# Cartan Formalism and some computations

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In this text we will briefly discuss Cartan's formalism of differential forms in the setting of vector bundles. We will:

- recall some definitions along the way;
- see how to apply this machinery for the tangent bundle of a pseudo-Riemannian manifold;
- explicitly carry out the computations for several examples;
- see one of the most general setting on which the notion of *second fundamental form* makes sense, deduce general versions of the Gauss and Codazzi equations, and;
- set up the Cartan formalism for the study of non-degenerate submanifolds of a pseudo-Riemannian manifold.

This is by no means is self-contained, as I have written it mainly to try and organize some of this material for myself (so don't expect these notes to be that great – I haven't organized everything neatly in definitions, lemmas, etc.). We will adopt Einstein's summation convention in full force. And even though this text has "Cartan Formalism" in its title, I will try to achieve a healthy balance between forms and non-forms computations – the goal is just to flesh out some basic Riemannian geometry examples. The conclusion I have reached writing this is that Cartan computations are efficient for studying the geometry of a manifold "on its own", but as far the extrinsic geometry goes, the fundamental equations are a more powerful tool, as that the most standard examples of submanifolds for which we can actually do calculations are usually found "friendlier" ambients, such as flat vector spaces, or space forms. I have also added some references in the end, from which I studied bits and pieces.

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# 1 General definitions

Let $M^n$ be a differentiable manifold $\pi\colon E \to M$ be a rank $k$ real vector bundle over $M$. A *Koszul connection* in $E$ is a map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ such that

(i) $\nabla_{\boldsymbol{X}_1 + f\boldsymbol{X}_2}\psi = \nabla_{\boldsymbol{X}_1}\psi + f\nabla_{\boldsymbol{X}_2}\psi$;

(ii) $\nabla_{\boldsymbol{X}}(f\psi) = f\nabla_{\boldsymbol{X}}\psi + \boldsymbol{X}(f)\psi$.

These conditions say that the value of $\nabla_{\boldsymbol{X}}\psi$ at a point $p \in M$ depends on the value $\boldsymbol{X}_p$ and on the values of $\psi$ in a neighborhood of $p$. So we may naturally restrict $\nabla$ to open subsets of $M$. Usually, if $(x^j)$ is a coordinate system for $M$ and $(e_a)$ are local trivializing sections for $E$, one may write $\nabla_{\partial_j} e_a = \Gamma_{ja}^b e_b$ (Einstein's convention in force), where the functions $\Gamma_{ja}^b$ are called the *connection coefficients* of $\nabla$ relative to $(x^j)$ and $(e_a)$. Moreover, the *curvature* of $\nabla$ is $R^\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defined by

$$R^\nabla(\boldsymbol{X}, \boldsymbol{Y})\psi = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi,$$

where $[\boldsymbol{X}, \boldsymbol{Y}]$ denotes the Lie bracket of $\boldsymbol{X}$ and $\boldsymbol{Y}$. It turns out that $R^\nabla$ is a tensor (precisely because of the perhaps artificial term $-\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi$), which measures the non-integrability of the horizontal distribution in $TE$ associated to $\nabla$. With respect to $(x^j)$ and $(e_a)$, we may write $R^\nabla(\partial_j, \partial_k)e_a = R_{jka}{}^b e_b$, where

$$R_{jka}{}^b = \partial_j \Gamma_{ka}^b - \partial_k \Gamma_{ja}^b + \Gamma_{ka}^c \Gamma_{jc}^b - \Gamma_{ja}^c \Gamma_{kc}^b.$$

We will not follow this approach, but instead will try to understand Cartan's formalism with differential forms. Fixed $\boldsymbol{X} \in \mathfrak{X}(M)$, we write

$$\nabla_{\boldsymbol{X}} e_a = \omega^b{}_a(\boldsymbol{X}) e_b \quad \text{and} \quad R^\nabla(\boldsymbol{X}, \boldsymbol{Y}) e_a = \Omega^b{}_a(\boldsymbol{X}, \boldsymbol{Y}) e_b,$$

where $\omega^b{}_a$ and $\Omega^b{}_a$ are called the *connection* 1*-forms* and *curvature* 2*-forms* of $\nabla$ relative to $(e_a)$. Observe that $\omega^b{}_a(\partial_j) = \Gamma_{ja}^b$ and $\Omega^b{}_a(\partial_j, \partial_k) = R_{jka}{}^b$. The $\mathscr{C}^\infty(M)$-linearity of $\nabla$ in the vector argument and the tensoriality and skew-(12) symmetry of $R^\nabla$ ensure that these objects are indeed differential forms. In the same way that one may express $R_{jka}{}^b$ in terms of $\Gamma_{ja}^b$'s, there is an analogous relation between the connection 1-forms and curvature 2-forms. The derivatives taken with respect to the coordinate vector fields of $(x^j)$ are then replaced by the single exterior derivative operation. Here's how:

**Proposition** (Second structure equations). $\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$.

**Remark.** One may write this simply as $\Omega = \mathrm{d}\omega + \omega \wedge \omega$, thinking of matrices.

**Proof:** Compute

$$\begin{aligned}
&R^\nabla(\boldsymbol{X}, \boldsymbol{Y}) e_b = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}} e_b - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}} e_b - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]} e_b \\
&= \nabla_{\boldsymbol{X}}\left(\omega^c{}_b(\boldsymbol{Y}) e_c\right) - \nabla_{\boldsymbol{Y}}\left(\omega^c{}_b(\boldsymbol{X}) e_c\right) - \omega^a{}_b([\boldsymbol{X}, \boldsymbol{Y}]) e_a \\
&= \boldsymbol{X}(\omega^a{}_b(\boldsymbol{Y})) e_a + \omega^c{}_b(\boldsymbol{Y}) \omega^a{}_c(\boldsymbol{X}) e_a - \boldsymbol{Y}(\omega^a{}_b(\boldsymbol{X})) e_a - \omega^c{}_b(\boldsymbol{X}) \omega^a{}_c(\boldsymbol{Y}) e_a - \omega^a{}_b([\boldsymbol{X}, \boldsymbol{Y}]) e_a \\
&= \left(\mathrm{d}\omega^a{}_b(\boldsymbol{X}, \boldsymbol{Y}) + \omega^a{}_c(\boldsymbol{X}) \omega^c{}_b(\boldsymbol{Y}) - \omega^c{}_b(\boldsymbol{X}) \omega^a{}_c(\boldsymbol{Y})\right) e_a.
\end{aligned}$$

$\square$

As a consequence, keeping the same notation, we have the:

**Proposition** (Bianchi identity). $\mathrm{d}\Omega^a{}_b = \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b$.

**Remark.** Or $\mathrm{d}\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

**Proof:** Compute:

$$\begin{aligned}
\mathrm{d}\Omega^a{}_b &= \mathrm{d}\left(\mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\right) \\
&= 0 + \mathrm{d}\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \mathrm{d}\omega^c{}_b \\
&= (\Omega^a{}_c - \omega^a{}_d \wedge \omega^d{}_c) \wedge \omega^c{}_b - \omega^a{}_c \wedge (\Omega^c{}_b - \omega^c{}_d \wedge \omega^d{}_b) \\
&= \Omega^a{}_c - \omega^a{}_d \wedge \omega^d{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b + \omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b \\
&= \Omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \Omega^c{}_b,
\end{aligned}$$

by using that $c$ and $d$ are dummy indices in the triple wedge products. □

Now, assume further that our vector bundle has been equipped with a pseudo-Euclidean fiber metric $g \in \Gamma(E^* \otimes E^*)$, that is, a smooth assignment to each fiber $E_p$ of a non-degenerate symmetric $\mathbb{R}$-bilinear form $g_p\colon E_p \times E_p \to \mathbb{R}$. We will also write $g = \langle\cdot,\cdot\rangle$ for this fiber metric. With respect to local trivializing sections $(e_a)$ of $E$, we set $g_{ab} = g(e_a, e_b) = \langle e_a, e_b\rangle$, and the non-degeneracy of $g$ ensures the existance of the inverse matrix to $(g_{ab})$, to be denoted $(g^{ab})$. This means that $g^{ac}g_{cb} = \delta^a_b$ holds. With those, we obtain natural identifications between the fibers of $E$ and the fibers of the dual bundle $E^*$, which rise to the level of sections: given $\psi \in \Gamma(E)$, we define $\psi_\flat \in \Gamma(E^*)$ by $\psi_\flat = g(\psi, \cdot)$. The inverse assigns to $\xi \in \Gamma(E^*)$ the unique section $\xi^\sharp \in \Gamma(E)$ with $\xi = g(\xi^\sharp, \cdot)$. If $(e^a)$ are the local trivializing sections for $E^*$ dual to $(e_a)$ (characterized by $e^a(e_b) = \delta^a_b$), these so-called *musical isomorphisms* read

$$\psi = \psi^a e_a \to \psi_\flat = g_{ab}\psi^b e^a \quad \text{and} \quad \xi = \xi_a e^a \to \xi^\sharp = g^{ab}\xi_b e_a.$$

One usually drops the symbols $\flat$ and $\sharp$ from the notation, writing simply $\psi_a = g_{ab}\psi^b$ and $\xi^a = g^{ab}\xi_b$. The similar relations $(e^a)^\sharp = g^{ab}e_b$ and $(e_a)_\flat = g_{ab}e^b$ also hold.

This process of raising and lowering indices using $g$ (which is also done for sections of the tensor bundles associated to $E$) can be also done for the connection and curvature forms. For the record, we set $\omega_{ab} = g_{ac}\omega^c{}_b$, and $\Omega_{ab} = g_{ac}\Omega^c{}_b$. Intrinsically, we have that $\omega_{ab}(\boldsymbol{X}) = g(e_a, \nabla_{\boldsymbol{X}} e_b)$ (proof: lower the index $a$ in $\omega^a{}_b(\boldsymbol{X}) = g^{ac}g(e_c, \nabla_{\boldsymbol{X}} e_b)$). Similarly, $\Omega_{ab}(\boldsymbol{X}, \boldsymbol{Y}) = g(R^\nabla(\boldsymbol{X}, \boldsymbol{Y})e_b, e_a) = R^\nabla(\boldsymbol{X}, \boldsymbol{Y}, e_b, e_a)$, where the latter $R^\nabla$ is obtained from the first one with the aid of $g$. In other words, if one chooses a coordinate system $(x^j)$ for $M$, then $\Omega_{ab}(\partial_j, \partial_k) = R_{jkba}$.

At this point, it is natural to ask if $\nabla$ plays along well with the musical isomorphisms. Namely, if $\boldsymbol{X} \in \mathfrak{X}(M)$, $\psi \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$ are given, one may form four objects:

$$\nabla_{\boldsymbol{X}}(\psi_\flat), \quad (\nabla_{\boldsymbol{X}}\psi)_\flat, \quad \nabla_{\boldsymbol{X}}(\xi^\sharp) \quad \text{and} \quad (\nabla_{\boldsymbol{X}}\xi)^\sharp,$$

where the connection induced by $\nabla$ in $E^*$, also denoted by $\nabla$, is characterized by the Leibniz rule $(\nabla_{\boldsymbol{X}}\xi)(\psi) = \boldsymbol{X}(\xi(\psi)) - \xi(\nabla_{\boldsymbol{X}}\psi)$. One may also form the covariant derivatives of the metric $g$ itself, again by emulating a Leibniz rule:

$$(\nabla_{\boldsymbol{X}} g)(\psi_1, \psi_2) = \boldsymbol{X}(g(\psi_1, \psi_2)) - g(\nabla_{\boldsymbol{X}}\psi_1, \psi_2) - g(\psi_1, \nabla_{\boldsymbol{X}}\psi_2),$$

Then:

**Proposition.** *$(\nabla_X \psi)_\flat = \nabla_X(\psi_\flat)$ and $(\nabla_X \xi)^\sharp = \nabla_X(\xi^\sharp)$ for all $X \in \mathfrak{X}(M)$, $\psi \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$ if and only if $\nabla g = 0$. In this case, $\nabla$ is called a* metric connection.

**Remark.** $\nabla g = 0$ reads as a simple product rule: $X\langle \psi_1, \psi_2\rangle = \langle \nabla_X \psi_1, \psi_2\rangle + \langle \psi_1, \nabla_X \psi_2\rangle$.

**Proof:** Assume $\nabla g = 0$. Let's check, for example, the first identity $(\nabla_X \psi)_\flat = \nabla_X(\psi_\flat)$ (the other one being treated analogously). Let $\phi \in \Gamma(E)$ be any test section. We have:

$$(\nabla_X \psi)_\flat(\phi) = \langle \nabla_X \psi, \phi\rangle = X\langle \psi, \phi\rangle - \langle \psi, \nabla_X \phi\rangle = X(\psi_\flat(\phi)) - \psi_\flat(\nabla_X \phi) = (\nabla_X(\psi_\flat))\,(\phi).$$

For the converse, assume that $\nabla$ is compatible with the musical isomorphisms. We compute

$$\begin{aligned}(\nabla_X g)(\psi_1, \psi_2) &= X\langle \psi_1, \psi_2\rangle - \langle \nabla_X \psi_1, \psi_2\rangle - \langle \psi_1, \nabla_X \psi_2\rangle \\ &= X((\psi_1)_\flat(\psi_2)) - \langle \nabla_X \psi_1, \psi_2\rangle - (\psi_1)_\flat(\nabla_X \psi_2) \\ &= \nabla_X((\psi_1)_\flat)(\psi_2) - \langle \nabla_X \psi_1, \psi_2\rangle \\ &= (\nabla_X \psi_1)_\flat(\psi_2) - \langle \nabla_X \psi_1, \psi_2\rangle \\ &= 0,\end{aligned}$$

as wanted. $\square$

In terms of local trivializing sections and connection 1-forms, metric compatibility reads as

$$\mathrm{d}g_{ab} = g_{cb}\omega^c{}_a + g_{ac}\omega^c{}_b = \omega_{ab} + \omega_{ba}.$$

This can be seen by making $\psi_1 = e_a$ and $\psi_2 = e_b$ in the expression given in the last remark, and lowering the upper indices in the connection 1-forms. The middle expression can be written in matrix form as $\mathrm{d}g = g\omega + (g\omega)^\top$. In general, there are plenty of metric connections in a given vector bundle. To proceed further and obtain any sort of classification, we need a bit more of structure.

# 2 What happens in a tangent bundle

Assume now that $E = TM$, and that $\nabla$ is any Koszul connection in the tangent bundle $TM$. The *torsion tensor* of $\nabla$ is the map $\tau^\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X},\boldsymbol{Y}]$. Just like what happened with the curvature $R^\nabla$, the tensorial character of $\tau^\nabla$ is due to the last term $-[\boldsymbol{X},\boldsymbol{Y}]$. Recalling that given $f \in \mathscr{C}^\infty(M)$, the *covariant Hessian* of $f$ according to $\nabla$ is the twice covariant tensor field defined by

$$\mathrm{Hess}^\nabla(f)(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}(\mathrm{d}f)(\boldsymbol{Y}) = \boldsymbol{X}(\boldsymbol{Y}(f)) - \mathrm{d}f(\nabla_{\boldsymbol{X}}\boldsymbol{Y}),$$

a simple interpretation of $\tau^\nabla$ is given in the following:

**Proposition.** $\tau^\nabla = \boldsymbol{0}$ *if and only if* $\mathrm{Hess}^\nabla(f)$ *is a symmetric tensor for every* $f \in \mathscr{C}^\infty(M)$.

**Proof:** $\mathrm{Hess}^\nabla(f)(\boldsymbol{X},\boldsymbol{Y}) - \mathrm{Hess}^\nabla(f)(\boldsymbol{Y},\boldsymbol{X}) = -\tau^\nabla(\boldsymbol{X},\boldsymbol{Y})(f)$. □

Now we turn back our attention to the connection 1-forms again. This torsion is not only an obstacle for the symmetry of Hessian tensors, but also appears as a defect in another set of Cartan's structure equations:

**Proposition** (First structure equations)**.** *Let* $(\boldsymbol{E}_i)$ *be a local frame for* $M$*, and* $(\theta^i)$ *be the dual coframe. Then* $\mathrm{d}\theta^i = \theta^i \circ \tau^\nabla + \theta^j \wedge \omega^i{}_j$.

**Remark.** Or $\mathrm{d}\theta = \tau - \omega \wedge \theta$. Here, we understand $\tau$ as the column vector formed by the 2-forms $\tau^i$ satisfying $\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) = \tau^i(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{E}_i$.

**Proof:** Write $\boldsymbol{Y} = \theta^k(\boldsymbol{Y})\boldsymbol{E}_k$, and apply $\nabla_{\boldsymbol{X}}$ on both sides to obtain

$$\nabla_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}(\theta^k(\boldsymbol{Y})\boldsymbol{E}_k) = \boldsymbol{X}(\theta^k(\boldsymbol{Y}))\boldsymbol{E}_k + \theta^k(\boldsymbol{Y})\omega^i{}_k(\boldsymbol{X})\boldsymbol{E}_i.$$

Follow through with $\theta^i$, so that $\boldsymbol{X}(\theta^i(\boldsymbol{Y})) = \theta^i(\nabla_{\boldsymbol{X}}\boldsymbol{Y}) - \theta^j(\boldsymbol{Y})\omega^i{}_j(\boldsymbol{X})$. Now compute the exterior derivative:

$$\begin{aligned}\mathrm{d}\theta^i(\boldsymbol{X},\boldsymbol{Y}) &= \boldsymbol{X}(\theta^i(\boldsymbol{Y})) - \boldsymbol{Y}(\theta^i(\boldsymbol{X})) - \theta^i([\boldsymbol{X},\boldsymbol{Y}]) \\ &= \theta^i(\nabla_{\boldsymbol{X}}\boldsymbol{Y}) - \theta^j(\boldsymbol{Y})\omega^i{}_j(\boldsymbol{X}) - \theta^i(\nabla_{\boldsymbol{Y}}\boldsymbol{X}) + \theta^j(\boldsymbol{X})\omega^i{}_j(\boldsymbol{Y}) - \theta^i([\boldsymbol{X},\boldsymbol{Y}]) \\ &= \theta^i(\tau^\nabla(\boldsymbol{X},\boldsymbol{Y})) + (\theta^j \wedge \omega^i{}_j)(\boldsymbol{X},\boldsymbol{Y}).\end{aligned}$$

□

To relate our structure equations and metric compatibility, we assume now that $(M,g)$ is a pseudo-Riemannian manifold.

**Theorem.** *There is a unique torsion-free metric connection for* $(M,g)$*. It is called the* Levi-Civita connection *of* $(M,g)$.

