie-Rinehart algebra Y under q together with the fact that **LR** is closed under images make $q : Y \to C$ a coequaliser in **LR**. Hence the category of Lie-Rinehart algebras is a regular category

To see that **LR** is Barr-exact, note that an equivalence relation $\binom{k}{l} : X \to Y \times Y$ in **LR** has a coequaliser $q : Y \to C$ in $\mathscr{A}/\mathrm{Der}(A)$, and that $\binom{k}{l}$ is the kernel pair of this coequaliser q in $\mathscr{A}/\mathrm{Der}(A)$. Again, the fact that **LR** is closed under images makes $q : Y \to C$ a coequaliser in **LR**. Then the fact that **LR** is closed under pullbacks ensures that $\binom{k}{l}$ is the kernel pair of q in **LR**. □

The category of Lie-Rinehart algebras is protomodular

We now know that the category of Lie-Rinehart algebras has finite coproducts, and is Barr-exact. In this section, we will show that **LR** is protomodular. This will allow us to apply Definition 2.3.2 and Theorem 2.3.3 to **LR**.

Lemma 3.1.6. *If $\mathscr{E}$ is a category with pullbacks, $\mathscr{B}$ is a protomodular category, and there is a functor $P : \mathscr{E} \to \mathscr{B}$ that preserves pullbacks and reflects isomorphisms, then $\mathscr{E}$ is protomodular.*

Proof. Let $\mathscr{E}$ be a category with pullbacks, let $\mathscr{B}$ be a protomodular category, and let $P : \mathscr{E} \to \mathscr{B}$ be a functor that preserves pullbacks and reflects isomorphisms. Let $f : A \to B$ be a morphism of $\mathscr{E}$, and let $g : (E, B, p, s) \to (E', B, p', s')$ be a morphism of $\mathbf{SplEp}_B(\mathscr{E})$ for which $!_B^*(g)$ is an isomorphism. Note that $!_B^*(g)$ is the unique morphism making

$$\begin{tikzcd}
\mathrm{Ker}\, p \arrow[r, "k_p"] \arrow[d, dashed, "!_B^*(g)"] \arrow[dd, bend right, "\kappa_p"'] & E \arrow[d, "g"] \arrow[dd, bend left, "p"] \\
\mathrm{Ker}\, p' \arrow[r, "k_p"] \arrow[d, "\kappa_{p'}"] & E' \arrow[d, "p'"] \\
0 \arrow[r, "!_B"] & B
\end{tikzcd}$$

commute. By the pullback lemma, the top square is a pullback. Then applying P to everything gives a pullback in $\mathscr{B}$. The isomorphism $P(!_B^*(g)) = (P(!_B))^*(P(g))$. Since $\mathscr{B}$ is protomodular, $P(g)$ is an isomorphism. Since P reflects isomorphisms, g is an isomorphism. Thus, for every $B \in \mathscr{E}$, the functor $!_B^*$ reflects isomorphisms, by Proposition (2.1.11), we conclude that $\mathscr{E}$ is protomodular. □

Lemma 3.1.7. *The forgetful functor $G : \mathbf{LR} \to \mathbf{Vect}_k$ preserves pullbacks and reflects isomorphims.*

Proof. Let
$$f : (L \xrightarrow{\alpha_L} \mathrm{Der}(A)) \to (N \xrightarrow{\alpha_N} \mathrm{Der}(A))$$
and
$$g : (M \xrightarrow{\alpha_M} \mathrm{Der}(A)) \to (N \xrightarrow{\alpha_N} \mathrm{Der}(A))$$
be morphisms of **LR**. At the level of vector spaces (and modules), the pullback is

$$\begin{tikzcd} L \times_N M \arrow[r, "\pi_M"] \arrow[d, "\pi_L"'] & M \arrow[d, "g"] \\ L \arrow[r, "f"'] & N, \end{tikzcd}$$

where $L \times_N M = \{\binom{l}{m} : l \in L, m \in M, \text{ and } f(l) = g(m)\}$ is a subspace (and submodule) of the direct product, defined with componentwise addition and scalar multiplication, and π_L and π_M are the projections. The bracket $[\binom{l}{m}, \binom{l'}{m'}] = \binom{[l,l']}{[m,m']}$ makes $L \times_N M$ a Lie algebra, and the morphism $\alpha_N f \pi_L$ makes $L \times_N M$ a Lie-Rinehart algebra. It can be checked that the projections are Lie-Rinehart algebra morphisms, and that, with these projections, the Lie-Rinehart algebra $L \times_N M$ satisfies the universal property of the pullback. Thus G preserves pullbacks.

