DIMENSIONAL REDUCTION IN FIELD THEORY
AND HIDDEN SYMMETRIES IN EXTENDED SUPERGRAVITY

E. CREMMER
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure +

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Postal address : 24, rue Lhomond, 75231 PARIS CEDEX 05 FRANCE

The idea of a 4 + N dimensional space-time is by no means a new one. In 1921, already, Kaluza suggested that gravitation and electromagnetism could be unified in a 5 dimensional theory of gravity. This idea has been revived several times, in particular, in connection with the possible unification of gravitation with gauge fields, or in the context of the fiber bundle approach to Yang-Mills theories trying to associate the extra coordinates with the group space. It has also been put forward in the context of dual models which are consistent only in a precise space-time dimension (D= 10 or D = 26 for the known models) (Scherk & Schwarz 1975 ; Cremmer & Scherk 1976 a). It is then possible to associate to the extra coordinates a compact space (product of N torus in this case) such that the ordinary 4-dimensional physics is a low energy approximation of a bigger theory. We call the 4-dimensional theory the dimensional reduction of the 4 + N ones. We shall concentrate in these lectures essentially on this aspect forgetting about the possible interpretation (or existence) of the extra dimensions. This applies to most of the attempts to unify gravity and Yang-Mills, the concept of low energy approximation being most of the time replaced by the requirement of specific symmetries of the solution of field equations.

The simplest dimensional reduction has been particularly fruitful for supersymmetric theories, especially extended supersymmetric Yang-Mills or extended supergravities. It has made connection between N = 4 Yang-Mills in 4 dimensions and N = 1 Yang-Mills in 10 dimensions, and N = 8 supergravity in 4 dimensions and N = 1 supergravity in 11 dimensions. Moreover, the dimensional reduction explains part of the hidden symmetries in extended supergravities.

In these lectures, we shall discuss first the dimensional reduction of theories which do not include gravitation and then proceed in the second part with the dimensional reduction of theories including gravitation. In particular, we shall describe the 11-dimensional supergravity and its reduction to 4 dimensions. The hunt for the hidden symmetries, global E_7 and local SU(8) of the N = 8 supergravity in 4 dimensions will be described in part III. These hidden symmetries shall provide geometrical meaning to scalar fields. This will be a property of all extended supergravities and will be discussed in part IV. Finally in part V we shall summarize what we know or would like to know about N = 8 supergravity at the classical and quantum level. We shall discuss the possible implications of these hidden symmetries.

I DIMENSIONAL REDUCTION WITHOUT GRAVITATION

The simplest example of dimensional reduction arises when we try to interpret a theory in 4 + N dimensions whose Lagrangian is Poincaré invariant. An interesting case is that of Yang-Mills theory in 4 + N dimensions which leads, via dimensional reduction, to a Yang-Mills + Higgs scalars coupled theory with specific couplings. However, this will lead to unified theory only in the case of supersymmetric theories which requires the study of the supersymmetry algebra in D dimensions. The dimensional reduction of supersymmetric Yang-Mills in 10 dimensions leads to the well-known N = 4 supersymmetric Yang-Mills in 4 dimensions.

1 Interpretation of extra dimensions

Let us start with a scalar theory in 4 + N dimensions whose Lagrangian $\mathcal{L}$ is Poincaré invariant (Cremmer & Scherk 1976 a)

$$S = \int d^{4+N}x \left[\tfrac{1}{2} \partial_M \phi \, \partial^M \phi - \tfrac{1}{2} \mu_o^2 \phi^2 + \mathcal{V}(\phi) \right]$$

It is consistent with the metric $\eta_{MN} = (+-\ldots.-)$ to assume that the extra dimensions are circles of length $L_1 \ldots. L_N$ or, denoting $x_M = (x_\mu, y_i)$ ($i = 1 \ldots N$), to assume that

$$\phi(x_\mu, y_i + L_i) = \phi(x_\mu, y_i)$$

This breaks "spontaneously" the Poincaré invariance of the action S P_{4+N} to $P_4 \times U(1)^N$. $U(1)^N$ will be associated with the conservation of N "heaviness" numbers. We can now expand $\phi(x_\mu, y_i)$ in Fourier series

$$\phi(x_\mu, y_i) = \frac{1}{(L_1 \ldots L_N)^{1/2}} \sum_{\{n_i\}} \phi_{\{n_i\}}(x_\mu) \exp\left(2i\pi \sum_i \frac{y_i n_i}{L_i}\right)$$

with n_i integer. If ϕ is real, we have $\phi_{\{n_i\}} = \phi^*_{\{-n_i\}}$. We can integrate over y_i and obtain a 4-dimensional description of this theory

$$S = \int d_4x \left\{ \sum_{\{n_i\}} \left[\tfrac{1}{2} \partial_\mu \phi^*_{\{n_i\}} \partial^\mu \phi_{\{n_i\}} - \tfrac{1}{2} m^2_{\{n_i\}} \phi^*_{\{n_i\}} \phi_{\{n_i\}} + \text{"} \mathcal{V}(\phi_{\{n_i\}}) \text{"} \prod_i \delta\left(\sum_\alpha n_i^\alpha\right) \text{"} \right] \right\}$$

with $\quad m^2_{\{n_i\}} = \mu_o^2 + 4\pi^2 \sum_i \frac{n_i^2}{L_i^2}$

"$\prod_i \delta(\sum_\alpha n_i^\alpha)$" symbolizes the conservation of the N heaviness numbers. S describes now an infinite number of interacting scalar particles.