One can give a coordinate-free proof of this theorem by showing that such a connection $\nabla$ must satisfy the so-called *Koszul formula*

$$2\langle\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\rangle = \boldsymbol{X}\langle\boldsymbol{Y},\boldsymbol{Z}\rangle + \boldsymbol{Y}\langle\boldsymbol{X},\boldsymbol{Z}\rangle - \boldsymbol{Z}\langle\boldsymbol{X},\boldsymbol{Y}\rangle + \langle[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\rangle - \langle[\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\rangle - \langle[\boldsymbol{X},\boldsymbol{Z}],\boldsymbol{Y}\rangle,$$

which proves uniqueness, and then defining the connection by this formula (possible due to the non-degeneracy of the metric) and checking that it satisfies everything needed. But we wish to illustrate how this can also be achieved by using Cartan's formalism. So we'll work with a slightly different statement:

**Theorem.** *Let $(\boldsymbol{E}_i)$ be a local frame for $(M, g)$ and $(\theta^i)$ be the dual coframe. The connection 1-forms $\omega^i{}_j$ for the Levi-Civita connection of $(M, g)$ are completely determined by the relations*

$$\mathrm{d}\theta^i = \theta^j \wedge \omega^i{}_j \quad \textit{and} \quad \mathrm{d}g_{ij} = \omega_{ij} + \omega_{ji}.$$

**Proof:** First observe that $\mathrm{d}\theta^i(\boldsymbol{E}_j, \boldsymbol{E}_k) = -\theta^i([\boldsymbol{E}_j, \boldsymbol{E}_k])$. Now use the structure equations to compute

$$-\theta^i([\boldsymbol{E}_j, \boldsymbol{E}_k]) = (\theta^r \wedge \omega^i{}_r)(\boldsymbol{E}_j, \boldsymbol{E}_k) = \theta^r(\boldsymbol{E}_j)\omega^i{}_r(\boldsymbol{E}_k) - \theta^r(\boldsymbol{E}_k)\omega^i{}_r(\boldsymbol{E}_j) = \omega^i{}_j(\boldsymbol{E}_k) - \omega^i{}_k(\boldsymbol{E}_j).$$

Now lower the index $i$ and write $-\langle \boldsymbol{E}_i, [\boldsymbol{E}_j, \boldsymbol{E}_k]\rangle = \omega_{ij}(\boldsymbol{E}_k) - \omega_{ik}(\boldsymbol{E}_j)$. Consider cyclic permutations of $(ijk)$:

$$\begin{cases} -\langle \boldsymbol{E}_i, [\boldsymbol{E}_j, \boldsymbol{E}_k]\rangle = \omega_{ij}(\boldsymbol{E}_k) - \omega_{ik}(\boldsymbol{E}_j) \\ -\langle \boldsymbol{E}_j, [\boldsymbol{E}_k, \boldsymbol{E}_i]\rangle = \omega_{jk}(\boldsymbol{E}_i) - \omega_{ji}(\boldsymbol{E}_k) \\ -\langle \boldsymbol{E}_k, [\boldsymbol{E}_i, \boldsymbol{E}_j]\rangle = \omega_{ki}(\boldsymbol{E}_j) - \omega_{kj}(\boldsymbol{E}_i) \end{cases}$$

We add the first two equations and subtract the last one. The left side is a combination with Lie brackets that we'll address shortly, while the right side becomes just

$$2\omega_{ij}(\boldsymbol{E}_k) - \mathrm{d}g_{ij}(\boldsymbol{E}_k) + \mathrm{d}g_{jk}(\boldsymbol{E}_i) - \mathrm{d}g_{ik}(\boldsymbol{E}_j).$$

Solve for $2\omega_{ij}(\boldsymbol{E}_k)$ to recover the Koszul formula:

$$2\omega_{ij}(\boldsymbol{E}_k) = -\langle \boldsymbol{E}_i, [\boldsymbol{E}_j, \boldsymbol{E}_k]\rangle - \langle \boldsymbol{E}_j, [\boldsymbol{E}_k, \boldsymbol{E}_i]\rangle + \langle \boldsymbol{E}_k, [\boldsymbol{E}_i, \boldsymbol{E}_j]\rangle + \boldsymbol{E}_k\langle \boldsymbol{E}_i, \boldsymbol{E}_j\rangle - \boldsymbol{E}_i\langle \boldsymbol{E}_j, \boldsymbol{E}_k\rangle + \boldsymbol{E}_j\langle \boldsymbol{E}_i, \boldsymbol{E}_k\rangle.$$

The explicit expression for $\omega^i{}_j$ may then be obtained by raising $i$ on both sides. We are done. $\square$

The advantage here is that in some situations one may guess what the connection 1-forms are, and also that we are no longer bound to coordinate frames. In other words, checking that the $\omega^i{}_j$ satisfy the first structure equation is checking that the connection defined on the domain of the $\omega^i{}_j$ is torsion-free. Similarly, checking the other relation with $\mathrm{d}g_{ij}$ is checking that said connection is metric compatible – it follows that it is the Levi-Civita connection. As for the curvature 2-forms, we may consider the extra symmetries of the curvature tensor of the Levi-Civita connection of a pseudo-Riemannian manifold $(M, g)$ with a local frame $(\boldsymbol{E}_i)$:

- the skew-(34) symmetry $R(\boldsymbol{E}_i, \boldsymbol{E}_j, \boldsymbol{E}_k, \boldsymbol{E}_\ell) = -R(\boldsymbol{E}_i, \boldsymbol{E}_j, \boldsymbol{E}_\ell, \boldsymbol{E}_k)$ then reads as $\Omega_{\ell k}(\boldsymbol{E}_i, \boldsymbol{E}_j) = -\Omega_{k\ell}(\boldsymbol{E}_i, \boldsymbol{E}_j)$, and since the indices $i$ and $j$ are arbitrary, we get an equality between two forms. Renaming back $(\ell, k) \to (i, j)$ we obtain that $\Omega_{ij} = -\Omega_{ji}$ for all $i$ and $j$.

- the Bianchi identity[1] $R(\boldsymbol{E}_i, \boldsymbol{E}_j)\boldsymbol{E}_k + R(\boldsymbol{E}_j, \boldsymbol{E}_k)\boldsymbol{E}_i + R(\boldsymbol{E}_k, \boldsymbol{E}_i)\boldsymbol{E}_j = \boldsymbol{0}$ reads as

  $$\Omega^\ell{}_k(\boldsymbol{E}_i, \boldsymbol{E}_j) + \Omega^\ell{}_i(\boldsymbol{E}_j, \boldsymbol{E}_k) + \Omega^\ell{}_j(\boldsymbol{E}_k, \boldsymbol{E}_i) = 0,$$

  for all choices of indices. In particular, one can lower $\ell$ everywhere and also conclude that $\Omega_{\ell k}(\boldsymbol{E}_i, \boldsymbol{E}_j) + \Omega_{\ell i}(\boldsymbol{E}_j, \boldsymbol{E}_k) + \Omega_{\ell j}(\boldsymbol{E}_k, \boldsymbol{E}_i) = 0$.

[1] We have $(\mathrm{d}^\nabla \tau)(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} + R(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} + R(\boldsymbol{Z}, \boldsymbol{X})\boldsymbol{Y}$, where $\mathrm{d}^\nabla$ is the *covariant exterior derivative* operator. So not only we see that this identity holds for every torsion-free connection, but it's content is trivial: the derivative of zero is zero. The second Bianchi identity, in turn, says that $\mathrm{d}^\nabla R^\nabla = 0$.

# 3 Further notions of curvature and some useful things

Throughout this section, let $(M, g)$ be a pseudo-Riemannian manifold.

## 3.1 Ricci curvature

The *Ricci tensor* of $(M, g)$ is the map $\mathrm{Ric}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathscr{C}^\infty(M)$ given by $\mathrm{Ric}(\boldsymbol{X}, \boldsymbol{Y}) = \mathrm{tr}(\boldsymbol{V} \mapsto R(\boldsymbol{V}, \boldsymbol{X})\boldsymbol{Y})$. It can also be defined as $\mathrm{tr}_g R(\cdot, \boldsymbol{X}, \boldsymbol{Y}, \cdot)$, and with respect to any local frame $(\boldsymbol{E}_i)$, can be expressed by

$$R_{ij} = \mathrm{Ric}(\boldsymbol{E}_i, \boldsymbol{E}_j) = R_{kij}{}^k = \Omega^k{}_j(\boldsymbol{E}_k, \boldsymbol{E}_i),$$

by using the symmetries of $R$. Considering also the dual coframe $(\theta^i)$, we may also express this quantity in terms of the modified connection 2-forms (or, in other words, lowering index $k$ above), as

$$R_{ij} = g^{k\ell} R_{ki\ell j} = g^{k\ell} \Omega_{j\ell}(\boldsymbol{E}_k, \boldsymbol{E}_i) = \Omega_{j\ell}((\theta^\ell)^\sharp, \boldsymbol{E}_i),$$

where $(\theta^\ell)^\sharp$ is $\theta^\ell$ turned into a vector field with the aid of $g$. The Ricci tensor of the curvature of the Levi-Civita connection of a pseudo-Riemannian manifold is always symmetric. It is also possible to define the Ricci tensor of any Koszul connection in $TM$, but we might lose this symmetry.

## 3.2 Scalar curvature

The *scalar curvature* of $(M, g)$ is the smooth map $\mathrm{s}\colon M \to \mathbb{R}$ defined as the contraction of the Ricci tensor: $\mathrm{s} = \mathrm{tr}_g \mathrm{Ric}$. Again considering a frame $(\boldsymbol{E}_i)$ and its dual coframe $(\theta^i)$, one may write (by employing symmetries of $R$) that

$$\mathrm{s} = g^{ij} R_{ij} = g^{ij} \Omega_{j\ell}(\boldsymbol{E}_i, (\theta^\ell)^\sharp) = \Omega_{ij}((\theta^i)^\sharp, (\theta^j)^\sharp),$$

where in the last step we rename $\ell \to j$ after using the dummy index in the summation forming $(\theta^i)^\sharp$. Using the curvature 2-forms, we see this funny thing: it is easier to compute s directly, completely bypassing the computation of Ric.

We'll say that $(M, g)$ is an *Einstein manifold* if there is $\lambda \in \mathbb{R}$ such that $\mathrm{Ric} = \lambda g$ If $(M, g)$ is Einstein, we know exactly what $\lambda$ should be: apply $\mathrm{tr}_g$ on both sides and solve for $\lambda = \mathrm{s}/n$. On the other hand, we see that s is necessarily constant. But what if $\lambda$ was not a constant? If $\dim M \geq 3$, it does not matter.

**Theorem** (Schur). *Let $(M^n, g)$ be a connected pseudo-Riemannian manifold with $n \geq 3$, and $f \in \mathscr{C}^\infty(M)$ be such that* $\mathrm{Ric} = fg$. *Then $f$ is constant and $(M, g)$ is Einstein (hence with constant scalar curvature).*

**Proof:** Start with the *second Bianchi identity*, expressed in coordinates as

$$R_{jk\ell m;i} + R_{ki\ell m;j} + R_{ij\ell m;k} = 0.$$

Raise the index $m$ to get $R_{jk\ell}{}^{m}{}_{;i} + R_{ki\ell}{}^{m}{}_{;j} + R_{ij\ell}{}^{m}{}_{;k} = 0$. Make $m = k$ to obtain

$$-R_{j\ell;i} + R_{i\ell;j} + R_{ij\ell}{}^{m}{}_{;m} = 0.$$

Note here that since the Ricci tensor is symmetric, we may indeed write $R^i_j$ instead of $R^i{}_j$ or $R_j{}^i$, as they're all equal. Now, attack that expression with $g^{i\ell}$ and use that metric contractions commute with covariant derivatives (as $\nabla g = 0$) to obtain

$$-R^i_{j;i} + \mathrm{s}_{;j} + g^{i\ell} R_{ij\ell}{}^{m}{}_{;m} = 0.$$

Now, using the first Bianchi identity for that last term, we get

$$g^{i\ell} R_{ij\ell}{}^{m}{}_{;m} = -g^{i\ell} R_{ji\ell}{}^{m}{}_{;m} - g^{i\ell} R_{\ell ij}{}^{m}{}_{;m} = -R^m_{j;m} - 0 = -R^m_{j;m},$$

by using symmetries of $R$. So $\mathrm{s}_{;j} = 2R^m_{j;m}$, which is to say, $\mathrm{ds} = 2\,\mathrm{div}\,\mathrm{Ric}$. With that expression in place, let's compute $\mathrm{div}(fg)$, as follows[2]:

$$(fg)_j{}^{m}{}_{;m} = g^{im}(fg)_{ji;m} = g^{im}(f_{;m}g_{ji} + fg_{ji;m}) = \delta^m_j f_{;m} + 0 = f_{;j} = \partial_j f.$$

So $\mathrm{div}(fg) = \mathrm{d}f$. With this, we have

$$\begin{aligned} \mathrm{ds} &= 2\,\mathrm{div}\,\mathrm{Ric} = 2\,\mathrm{div}(fg) = 2\,\mathrm{d}f, \\ \mathrm{ds} &= \mathrm{d}(\mathrm{tr}_g \mathrm{Ric}) = \mathrm{d}\left(\mathrm{tr}_g(fg)\right) = \mathrm{d}(nf) = n\,\mathrm{d}f. \end{aligned}$$

As $n > 2$, it follows that $\mathrm{d}f = 0$. $\square$

### 3.3 Sectional curvature

If $x \in M$ and $\boldsymbol{v}, \boldsymbol{w} \in T_xM$ span a non-degenerate plane $\Pi \subseteq T_xM$, the *sectional curvature* of $\Pi$ as

$$K(\Pi) = \frac{R_x(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v})}{\langle \boldsymbol{v}, \boldsymbol{v}\rangle_x \langle \boldsymbol{w}, \boldsymbol{w}\rangle_x - \langle \boldsymbol{v}, \boldsymbol{w}\rangle_x^2}.$$

This definition indeed does not depend on the choice of basis for $\Pi$, as replacing $\boldsymbol{v} \mapsto a\boldsymbol{v} + c\boldsymbol{w}$ and $\boldsymbol{w} \to b\boldsymbol{v} + d\boldsymbol{w}$ produces the non-zero determinant $ad - bc$ both in numerator and denominator of the above expression. So, if we have linearly independent vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$, at least on some open subset of $M$, we may make sense of $K(\boldsymbol{X}, \boldsymbol{Y})$ there. So, if $(\boldsymbol{E}_i)$ is a local frame for $M$, we may write

$$K_{ij} = K(\boldsymbol{E}_i, \boldsymbol{E}_j) = \frac{R(\boldsymbol{E}_i, \boldsymbol{E}_j, \boldsymbol{E}_j, \boldsymbol{E}_i)}{g_{ii}g_{jj} - g_{ij}^2} = \frac{\Omega_{ij}(\boldsymbol{E}_i, \boldsymbol{E}_j)}{g_{ii}g_{jj} - g_{ij}^2}.$$

[2] A coordinate-free computation is as follows: note that, by definition, we have

$$\mathrm{div}(fg)(\boldsymbol{Z}) = \mathrm{tr}((\boldsymbol{X}, \boldsymbol{Y}) \mapsto \nabla_{\boldsymbol{X}}(fg)(\boldsymbol{Y}, \boldsymbol{Z})).$$

Now, we have $\nabla_{\boldsymbol{X}}(fg)(\boldsymbol{Y}, \cdot) = (\mathrm{d}f(\boldsymbol{X})g + f\nabla_{\boldsymbol{X}}g)(\boldsymbol{Y}, \cdot) = \mathrm{d}f(\boldsymbol{X})g(\boldsymbol{Y}, \cdot)$, since $\nabla g = 0$. Tracing gives that $\mathrm{div}(fg) = \langle \nabla f, \cdot \rangle = \mathrm{d}f$, as wanted.

Einstein's convention behaves poorly because $K_{ij}$ is not a tensor. In particular, if the chosen frame is orthonormal, then $K_{ij} = \varepsilon\Omega_{ij}(\boldsymbol{E}_i, \boldsymbol{E}_j)$, where $\varepsilon$ is 1 or $-1$ according whether the plane spanned by $\boldsymbol{E}_i$ and $\boldsymbol{E}_j$ is spacelike or timelike, respectively. If $\dim M = 2$, there is only one plane to be considered in $T_xM$: $T_xM$ itself. The quantity $K(x)$ is then called the *Gaussian curvature* of $M$ at $x$. Two important results to know about $K$ are:

**Proposition.** *If $(M, g)$ is a pseudo-Riemannian manifold with constant sectional curvature $K$, then*

$$R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} = K(\langle \boldsymbol{Y}, \boldsymbol{Z}\rangle \boldsymbol{X} - \langle \boldsymbol{X}, \boldsymbol{Z}\rangle \boldsymbol{Y}),$$

*for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M)$. In particular, we see that $R = 0$ if and only if $K = 0$.*

**Remark.**

- The above formula is easy to remember, at least: $R(\boldsymbol{X}, \boldsymbol{Y})$ evaluated at any $\boldsymbol{Z}$ should produce a combination of $\boldsymbol{X}$ and $\boldsymbol{Y}$. As for the coefficients, we didn't really have much choice.

- Bearing in mind the Lagrange identity in an arbitrary pseudo-Euclidean vector space $(V^n, g)$, $g(u_1 \times \cdots \times u_{n-1}, v_1 \times \cdots \times v_{n-1}) = \det(g(u_i, v_j))_{i,j=1}^{n-1}$, for any vectors $u_i, v_j \in V$, we may identify the bilinear map $u_1 \wedge u_2$ taking $(v_1, v_2)$ to

$$\begin{vmatrix} g(u_1, v_1) & g(u_1, v_2) \\ g(u_2, v_1) & g(u_2, v_2) \end{vmatrix} = g(u_1, v_1)g(u_2, v_2) - g(u_2, v_1)g(u_1, v_2)$$

  with a linear endomorphism $u_1 \wedge u_2$ of $V$ acting on $v_1$, by using the non-degeneracy of $g$ to write $(u_1 \wedge u_2)(v) = \langle u_2, v\rangle u_1 - \langle u_1, v\rangle u_2$. With this notation, we can say that if a pseudo-Riemannian manifold $(M, g)$ has constant sectional curvature $K$, then $R(\boldsymbol{X}, \boldsymbol{Y}) = K\boldsymbol{X} \wedge \boldsymbol{Y}$, for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$.