Now we show that G also reflects isomorphisms. If
$$h : (V \xrightarrow{\alpha_V} \mathrm{Der}(A)) \to (W \xrightarrow{\alpha_W} \mathrm{Der}(A))$$
is both a morphism of **LR** and an isomorphism of vector spaces, then for all $x, y \in V$,
$$h^{-1}[x, y] = h^{-1}[hh^{-1}(x), hh^{-1}(y)] = h^{-1}h[h^{-1}(x), h^{-1}(y)] = [h^{-1}(x), h^{-1}(y)],$$
so that h^{-1} is a morphism of Lie algebras. Furthermore, for all $a \in A$,
$$h^{-1}(ax) = h^{-1}(ahh^{-1}(x)) = h^{-1}h(ah^{-1}(x)) = ah^{-1}(x),$$
making h^{-1} a morphism of A-modules. From $\alpha_W h = \alpha_V$, we get $\alpha_W = \alpha_V h^{-1}$. Thus h^{-1} is a morphism of **LR**, making h an isomorphism of **LR**, and proving that G reflects isomorphisms. □

Proposition 3.1.8. *The category* **LR** *of Lie-Rinehart algebras is protomodular.*

Proof. By Lemma (3.1.7), the forgetful functor $G : \mathbf{LR} \to \mathbf{Vect}_k$ preserves pullbacks and reflects isomorphisms. Since $\mathbf{Vect}_k$ is abelian, and therefore protomodular, Lemma (3.1.6) allows us to conclude that **LR** is protomodular. □

3.2 Crossed modules of Lie-Rinehart algebras

We now know that the category **LR** is protomodular, Barr-exact, and has finite coproducts. Thus, by Theorem 2.3.3, we know that the category of internal categories in **LR** is equivalent to the category **XMod**(**LR**) of internal crossed modules in **LR**, obtained by applying Definition 2.3.2 to **LR**. We then compare the category **XMod**(**LR**), thus defined, to the classically defined category of crossed modules of Lie-Rinehart algebras.

Classical Definitions

We begin by stating the classical definitions of action, semi-direct product, pre-crossed module, and crossed module of Lie Rinehart algebras. We take the following standard definitions from the paper [11] by Casas et al.

Definition 3.2.1 (Lie-Rinehart algebra action [11])**.** Let

$$L \xrightarrow{\alpha} \mathrm{Der}(A)$$

be a Lie-Rinehart algebra, and let R be a Lie A-algebra, so that

$$R \xrightarrow{0} \mathrm{Der}(A)$$

is Lie-Rinehart algebra. We say that L *acts* on R if there is a k-bilinear map

$$L \times R \to R : (x, r) \mapsto [x, r]$$

such that for all $x, y \in L, r, s \in R, a \in A$, we have

(A.1) $[[x, y], r] = [x, [y, r]] - [y, [x, r]]$,

(A.2) $[x, [r, s]] = [[x, r], s] - [[x, s], r]$,

(A.3) $[ax, r] = a[x, r]$,

(A.4) $[x, ar] = a[a, r] + \alpha_x(a)r$.

Conditions (A.1) and (A.2) together ensure that L acts on R at the level of Lie k-algebras. We will now see that conditions (A.3) and (A.4) will enable us to define a new Lie-Rinehart algebra: the semi-direct product $L \rtimes R$. Let

$$L \xrightarrow{\alpha} \mathrm{Der}(A)$$

be a Lie-Rinehart algebra that acts on a Lie A-algebra

$$R \xrightarrow{0} \mathrm{Der}(A).$$

We can form the direct sum of vector spaces $L \oplus R$ and define the bilinear map

$$L \oplus R \times L \oplus R \to L \oplus R : ((x, r), (y, s)) \mapsto ([x, y], [r, s] + [x, s] - [y, r]), \tag{3.1}$$

which is a Lie-bracket, and thus makes $L \oplus R$ into a Lie-k algebra: it is called the *semi-direct product* $L \rtimes R$ of the Lie k-algebras L and R. Being the direct sum of the A-modules L and R, it is also an A-module, with $a(x, r)$ defined to be (ax, ar). Also, observe that the projection $L \rtimes R \to L$ preserves both the Lie algebra and the A-module structure. In order for $L \rtimes R$ to be a Lie-Rinehart algebra, we need a morphism $\gamma : L \rtimes R \to \mathrm{Der}(A)$ that is a morphism both of A-modules and of Lie k-algebras, and satisfies