The ultraviolet behaviour is the same as in 4 + N dimensions. This can be shown by using properties of Jacobi θ-functions. The infrared behaviour is the same as in 4 dimensions : if we start with 1 massless particle in 4 + N dimensions, we get 1 massless particle in 4 dimensions plus an infinite number of massive particles.

There are two limiting cases : if all the L_i's $\to \infty$: we recover the original theory in 4 + N dimensions. If all the L_i's $\to 0$: only one state keeps a finite mass classically. This limit is associated with the process called "dimensional reduction". This remaining state, ϕ, is described by the "reduced action"

$$S_{(4)} = \int d_4x \left[\tfrac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \tfrac{1}{2} \mu_o^2 \phi^2 + V(\phi) \right]$$

where the coupling constants have been rescaled before the limit $L_i \to 0$ according to their canonical dimensions.

This is equivalent to retaining only the mode independent of y_i or to impose the "symmetry conditions"

$$\frac{\partial \phi}{\partial y_i} = 0$$

This ansatz, together with the equations of motion in 4 + N dimensions, leads to equations which are derivable from a reduced Lagrangian L. L is identical with $\mathcal{L}$ where we have dropped the y_i dependence and we have done some canonical rescaling to the scalar fields and the coupling constants.

Remarks : we could have started with a field theory in curved space, for example, S x M_4 where S is a compact space of dimension N. If G is the symmetry group of S, we can expand a field $\phi(x_\mu, y_i)$ in "G harmonics" on S $Y_N(y_i)$

$$\phi(x_\mu, y_i) = \sum_N Y_N(y_i) \, \phi_N(x_\mu)$$

Φ_N will describe a scalar particle of mass $\mu_0^2 + \frac{C(N)}{L^2}$ where C(N) is the eigenvalue of the Laplace-Beltrami operator for S in the representation N, L being some length which characterizes S. If some of the C(N)'s are zero we can make a consistent truncation by letting $L_i \to 0$. The corresponding ansatz on ϕ

$$\phi(x_\mu, y_i) = \sum_{N_0} Y_{N_0}(y_i)\,\Phi_{N_0}(x_\mu) \qquad \text{with } C(N_0) = 0$$

is such that the y_i dependence factorizes partially from the 4 + N-dimensional equations of motion implying several 4-dimensional equations. The groupe G is then interpreted as an internal symmetry group, S can be the space of G itself or a coset space G/H (dim S = dim G − dim H)

2 <u>Dimensional reduction of non abelian Yang-Mills theory</u>

Let us consider the non abelian Yang-Mills theory in 4 + N dimensions described by the Lagrangian

$$S = -\frac{1}{4}\int d^{4+N}x \;\; \mathrm{Tr}\,(F_{MN}^2)$$

where η_{NM} the metric is (+−...−) and F_{MN} the field strength

$$F_{MN} = \partial_M A_N - \partial_N A_M + i g_0 [A_M, A_N]$$

A_M and F_{MN} are in the Lie algebra of the group G.

Let us now apply the process of dimensional reduction by implementing the condition

$$\partial A_M / \partial y_i = 0$$

The 4 + N components of A_M will split into two parts : a 4-dimensional vector A_μ and N scalars A_i. After canonical rescalings, we obtain the corresponding 4-dimensional action

$$S_4 = \int d_4x \; \mathrm{Tr}\left\{ -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(D_\mu A_i)^2 + \frac{1}{4} g^2 [A_i, A_j]^2 \right\}$$

where $D_\mu A_i = \partial_\mu A_i + i g [A_\mu, A_i]$

<u>Properties of the reduced theory</u> :

(1) It contains both vector and scalar particles (in the adjoint representation of G)

(2) The global internal symmetry is O(N) instead of $U(1)^N$ if

we had kept all the modes of 4 + N dimensions. We have now the breaking

$$P_{4+N} \longrightarrow P_4 \otimes O(N)$$

(3) There is a specific Higgs coupling g^2 related to the gauge coupling g as well as a very specific structure of the potential

(4) The signature of the extra dimensions of space time (space like) is such that we obtain the right sign for the kinetic term of the scalar fields A_i

<u>Important remark : This is not a unified theory of vector and scalar particles.</u> There is no symmetry relating A_μ and A_i (No go theorem of Coleman-Mandula) which implies the relation between the Higgs coupling constant and the gauge coupling constant. Explicit calculations at the quantum level show that this relation is not preserved by radiative corrections.

This problem will be overcome if we start with supersymmetric theories which can relate scalar, vector and spinor particles.