**Proof:** This is a linear algebra fact: let $(V, g)$ be a pseudo-Euclidean vector space and $R, \widetilde{R}$ be two curvaturelike $(0,4)$-tensors on $V$ determining the same sectional curvature function: $K = \widetilde{K}$. We will prove that $R = \widetilde{R}$. This concludes the proof of the proposition, since the formula for $R$ given in the statement indeed produces constant sectional curvature $K$. To wit,

$$K(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = \frac{\widetilde{R}(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = \widetilde{K}(X, Y)$$

readily implies that $R(X, Y, Y, X) = \widetilde{R}(X, Y, Y, X)$ for all pairs $(X, Y)$ of vectors in $V$ spanning a non-degenerate plane. By a simple continuity argument, we have that $R(X, Y, Y, X) = \widetilde{R}(X, Y, Y, X)$ holds for all $X, Y \in V$. Since the difference of curvaturelike tensors is again curvaturelike, we may assume that $\widetilde{R} = 0$ and show that $R = 0$. This is done by polarizing twice. Namely, the first polarization gives

$$2R(X, Y, Y, Z) = R(X + Z, Y, Y, X + Z) = 0 \implies R(X, Y, Y, Z) = 0.$$

The second one goes in two parts:

$$0 = R(X, Y + W, Y + W, Z) = R(X, Y, W, Z) + R(X, W, Y, Z),$$

so that $R$ is skew on the middle pair. Apply symmetries of $R$ to bring $X$ to the last argument, obtaining

$$R(W,Z,Y,X) + R(Y,Z,W,X) = 0.$$

Now we apply the Bianchi identity on the first term (cycling through the first three entries), and the middle-pair skew symmetry on the second term to finally get $3R(Y,Z,W,X) = 0$. We are done. $\square$

**Lemma** (Schur)**.** *If $(M,g)$ is a connected pseudo-Riemannian manifold with $\dim M \geq 3$ for which the sectional curvature is pointwise constant (that is, $K(\boldsymbol{v},\boldsymbol{w})$ depends only on the point $x \in M$ but not on the vectors $\boldsymbol{v},\boldsymbol{w} \in T_xM$), then the sectional curvature of $(M,g)$ is in fact constant.*

**Proof:** The previous result applied pointwise says that (say, in $(0,4)$ form) we have $R = KR_0$, where $R_0(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = \langle \boldsymbol{Y},\boldsymbol{Z}\rangle\langle \boldsymbol{X},\boldsymbol{W}\rangle - \langle \boldsymbol{X},\boldsymbol{Z}\rangle\langle \boldsymbol{Y},\boldsymbol{W}\rangle$. Now, since $\nabla g = 0$, it also follows that $\nabla R_0 = 0$, and so $\nabla_{\boldsymbol{V}} R = \boldsymbol{V}(K)R_0$ for all $\boldsymbol{V} \in \mathfrak{X}(M)$. With this, the differential Bianchi identity

$$(\nabla_{\boldsymbol{X}} R)(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{V},\boldsymbol{W}) + (\nabla_{\boldsymbol{Y}} R)(\boldsymbol{Z},\boldsymbol{X},\boldsymbol{V},\boldsymbol{W}) + (\nabla_{\boldsymbol{Z}} R)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{V},\boldsymbol{W}) = 0$$

becomes

$$\boldsymbol{X}(K)R_0(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{V},\boldsymbol{W}) + \boldsymbol{Y}(K)R_0(\boldsymbol{Z},\boldsymbol{X},\boldsymbol{V},\boldsymbol{W}) + \boldsymbol{Z}(K)R_0(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{V},\boldsymbol{W}) = 0,$$

for *any* fields $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{V}$ and $\boldsymbol{W} \in \mathfrak{X}(M)$. The proof is concluded once we verify that $\boldsymbol{X}(K) = 0$. For this end, we may make suitable choices for the other fields, since this equality has a tensorial character and $\dim M \geq 3$. More precisely, fix $x \in M$, assume that $\boldsymbol{X}_x \neq 0$ and $\boldsymbol{Y}_x \neq 0$ are linearly independent, that $\boldsymbol{X}_x$, $\boldsymbol{Y}_x$ and $\boldsymbol{Z}_x$ are pairwise orthogonal, $\boldsymbol{Z}_x$ is a unit vector (hence with constant causal type in a neighborhood of $x$, say $\varepsilon = \pm 1$, and $\boldsymbol{W}_x = \boldsymbol{Z}_x$. Evaluating the Bianchi identity with these choices yields

$$-\varepsilon \boldsymbol{X}_x(K)\langle \boldsymbol{Y}_x,\boldsymbol{V}_x\rangle + \varepsilon \boldsymbol{Y}_x(K)\langle \boldsymbol{X}_x,\boldsymbol{V}_x\rangle + 0 = 0,$$

and since $\boldsymbol{V}$ was still arbitrary, it follows that $-\boldsymbol{X}_x(K)\boldsymbol{Y}_x + \boldsymbol{Y}_x(K)\boldsymbol{X}_x = \boldsymbol{0}$. By linear independence, $\boldsymbol{X}_x(K) = 0$. Since $\boldsymbol{X}_x \in T_xM$ and $x \in M$ were arbitrary, $\mathrm{d}K = 0$ and by connected of $M$ we get that $K$ is constant, as wanted. $\square$

**Corollary.** *Let $(M^n,g)$ be a pseudo-Riemannian manifold with constant sectional curvature $K$. Then $\mathrm{Ric} = (n-1)Kg$ and $\mathrm{s} = n(n-1)K$. In particular, $(M,g)$ is Einstein.*

**Proof:** Trace $R(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = K(\langle \boldsymbol{Y},\boldsymbol{Z}\rangle\langle \boldsymbol{X},\boldsymbol{W}\rangle - \langle \boldsymbol{X},\boldsymbol{Z}\rangle\langle \boldsymbol{Y},\boldsymbol{W}\rangle)$ twice. $\square$

With those results set in place, we may also explore the direct relation between $K$ and s for surfaces:

**Proposition.** *Let $(M^2,g)$ be a pseudo-Riemannian surface. Then:*

*(i)* $\mathrm{s} = 2K$.

*(ii)* *If $(\boldsymbol{E}_1,\boldsymbol{E}_2)$ is an orthonormal frame with $\boldsymbol{E}_1$ spacelike and $g(\boldsymbol{E}_2,\boldsymbol{E}_2) = \varepsilon \in \{-1,1\}$, then we have $\mathrm{d}\omega^1{}_2 = \varepsilon K\theta^1 \wedge \theta^2$.*

*(iii) If $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ is a Penrose frame, i.e., both $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are lightlike with $g(\boldsymbol{E}_1, \boldsymbol{E}_2) = 1$, then we have $\mathrm{d}\omega^2{}_2 = -K\,\theta^1 \wedge \theta^2$.*

*(iv) $\mathrm{Ric} \propto g$ (i.e., $\mathrm{Ric}$ is a function multiple of $g$).*

**Proof:**

(i) Raising and lowering the index 1 does not change any expression, while raising and lowering 2 amounts to multiplying by $\varepsilon$. We may write the Gaussian curvature as $K = \varepsilon R_{2112} = \varepsilon R_{1221} = \varepsilon R_{122}{}^1$, and then

$$\mathrm{s} = R_{11} + \varepsilon R_{22} = R_{211}{}^2 + \varepsilon R_{122}{}^1 = \varepsilon R_{2112} + K = K + K = 2K.$$

(ii) We have that $\mathrm{d}\omega^1{}_2 = \mathrm{d}\omega^1{}_2(\boldsymbol{E}_1, \boldsymbol{E}_2)\,\theta^1 \wedge \theta^2$, where $(\theta^1, \theta^2)$ is the dual coframe to $(\boldsymbol{E}_1, \boldsymbol{E}_2)$. But the metric compatibility $\omega_{ij} + \omega_{ji} = 0$ implies (by raising $i$) that $\omega^1{}_1 = \omega^2{}_2 = 0$. Then $\Omega^1{}_2 = \mathrm{d}\omega^1{}_2$, by the second structure equation. It follows that $\Omega^1{}_2(\boldsymbol{E}_1, \boldsymbol{E}_2) = \Omega_{12}(\boldsymbol{E}_1, \boldsymbol{E}_2) = \varepsilon K$, as wanted.

(iii) Let $(\theta^1, \theta^2)$ again denote the dual coframe to $(\boldsymbol{E}_1, \boldsymbol{E}_2)$. For a Penrose frame, lowering or raising an index amounts to replacing it by the other one. Write $\mathrm{d}\omega^2{}_2 = \mathrm{d}\omega^2{}_2(\boldsymbol{E}_1, \boldsymbol{E}_2)\theta^1 \wedge \theta^2$. By definition of Gaussian curvature and noting that $g_{11}g_{22} - g_{12}^2 = -1$, we have that $-K = \Omega_{12}(\boldsymbol{E}_1, \boldsymbol{E}_2) = \Omega^2{}_2(\boldsymbol{E}_1, \boldsymbol{E}_2)$. Now, the metric compatibility $\omega_{ij} + \omega_{ji} = 0$ allows us to write $\Omega^2{}_2 = \mathrm{d}\omega^2{}_2$ (since $\omega^1{}_2 = \omega_{22} = 0$ and $\omega^2{}_2 = \omega_{12}$, etc.).

(iv) The full force of Schur's lemma won't apply as $\dim M = 2 < 3$, but we still have the formula $R(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) = K(\langle \boldsymbol{Y}, \boldsymbol{Z}\rangle\langle \boldsymbol{X}, \boldsymbol{W}\rangle - \langle \boldsymbol{X}, \boldsymbol{Z}\rangle\langle \boldsymbol{Y}, \boldsymbol{W}\rangle)$, where the Gaussian curvature $K$ may be nonconstant. Then $\mathrm{Ric} = Kg$ by tracing.

$\square$

**Remark.** The above result means that for surfaces, the only quantity we really care about is the Gaussian curvature $K$, which can be computed through the connection forms only.

# 4 Cartan Computations

## 4.1 Constant coefficients metric in $\mathbb{R}^n$

Fix a nonsingular symmetric matrix $(a_{ij})_{i,j=1}^n$, and take in $\mathbb{R}^n$ the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle = a_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$. Consider the natural coordinate frame $(\partial_1,\ldots,\partial_n)$ and the dual coframe $(\mathrm{d}x^1,\ldots,\mathrm{d}x^n)$. The (global) connection forms $\omega^i{}_j$ are characterized by $0 = \omega_{ij}+\omega_{ji}$ and $0 = \mathrm{d}x^j \wedge \omega^i{}_j$. Since $\omega^i{}_j = 0$ for all choices of $i$ and $j$ fits the bill, this *must* be the case (by the uniqueness previously proven). It also follows that $\Omega^i{}_j = 0$ and hence the Levi-Civita connection of this metric is flat. Hence $\mathrm{Ric} = 0$, $\mathrm{s} = 0$, the metric is Einstein, etc..

## 4.2 Schwarzschild half-plane with mass $\mathrm{M} > 0$

In $P_I = \{(t,r) \in \mathbb{R}^2 \mid r > 2\mathrm{M}\}$, take the Lorentzian metric

$$\langle\cdot,\cdot\rangle = -\hbar(r)\,\mathrm{d}t\otimes\mathrm{d}t + \hbar(r)^{-1}\,\mathrm{d}r\otimes\mathrm{d}r,$$

where $\hbar(r) = 1 - 2\mathrm{M}/r$ is the Schwarzschild horizon function. Consider the natural coordinate frame $(\partial_t,\partial_r)$ and the dual coframe $(\mathrm{d}t,\mathrm{d}r)$. We can already observe from the metric expression that lowering and raising indices amounts to multiplying or dividing by $\pm\hbar(r)$. Also, we have that

$$(\mathrm{d}t)^\sharp = -\frac{1}{\hbar(r)}\partial_t \quad \text{and} \quad (\mathrm{d}r)^\sharp = \hbar(r)\partial_r.$$

**Connection 1-forms:** We have four connection 1-forms here, $\omega^t{}_t$, $\omega^r{}_t$, $\omega^t{}_r$ and $\omega^r{}_r$ (we're dropping Einstein's convention). It is easier to find first the versions with both lower indices. We immediately have

$$2\omega_{tt} = \omega_{tt}+\omega_{tt} = \mathrm{d}g_{tt} = -\hbar'(r)\,\mathrm{d}r \implies \omega_{tt} = -\frac{\hbar'(r)}{2}\,\mathrm{d}r$$

and similarly

$$2\omega_{rr} = \omega_{rr}+\omega_{rr} = \mathrm{d}g_{rr} = -\frac{\hbar'(r)}{\hbar(r)^2}\,\mathrm{d}r \implies \omega_{rr} = -\frac{\hbar'(r)}{2\hbar(r)^2}\,\mathrm{d}r.$$

Raising the indices, we get

$$\omega^t{}_t = g^{tt}\omega_{tt} + g^{tr}\omega_{rt} = \frac{-1}{\hbar(r)}\left(-\frac{\hbar'(r)}{2}\,\mathrm{d}r\right) = \frac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}r$$

and also

$$\omega^r{}_r = g^{rt}\omega_{tr} + g^{rr}\omega_{rr} = \hbar(r)\left(-\frac{\hbar'(r)}{2\hbar(r)^2}\,\mathrm{d}r\right) = -\frac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}r$$

Since $\omega_{rt} + \omega_{tr} = 0$, to find the remaining 1-form we use the structure equations. Lowering $t$ and $r$ on the following first and second equations, respectively, gives

$$\begin{cases} 0 = \mathrm{d}(\mathrm{d}t) = \mathrm{d}t \wedge \omega^t{}_t + \mathrm{d}r \wedge \omega^t{}_r \\ 0 = \mathrm{d}(\mathrm{d}r) = \mathrm{d}t \wedge \omega^r{}_t + \mathrm{d}r \wedge \omega^r{}_r \end{cases} \implies \begin{cases} \mathrm{d}t \wedge \omega_{tt} + \mathrm{d}r \wedge \omega_{tr} = 0 \\ \mathrm{d}t \wedge \omega_{rt} + \mathrm{d}r \wedge \omega_{rr} = 0, \end{cases}$$

which simplifies to $\mathrm{d}t \wedge \omega_{rt} = 0$ and $\mathrm{d}r \wedge \omega_{rt} = \mathrm{d}t \wedge \omega_{tt} = -(\hbar'(r)/2)\mathrm{d}t \wedge \mathrm{d}r$. This means that writing $\omega_{rt} = \omega_{rtt}\,\mathrm{d}t + \omega_{rtr}\,\mathrm{d}r$, we have

$$0 = \mathrm{d}t \wedge \omega_{rt} = \omega_{rtr}\,\mathrm{d}t \wedge \mathrm{d}r \quad \text{and} \quad -\frac{\hbar'(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r = \mathrm{d}r \wedge \omega_{rt} = -\omega_{rtt}\mathrm{d}t \wedge \mathrm{d}r,$$

and so $\omega_{rt} = (\hbar'(r)/2)\,\mathrm{d}t$. Raising the index $r$, we obtain that

$$\omega^r{}_t = g^{rt}\omega_{tt} + g^{rr}\omega_{rt} = \frac{\hbar(r)\hbar'(r)}{2}\,\mathrm{d}t.$$

Similarly

$$\omega^t{}_r = g^{tt}\omega_{tr} + g^{tr}\omega_{rr} = -g^{tt}\omega_{rt} = \frac{1}{\hbar(r)}\frac{\hbar'(r)}{2}\,\mathrm{d}t = \frac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}t.$$

Let's organize our results in tables:

| | |
|---|---|
| $\omega^t{}_t = \dfrac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}r$ | $\omega^t{}_r = \dfrac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}t$ |
| $\omega^r{}_t = \dfrac{\hbar(r)\hbar'(r)}{2}\,\mathrm{d}t$ | $\omega^r{}_r = -\dfrac{\hbar'(r)}{2\hbar(r)}\,\mathrm{d}r$ |

| | |
|---|---|
| $\omega_{tt} = -\dfrac{\hbar'(r)}{2}\,\mathrm{d}r$ | $\omega_{tr} = -\dfrac{\hbar'(r)}{2}\,\mathrm{d}t$ |
| $\omega_{rt} = \dfrac{\hbar'(r)}{2}\,\mathrm{d}t$ | $\omega_{rr} = -\dfrac{\hbar'(r)}{2\hbar(r)^2}\,\mathrm{d}r$ |

**Curvature 2-forms:** For the first one, we have

$$\Omega^t{}_t = \mathrm{d}\omega^t{}_t + \omega^t{}_t \wedge \omega^t{}_t + \omega^t{}_r \wedge \omega^r{}_t = 0 + 0 + 0 = 0.$$

Then

$$\begin{aligned} \Omega^t{}_r &= \mathrm{d}\omega^t{}_r + \omega^t{}_t \wedge \omega^t{}_r + \omega^t{}_r \wedge \omega^r{}_r \\ &= \frac{2\hbar'(r)^2 - 2\hbar(r)\hbar''(r)}{4\hbar(r)^2}\mathrm{d}t \wedge \mathrm{d}r - \frac{\hbar'(r)^2}{4\hbar(r)^2}\,\mathrm{d}t \wedge \mathrm{d}r - \frac{\hbar'(r)^2}{4\hbar(r)^2}\,\mathrm{d}t \wedge \mathrm{d}r \\ &= \frac{-\hbar''(r)}{2\hbar(r)}\,\mathrm{d}t \wedge \mathrm{d}r. \end{aligned}$$

The next one is

$$\begin{aligned} \Omega^r{}_t &= \mathrm{d}\omega^r{}_t + \omega^r{}_t \wedge \omega^t{}_t + \omega^r{}_r \wedge \omega^r{}_t \\ &= -\frac{\hbar'(r)^2 + \hbar(r)\hbar''(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r + \frac{\hbar'(r)^2}{4}\,\mathrm{d}t \wedge \mathrm{d}r + \frac{\hbar'(r)^2}{4}\,\mathrm{d}t \wedge \mathrm{d}r \\ &= -\frac{\hbar(r)\hbar''(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r, \end{aligned}$$

and finally $\Omega^r{}_r = \mathrm{d}\omega^r{}_r + \omega^r{}_t \wedge \omega^t{}_r + \omega^r{}_r \wedge \omega^r{}_r = 0 + 0 + 0 = 0$. Let's summarize it:

| | |
|---|---|
| $\Omega^t{}_t = 0$ | $\Omega^t{}_r = -\dfrac{\hbar''(r)}{2\hbar(r)}\,\mathrm{d}t \wedge \mathrm{d}r$ |
| $\Omega^r{}_t = -\dfrac{\hbar(r)\hbar''(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r$ | $\Omega^r{}_r = 0$ |

With this in hand, we can compute further curvature invariants. Lowering all the indices, we obtain also the following table:

| | |
|---|---|
| $\Omega_{tt} = 0$ | $\Omega_{tr} = \dfrac{\hbar''(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r$ |
| $\Omega_{rt} = -\dfrac{\hbar''(r)}{2}\,\mathrm{d}t \wedge \mathrm{d}r$ | $\Omega_{rr} = 0$ |

**Ric and** s : First, we have that

$$R_{tt} = \Omega_{tr}((\mathrm{d}r)^\sharp, \partial_t) = \frac{\hbar''(r)}{2}(\mathrm{d}t \wedge \mathrm{d}r)(\hbar(r)\partial_r, \partial_t) = \frac{\hbar''(r)}{2}\begin{vmatrix} 0 & 1 \\ \hbar(r) & 0 \end{vmatrix} = -\frac{\hbar(r)\hbar''(r)}{2}.$$

Then we have

$$R_{tr} = \Omega_{rt}((\mathrm{d}t^\sharp), \partial_t) = -\frac{\hbar''(r)}{2}(\mathrm{d}t \wedge \mathrm{d}r)\left(-\frac{1}{\hbar(r)}\partial_t, \partial_t\right) = 0.$$

By symmetry it also follows that $R_{rt} = 0$. Lastly, we obtain that

$$R_{rr} = \Omega_{rt}((\mathrm{d}t)^\sharp, \partial_r) = -\frac{\hbar''(r)}{2}(\mathrm{d}t \wedge \mathrm{d}r)\left(-\frac{1}{\hbar(r)}\partial_t, \partial_r\right) = \frac{\hbar''(r)}{2\hbar(r)}.$$

Putting all of this together, we conclude that

$$\mathrm{Ric} = -\frac{\hbar(r)\hbar''(r)}{2}\,\mathrm{d}t \otimes \mathrm{d}t + \frac{\hbar''(r)}{2\hbar(r)}\,\mathrm{d}r \otimes \mathrm{d}r = \frac{\hbar''(r)}{2}\langle \cdot, \cdot \rangle$$

Then $\mathrm{s} = \hbar''(r) = -4\mathrm{M}/r^3 < 0$. We also get $K = \hbar''(r)/2$. All of those computations hold for half-planes equipped with more sophisticated horizon functions (e.g., Reissner-Nördstrom, Kerr-Newman horizons, etc.).