$$[(x, r), a(y, s)] = a[(x, r)] + \gamma_{(x,r)}(a)(y, s).$$

Letting $\gamma_{(x,r)} = \alpha_x$, we have

$$\begin{aligned}
[(x, r), a(y, s)] &= [(x, r), (ay, as)] \\
&= ([x, ay], [r, as] + [x, as] - [ay, r]) \\
&= (a[x, y] + \alpha_x(a)y, a[r, s] + 0 + a[x, s] + \alpha_x(a)s - a[y, r]) \\
&= (a[x, y], a[r, s] + a[x, s] - a[y, r]) + (\alpha_x(a)y, \alpha_x(a)s) \\
&= a([x, y], [r, s] + [x, s] - [y, r]) + \alpha_x(a)(y, s) \\
&= a[(x, r), (y, s)] + \gamma_{(x,r)}(a)(y, s).
\end{aligned}$$

Thus we can make the following definition

Definition 3.2.2 (Semi-direct product of Lie-Rinehart algebras [11]). Let

$$L \xrightarrow{\alpha} \mathrm{Der}(A)$$

be a Lie-Rinehart algebra that acts on a Lie A-algebra

$$R \xrightarrow{0} \mathrm{Der}(A)$$

by $(x, r) \mapsto [x, r]$. We define the *semi-direct product* $L \rtimes R \xrightarrow{\gamma} \mathrm{Der}(A)$ of Lie-Rinehart algebras to be the direct sum of A-modules $L \oplus R$ along with the Lie bracket

$$L \oplus R \times L \oplus R \to L \oplus R : ((x, r), (y, s)) \mapsto ([x, y], [r, s] + [x, s] - [y, r]),$$

with $\gamma_{(x,r)} = \alpha_x$.

Definition 3.2.3 (Crossed module of Lie-Rinehart algebras [11]). A *crossed module* of Lie-Rinehart algebras consists of a Lie-Rinehart algebra L, a Lie A-algebra R, a Lie-Rinehart algebra action $\phi : L \times R \to R$, and a Lie-Rinehart algebra homomorphism $\partial : R \to L$ such that

(CM.1) $\partial([x, r]) = [x, \partial(r)]$

(CM.2) $[\partial(r'), r] = [r', r]$

We call equation (CM.1) *equivariance*, and we call equation (CM.2) the *Peiffer condition*. Whenever we have a quadruple (L, R, ϕ, ∂) that satisfies condition (CM.1) we say that (L, R, ϕ, ∂) is a *pre-crossed module*.

Internal crossed modules in the category of Lie-Rinehart algebras

We will now apply definition (2.3.2) to the category **LR** and compare the results to the classical definition of the category of crossed modules. Recall that our general definition says that for any protomodular, Barr-exact category $\mathscr{C}$ with finite coproducts, an *internal crossed module* of $\mathscr{C}$ is defined to be an internal pre-crossed module

$$(X \xrightarrow{x} 0, A, \xi, g)$$

such that the diagram

$$\begin{tikzcd} (A + X)\flat X \arrow[r, "(1_A,\, g)\flat 1_X"] \arrow[d, "v"'] & A\flat X \arrow[d, "\xi"] \\ A\flat X \arrow[r, "\xi"'] & X \end{tikzcd}$$

commutes, where v is defined as in (2.16).

Let

$$L \xrightarrow{\alpha} \mathrm{Der}(A)$$

be a Lie-Rinehart algebra, let

$$R \xrightarrow{0} \mathrm{Der}(A)$$

be a Lie A-algebra, and let

$$L + R \xrightarrow{\beta} \mathrm{Der}(A)$$

be their coproduct in **LR**. Suppose that we are given an internal action

$$(R \xrightarrow{0} 0, L, \xi : L\flat R \to R)$$

of L on R. Recall that this means that ξ is an algebra structure map for the monad $UF = \flat$. Let $x \in L$, $r \in R$. Then $[i_L(x), i_R(r)] \in L + R$, and

$$(1_L + 0)([i_L(x), i_R(r)] = [(1_L + 0)i_L(x), (1_L + 0)i_R(r)] = [1_L(x), 0(r)] = [x, 0] = 0,$$

so that for all $(x, r) \in L \times R$, we have $[i_L(x), i_R(r)] \in L\flat R$.

We claim that we can define an action (in the classical sense) of L on R by

$$[x, r] := \xi([i_L(x), i_R(r)]).$$

We now check that this is indeed an action in the classical sense.

Condition (A.1) of Definition 3.2.1 translates to

$$\xi([i_L[x, x'], i_R(r)]) = \xi([i_L(x), \xi[i_L(x'), i_R(r)]]) - \xi([i_L(x'), \xi[i_L(x), i_R(r)]]). \tag{3.2}$$

Now, using the Jacobi condition,

$$\begin{aligned} [i_L[x, x'], i_R(r)] &= [[i_L(x), i_L(x')], i_R(r)] \\ &= -[i_R(r), [i_L(x), i_L(x')]] \\ &= [i_L(x), [i_L(x'), i_R(r)]] + [i_L(x'), [i_R(r), i_L(x)]] \\ &= [i_L(x), [i_L(x'), i_R(r)]] - [i_L(x'), [i_L(x), i_R(r)]]. \end{aligned}$$

And so

$$\begin{aligned} \xi([[i_L(x), i_L(x')], i_R(r)]) &= \xi([i_L(x), [i_L(x'), i_R(r)]] - [i_L(x'), [i_L(x), i_R(r)]]) \\ &= \xi([i_L(x), [i_L(x'), i_R(r)]]) - \xi([i_L(x'), [i_L(x), i_R(r)]]). \end{aligned} \tag{3.3}$$

Now note that, though we write $i_R(r)$, we are thinking of $i_R(r) \in L\flat R$, and since $k_{L,R}\eta_R = i_R$, we really mean $\eta_R(r)$, Thus, since $\xi\eta_R = 1_R$,

$$\begin{aligned} \xi([i_L(x), [i_L(x'), i_R(r)]]) &= [\xi i_L(x), \xi[i_L(x'), i_R(r)]] = [\xi i_L(x), \xi\eta\xi[i_L(x'), i_R(r)]] \\ &= [\xi i_L(x), \xi i_R\xi[i_L(x'), i_R(r)]] = \xi[i_L(x), i_R\xi[i_L(x'), i_R(r)]]. \end{aligned} \tag{3.4}$$

Similarly,

$$\xi([i_L(x'), [i_L(x), i_R(r)]]) = \xi[i_L(x'), i_R\xi[i_L(x), i_R(r)]]. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain (3.2). It follows that condition (A.1) holds.

Condition (A.2) of Definition 3.2.1 translates to

$$\xi([i_L(x), i_R[r, r']]) = [\xi([i_L(x), i_R(r)]), r'] - [\xi([i_L(x), i_R(r')]), r]. \tag{3.6}$$

Again, we can use the Jacobi condition to get

$$\begin{aligned} [i_L(x), i_R[r, r'] &= [i_L(x), [i_R(r), i_R(r')]] \\ &= [[i_L(x), i_R(r)], i_R(r')]] - [[i_L(x), i_R(r')], i_R(r)]] \end{aligned}$$

Applying ξ to both sides, we have

$$\begin{aligned} \xi([i_L(x), i_R[r, r']]) &= \xi([[i_L(x), i_R(r)], i_R(r')] - [[i_L(x), i_R(r')], i_R(r)]) \\ &= \xi([[i_L(x), i_R(r)], i_R(r')]) - \xi([[i_L(x), i_R(r')], i_R(r)]). \end{aligned} \tag{3.7}$$

Again, we use the fact that i_R really means η_R, and we have

$$\begin{aligned}\xi([[i_L(x), i_R(r)], i_R(r')]) &= [\xi([i_L(x), i_R(r)]), \xi i_R(r')] \\ &= [\xi([i_L(x), i_R(r)]), \xi\eta_R(r')] = [\xi([i_L(x), i_R(r)]), r'].\end{aligned} \tag{3.8}$$

Similarly, we have

$$\xi([[i_L(x), i_R(r')], i_R(r)]) = [\xi([i_L(x), i_R(r')]), r]. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) gives condition (A.2).