## 4.3 Hyperbolic half-space

In $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_{>0}$, take the Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{|\mathrm{d}\boldsymbol{x}|^2 + \mathrm{d}y^2}{y^2}$$

and consider the natural frame $(\partial_1, \ldots, \partial_n, \partial_y)$ with dual coframe $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n, \mathrm{d}y)$. For convenience, we may also set $y = x^{n+1}$ and work with indices

$$A, B, C, D = 1, \ldots, n \qquad \text{and} \qquad i, j, k, \ell = 1, \ldots, n+1.$$

We have $g_{ij} = \delta_{ij}/y^2$ and $g^{ij} = \delta^{ij}y^2$, and in particular, we see that raising or lowering an index amounts to multiplying or dividing by $y^2$. We also have $(\mathrm{d}x^i)^\sharp = \delta^{ij}y^2\partial_j$.

**Connection 1-forms:** Metric compatibility then reads

$$\omega_{ij} + \omega_{ji} = \frac{-2\delta_{ij}}{y^3}\mathrm{d}y$$

for all $i$ and $j$. The structure equations become $0 = \mathrm{d}(\mathrm{d}x^i) = \mathrm{d}x^j \wedge \omega^i{}_j$, and we lower the index $i$ to obtain $\mathrm{d}x^j \wedge \omega_{ij} = 0$. Write $\omega_{ij} = \omega_{ijk}\,\mathrm{d}x^k$. Then

$$\omega_{ijk}\mathrm{d}x^j \wedge \mathrm{d}x^k = 0 \implies \omega_{ijk} = \omega_{ikj}, \quad \text{for all } i, j, k.$$

Let's focus on the different types of connection 1-forms:

- We have $2\omega_{yy} = (-2/y^3)\mathrm{d}y$, so $\omega_{yy} = -\mathrm{d}y/y^3$ and hence $\omega^y{}_y = -\mathrm{d}y/y$.
- By metric compatibility, $\omega_{ABC} = -\omega_{BAC}$. And by the structure equations, we also have $\omega_{ABC} = \omega_{ACB}$. It follows[3] that $\omega_{ABC} = 0$ for all $A, B$ and $C$. So the coefficient left to be found is $\omega_{ABy}$. We have that

  $$0 = \mathrm{d}x^j \wedge \omega_{yj} = \mathrm{d}x^A \wedge \omega_{yA} + \mathrm{d}y \wedge \omega_{yy} = -\mathrm{d}x^A \wedge \omega_{Ay},$$

  from which

  $$0 = \mathrm{d}x^A \wedge (\omega_{AyB}\,\mathrm{d}x^B + \omega_{Ayy}\,\mathrm{d}y) = \omega_{AyB}\,\mathrm{d}x^A \wedge \mathrm{d}x^B + \omega_{Ayy}\,\mathrm{d}x^A \wedge \mathrm{d}y$$

  implies that $\omega_{AyB} = \omega_{ByA}$ and $\omega_{Ayy} = 0$. Thus

  $$\omega_{ABy} = \omega_{AyB} = \omega_{ByA} = \omega_{BAy} = -2\frac{\delta_{AB}}{y^3} - \omega_{ABy} \implies \omega_{ABy} = -\frac{\delta_{AB}}{y^3}.$$

  Hence $\omega_{AB} = -(\delta_{AB}/y^3)\,\mathrm{d}y$ and $\omega^A_B = -(\delta^A_B/y)\,\mathrm{d}y$.
- Again, metric compatibility says that $\omega_{Ay} + \omega_{yA} = 0$. So we'll focus on the first term. By the above, we have $\omega_{AyB} = -\delta_{AB}/y^3$, that together with the equality $\omega_{Ayy} = 0$ yields

  $$\omega_{Ay} = -\frac{\delta_{AB}}{y^3}\,\mathrm{d}x^B \implies \omega^A_y = -\frac{1}{y}\,\mathrm{d}x^A.$$

  Similarly, we get

  $$\omega_{yB} = -\omega_{By} = \frac{\delta_{AB}}{y^3}\,\mathrm{d}x^A \implies \omega^y{}_B = \frac{\delta_{AB}}{y}\,\mathrm{d}x^A$$

[3] $ABC = -BAC = -BCA = CBA = CAB = -ACB = -ABC \implies ABC = 0.$

Let's organize our results in tables:

| | |
|---|---|
| $\omega^A_{\ B} = -\dfrac{\delta^A_B}{y}\,\mathrm{d}y$ | $\omega^A_{\ y} = -\dfrac{1}{y}\,\mathrm{d}x^A$ |
| $\omega^y_{\ B} = \dfrac{\delta_{AB}}{y}\,\mathrm{d}x^A$ | $\omega^y_{\ y} = -\dfrac{1}{y}\,\mathrm{d}y$ |

| | |
|---|---|
| $\omega_{AB} = -\dfrac{\delta_{AB}}{y^3}\,\mathrm{d}y$ | $\omega_{Ay} = -\dfrac{\delta_{AB}}{y^3}\,\mathrm{d}x^B$ |
| $\omega_{yB} = \dfrac{\delta_{AB}}{y^3}\,\mathrm{d}x^A$ | $\omega_{yy} = -\dfrac{1}{y^3}\,\mathrm{d}y$ |

**Curvature 2-forms:** We'll have four types of curvature 2-forms, as follows:

$$\Omega^A_{\ B} = \mathrm{d}\omega^A_{\ B} + \omega^A_{\ C}\wedge\omega^C_{\ B} + \omega^A_{\ y}\wedge\omega^y_{\ B} = 0 + 0 - \frac{\delta_{BC}}{y^2}\mathrm{d}x^A\wedge\mathrm{d}x^C = -\frac{\delta_{BC}}{y^2}\mathrm{d}x^A\wedge\mathrm{d}x^C.$$

The next one is

$$\begin{aligned}\Omega^A_{\ y} &= \mathrm{d}\omega^A_{\ y} + \omega^A_{\ C}\wedge\omega^C_{\ y} + \omega^A_{\ y}\wedge\omega^y_{\ y}\\ &= -\frac{1}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y - \frac{1}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y + \frac{1}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y\\ &= -\frac{1}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y.\end{aligned}$$

Then

$$\begin{aligned}\Omega^y_{\ B} &= \mathrm{d}\omega^y_{\ B} + \omega^y_{\ C}\wedge\omega^C_{\ B} + \omega^y_{\ y}\wedge\omega^y_{\ B}\\ &= \frac{\delta_{AB}}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y - \frac{\delta_{BA}}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y + \frac{\delta_{BC}}{y^2}\,\mathrm{d}x^C\wedge\mathrm{d}y\\ &= \frac{\delta_{BC}}{y^2}\,\mathrm{d}x^C\wedge\mathrm{d}y\end{aligned}$$

and also $\Omega^y_{\ y} = \mathrm{d}\omega^y_{\ y} + \omega^y_{\ B}\wedge\omega^B_{\ y} + \omega^y_{\ y}\wedge\omega^y_{\ y} = 0+0+0 = 0$. Organizing it all, we have:

| | |
|---|---|
| $\Omega^A_{\ B} = -\dfrac{\delta_{BC}}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}x^C$ | $\Omega^A_{\ y} = -\dfrac{1}{y^2}\,\mathrm{d}x^A\wedge\mathrm{d}y$ |
| $\Omega^y_{\ B} = \dfrac{\delta_{BC}}{y^2}\,\mathrm{d}x^C\wedge\mathrm{d}y$ | $\Omega^y_{\ y} = 0$ |

Lower the indexes to get

| | |
|---|---|
| $\Omega_{AB} = -\dfrac{\delta_{AD}\delta_{BC}}{y^4}\,\mathrm{d}x^D\wedge\mathrm{d}x^C$ | $\Omega_{Ay} = -\dfrac{\delta_{AB}}{y^4}\,\mathrm{d}x^B\wedge\mathrm{d}y$ |
| $\Omega_{yB} = \dfrac{\delta_{BC}}{y^4}\,\mathrm{d}x^C\wedge\mathrm{d}y$ | $\Omega_{yy} = 0$ |

With this, we can compute $R$ explicitly. Namely, note that $\{y\partial_i\}_{i=1}^{n+1}$ is an orthonormal frame for $\mathbb{H}^{n+1}$. Then consider $R(y\partial_i, y\partial_j, y\partial_k, y\partial_\ell)$. It suffices to check three cases:

- when $y = x^{n+1}$ does not appear:

$$\begin{aligned}
R(y\partial_A, y\partial_B, y\partial_C, y\partial_D) &= y^4 R(\partial_A, \partial_B, \partial_C, \partial_D) = y^4 \Omega_{DC}(\partial_A, \partial_B) \\
&= -y^4 \frac{\delta_{DE}\delta_{CF}}{y^4}(\mathrm{d}x^E \wedge \mathrm{d}x^F)(\partial_A, \partial_B) \\
&= -\delta_{DE}\delta_{CF}(\delta^E_A \delta^F_B - \delta^E_B \delta^F_A) \\
&= -(\delta_{AD}\delta_{BC} - \delta_{BD}\delta_{AC})
\end{aligned}$$

- when $y = x^{n+1}$ appears exactly once: assume that it appears on the last argument (apply symmetries of $R$ and rename indices if needed) and compute

$$\begin{aligned}
R(y\partial_A, y\partial_B, y\partial_C, y\partial_y) &= y^4 R(\partial_A, \partial_B, \partial_C, \partial_y) = y^4 \Omega_{yC}(\partial_A, \partial_B) \\
&= y^4 \frac{\delta_{CD}}{y^4}(\mathrm{d}x^D \wedge \mathrm{d}y)(\partial_A, \partial_B) = 0.
\end{aligned}$$

- when $y = x^{n+1}$ appears twice, once in the first pair of arguments and once in the second pair: assume that it appears in the first and last entry. Write

$$\begin{aligned}
R(y\partial_y, y\partial_B, y\partial_C, y\partial_y) &= y^4 R(\partial_y, \partial_B, \partial_C, \partial_y) = y^4 \Omega_{yC}(\partial_y, \partial_B) \\
&= y^4 \frac{\delta_{CD}}{y^4}(\mathrm{d}x^D \wedge \mathrm{d}y)(\partial_y, \partial_B) \\
&= \delta_{CD}(-\delta^D_B) \\
&= -\delta_{BC}
\end{aligned}$$

We conclude that $\mathbb{H}^{n+1}$ has constant sectional curvature $-1$. It follows from this that $\mathrm{Ric} = -n\langle\cdot,\cdot\rangle$ and $\mathrm{s} = -n(n+1)$.

### 4.4 Surfaces of Revolution in $\mathbb{R}^3$

Consider an open interval $I \subseteq \mathbb{R}$, a smooth function $f\colon I \to \mathbb{R}_{>0}$, and the warped product $I \times_f \mathbb{S}^1$ with Riemannian metric given by

$$\langle\cdot,\cdot\rangle = \mathrm{d}s^2 + f(s)^2\,\mathrm{d}\theta^2.$$

Do note the abuse of notation: this $\mathrm{d}\theta \in \Omega^1(\mathbb{S}^1)$ is not an exact form – it is just the usual angle form, in the same way that $\partial_\theta$ just denotes the rotation field tangent to $\mathbb{S}^1$. This time we will not work with coordinate frames, and we'll omit the point of application $s$ from $f(s)$. Let's define an orthonormal frame field by setting $\boldsymbol{E}_s = \partial_s$ and $\boldsymbol{E}_\theta = (1/f)\partial_\theta$, so that the dual 1-forms are given by $\theta^s = \mathrm{d}s$ and $\theta^\theta = f\mathrm{d}\theta$, and $\langle\cdot,\cdot\rangle = \theta^s \otimes \theta^s + \theta^\theta \otimes \theta^\theta$. As a consequence from this last expression, we may freely raise and lower indexes without effectively changing anything.

**Connection 1-forms:** We immediately get $\omega_{ss} = \omega_{\theta\theta} = 0$, and so $\omega^s{}_\theta = \omega^\theta{}_\theta = 0$. Since $\omega_{s\theta} + \omega_{\theta s} = 0$, we may focus on the first term. From the structure equations, we see that

$$\begin{cases} \mathrm{d}\theta^s = \theta^s \wedge \omega^s{}_s + \theta^\theta \wedge \omega^s{}_\theta \\ \mathrm{d}\theta^\theta = \theta^s \wedge \omega^\theta{}_s + \theta^\theta \wedge \omega^\theta{}_\theta \end{cases} \implies \begin{cases} 0 = \theta^\theta \wedge \omega^s{}_\theta \\ \dfrac{f'}{f}\,\theta^s \wedge \theta^\theta = -\theta^s \wedge \omega^s{}_\theta. \end{cases}$$

So, write $\omega^s{}_\theta = \omega^s{}_{\theta s}\theta^s + \omega^s{}_{\theta\theta}\theta^\theta$. The first equation gives $\omega^s{}_{\theta s} = 0$, while the second yields $\omega^s{}_{\theta\theta} = -f'/f$. So $\omega^s{}_\theta = (-f'/f)\theta^\theta$ and $\omega^\theta{}_s = (f'/f)\theta^\theta$. Let's register this in a table:

| | |
|---|---|
| $\omega^s{}_s = 0$ | $\omega^s{}_\theta = -\dfrac{f'}{f}\,\theta^\theta$ |
| $\omega^\theta{}_s = \dfrac{f'}{f}\,\theta^\theta$ | $\omega^\theta{}_\theta = 0$ |

Even though it is easy to express these forms in terms of $\mathrm{d}s$ and $\mathrm{d}\theta$, one must remember that the connection forms found are the ones associated to $(\boldsymbol{E}_s, \boldsymbol{E}_\theta)$ instead of the coordinate frame $(\partial_s, \partial_\theta)$.

**Further curvatures:** Let's explore the fact that we're dealing with a surface, compute $K$ without computing the curvature 2-forms, and from there obtain Ric and s. We have that

$$\begin{aligned} \mathrm{d}\omega^s{}_\theta &= \mathrm{d}\left(-\frac{f'}{f}\right) \wedge \theta^\theta - \frac{f'}{f}\,\mathrm{d}\theta^\theta \\ &= \frac{(f')^2 - ff''}{f^2}\,\mathrm{d}s \wedge \theta^\theta - \frac{f'}{f}(f'\,\mathrm{d}s \wedge \mathrm{d}\theta) \\ &= \frac{(f')^2 - ff''}{f^2}\,\theta^s \wedge \theta^\theta - \frac{(f')^2}{f^2}\theta^s \wedge \theta^\theta \\ &= -\frac{f''}{f}\,\theta^s \wedge \theta^\theta. \end{aligned}$$

It follows that $K = -f''/f$, the Ricci tensor is $\mathrm{Ric} = -(f''/f)\langle\cdot,\cdot\rangle$ and $\mathrm{s} = -2f''/f$.

## 4.5 A thickening of $\mathbb{S}^2$

Let $I \subseteq \mathbb{R}$ be an open interval with natural coordinate $r$, and let $(\theta, \varphi)$ be spherical coordinates on $\mathbb{S}^2$. Consider in the product $I \times \mathbb{S}^2$ the Riemannian metric

$$\langle\cdot,\cdot\rangle = A(r)^2\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2,$$

where $A\colon I \to \mathbb{R}_{>0}$ is smooth. We will work with the orthonormal frame $(\boldsymbol{E}_r, \boldsymbol{E}_\theta, \boldsymbol{E}_\theta)$ given by

$$\boldsymbol{E}_r = \frac{1}{A(r)}\partial_r, \qquad \boldsymbol{E}_\theta = \frac{1}{r}\partial_\theta, \qquad \boldsymbol{E}_\varphi = \frac{1}{r\sin\theta}\partial_\varphi.$$

The corresponding dual coframe is then

$$\theta^r = A(r)\,\mathrm{d}r, \qquad \theta^\theta = r\,\mathrm{d}\theta, \qquad \theta^\varphi = r\sin\theta\,\mathrm{d}\varphi.$$

We may raise and lower indices at will.

**Connection 1-forms:** Since $\omega_{ij} + \omega_{ji} = 0$, it immediately follows (by raising $i$) that ${\omega^r}_r = {\omega^\theta}_\theta = {\omega^\varphi}_\varphi = 0$. We also have skew-symmetry for distinct indices. The first structure equations are

$$\begin{cases} \mathrm{d}\theta^r = \theta^\theta \wedge {\omega^r}_\theta + \theta^\varphi \wedge {\omega^r}_\varphi \\ \mathrm{d}\theta^\theta = \theta^r \wedge {\omega^\theta}_r + \theta^\varphi \wedge {\omega^\theta}_\varphi \\ \mathrm{d}\theta^\varphi = \theta^r \wedge {\omega^\varphi}_r + \theta^\theta \wedge {\omega^\varphi}_\theta \end{cases}$$

We will solve for ${\omega^r}_\theta$, ${\omega^r}_\varphi$ and ${\omega^\theta}_\varphi$ (the remaining ones are determined by symmetries). Focusing on those forms and rewriting the left side of the above equations in terms of the dual coframe, we get

$$\begin{cases} 0 = \theta^\theta \wedge {\omega^r}_\theta + \theta^\varphi \wedge {\omega^r}_\varphi \\ \dfrac{1}{rA(r)}\theta^r \wedge \theta^\theta = -\theta^r \wedge {\omega^r}_\theta + \theta^\varphi \wedge {\omega^\theta}_\varphi \\ \dfrac{1}{rA(r)}\theta^r \wedge \theta^\varphi + \dfrac{1}{r\tan\theta}\theta^\theta \wedge \theta^\varphi = -\theta^r \wedge {\omega^r}_\varphi - \theta^\theta \wedge {\omega^\theta}_\varphi \end{cases}$$

For $i,j,k \in \{r,\theta,\varphi\}$, write ${\omega^i}_j = {\omega^i}_{jk}\theta^k$ as usual. We'll solve a system for all these components. In terms of the components, it becomes

$$\begin{cases} 0 = -{\omega^r}_{\theta r}\theta^r \wedge \theta^\theta - {\omega^r}_{\varphi r}\theta^r \wedge \theta^\varphi + ({\omega^r}_{\theta\varphi} - {\omega^r}_{\varphi\theta})\theta^\theta \wedge \theta^\varphi \\ \dfrac{1}{rA(r)}\theta^r \wedge \theta^\theta = -{\omega^r}_{\theta\theta}\theta^r \wedge \theta^\theta + (-{\omega^r}_{\theta\varphi} - {\omega^\theta}_{\varphi r})\theta^r \wedge \theta^\varphi - {\omega^\theta}_{\varphi\theta}\theta^\theta \wedge \theta^\varphi \\ \dfrac{1}{rA(r)}\theta^r \wedge \theta^\varphi + \dfrac{1}{r\tan\theta}\theta^\theta \wedge \theta^\varphi = ({\omega^\theta}_{\varphi r} - {\omega^r}_{\varphi\theta})\theta^r \wedge \theta^\theta - {\omega^r}_{\varphi\varphi}\theta^r \wedge \theta^\varphi - {\omega^\theta}_{\varphi\varphi}\theta^\theta \wedge \theta^\varphi. \end{cases}$$

Extracting coefficients, we get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} {\omega^r}_{\theta r} \\ {\omega^r}_{\theta\theta} \\ {\omega^r}_{\theta\varphi} \\ {\omega^r}_{\varphi r} \\ {\omega^r}_{\varphi\theta} \\ {\omega^r}_{\varphi\varphi} \\ {\omega^\theta}_{\varphi r} \\ {\omega^\theta}_{\varphi\theta} \\ {\omega^\theta}_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{rA(r)} \\ 0 \\ 0 \\ 0 \\ \frac{1}{rA(r)} \\ \frac{1}{r\tan\theta} \end{pmatrix}$$

As complicated as this looks, it is just a linear system. From the first row we get ${\omega^r}_{\theta r} = 0$, and from the fourth that ${\omega^r}_{\theta\varphi} = -1/(rA(r))$. Combining rows 3, 7 and 5 (in this order) we get that ${\omega^r}_{\theta\varphi} = {\omega^r}_{\varphi\theta} = {\omega^\theta}_{\varphi r} = -{\omega^r}_{\theta\varphi}$, so all of those coefficients vanish. In particular, we know now that

$$ {\omega^r}_\theta = -\frac{1}{rA(r)}\theta^\theta. $$

We proceed: row 2 gives ${\omega^r}_{\varphi r} = 0$. By the above, ${\omega^r}_{\varphi\theta} = 0$, and row 8 says that ${\omega^r}_{\varphi\varphi} = -1/(rA(r))$. Thus

$$ {\omega^r}_\varphi = -\frac{1}{rA(r)}\theta^\varphi. $$

Finally, we already had ${\omega^\theta}_{\varphi r} = 0$. Now row 6 gives ${\omega^\theta}_{\varphi\theta} = 0$, and row 9 says that ${\omega^\theta}_{\varphi\varphi} = -1/(r\tan\theta)$. So

$$ {\omega^\theta}_\varphi = -\frac{1}{r\tan\theta}\theta^\varphi, $$

and we may organize our results in a table:

| | | |
|---|---|---|
| ${\omega^r}_\theta = -\frac{1}{rA(r)}\theta^\theta$ | ${\omega^r}_\varphi = -\frac{1}{rA(r)}\theta^\varphi$ | ${\omega^\theta}_\varphi = -\frac{1}{r\tan\theta}\theta^\varphi$ |

**Curvature 2-forms:** Again we may exploit symmetry and compute just the three relevant curvature forms ${\Omega^r}_\theta$, ${\Omega^r}_\varphi$ and ${\Omega^\theta}_\varphi$. First

$$ \begin{aligned} {\Omega^r}_\theta &= \mathrm{d}{\omega^r}_\theta + {\omega^r}_\varphi \wedge {\omega^\varphi}_\theta \\ &= \mathrm{d}\left(-\frac{1}{rA(r)}\theta^\theta\right) + \left(-\frac{1}{rA(r)}\theta^\varphi\right) \wedge \left(\frac{1}{r\tan\theta}\theta^\varphi\right) \\ &= \frac{A(r) + rA'(r)}{r^2A(r)^2}\,\mathrm{d}r \wedge \theta^\theta - \frac{1}{rA(r)}\,\mathrm{d}\theta^\theta \\ &= \frac{A(r) + rA'(r)}{r^2A(r)^3}\,\theta^r \wedge \theta^\theta - \frac{1}{rA(r)}\,\frac{1}{rA(r)}\theta^r \wedge \theta^\theta \\ &= \frac{A'(r)}{rA(r)^3}\,\theta^r \wedge \theta^\theta, \end{aligned} $$

then

$$ \begin{aligned} {\Omega^r}_\varphi &= \mathrm{d}{\omega^r}_\varphi + {\omega^r}_\theta \wedge {\omega^\theta}_\varphi \\ &= \mathrm{d}\left(-\frac{1}{rA(r)}\theta^\varphi\right) + \left(-\frac{1}{rA(r)}\theta^\theta\right) \wedge \left(-\frac{1}{r\tan\theta}\theta^\varphi\right) \\ &= \frac{A(r) + rA'(r)}{r^2A(r)^2}\,\mathrm{d}r \wedge \theta^\varphi - \frac{1}{rA(r)}\,\mathrm{d}\theta^\varphi + \frac{1}{r^2A(r)\tan\theta}\theta^\theta \wedge \theta^\varphi \\ &= \frac{A(r) + rA'(r)}{r^2A(r)^3}\,\theta^r \wedge \theta^\varphi - \frac{1}{rA(r)}\left(\frac{1}{rA(r)}\theta^r \wedge \theta^\varphi + \frac{1}{r\tan\theta}\theta^\theta \wedge \theta^\varphi\right) + \frac{1}{r^2A(r)\tan\theta}\theta^\theta \wedge \theta^\varphi \\ &= \frac{A'(r)}{rA(r)^3}\,\theta^r \wedge \theta^\varphi, \end{aligned} $$

and lastly

$$\begin{aligned}
\Omega^\theta{}_\varphi &= \mathrm{d}\omega^\theta{}_\varphi + \omega^\theta{}_r \wedge \omega^r{}_\varphi \\
&= \mathrm{d}\left(-\frac{1}{r\tan\theta}\theta^\varphi\right) + \left(\frac{1}{rA(r)}\theta^\theta\right) \wedge \left(-\frac{1}{rA(r)}\theta^\varphi\right) \\
&= \frac{1}{r^2\tan\theta}\,\mathrm{d}r\wedge\theta^\varphi + \frac{\sec^2\theta}{r\tan^2\theta}\,\mathrm{d}\theta\wedge\theta^\varphi - \frac{1}{r\tan\theta}\,\mathrm{d}\theta^\varphi - \frac{1}{r^2A(r)^2}\,\theta^\theta\wedge\theta^\varphi \\
&= \frac{1}{r^2A(r)\tan\theta}\,\theta^r\wedge\theta^\varphi + \frac{1}{r^2\sin^2\theta}\theta^\theta\wedge\theta^\varphi - \frac{1}{r\tan\theta}\left(\frac{1}{rA(r)}\theta^r\wedge\theta^\varphi + \frac{1}{r\tan\theta}\theta^\theta\wedge\theta^\varphi\right) - \frac{1}{r^2A(r)^2}\theta^\theta\wedge\theta^\varphi \\
&= \left(\frac{1}{r^2\sin^2\theta} - \frac{1}{r^2\tan^2\theta} - \frac{1}{r^2A(r)^2}\right)\theta^\theta\wedge\theta^\varphi \\
&= \frac{1}{r^2}\left(1 - \frac{1}{A(r)^2}\right)\theta^\theta\wedge\theta^\varphi.
\end{aligned}$$

Thus we obtain another table:

| | | |
|---|---|---|
| $\Omega^r{}_\theta = \dfrac{A'(r)}{rA(r)^3}\,\theta^r\wedge\theta^\theta$ | $\Omega^r{}_\varphi = \dfrac{A'(r)}{rA(r)^3}\,\theta^r\wedge\theta^\varphi$ | $\Omega^\theta{}_\varphi = \dfrac{1}{r^2}\left(1-\dfrac{1}{A(r)^2}\right)\theta^\theta\wedge\theta^\varphi$ |

**Ric and** s**:** For the Ricci curvature, we use the formula $R_{ij} = \Omega^k{}_j(\boldsymbol{E}_k, \boldsymbol{E}_i)$ to obtain:

$$\begin{aligned}
R_{rr} &= \Omega^\theta{}_r(\boldsymbol{E}_\theta,\boldsymbol{E}_r) + \Omega^\varphi{}_r(\boldsymbol{E}_\varphi,\boldsymbol{E}_r) = \frac{A'(r)}{rA(r)^3} + \frac{A'(r)}{rA(r)^3} = \frac{2A'(r)}{rA(r)^3} \\
R_{r\theta} &= \Omega^\varphi{}_\theta(\boldsymbol{E}_\varphi,\boldsymbol{E}_r) = 0 \\
R_{r\varphi} &= \Omega^\theta{}_\varphi(\boldsymbol{E}_\theta,\boldsymbol{E}_r) = 0 \\
R_{\theta r} &= 0 \quad \text{(by symmetry)} \\
R_{\theta\theta} &= \Omega^r{}_\theta(\boldsymbol{E}_r,\boldsymbol{E}_\theta) + \Omega^\varphi{}_\theta(\boldsymbol{E}_\varphi,\boldsymbol{E}_\theta) = \frac{A'(r)}{rA(r)^3} + \frac{1}{r^2}\left(1-\frac{1}{A(r)^2}\right) \\
R_{\theta\varphi} &= \Omega^r{}_\varphi(\boldsymbol{E}_r,\boldsymbol{E}_\theta) = 0 \\
R_{\varphi r} &= 0 \quad \text{(by symmetry)} \\
R_{\varphi\theta} &= 0 \quad \text{(by symmetry)} \\
R_{\varphi\varphi} &= \Omega^r{}_\varphi(\boldsymbol{E}_r,\boldsymbol{E}_\varphi) + \Omega^\theta{}_\varphi(\boldsymbol{E}_\theta,\boldsymbol{E}_\varphi) = \frac{A'(r)}{rA(r)^3} + \frac{1}{r^2}\left(1-\frac{1}{A(r)^2}\right).
\end{aligned}$$

As for the scalar curvature, note that since the frame we're working with is already orthonormal, we have

$$\mathrm{s} = R_{rr} + R_{\theta\theta} + R_{\varphi\varphi} = \frac{4A'(r)}{rA(r)^3} + \frac{2}{r^2}\left(1-\frac{1}{A(r)^2}\right).$$

**Particular cases ($\mathbb{S}^3$ and $\mathbb{H}^3$):**

- Consider the unit sphere $\mathbb{S}^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\}$, with the Riemannian metric induced from $\mathbb{R}^4$. Write $x^2+y^2+z^2 = 1-w^2$ and use spherical coordinates for the slices of radius $r = \sqrt{1-w^2}$. More precisely, consider in an adequate domain the relations

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ w = \sqrt{1-r^2}. \end{cases}$$

With this, we compute all the products between the coordinate vector fields

$$\begin{cases} \partial_r = \sin\theta\cos\varphi\,\partial_x + \sin\theta\sin\varphi\,\partial_y + \cos\theta\,\partial_z - \dfrac{r}{\sqrt{1-r^2}}\partial_w \\ \partial_\theta = r\cos\theta\cos\varphi\,\partial_x + r\cos\theta\sin\varphi\,\partial_y - r\sin\theta\,\partial_z \\ \partial_\varphi = -r\sin\theta\sin\varphi\,\partial_x + r\sin\theta\cos\varphi\,\partial_y \end{cases}$$

to obtain

$$g_{\mathbb{S}^3} = \iota^*(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2) = \frac{1}{1-r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi.$$

So the above calculations apply for $A(r) = (1-r^2)^{-1/2}$. We obtain, after simplifications, that $\mathrm{s} = 2+2+2 = 6$.

- Consider the hyperbolic space $\mathbb{H}^3 = \{(x,y,z,w) \in \mathbb{L}^4 \mid x^2+y^2+z^2-w^2 = -1\}$, with Riemannian metric induced from Lorentz-Minkowski space $\mathbb{L}^4$. Repeat the strategy adopted for $\mathbb{S}^3$ and write $x^2+y^2+z^3 = 1+w^2$. Take spherical coordinates for the slices of radius $r = \sqrt{1+w^2}$. We obtain coordinates

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ w = \sqrt{1+r^2}, \end{cases}$$

for which

$$\begin{cases} \partial_r = \sin\theta\cos\varphi\,\partial_x + \sin\theta\sin\varphi\,\partial_y + \cos\theta\,\partial_z + \dfrac{r}{\sqrt{1+r^2}}\partial_w \\ \partial_\theta = r\cos\theta\cos\varphi\,\partial_x + r\cos\theta\sin\varphi\,\partial_y - r\sin\theta\,\partial_z \\ \partial_\varphi = -r\sin\theta\sin\varphi\,\partial_x + r\sin\theta\cos\varphi\,\partial_y, \end{cases}$$

and the same calculations done for $\mathbb{S}^3$ show that

$$g_{\mathbb{H}^3} = \iota^*(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}w^2) = \frac{1}{1+r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2.$$

So the general calculations apply with $A(r) = (1+r^2)^{1/2}$, and in particular we get $\mathrm{s} = -2-2-2 = -6$, after some simplifications.

# 5 Subbundles and submanifolds

## 5.1 Fundamental equations

Let $E \to M$ be a vector bundle equipped with a Koszul connection $\nabla$, and consider two subbundles $E^\pm$ of $E$ such that $E = E^+ \oplus E^-$. That is to say, every $\psi \in \Gamma(E)$ decomposes uniquely as $\psi = \psi^+ + \psi^-$, where $\psi^\pm \in \Gamma(E^\pm)$. We may use $\nabla$ to define connections in $E^\pm$ as follows: if $\phi \in \Gamma(E^\pm)$, then $\phi$ is also a section of $E$, and so $\nabla_{\boldsymbol{X}}\phi$ makes sense, for every vector field $\boldsymbol{X} \in \mathfrak{X}(M)$. To define a connection in $E^\pm$, we'd like our end result to be a section of $E^\pm$, and so we project: $\nabla^\pm_{\boldsymbol{X}}\phi = (\nabla_{\boldsymbol{X}}\phi)^\pm$, where $(\_)^\pm \colon E \to E^\pm$ are the projections relative to the given decomposition (also in the section level).

Now, once we have the connections $\nabla^+$ and $\nabla^-$ in $E^+$ and $E^-$, we could consider the *van der Waerden-Bortolotti* (direct sum) connection $\overline{\nabla} = \nabla^+ \oplus \nabla^-$ in $E$. The natural thing one could wonder at this point is whether $\overline{\nabla} = \nabla$. In general, there is no reason whatsoever for this to happen. So we look at the difference between those connections (which is then a tensor). Fixed $\boldsymbol{X} \in \mathfrak{X}(M)$, we only need to understand how this acts on $E^+$ and on $E^-$ separately:

**Definition.** The *second fundamental form* of the decomposition $E = E^+ \oplus E^-$ relative to the connection $\nabla$ is the $\mathscr{C}^\infty(M)$-bilinear map $\alpha \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying the relations

$$\nabla_{\boldsymbol{X}}\phi = \nabla^\pm_{\boldsymbol{X}}\phi + \alpha(\boldsymbol{X}, \phi),$$

for all $\boldsymbol{X} \in \mathfrak{X}(M)$ and $\phi \in \Gamma(E^\pm)$.

Note indeed that given $\psi \in \Gamma(E)$, we have

$$\begin{aligned}
\overline{\nabla}_{\boldsymbol{X}}\psi + \alpha(\boldsymbol{X}, \psi) &= \overline{\nabla}_{\boldsymbol{X}}(\psi^+ + \psi^-) + \alpha(\boldsymbol{X}, \psi^+ + \psi^-) \\
&= \nabla^+_{\boldsymbol{X}}(\psi^+) + \nabla^-_{\boldsymbol{X}}(\psi^-) + \alpha(\boldsymbol{X}, \psi^+) + \alpha(\boldsymbol{X}, \psi^-) \\
&= \nabla_{\boldsymbol{X}}(\psi^+) + \nabla_{\boldsymbol{X}}(\psi^-) \\
&= \nabla_{\boldsymbol{X}}\psi,
\end{aligned}$$

so that $\nabla = \overline{\nabla} + \alpha$. In particular, setting $\psi^- = 0$ and taking the $(\_)^+$ projection, we see that $\alpha(\boldsymbol{X}, \psi^+)^+ = 0$, and so $\alpha(\boldsymbol{X}, \psi^+) \in \Gamma(E^-)$. Similarly, $\alpha(\boldsymbol{X}, \psi^-) \in \Gamma(E^+)$. This means that $\alpha$ carries $E^+$ and $E^-$ into each other, and so we may consider the restrictions $\alpha^\pm \colon \mathfrak{X}(M) \times \Gamma(E^\mp) \to \Gamma(E^\pm)$.