Condition (A.3) of Definition 3.2.1 translates to

$$\xi([i_L(ax), i_R(r)]) = a\xi([i_L(x), i_R(r)]). \tag{3.10}$$

We have

$$\begin{aligned}\xi([i_L(ax), i_R(r)]) &= \xi([ai_L(x), i_R(r)]) = \xi(-[i_R(r), ai_L(x)]) \\ &= -\xi(a[i_R(r), i_L(x)]) + \beta_{i_R(r)}(a)i_L(x) \\ &= -a\xi([i_R(r), i_L(x)]) + (\beta i_R)_r(a)i_L(x) \\ &= -a\xi([i_R(r), i_L(x)]) + 0 \\ &= a\xi([i_L(x), i_R(r)]).\end{aligned}$$

Condition (A.4) of Definition 3.2.1 translates to

$$\xi([i_L(x), i_R(ar)]) = a\xi([i_L(x), i_R(r)]) + \alpha_x(a)r.$$

Now

$$\begin{aligned}\xi([i_L(x), i_R(ar)]) &= \xi(a[i_L(x), i_R(r)] + \beta_{i_L(x)}(a)i_R(r)) \\ &= \xi(a[i_L(x), i_R(r)]) + \xi((\beta i_L)_x(a)i_R(r)) \\ &= \xi(a[i_L(x), i_R(r)]) + \xi(\alpha_x(a)i_R(r)) \\ &= \xi(a[i_L(x), i_R(r)]) + \alpha_x(a)\xi i_R(r) \\ &= \xi(a[i_L(x), i_R(r)]) + \alpha_x(a)\xi\eta(r) \\ &= \xi(a[i_L(x), i_R(r)]) + \alpha_x(a)r\end{aligned}$$

as required.

We have just seen that every internal action of Lie-Rinehart algebras gives rise to a “classical action" of Lie-Rinehart algebras. On the other hand, given a “classical action”, we can form the “classical” semi-direct product and thus obtain the split epimorphism given by the projection onto L, which, under the equivalence **SplEp**(**LR**) $\simeq$ **Act**(**LR**), gives us an internal action. It can be checked that if we start with a classical action, form the internal action corresponding to the classical semi-direct product projection, and finally form from this the induced classical action, we obtain the same action we started with. On the other hand, if we start with an internal action ξ, form the induced classical action and the corresponding semi-direct product projection, then this turns out to be the same as the split epimorphism corresponding to ξ under the equivalence **SplEp**(**LR**) $\simeq$ **Act**(**LR**). It follows that the categories of split epimorphisms in **LR** and (classical) actions in **LR** are equivalent.

Given an internal pre-crossed module, the morphism $g : R \to L$ satisfies the internal equivariance condition. The “classical" action induced by the underlying internal action is really a special case of internal action, formed by restricting to $L \times R$. It follows that an internal pre-crossed module is also a “classical” pre-crossed module. On the other hand, given a “classical” pre-crossed module, the morphism $g : R \to L$ gives a morphism $(1, g) : L \oplus R \to L$. It can be checked that $(1, g)$ is indeed a Lie-Rinehart algebra morphism, and thus gives a reflexive graph in **LR**. Under the equivalence **RefGph**(**LR**) $\simeq$ **PXMod**(**LR**), this is a pre-crossed module.

Similarly for crossed modules, internal crossed modules can be seen to give "classical" crossed modules by restricting to $L \times R$. On the other hand, given a "classical" crossed module, we can form the reflexive graph just described from the underlying pre-crossed module. We can then form an internal category at the level of A-modules. It can be checked that the composition m, defined by

$$L \oplus R \oplus R \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}} L \oplus R,$$

is actually a Lie-Rinehart algebra morphism. This gives an internal category of Lie-Rinehart algebras, and therefore gives the corresponding internal crossed module.

Chapter. 4

Conclusion

We have given new definitions of internal action, internal pre-crossed module, and internal crossed module in protomodular, Barr-exact categories $\mathscr{C}$ with finite coproducts; and we have shown that the resulting categories of internal actions in $\mathscr{C}$, pre-crossed modules in $\mathscr{C}$, and crossed modules in $\mathscr{C}$ are equivalent to the categories of split epimorphisms in $\mathscr{C}$, reflexive graphs in $\mathscr{C}$, and internal categories in $\mathscr{C}$ respectively. These results generalise Janelidze's corresponding results for semi-abelian categories in [16].