The next step would be to relate the curvature tensors of all of those connections. The first one is clear: $R^{\overline{\nabla}} = R^+ \oplus R^-$, where $R^\pm$ is the curvature of $\nabla^\pm$. Namely, we have

$$R^{\overline{\nabla}}(\boldsymbol{X}, \boldsymbol{Y})\psi = R^+(\boldsymbol{X}, \boldsymbol{Y})(\psi^+) + R^-(\boldsymbol{X}, \boldsymbol{Y})(\psi^-),$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ and $\psi \in \Gamma(E)$. To avoid the initial clutter with $\pm$'s and $\alpha^\pm$'s, we organize ourselves with the general:

**Lemma.** *Let $E \to M$ be a vector bundle with a connection $\nabla$, and $F \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ be a tensor. Consider the connection $\nabla' = \nabla + F$. The relation between $R'$ and $R$ is given by*

$$\begin{aligned}
R'(\boldsymbol{X}, \boldsymbol{Y})\psi = R(\boldsymbol{X}, \boldsymbol{Y})\psi &+ \nabla_{\boldsymbol{X}}(F(\boldsymbol{Y}, \psi)) - F(\boldsymbol{Y}, \nabla_{\boldsymbol{X}}\psi) - \nabla_{\boldsymbol{Y}}(F(\boldsymbol{X}, \psi)) + F(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\psi) \\
&+ F(\boldsymbol{X}, F(\boldsymbol{Y}, \psi)) - F(\boldsymbol{Y}, F(\boldsymbol{X}, \psi)) - F([\boldsymbol{X}, \boldsymbol{Y}], \psi).
\end{aligned}$$

*If one fixes any connection $\nabla^{TM}$ in the tangent bundle of $M$ and uses $\nabla^{TM} \oplus \nabla$ to form covariant derivatives of $F$ (which we'll abuse notation and still denote by $\nabla F$, the dependence on $\nabla^{TM}$ being understood), the above relation can be rewritten as*

$$\begin{aligned} R'(\boldsymbol{X},\boldsymbol{Y})\psi = R(\boldsymbol{X},\boldsymbol{Y})\psi + (\nabla_{\boldsymbol{X}} F)(\boldsymbol{Y},\psi) - (\nabla_{\boldsymbol{Y}} F)(\boldsymbol{X},\psi) \\ + F(\boldsymbol{X}, F(\boldsymbol{Y},\psi)) - F(\boldsymbol{Y}, F(\boldsymbol{X},\psi)) + F(\tau(\boldsymbol{X},\boldsymbol{Y}),\psi), \end{aligned}$$

*where $\tau$ is the torsion of $\nabla^{TM}$. Usually, we take $\nabla^{TM}$ to be the Levi-Civita connection of some pseudo-Riemannian metric in $M$.*

**Proof:** This is just brute force: compute

$$\begin{aligned} \nabla'_{\boldsymbol{X}}\nabla'_{\boldsymbol{Y}}\psi &= \nabla_{\boldsymbol{X}}\nabla'_{\boldsymbol{Y}}\psi + F(\boldsymbol{X}, \nabla'_{\boldsymbol{Y}}\psi) \\ &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi + \nabla_{\boldsymbol{X}}(F(\boldsymbol{Y},\psi)) + F(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\psi) + F(\boldsymbol{X}, F(\boldsymbol{Y},\psi)), \end{aligned}$$

switch the roles of $\boldsymbol{X}$ and $\boldsymbol{Y}$, subtract, and subtract $\nabla'_{[\boldsymbol{X},\boldsymbol{Y}]}\psi = \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi + F([\boldsymbol{X},\boldsymbol{Y}],\psi)$. For the second part, write $[\boldsymbol{X},\boldsymbol{Y}] = \nabla^{TM}_{\boldsymbol{X}}\boldsymbol{Y} - \nabla^{TM}_{\boldsymbol{Y}}\boldsymbol{X} - \tau(\boldsymbol{X},\boldsymbol{Y})$, and use the definition

$$(\nabla_{\boldsymbol{X}} F)(\boldsymbol{Y},\psi) = \nabla_{\boldsymbol{X}}(F(\boldsymbol{Y},\psi)) - F(\nabla^{TM}_{\boldsymbol{X}}\boldsymbol{Y},\psi) - F(\boldsymbol{X}, \nabla_{\boldsymbol{X}}\psi).$$

□

**Remark.** We might as well register what the relation between $R'$ and $R$ is, with respect to a coordinate system $(x^j)$ on $M$ and local trivializing sections $(e_a)$ of $E$. Writing $F(\partial_j, e_a) = F^b_{ja} e_b$, we have

$$R'_{jka}{}^b = R_{jka}{}^b + \partial_j F^b_{ka} - \partial_k F^b_{ja} + F^c_{ka}\Gamma^b_{jc} - F^c_{ja}\Gamma^b_{kc} - F^b_{kc}\Gamma^c_{ja} + F^b_{jc}\Gamma^c_{ka} + F^c_{ka}F^c_{jb} - F^c_{ja}F^b_{kc}.$$

If one sets $F_{jka}{}^b \doteq \partial_j F^b_{ka} - \partial_k F^b_{ja} + F^c_{ka}F^c_{jb} - F^c_{ja}F^b_{kc}$, mimicking the explicit expression for $R$, the formula writes simply as

$$R'_{jka}{}^b = R_{jka}{}^b + F_{jka}{}^b + F^c_{ka}\Gamma^b_{jc} - F^c_{ja}\Gamma^b_{kc} - \Gamma^c_{ja}F^b_{kc} + \Gamma^c_{ka}F^b_{jc},$$

emphasizing that the new curvature is the sum of the curvatures of $R$ and $F$, with an interplay between them (perhaps seen as a difference of anticommutators of $F$ and $\Gamma$).

Back to the main discussion on $E = E^+ \oplus E^-$, we apply the lemma for the decomposition $\nabla = (\nabla^+ \oplus \nabla^-) + \alpha$, to obtain

$$\begin{aligned} R^{\nabla}(\boldsymbol{X},\boldsymbol{Y})\psi = R^+(\boldsymbol{X},\boldsymbol{Y})(\psi^+) + R^-(\boldsymbol{X},\boldsymbol{Y})(\psi^-) + (\overline{\nabla}_{\boldsymbol{X}}\alpha)(\boldsymbol{Y},\psi) - (\overline{\nabla}_{\boldsymbol{Y}}\alpha)(\boldsymbol{X},\psi) \\ + \alpha(\boldsymbol{X}, \alpha(\boldsymbol{Y},\psi)) - \alpha(\boldsymbol{Y}, \alpha(\boldsymbol{X},\psi)) + \alpha(\tau(\boldsymbol{X},\boldsymbol{Y}),\psi), \end{aligned}$$

for all $\psi \in \Gamma(E)$, where $\tau$ is the torsion of a connection in $TM$, also used to form the covariant derivatives of $\alpha$. Using that the above is linear in $\psi = \psi^+ + \psi^-$ and that $\alpha$ "switches" $E^+$ and $E^-$, we may set $\psi = \psi^\pm$, to get

$$\begin{aligned} R^{\nabla}(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) = R^\pm(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) + (\overline{\nabla}_{\boldsymbol{X}}\alpha^\mp)(\boldsymbol{Y},\psi^\pm) - (\overline{\nabla}_{\boldsymbol{Y}}\alpha^\mp)(\boldsymbol{X},\psi^\pm) \\ + \alpha^\pm(\boldsymbol{X}, \alpha^\mp(\boldsymbol{Y},\psi^\pm)) - \alpha^\pm(\boldsymbol{Y}, \alpha^\mp(\boldsymbol{X},\psi^\pm)) + \alpha^\mp(\tau(\boldsymbol{X},\boldsymbol{Y}),\psi), \end{aligned}$$

where we use that $\overline{\nabla}_{\boldsymbol{X}}(\psi^\pm) = \nabla^\pm_{\boldsymbol{X}}(\psi^\pm)$ (by definition) to conclude that the restriction commutes with the covariant derivatives: $(\overline{\nabla}_{\boldsymbol{X}}\alpha)(\boldsymbol{Y},\psi^\pm) = (\overline{\nabla}_{\boldsymbol{X}}\alpha^\mp)(\boldsymbol{Y},\psi^\pm)$. The fundamental equations of pseudo-Riemannian geometry are obtained by taking the components of this last expression:

- The *Gauss formula*:

$$R^{\pm}(\boldsymbol{X},\boldsymbol{Y})(\psi^{\pm}) = \left(R^{\nabla}(\boldsymbol{X},\boldsymbol{Y})(\psi^{\pm})\right)^{\pm} - \alpha^{\pm}(\boldsymbol{X},\alpha^{\mp}(\boldsymbol{Y},\psi^{\pm})) + \alpha^{\pm}(\boldsymbol{Y},\alpha^{\mp}(\boldsymbol{X},\psi^{\pm})).$$

  Here, we have solved for $R^{\pm}$, thinking that the curvature of the subbundle should be computed in terms of the curvature of the ambient bundle and the second fundamental form

- The *Codazzi equation:*

$$\left(R^{\nabla}(\boldsymbol{X},\boldsymbol{Y})(\psi^{\pm})\right)^{\mp} = (\overline{\nabla}_{\boldsymbol{X}}\alpha^{\mp})(\boldsymbol{Y},\psi^{\pm}) - (\overline{\nabla}_{\boldsymbol{Y}}\alpha^{\mp})(\boldsymbol{X},\psi^{\pm}) + \alpha^{\mp}(\tau(\boldsymbol{X},\boldsymbol{Y}),\psi^{\pm}).$$

A very particular situation is when $E$ is also equipped with a parallel pseudo-Euclidean fiber metric $g \in \Gamma(E^* \otimes E^*)$ (that is, $\nabla g = 0$), and $E^+ \perp E^-$. In this case, we have two useful properties:

- $\nabla^{\pm} g = 0$, where we again denote by $g$ its restriction to $E^{\pm}$: let $\psi_1^{\pm}, \psi_2^{\pm} \in \Gamma(E^{\pm})$. So we have:

$$\begin{aligned}
\boldsymbol{X}(g(\psi_1^{\pm},\psi_2^{\pm})) &= g(\nabla_{\boldsymbol{X}}(\psi_1^{\pm}),\psi_2^{\pm}) + g(\psi_1^{\pm},\nabla_{\boldsymbol{X}}(\psi_2^{\pm})) \\
&= g(\nabla^{\pm}_{\boldsymbol{X}}(\psi_1^{\pm}) + \alpha^{\mp}(\boldsymbol{X},\psi_1^{\pm}),\psi_2^{\pm}) + g(\psi_1^{\pm},\nabla^{\pm}_{\boldsymbol{X}}(\psi_2^{\pm}) + \alpha^{\mp}(\boldsymbol{X},\psi_2^{\pm})) \\
&= g(\nabla^{\pm}_{\boldsymbol{X}}(\psi_1^{\pm}),\psi_2^{\pm}) + g(\psi_1^{\pm},\nabla^{\pm}_{\boldsymbol{X}}(\psi_2^{\pm})),
\end{aligned}$$

  as wanted.

- $\alpha^+(\boldsymbol{X},\cdot)$ and $\alpha^-(\boldsymbol{X},\cdot)$ are negative adjoints: let $\psi^+ \in \Gamma(E^+)$ and $\psi^- \in \Gamma(E^-)$, so that $g(\psi^+,\psi^-) = 0$. Apply $\boldsymbol{X}$ to get

$$\begin{aligned}
0 &= \boldsymbol{X}(g(\psi^+,\psi^-)) \\
&= g(\nabla_{\boldsymbol{X}}(\psi^+),\psi^-) + g(\psi^+,\nabla_{\boldsymbol{X}}(\psi^-)) \\
&= g(\nabla^{+}_{\boldsymbol{X}}(\psi^+) + \alpha^-(\boldsymbol{X},\psi^+),\psi^-) + g(\psi^+,\nabla^{-}_{\boldsymbol{X}}(\psi^-) + \alpha^+(\boldsymbol{X},\psi^-)) \\
&= g(\alpha^-(\boldsymbol{X},\psi^+),\psi^-) + g(\psi^+,\alpha^+(\boldsymbol{X},\psi^-)).
\end{aligned}$$

Using the last condition above, we may rewrite the Gauss formula by considering the fully covariant $(0,4)$-curvature tensors. Namely, consider also $\phi^{\pm} \in \Gamma(E^{\pm})$ and set $g = \langle \cdot,\cdot \rangle$, so that

$$\begin{aligned}
R^{\pm}(\boldsymbol{X},\boldsymbol{Y},\psi^{\pm},\phi^{\pm}) = R^{\nabla}&(\boldsymbol{X},\boldsymbol{Y},\psi^{\pm},\phi^{\pm}) \\
&+ \langle \alpha^{\mp}(\boldsymbol{Y},\psi^{\pm}),\alpha^{\mp}(\boldsymbol{X},\phi^{\pm})\rangle - \langle \alpha^{\mp}(\boldsymbol{X},\psi^{\pm}),\alpha^{\mp}(\boldsymbol{Y},\phi^{\pm})\rangle.
\end{aligned}$$

As a concrete example of the above situation, let $(\overline{M},g)$ be a pseudo-Riemannian manifold, and $M \subseteq \overline{M}$ be a non-degenerate submanifold. We have the orthogonal decomposition into non-degenerate subbundles, $T\overline{M}\big|_M = TM \oplus TM^{\perp}$. Here we'll slightly change notation: let $\overline{\nabla}$ be the Levi-Civita connection of $(\overline{M},g)$. Then the Levi-Civita connection $\nabla$ of $M$ with the induced metric is obtained by projecting $\overline{\nabla}$ to $TM$, as it is metric compatible (by the above), while being torsion-free is clear. Let's translate what we discussed above in this setting. First, we'll denote by $\nabla^{\perp}$ the projection of

$\overline{\nabla}$ in $TM^\perp$. Given $\boldsymbol{Y} \in \mathfrak{X}(M)$ and $\boldsymbol{\xi} \in \mathfrak{X}^\perp(M) \doteq \Gamma(TM^\perp)$, we set $A_{\boldsymbol{\xi}}(\boldsymbol{X}) = -\alpha^+(\boldsymbol{X}, \boldsymbol{\xi})$, so that

$$\overline{\nabla}_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \alpha(\boldsymbol{X}, \boldsymbol{Y}) \quad \text{and} \quad \overline{\nabla}_{\boldsymbol{X}}\boldsymbol{\xi} = -A_{\boldsymbol{\xi}}(\boldsymbol{X}) + \nabla^\perp_{\boldsymbol{X}}\boldsymbol{\xi}.$$

In particular, we observe that in contrast to what we see in most Riemannian geometry books, the second fundamental form actually is a map $\mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \to \overline{\mathfrak{X}}(M)$ (here $\overline{\mathfrak{X}}(M) = \Gamma(T\overline{M}|_M)$) instead of a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^\perp(M)$, the latter being just a convenient restriction. Moreover, the *shape operators* $A_{\boldsymbol{\xi}}$ are part of such object. The fact that $\alpha^+(\boldsymbol{X}, \cdot)$ and $\alpha^-(\boldsymbol{X}, \cdot)$ are negative adjoints becomes $\langle \alpha(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{\xi} \rangle = \langle A_{\boldsymbol{\xi}}(\boldsymbol{X}), \boldsymbol{Y} \rangle$, showing us that the negative sign in the definition of $A_{\boldsymbol{\xi}}$ is indeed natural. Next, by making the correct choices of $\pm$, the fundamental equations read:

- The Gauss formulas: given $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \mathfrak{X}(M)$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}^\perp(M)$, we have

$$\begin{aligned} R(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) &= \overline{R}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) + \langle \alpha(\boldsymbol{Y}, \boldsymbol{Z}), \alpha(\boldsymbol{X}, \boldsymbol{W}) \rangle - \langle \alpha(\boldsymbol{X}, \boldsymbol{Z}), \alpha(\boldsymbol{Y}, \boldsymbol{W}) \rangle, \\ R^\perp(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) &= \overline{R}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) + \langle A_{\boldsymbol{\xi}}(\boldsymbol{Y}), A_{\boldsymbol{\eta}}(\boldsymbol{X}) \rangle - \langle A_{\boldsymbol{\xi}}(\boldsymbol{X}), A_{\boldsymbol{\eta}}(\boldsymbol{Y}) \rangle. \end{aligned}$$

  Since $\alpha$ is symmetric in this case, each shape operator is self-adjoint, and thus we may rewrite the second formula (also known as the *Ricci equation*) as

$$R^\perp(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \overline{R}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) + \langle [A_{\boldsymbol{\xi}}, A_{\boldsymbol{\eta}}](\boldsymbol{X}), \boldsymbol{Y} \rangle,$$

  where $[\cdot, \cdot]$ denotes the commutator of endomorphisms in $TM$.

- The Codazzi equations: here we're considering Levi-Civita connections, so the torsion term vanishes and we obtain

$$\begin{aligned} \big(\overline{R}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z}\big)^\perp &= (\nabla_{\boldsymbol{X}}\alpha)(\boldsymbol{Y}, \boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}\alpha)(\boldsymbol{X}, \boldsymbol{Z}) \\ \big(\overline{R}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{\xi}\big)^\top &= A_{\nabla^\perp_{\boldsymbol{X}}\boldsymbol{\xi}}(\boldsymbol{Y}) - (\nabla_{\boldsymbol{X}}A_{\boldsymbol{\xi}})(\boldsymbol{Y}) - A_{\nabla^\perp_{\boldsymbol{Y}}\boldsymbol{\xi}}(\boldsymbol{X}) + (\nabla_{\boldsymbol{Y}}A_{\boldsymbol{\xi}})(\boldsymbol{X}). \end{aligned}$$

  For the second relation, just recall that $(\nabla_{\boldsymbol{X}}A_{\boldsymbol{\xi}})(\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}(A_{\boldsymbol{\xi}}(\boldsymbol{Y})) - A_{\boldsymbol{\xi}}(\nabla_{\boldsymbol{X}}\boldsymbol{Y})$, and use the definition of the covariant derivative of $\alpha$.

In particular, if $\boldsymbol{X}, \boldsymbol{Y}$ are vector fields tangent to $M$ along some open subset, spanning non-degenerate 2-planes, we have

$$K(\boldsymbol{X}, \boldsymbol{Y}) = \overline{K}(\boldsymbol{X}, \boldsymbol{Y}) + \frac{\langle \alpha(\boldsymbol{X}, \boldsymbol{X}), \alpha(\boldsymbol{Y}, \boldsymbol{Y}) \rangle - \langle \alpha(\boldsymbol{X}, \boldsymbol{Y}), \alpha(\boldsymbol{X}, \boldsymbol{Y}) \rangle}{\langle \boldsymbol{X}, \boldsymbol{X} \rangle \langle \boldsymbol{Y}, \boldsymbol{Y} \rangle - \langle \boldsymbol{X}, \boldsymbol{Y} \rangle^2}.$$

Another very important object is *mean curvature vector* of $M^n$, defined as

$$\boldsymbol{H} = \frac{1}{n}\,\mathrm{tr}_g\alpha = \frac{1}{n}g^{ij}\alpha_{ij},$$

where $\alpha_{ij} = \alpha(\boldsymbol{E}_i, \boldsymbol{E}_j)$ and $(\boldsymbol{E}_i)$ is any local tangent frame for $M$. We'll say that $M$ is *critical* if $\boldsymbol{H} = \boldsymbol{0}$, and *marginally trapped* in $\overline{M}^{n+k}$ if $\boldsymbol{H}$ is lightlike. If $(\boldsymbol{\xi}_\lambda)$ is a local orthonormal normal frame along $M$, we may also write

$$\boldsymbol{H} = \frac{1}{n}(\varepsilon_1\mathrm{tr}(A_{\boldsymbol{\xi}_1}) + \cdots + \varepsilon_k\mathrm{tr}(A_{\boldsymbol{\xi}_k})),$$

where $\varepsilon_\lambda = g(\boldsymbol{\xi}_\lambda, \boldsymbol{\xi}_\lambda) \in \{-1, 1\}$.

### 5.2 In terms of differential forms

Assume again the same setting, the direct sum decomposition of a vector bundle $E = E^+ \oplus E^-$ over $M$, equipped with a Koszul connection $\nabla$, and now consider: a local coordinate system for $M$, and local trivializing sections $(e_a^+, e_\lambda^-)$ adapted to the decomposition, i.e., where $e_a^+ \in \Gamma(E^+)$ and $e_\lambda^- \in \Gamma(E^-)$, for all $a$ and $\lambda$. The relation $\nabla_{\boldsymbol{X}}(\psi^\pm) = \nabla^\pm_{\boldsymbol{X}}(\psi^\pm) + \alpha(\boldsymbol{X}, \psi^\pm)$ gives us the four relations

$$\nabla^+_{\partial_j} e_a^+ = \Gamma^b_{ja} e_b^+, \quad \alpha(\partial_j, e_a^+) = \Gamma^\mu_{ja} e_\mu^-, \quad \nabla^-_{\partial_j} e_\lambda^- = \Gamma^\mu_{j\lambda} e_\mu^- \quad \text{and} \quad \alpha(\partial_j, e_\lambda^-) = \Gamma^b_{j\lambda} e_b^+,$$

where all of those connection components are relative to the original connection $\nabla$. In terms of the connection 1-forms of $\nabla$, we have

$$\nabla^+_{\boldsymbol{X}} e_b^+ = \omega^a{}_b(\boldsymbol{X}) e_a^+, \quad \alpha(\boldsymbol{X}, e_b^+) = \omega^\lambda{}_b(\boldsymbol{X}) e_\lambda^-, \quad \nabla^-_{\boldsymbol{X}} e_\mu^- = \omega^\lambda{}_\mu(\boldsymbol{X}) e_\lambda^-, \quad \alpha(\boldsymbol{X}, e_\mu^-) = \omega^a{}_\mu(\boldsymbol{X}) e_a^+,$$

and we see that the second fundamental form essentially carries information about the "mixed" coefficients. The issue, however, is that the curvature 2-forms of $R^\pm$ are not the curvature 2-forms of $\nabla$. To illustrate this, note that if $E^- = L$ is a line bundle and we take a local trivializing section $e_L$ for $L$, then the curvature 2-form of $\nabla^L$ is just $\mathrm{d}\omega^L{}_L$, without the extra terms coming from the structure equations. This can be improved:

**Proposition.** *Let $E = E^+ \oplus L$ be a vector bundle over $M$ equipped with a Koszul connection $\nabla$ and a parallel pseudo-Euclidean fiber metric $\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes E^*)$ such that $L$ is a non-degenerate subbundle and $E^+ \perp L$. Then the projected connection $\nabla^L$ is flat.*

**Proof:** Let $e_L$ be a unit local trivializing section for $L$ and write $\nabla^L_{\boldsymbol{X}} e_L = \omega^L{}_L(\boldsymbol{X}) e_L$ (no summation). We will show that $\omega^L{}_L = 0$. Since $\langle e_L, e_L \rangle$ is constant, apply any $\boldsymbol{X}$ to obtain $0 = 2\langle \nabla^L_{\boldsymbol{X}} e_L, e_L \rangle = 2\omega^L{}_L(\boldsymbol{X})\langle e_L, e_L \rangle$. So $\omega^L{}_L = 0$, and the (single) curvature 2-form of $\nabla^L$ is $\mathrm{d}\omega^L{}_L = 0$. Thus $R^L = 0$. $\square$

**Remark.** In the above proof, we are *not* claiming that $\nabla^L = 0$. If $\psi$ is any section of $L$ and $\psi = f e_L$, then $\nabla^L_{\boldsymbol{X}} \psi = \nabla^L_{\boldsymbol{X}}(f e_L) = \mathrm{d}f(\boldsymbol{X}) e_L$, which has no reason to vanish. We do again recognize, though, that $\nabla^L$ is a flat connection from this formula (by using the definition of $R^L$, for example). This has a funny consequence: if we have such a decomposition of $E$ with $\nabla^L$ non-flat, then every pseudo-Euclidean fiber metric on $E$ is either non-parallel, or degenerates $L$.

**Corollary.** *Every non-degenerate hypersurface in a pseudo-Riemannian manifold has flat normal bundle.*

The relations between the curvature forms of $\nabla$ and its projections on the factors $E^\pm$ is, not surprinsingly, given by the fundamental equations discussed in the previous section. Here's how to translate the Gauss formulas: they are a straight consequence of the curvature structure equations, again using the fact that the connection 1-forms of $\nabla^\pm$ are a subset of the connection 1-forms of $\nabla$:

$$\begin{aligned}
\Omega^a{}_b &= \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b + \omega^a{}_\lambda \wedge \omega^\lambda{}_b = (\Omega^+)^a{}_b + \omega^a{}_\lambda \wedge \omega^\lambda{}_b \\
\Omega^\lambda{}_\mu &= \mathrm{d}\omega^\lambda{}_\mu + \omega^\lambda{}_c \wedge \omega^c{}_\mu + \omega^\lambda{}_\nu \wedge \omega^\nu{}_\mu = (\Omega^-)^\lambda{}_\mu + \omega^\lambda{}_c \wedge \omega^c{}_\mu
\end{aligned}$$

As an example of this technique, we have the:

**Theorem** (Theorema Egregium). *Let $(\overline{M}^3, g)$ be a pseudo-Riemannian manifold and consider $M^2 \subseteq \overline{M}$ be a non-degenerate surface. Let $\boldsymbol{N}$ be a (Gauss) unit normal vector field along $M$, and write $g(\boldsymbol{N},\boldsymbol{N}) = \eta \in \{-1,1\}$. Then for all $p \in M$ we have*

$$K(p) = \overline{K}(T_pM) + \eta \det(A_{\boldsymbol{N}(p)}).$$

**Proof:** Consider a local orthonormal frame $(\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3 = \boldsymbol{N})$ to $M$, with $\boldsymbol{E}_1$ spacelike and $g(\boldsymbol{E}_2,\boldsymbol{E}_2) = \varepsilon \in \{-1,1\}$. We have seen that since $\boldsymbol{N}$ is a unit vector field, then $\nabla^{\perp}_{\boldsymbol{X}}\boldsymbol{N} = \boldsymbol{0}$, and so $A_{\boldsymbol{N}}(\boldsymbol{X}) = -(\overline{\nabla}_{\boldsymbol{X}}\boldsymbol{N})^{\top}$, where $\overline{\nabla}$ denotes the Levi-Civita connection of $(\overline{M}, g)$ (the tangent projection here being superfluous and written for clarity). If $({\omega^i}_j)_{i,j=1}^3$ denotes the matrix of connection 1-forms of $\overline{\nabla}$, then the block $({\omega^i}_j)_{i,j=1}^2$ is the matrix of connection 1-forms for the Levi-Civita connection $\nabla$ of $M$.

Note here the raising/lowering the index 2 amounts to multiplying by $\varepsilon$, and same for the index 3 with $\eta$. Thus we write

$$A_{\boldsymbol{N}}(\boldsymbol{E}_i) = -{\omega^1}_3(\boldsymbol{E}_i)\boldsymbol{E}_1 - {\omega^2}_3(\boldsymbol{E}_i)\boldsymbol{E}_2, \quad i = 1,2,$$

and so

$$\det(A_{\boldsymbol{N}}) = \begin{vmatrix} -{\omega^1}_3(\boldsymbol{E}_1) & -{\omega^1}_3(\boldsymbol{E}_2) \\ -{\omega^2}_3(\boldsymbol{E}_1) & -{\omega^2}_3(\boldsymbol{E}_2) \end{vmatrix} = ({\omega^1}_3 \wedge {\omega^2}_3)(\boldsymbol{E}_1,\boldsymbol{E}_2) = \varepsilon(\omega_{13} \wedge \omega_{23})(\boldsymbol{E}_1,\boldsymbol{E}_2).$$

With this set in place, we use the structure equation for the curvature 2-form ${\Omega^1}_2$ of $\overline{\nabla}$:

$$\begin{aligned} \Omega_{12} = {\Omega^1}_2 &= \mathrm{d}{\omega^1}_2 + {\omega^1}_1 \wedge {\omega^1}_2 + {\omega^1}_2 \wedge {\omega^2}_2 + {\omega^1}_3 \wedge {\omega^3}_2 \\ &= \mathrm{d}{\omega^1}_2 + {\omega^1}_3 \wedge {\omega^3}_2 \\ &= \mathrm{d}{\omega^1}_2 + \eta\, \omega_{13} \wedge \omega_{32} \\ &= \mathrm{d}{\omega^1}_2 - \eta\, \omega_{13} \wedge \omega_{23}. \end{aligned}$$

Here, we use that since $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ is an orthonormal frame tangent to $M$, the middle two terms in the first line of the above calculation vanish. Now evaluate at $(\boldsymbol{E}_1, \boldsymbol{E}_2)$, at a point $p \in M$, to get $\varepsilon\overline{K}(T_pM) = \varepsilon K(p) - \varepsilon\eta \det(A_{\boldsymbol{N}(p)})$. Cancel $\varepsilon$ and reorganize to conclude that $K(p) = \overline{K}(T_pM) + \eta \det(A_{\boldsymbol{N}(p)})$, as wanted. $\square$

Now, assume again the case that we have a pseudo-Riemannian manifold $(\overline{M}, g)$ and a non-degenerate submanifold $M$. Assume that $(\boldsymbol{E}_i, \boldsymbol{\xi}_\lambda)$ is a local frame for $T\overline{M}|_M$ adapted to $M$, i.e., all the $\boldsymbol{E}_i$ are tangent to $M$ while the $\boldsymbol{\xi}_\lambda$ are normal. Let also $(\theta^i, \Xi^\lambda)$ be the dual coframe. The connection 1-forms carry all the information about the shape operators: we that

$$A_{\boldsymbol{\xi}_\lambda}(\boldsymbol{E}_j) = -\alpha^{+}(\boldsymbol{E}_j, \boldsymbol{\xi}_\lambda) = -{\omega^i}_\lambda(\boldsymbol{E}_j)\boldsymbol{E}_i \implies \operatorname{tr}(A_{\boldsymbol{\xi}_\lambda}) = -{\omega^i}_\lambda(\boldsymbol{E}_i) = -\omega_{i\lambda}((\theta^i)^\sharp).$$

Also, we have

$$\boldsymbol{H} = \frac{1}{n}\operatorname{tr}_g \alpha = \frac{1}{n} g^{ij}\alpha_{ij} = \frac{1}{n} g^{ij}{\omega^\lambda}_j(\boldsymbol{E}_i)\boldsymbol{\xi}_\lambda = \frac{1}{n}{\omega^\lambda}_j((\theta^j)^\sharp)\boldsymbol{\xi}_\lambda = \frac{1}{n}\omega_{\lambda i}((\theta^i)^\sharp)(\Xi^\lambda)^\sharp,$$

and putting all of it together with $\omega_{i\lambda} + \omega_{\lambda i} = 0$ gives

$$\boldsymbol{H} = \frac{1}{n}\mathrm{tr}\,(A_{\boldsymbol{\xi}_\lambda})(\Xi^\lambda)^\sharp.$$

Usually $(\boldsymbol{\xi}_\lambda)$ will be an orthonormal normal frame, so that $(\Xi^\lambda)^\sharp = \pm\boldsymbol{\xi}_\lambda$, which will make the calculations easier. For completeness, another point of view/notation (obviously equivalent to the previous one) which can be adopted when studying this: write $\alpha_{ij} = \alpha^\lambda{}_{ij}\boldsymbol{\xi}_\lambda$ and $A_{\boldsymbol{\xi}_\lambda}(\boldsymbol{E}_j) = \alpha_\lambda{}^i{}_j\boldsymbol{E}_i$. These coefficients are related by the components of the $g$ and their inverses. So

$$n\boldsymbol{H} = g^{ij}\alpha_{ij} = g^{ij}\alpha^\lambda{}_{ij}\boldsymbol{\xi}_\lambda = g^{ij}g^{\lambda\mu}g_{ik}\alpha_\mu{}^k{}_j\boldsymbol{\xi}_\lambda = \delta^j_k\alpha_\mu{}^k{}_j(\Xi^\mu)^\sharp = \alpha_\lambda{}^j{}_j(\Xi^\lambda)^\sharp = \mathrm{tr}(A_{\boldsymbol{\xi}_\lambda})(\Xi^\lambda)^\sharp,$$

as wanted again. In particular, if $(\overline{M}^{n+1}, g)$ is a given pseudo-Riemannian manifold, $M^n \subseteq \overline{M}^{n+1}$ is a non-degenerate hypersurface, and $\boldsymbol{N}$ is a Gauss unit normal vector field along $M$ with $g(\boldsymbol{N},\boldsymbol{N}) = \eta \in \{-1, 1\}$, we have that the vector equivalent to the dual one-form to $\boldsymbol{N}$ with respect to any local tangent frame $(\boldsymbol{E}_i)$ to $M$, is just $\eta\boldsymbol{N}$. So

$$\boldsymbol{H} = \frac{\eta}{n}\,\mathrm{tr}(A_{\boldsymbol{N}})\boldsymbol{N} = \frac{-\eta}{n}\omega^i{}_{n+1}(\boldsymbol{E}_i)\boldsymbol{N}.$$

# 6 Computations for submanifolds (Cartan and non-Cartan)

## 6.1 Pseudo-spheres (de Sitter, anti-de Sitter, hyperbolic spaces, etc.)

Let $(V, g)$ be a pseudo-Euclidean vector space, and $\Sigma = \{p \in V \mid \langle p, p\rangle = c\}$, where $c \neq 0$. Then $\Sigma$ is a regular hypersurface in $V$. As a pseudo-Riemannian manifold, the Levi-Civita connection $\overline{\nabla}$ is the standard flat one, for which all connection 1-forms vanish when taken with respect to any orthonormal basis for $V$. To wit, if $(e_i)$ is a basis for $V$, then we have a global coordinate system $V \ni v = v^i e_i \mapsto (v^i) \in \mathbb{R}^n$ for which all the coordinate fields (given by $\partial_i|_p = e_i$ for all $p \in V$) are parallel. Now, consider the unit normal vector field $\boldsymbol{N}\colon \Sigma \to V$ given by $\boldsymbol{N}(p) = p/\sqrt{|c|}$. Let's show that $\Sigma$ has constant sectional curvature. Consider the second fundamental form $\alpha$ of $\Sigma$. Since $\Sigma$ has codimension 1, we get

$$\begin{aligned}\alpha(\boldsymbol{X}, \boldsymbol{Y}) &= \langle \boldsymbol{N}, \boldsymbol{N}\rangle \langle \alpha(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{N}\rangle \boldsymbol{N} = \frac{c}{|c|}\langle A_{\boldsymbol{N}}(\boldsymbol{X}), \boldsymbol{Y}\rangle \boldsymbol{N} \\ &= -\frac{c}{|c|}\langle \overline{\nabla}_{\boldsymbol{X}}\boldsymbol{N}, \boldsymbol{Y}\rangle \boldsymbol{N} = -\frac{c}{|c|^{3/2}}\langle \boldsymbol{X}, \boldsymbol{Y}\rangle \boldsymbol{N}.\end{aligned}$$

So, if we take an orthogonal basis $(\boldsymbol{X}, \boldsymbol{Y})$ for any non-degenerate 2-plane tangent to $\Sigma$, we have $\alpha(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{0}$ and so

$$K(\boldsymbol{X}, \boldsymbol{Y}) = \frac{\langle \alpha(\boldsymbol{X}, \boldsymbol{X}), \alpha(\boldsymbol{Y}, \boldsymbol{Y})\rangle}{\langle \boldsymbol{X}, \boldsymbol{X}\rangle\langle \boldsymbol{Y}, \boldsymbol{Y}\rangle} = \frac{c^2}{|c|^3}\langle \boldsymbol{N}, \boldsymbol{N}\rangle = \frac{1}{|c|}\frac{c}{|c|} = \frac{c}{c^2} = \frac{1}{c}.$$

In particular, we see that in the concrete case where $\mathbb{R}^n_\nu$ is the space $\mathbb{R}^n$ equipped with the index $\nu$ metric tensor

$$\langle \cdot, \cdot\rangle_\nu = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^{n-\nu} \otimes \mathrm{d}x^{n-\nu} - \mathrm{d}x^{n-\nu+1} \otimes \mathrm{d}x^{n-\nu+1} - \cdots - \mathrm{d}x^n \otimes \mathrm{d}x^n,$$

the pseudo-spheres $\mathbb{S}^n_\nu = \{p \in \mathbb{R}^{n+1}_\nu \mid \langle p, p\rangle_\nu = 1\}$ and the pseudo-hyperbolic spaces $\mathbb{H}^n_\nu = \{p \in \mathbb{R}^{n+1}_{\nu+1} \mid \langle p, p\rangle_{\nu+1} = -1\}$ have constant curvatures equal to 1 and $-1$. This conclusion includes the $n$-dimensional *de Sitter* ($\mathbb{S}^n_1$) and *anti-de Sitter* ($\mathbb{H}^n_1$) spaces.

## 6.2 Marginally trapped pseudo-paraboloid

Let $(V, g)$ be a pseudo-Euclidean vector space, and $W \subseteq V$ a lightlike subpace for which $W \cap W^\perp = \mathbb{R}\boldsymbol{u}$, where $\boldsymbol{u} \in V$ is a lightlike vector. Choose a subspace $X \subseteq V$ such that $W = X \oplus \mathbb{R}\boldsymbol{u}$. Such $X$ must necessarily be non-degenerate, for the following reason: the $g$-radical of $X$ is contained in the $(g|_W)$-radical of $X$, and the latter is trivial, as $\langle x, w\rangle = 0$ for all $w \in W$ implies $x \in W \cap W^\perp = \mathbb{R}\boldsymbol{u}$, and so $x \in X \cap \mathbb{R}\boldsymbol{u} = \{0\}$.