We have shown that the category of Lie-Rinehart algebras is protomodular, Barr-exact, and has finite coproducts; thus we have shown that the categories of internal crossed modules and internal categories in the category of Lie-Rinehart algebras are equivalent. We have seen that the "classical" definitions of Lie-Rinehart algebra action and crossed module of Lie-Rinehart algebras agree with our definitions of internal action and internal crossed module in **LR**. Thus, we have shown that the categories of split epimorphisms in **LR** and crossed modules in **LR** are equivalent, respectively, to the categories of "classical" actions in **LR** and "classical" crossed modules in **LR**. These are, to our knowledge, new results.

However, there remains the minor frustration that the condition in our definition of an internal crossed module does not quite reduce to the Peiffer condition when the general definition is applied to concrete categories such as **LR** or **Grp**. In the semi-abelian setting, Janelidze raised the question as to when a simpler condition— one that does reduce to the Peiffer condition in **Grp**— is actually equivalent to the definition of an internal crossed module [16]. Martins-Ferriera and van der Linden showed, in [20], that under the so-called "Smith is Huq condition", the simpler condition—- the one that reduces to the Peiffer condition— is equivalent to the definition of an internal crossed module. Although pointedness was assumed in proving this result, it is natural to ask whether pointedness is necessary. We plan to address this in the future.

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories: the joy of cats. *Reprints in Theory and Applications of Categories*, (17):1–507, 2006. Reprint of the 1990 original [Wiley, New York].

[2] Michael Barr and Charles Wells. Toposes, triples and theories. *Reprints in Theory and Applications of Categories*, (12):x+288, 2005. Corrected reprint of the 1985 original.

[3] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[4] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[5] Francis Borceux and Dominique Bourn. *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2004.

[6] D. Bourn and G. Janelidze. Protomodularity, descent, and semidirect products. *Theory and Applications of Categories*, 4:No. 2, 37–46, 1998.

[7] Dominique Bourn. Mal'cev categories and fibration of pointed objects. *Applied Categorical Structures*, 4(2):307–327, Jun 1996.

[8] Ronald Brown and Christopher B Spencer. G-groupoids, crossed modules and the fundamental groupoid of a topological group. *Indagationes Mathematicae (Proceedings)*, 79(4):296 – 302, 1976.

[9] A. Carboni, G. M. Kelly, and M. C. Pedicchio. Some remarks on maltsev and goursat categories. *Applied Categorical Structures*, 1(4):385–421, Dec 1993.

[10] A. Carboni, J. Lambek, and M. C. Pedicchio. Diagram chasing in Mal'cev categories. *Journal of Pure and Applied Algebra*, 69(3):271–284, 1991.

[11] J.M. Casas, M. Ladra, and T. Pirashvili. Crossed modules for lie-rinehart algebras. *Journal of Algebra*, 274(1):192 – 201, 2004.

[12] Peter Freyd. *Abelian categories. An introduction to the theory of functors*. Harper's Series in Modern Mathematics. Harper & Row, Publishers, New York, 1964.

[13] Peter J. Freyd and Andre Scedrov. *Categories, allegories*, volume 39 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1990.

[14] Marino Gran and Jiří Rosický. Semi-abelian monadic categories. *Theory and Applications of Categories*, 13:No. 6, 106–113, 2004.

[15] Johannes Huebschmann. Crossed n-fold extensions of groups and cohomology. *Comment. Math. Helv.*, 55(2):302–313, 1980.

[16] G. Janelidze. Internal crossed modules. *Georgian Mathematical Journal*, 10(1):99–114, 2003.

[17] George Janelidze, László Márki, and Walter Tholen. Semi-abelian categories. *Journal of Pure and Applied Algebra*, 168(2-3):367–386, 2002. Category theory 1999 (Coimbra).

[18] C. Kassel and J.-L. Loday. Extensions centrales d'algèbres de Lie. *Université de Grenoble. Annales de l'Institut Fourier*, 32(4):119–142 (1983), 1982.

[19] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1998.

[20] Nelson Martins-Ferreira and Tim Van der Linden. A note on the "Smith is Huq" condition. *Applied Categorical Structures*, 20(2):175–187, 2012.

[21] Barry Mitchell. *Theory of categories*. Pure and Applied Mathematics, Vol. XVII. Academic Press, New York-London, 1965.

[22] J. H. C. Whitehead. Combinatorial homotopy. II. *Bulletin of the American Mathematical Society*, 55:453–496, 1949.