Now, consider $f\colon X \to \mathbb{R}$ given by $f(x) = \langle x - o, x - o\rangle + a$, where $o \in X$ is a fixed "origin" and $a > 0$. Note that $\mathrm{d}f_x(\boldsymbol{v}) = \langle \boldsymbol{v}, x - o\rangle$. Let $M = \{x + f(x)\boldsymbol{u} \mid x \in X\}$. Let's show that $M$ is marginally trapped in $V$. Consider the global parametrization $\Phi\colon X \to M$ given by $\Phi(x) = x + f(x)\boldsymbol{u}$, and note that $\mathrm{d}\Phi_x(\boldsymbol{v}) = \boldsymbol{v} + \langle \boldsymbol{v}, x - o\rangle \boldsymbol{u}$ for all $\boldsymbol{v} \in X$. From this, we see that

$$\langle \mathrm{d}\Phi_x(\boldsymbol{v}), \mathrm{d}\Phi_x(\boldsymbol{w})\rangle = \langle \boldsymbol{v} + \langle v, x - o\rangle \boldsymbol{u}, \boldsymbol{w} + \langle \boldsymbol{w}, x - o\rangle \boldsymbol{u}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle,$$

so that $M$ is non-degenerate, by using that $\boldsymbol{u}$ is orthogonal to all vectors in $W$, and a fortiori to vectors in $X$. Moreover, if $\overline{\nabla}$ denotes the (standard flat) Levi-Civita connection of $V$, $\nabla$ denotes the Levi-Civita connection of $M$, $\alpha$ is the second fundamental form of $M$, and we take coordinates $(x^i)$ for $X$, then

$$\overline{\nabla}_{\mathrm{d}\Phi(\partial_i)}\mathrm{d}\Phi(\partial_j) = \frac{\partial^2\Phi}{\partial x^i\partial x^j}(x) = g_{ij}\boldsymbol{u},$$

and

$$\left\langle \frac{\partial^2\Phi}{\partial x^i\partial x^j}(x), \frac{\partial\Phi}{\partial x^k}(x)\right\rangle = \langle g_{ij}\boldsymbol{u}, \partial_k + \langle \partial_k, x - o\rangle \boldsymbol{u}\rangle = 0,$$

which says that $\partial^2\Phi/\partial x^i\partial x^j$ is always normal to $M$. Thus $\nabla_{\mathrm{d}\Phi(\partial_i)}\mathrm{d}\Phi(\partial_j) = \mathbf{0}$ and we conclude that $M$ is flat. Finally, we get that

$$\alpha(\partial_i, \partial_j) = \frac{\partial^2\Phi}{\partial x^i\partial x^j} = g_{ij}\boldsymbol{u} \implies \boldsymbol{H} = \frac{1}{\dim M} g^{ij} g_{ij}\boldsymbol{u} = \frac{1}{\dim M}(\dim M)\,\boldsymbol{u} = \boldsymbol{u},$$

and $M$ is totally umbilic and marginally trapped, as wanted.

## 6.3 Graphs of holomorphic functions

Let $U \subseteq \mathbb{C}$ be open, and $f = \varphi + i\psi \colon U \to \mathbb{C}$ be a holomorphic function. Consider the graph of $f$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ (with coordinates $(x, y, z, w)$):

$$\mathrm{gr}(f) = \{(x, y, \varphi(x, y), \psi(x, y)) \in \mathbb{R}^4 \mid (x, y) \in U\}.$$

Let's study $\mathrm{gr}(f)$, by looking at the global coordinate system given by

$$U \ni (u, v) \mapsto (u, v, \varphi(u, v), \psi(u, v)) \in \mathrm{gr}(f).$$

Since the Cauchy-Riemann equations read $\varphi_u = \psi_v$ and $\psi_u = -\varphi_v$, we may immediately obtain a tangent frame to $\mathrm{gr}(f)$ in terms only of $\varphi$:

$$\partial_u = \partial_x + \varphi_u\partial_z - \varphi_v\partial_w \quad \text{and} \quad \partial_v = \partial_y + \varphi_v\partial_z + \varphi_u\partial_w.$$

So, by inspection, we see that an orthonormal normal frame to $\mathrm{gr}(f)$ is given by

$$\boldsymbol{\xi}_3 = \frac{-\varphi_u\partial_x - \varphi_v\partial_y + \partial_z}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}} \quad \text{and} \quad \boldsymbol{\xi}_4 = \frac{\varphi_v\partial_x - \varphi_u\partial_y + \partial_w}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}}.$$

This is perhaps easier to see identifying vectors and points, so that $\partial_u = (1, 0, \varphi_u, -\varphi_v)$ and $\partial_v = (1, 0, \varphi_v, \varphi_u)$, and moving 1's and 0's to the last two components is something natural to try.

The geometry of $\mathrm{gr}(f)$ will be controlled by its second fundamental form $\alpha$. But since we already have the frame $(\boldsymbol{\xi}_3, \boldsymbol{\xi}_4)$, let's compute the corresponding shape operators. We organize ourselves:

- Computing $A_{\xi_3}$: first we have

$$
\begin{aligned}
A_{\xi_3}(\partial_u) &= -(\overline{\nabla}_{\partial_u}\xi_3)^\top \\
&= \left(\overline{\nabla}_{\partial_u}\frac{\varphi_u\partial_x + \varphi_v\partial_y - \partial_z}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\right)^\top \\
&\overset{(\dagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\overline{\nabla}_{\partial_u}(\varphi_u\partial_x + \varphi_v\partial_y - \partial_z)\right)^\top \\
&= \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\varphi_{uu}\partial_x + \varphi_{uv}\partial_y\right)^\top \\
&\overset{(\ddagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\frac{\varphi_{uu}}{1+\varphi_u^2+\varphi_v^2}\partial_u + \frac{\varphi_{uv}}{1+\varphi_u^2+\varphi_v^2}\partial_v\right) \\
&= \frac{\varphi_{uu}\partial_u + \varphi_{uv}\partial_v}{(1+\varphi_u^2+\varphi_v^2)^{3/2}}.
\end{aligned}
$$

In $(\dagger)$ we use the product rule for $\overline{\nabla}$ and that $\varphi_u\partial_x + \varphi_v\partial_y - \partial_z$ is normal to $\mathrm{gr}(f)$, while in $(\ddagger)$ we use that $\partial_u$ and $\partial_v$ are orthogonal, so that the projection in the tangent planes to $\mathrm{gr}(f)$ is the sum of the projections onto $\partial_u$ and $\partial_v$. We proceed:

$$
\begin{aligned}
A_{\xi_3}(\partial_v) &= -(\overline{\nabla}_{\partial_v}\xi_3)^\top \\
&= \left(\overline{\nabla}_{\partial_v}\frac{\varphi_u\partial_x + \varphi_v\partial_y - \partial_z}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\right)^\top \\
&\overset{(\dagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\overline{\nabla}_{\partial_v}(\varphi_u\partial_x + \varphi_v\partial_y - \partial_z)\right)^\top \\
&= \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}(\varphi_{uv}\partial_x + \varphi_{vv}\partial_y)^\top \\
&\overset{(\ddagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\frac{\varphi_{uv}}{1+\varphi_u^2+\varphi_v^2}\partial_u + \frac{\varphi_{vv}}{1+\varphi_u^2+\varphi_v^2}\partial_v\right) \\
&= \frac{\varphi_{uv}\partial_u + \varphi_{vv}\partial_v}{(1+\varphi_u^2+\varphi_v^2)^{3/2}}.
\end{aligned}
$$

- Computing $A_{\xi_4}$: we'll just repeat the strategy adopted for $A_{\xi_3}$. We start with

$$\begin{aligned}
A_{\xi_4}(\partial_u) &= -(\overline{\nabla}_{\partial_u}\boldsymbol{\xi}_4)^\top \\
&= \left(\overline{\nabla}_{\partial_u}\frac{-\varphi_v\partial_x + \varphi_u\partial_y - \partial_w}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\right)^\top \\
&\overset{(\dagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\overline{\nabla}_{\partial_u}(-\varphi_v\partial_x + \varphi_u\partial_y - \partial_w)\right)^\top \\
&= \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}(-\varphi_{uv}\partial_x + \varphi_{uu}\partial_y)^\top \\
&\overset{(\ddagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\frac{-\varphi_{uv}}{1+\varphi_u^2+\varphi_v^2}\partial_u + \frac{\varphi_{uu}}{1+\varphi_u^2+\varphi_v^2}\partial_v\right) \\
&= \frac{-\varphi_{uv}\partial_u + \varphi_{uu}\partial_v}{(1+\varphi_u^2+\varphi_v^2)^{3/2}}
\end{aligned}$$

and lastly:

$$\begin{aligned}
A_{\xi_4}(\partial_v) &= -(\overline{\nabla}_{\partial_v}\boldsymbol{\xi}_4)^\top \\
&= \left(\overline{\nabla}_{\partial_v}\frac{-\varphi_v\partial_x + \varphi_u\partial_y - \partial_w}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\right)^\top \\
&\overset{(\dagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\overline{\nabla}_{\partial_v}(-\varphi_v\partial_x + \varphi_u\partial_y - \partial_w)\right)^\top \\
&= \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}(-\varphi_{vv}\partial_x + \varphi_{uv}\partial_y)^\top \\
&\overset{(\ddagger)}{=} \frac{1}{\sqrt{1+\varphi_u^2+\varphi_v^2}}\left(\frac{-\varphi_{vv}}{1+\varphi_u^2+\varphi_v^2}\partial_u + \frac{\varphi_{uv}}{1+\varphi_u^2+\varphi_v^2}\partial_v\right) \\
&= \frac{-\varphi_{vv}\partial_u + \varphi_{uv}\partial_v}{(1+\varphi_u^2+\varphi_v^2)^{3/2}}
\end{aligned}$$

Now, let's put all of this together and find $\alpha$. We have

$$\begin{aligned}
\alpha(\partial_u,\partial_u) &= \langle A_{\xi_3}(\partial_u),\partial_u\rangle\boldsymbol{\xi}_3 + \langle A_{\xi_4}(\partial_u),\partial_u\rangle\boldsymbol{\xi}_4 = \frac{\varphi_{uu}\boldsymbol{\xi}_3 - \varphi_{uv}\boldsymbol{\xi}_4}{\sqrt{1+\varphi_u^2+\varphi_v^2}} \\
\alpha(\partial_u,\partial_v) &= \langle A_{\xi_3}(\partial_u),\partial_v\rangle\boldsymbol{\xi}_3 + \langle A_{\xi_4}(\partial_u),\partial_v\rangle\boldsymbol{\xi}_4 = \frac{\varphi_{uv}\boldsymbol{\xi}_3 + \varphi_{uu}\boldsymbol{\xi}_4}{\sqrt{1+\varphi_u^2+\varphi_v^2}} \\
\alpha(\partial_v,\partial_v) &= \langle A_{\xi_3}(\partial_v),\partial_v\rangle\boldsymbol{\xi}_3 + \langle A_{\xi_4}(\partial_v),\partial_v\rangle\boldsymbol{\xi}_4 = \frac{\varphi_{vv}\boldsymbol{\xi}_3 + \varphi_{uv}\boldsymbol{\xi}_4}{\sqrt{1+\varphi_u^2+\varphi_v^2}}
\end{aligned}$$

Thus

$$\boldsymbol{H} = \operatorname{tr}(A_{\xi_3})\boldsymbol{\xi}_3 + \operatorname{tr}(A_{\xi_4})\boldsymbol{\xi}_4 = \frac{\triangle\varphi\,\boldsymbol{\xi}_3 + 0\cdot\boldsymbol{\xi}_4}{(1+\varphi_u^2+\varphi_v^2)^{3/2}} = \boldsymbol{0},$$

since $f$ holomorphic implies that $\varphi$ is harmonic. In other words, *graphs of holomorphic functions are minimal surfaces*. Recall that in $\mathbb{R}^3$, minimal surfaces always had non-positive curvature. This phenomenon also happens here, since

$$\begin{aligned} K(\partial_u,\partial_v) &= \overline{K}(\partial_u,\partial_v) + \frac{\langle \alpha(\partial_u,\partial_u),\alpha(\partial_v,\partial_v)\rangle - \langle \alpha(\partial_u,\partial_v),\alpha(\partial_u,\partial_v)\rangle}{\langle \partial_u,\partial_u\rangle\langle \partial_v,\partial_v\rangle - \langle \partial_u,\partial_v\rangle^2} \\ &= \frac{1}{(1+\varphi_u^2+\varphi_v^2)^3}(\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2 - (\varphi_{uv}^2+\varphi_{uu}^2)) \\ &= \frac{-\|\nabla^2\varphi\|^2}{(1+\|\nabla\varphi\|^2)^3}, \end{aligned}$$

using one last time that $\varphi_{vv} = -\varphi_{uu}$.

## 6.4 Non-zero curvature planes in $\mathbb{H}^3$

Let $\mathbb{H}^3 = \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$ be the hyperbolic half-space, with the usual Riemannian metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{z^2}.$$

From our previous calculations, we have (identifying $(x,y,z) \leftrightarrow (1,2,3)$, as always) that the connection 1-forms for the Levi-Civita connection of $\mathbb{H}^3$ are

$$(\omega^i{}_j)_{i,j=1}^3 = \frac{1}{z}\begin{pmatrix} -\mathrm{d}z & 0 & -\mathrm{d}x \\ 0 & -\mathrm{d}z & -\mathrm{d}y \\ \mathrm{d}x & \mathrm{d}y & -\mathrm{d}z \end{pmatrix},$$

and it immediately follows that $\overline{\nabla}_{\partial_x}\partial_y = \mathbf{0}$, $\overline{\nabla}_{\partial_x}\partial_z = -(1/z)\partial_x$ and $\overline{\nabla}_{\partial_y}\partial_z = (-1/z)\partial_y$, etc..

- Consider the vertical plane $\Pi_{\mathrm{vert}} : x = 0, z > 0$. A unit normal field to $\Pi_{\mathrm{vert}}$ is given by $\boldsymbol{N}(0,y,z) = z\partial_y$. Let's compute the Gaussian curvature of $\Pi_{\mathrm{vert}}$. The computation

  $$A_{\boldsymbol{N}}(\partial_x) = -\overline{\nabla}_{\partial_x}(z\partial_y) = \mathbf{0} - z\overline{\nabla}_{\partial_x}\partial_y = \mathbf{0} - \mathbf{0} = \mathbf{0}$$

  says that $\partial_x \in \ker A_{\boldsymbol{N}}$, so that $\det(A_{\boldsymbol{N}}) = 0$ and so the Gaussian curvature of $\Pi_{\mathrm{vert}}$ is constant $K = \overline{K}(\Pi_{\mathrm{vert}}) = -1$. Since rotations about the $z$ axis and horizontal translations (with no vertical component) are isometries of $\mathbb{H}^3$, our calculation actually holds for *all* vertical planes in $\mathbb{H}^3$.

- Consider a horizontal plane $\Pi_{\mathrm{hor},z_0} : z = z_0 > 0$. Let's find the Gaussian curvature of $\Pi_{\mathrm{hor},z_0}$ in terms of the height $z_0$. This time, we see that $\boldsymbol{N}(x,y,z_0) = z_0\partial_z$ is a unit normal field to $\Pi_{\mathrm{hor},z_0}$. The metric induced in $\Pi_{\mathrm{hor},z_0}$ is a constant multiple of the standard flat metric, so we already see that the Gaussian curvature vanishes. But to ilustrate the technique, let's compute $A_{\boldsymbol{N}}$, by using the tangent frame $(\partial_x,\partial_y)$ to $\Pi_{\mathrm{hor},z_0}$. We have

  $$A_{\boldsymbol{N}}(\partial_x) = -\overline{\nabla}_{\partial_x}(z_0\partial_z) = -z_0\overline{\nabla}_{\partial_x}\partial_z = -z_0\left(-\frac{1}{z_0}\partial_x\right) = \partial_x$$

and

$$A_{\boldsymbol{N}}(\partial_y) = -\overline{\nabla}_{\partial_y}(z_0\partial_z) = -z_0\overline{\nabla}_{\partial_y}\partial_z = -z_0\left(-\frac{1}{z_0}\partial_y\right) = \partial_y$$

So $A_{\boldsymbol{N}}$ is the identity endomorphism, and we conclude from the Theorema Egregium that $K(\Pi_{\text{hor},z_0}) = -1+1 = 0$. Also, we see that in this case

$$\boldsymbol{H} = \frac{1}{2}\operatorname{tr}(A_{\boldsymbol{N}})\boldsymbol{N} = z_0\partial_z,$$

so this plane is not a minimal surface in $\mathbb{H}^3$.

- Let $m > 0$ and consider the slant plane $\Pi_m : z = my > 0$. A tangent frame to $\Pi_m$ is $(\partial_x, \partial_y + m\partial_z)$, and so a normal field to $\Pi_m$ at the point $(x, y, my)$ is $-m\partial_y + \partial_z$. Normalizing, we obtain the unit normal field

$$\boldsymbol{N}(x,y,my) = \frac{my}{\sqrt{1+m^2}}\left(-m\partial_y + \partial_z\right)$$

Let's compute $A_{\boldsymbol{N}}$. First, we have

$$\begin{aligned} A_{\boldsymbol{N}}(\partial_x) &= -\overline{\nabla}_{\partial_x}\left(\frac{my}{\sqrt{1+m^2}}\left(-m\partial_y+\partial_z\right)\right) \\ &= -\frac{my}{\sqrt{1+m^2}}\left(-m\overline{\nabla}_{\partial_x}\partial_y + \overline{\nabla}_{\partial_x}\partial_z\right) \\ &= -\frac{my}{\sqrt{1+m^2}}\left(-m\mathbf{0} - \frac{1}{my}\partial_x\right) \\ &= \frac{1}{\sqrt{1+m^2}}\,\partial_x, \end{aligned}$$

and then

$$\begin{aligned} &A_{\boldsymbol{N}}(\partial_y + m\partial_z) = -\overline{\nabla}_{\partial_y+m\partial_z}\left(\frac{my}{\sqrt{1+m^2}}\left(-m\partial_y+\partial_z\right)\right) \\ &= -\frac{m}{\sqrt{1+m^2}}(-m\partial_y+\partial_z) - \frac{my}{\sqrt{1+m^2}}(-m\overline{\nabla}_{\partial_y}\partial_y + \overline{\nabla}_{\partial_y}\partial_z) - \frac{m^2y}{\sqrt{1+m^2}}(-m\overline{\nabla}_{\partial_z}\partial_y + \overline{\nabla}_{\partial_z}\partial_z) \\ &= -\frac{m}{\sqrt{1+m^2}}(-m\partial_y+\partial_z) - \frac{my}{\sqrt{1+m^2}}\left(-m\left(\frac{1}{my}\partial_z\right) - \frac{1}{my}\partial_y\right) - \frac{m^2y}{\sqrt{1+m^2}}\left(-m\left(-\frac{1}{my}\partial_y\right) - \frac{1}{my}\partial_z\right) \\ &= -\frac{m}{\sqrt{1+m^2}}\left(-m\partial_y + \partial_z - \partial_z - \frac{1}{m}\partial_y + m\partial_y - \partial_z\right) \\ &= \frac{1}{\sqrt{1+m^2}}(\partial_y + m\partial_z). \end{aligned}$$

So:

$$A_{\boldsymbol{N}} = \frac{1}{\sqrt{1+m^2}}\operatorname{Id}_2 \implies K(\Pi_m) = -1 + \frac{1}{1+m^2} = \frac{-m^2}{1+m^2}.$$

Note that $\lim_{m\to+\infty} K(\Pi_m) = -1$ and $\lim_{m\to 0^+} K(\Pi_m) = 0$, keeping consistency with the previous two computations for $\Pi_{\text{vert}}$ and $\Pi_{\text{hor},z_0}$.

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