3 <u>Supersymmetry in D dimensions</u>

Unified theories and less trivial examples of dimensional reduction are obtained if we start with a theory invariant under simple supersymmetry algebra in D dimensions

$$\{ Q_{\hat\alpha}, \bar{Q}_{\hat\beta} \} = 2 (\Gamma^M)_{\hat\alpha\hat\beta} P_M$$

where the D matrices Γ^M have dimensions $2^{[D/2]}$ and satisfy the Clifford algebra

$$\{ \Gamma^M, \Gamma^N \} = 2 \eta^{NM}$$

They have the hermiticity properties

$$(\Gamma^0)^\dagger = \Gamma^0 \quad , \quad (\Gamma^m)^\dagger = -\Gamma^m \qquad \text{for } M \neq 0$$

or $$(\Gamma^M)^\dagger = \Gamma^0 \Gamma^M \Gamma^0 \quad \forall M$$

The possible properties of $Q_{\hat\alpha}$: Majorana, Weyl, Dirac or Majorana-Weyl will depend on the dimension D of space-time (Gliozzi et al.1977)

<u>D even</u> :

(1) Since Γ^M and $(\Gamma^M)^*$ satisfy the same Clifford algebra, there exists a matrix B such that

$$(\Gamma^M)^* = -B\Gamma^M B^{-1}$$

we can fix the phases such that $BB^* = \epsilon\, I$ with $\epsilon = \pm I$. A Majorana spinor will be defined as being its own antiparticle $B^{-1}\psi^* = \psi$ therefore it exists only if $\epsilon = +1$.

(2) Since Γ^M and $(\Gamma^M)^t$ satisfy also the same Clifford algebra, there exists a matrix C (charge conjugation) such that

$$(\Gamma^M)^t = -C\Gamma^M C^{-1}$$

Together with the hermiticity properties this implies (after a phase choice)

$$B^t = C\Gamma^0 \quad , \quad BB^\dagger = I$$

and consequently $\quad B = \epsilon B^t \quad , \quad C = -\epsilon C^t$

Defining $\Gamma^{(n)}$ as the antisymmetrized product of n matrices, we therefore have

$$(C\Gamma^{(n)})^t = \epsilon\,(-1)^{\frac{(n-1)(n-2)}{2}}\, C\Gamma^{(n)}$$

This allows us to count the number of antisymmetric matrices $2^{D/2} \times 2^{D/2}$ which should be $1/2\; 2^{D/2}\,(2^{D/2} - 1)$. This gives

$$\epsilon = -\sqrt{2}\cos\frac{\pi}{4}(D+1) \qquad \text{or} \begin{cases} \epsilon = +1 \text{ for } D = 2,4 \bmod 8 \\ \epsilon = -1 \text{ for } D = 0,6 \bmod 8 \end{cases}$$

For $D = 2,4 \bmod 8$, there exists a pure imaginary representation of Γ matrices. We can choose $B = 1$, $C = \Gamma^0$, then a Majorana spinor is simply a real spinor.

For $D = 0,6 \bmod 8$, we can however define "Majorana spinors" when there is an internal symmetry (extended supersymmetry).

The matrix Γ^{D+1} of square 1 and anticommuting with the D Γ matrices is given by

$$\Gamma^{D+1} = (-1)^{\frac{D-2}{4}}\, \Gamma^0\Gamma^1 \ldots \Gamma^{D-1}$$

A Weyl spinor λ is defined by

$$\Gamma^{D+1}\lambda = \pm\lambda$$

We can have a Majorana-Weyl spinor if Γ^{D+1} is real, i.e. for $D = 2 \bmod 8$ (in particular $D = 10$)

<u>D odd</u> : $D = d + 1$ with d even

A Clifford algebra for $d + 1$ is obtained from the one in d

dimensions by adding to the Γ^M ($M = 0, \ldots d - 1$) the matrix

$$\Gamma^d = i\,(-1)^{\frac{d-2}{4}}\, \Gamma^0\Gamma^1 \ldots \Gamma^{d-1} \quad (\equiv i\,\Gamma^{d+1})$$

such that $(\Gamma^d)^2 = -1$.

Therefore, we shall have the possibility of Majorana spinors if Γ^d is pure imaginary when $\Gamma^0, \ldots \Gamma^{d-1}$ are pure imaginary or equivalently if Γ^{d+1} is real : there are Majorana spinors in $d + 1$ dimensions if there are Majorana-Weyl spinors in d dimensions, namely $D = 3 \bmod 8$

Note : these properties of the spinors depend only on the signature of space-time s-t (the metric being $(\underbrace{++\ldots+}_{s}, \underbrace{-\ldots-}_{t})$)

<u>Dimensional reduction</u> : Starting from a Clifford algebra in $4 + N$ dimensions, we can always define it (up to an equivalence) as a tensor product of γ matrices 4×4 by "internal" matrices $\tilde{\gamma}$ $2^{[N/2]} \times 2^{[N/2]}$ such that

$$\Gamma^\mu = \gamma^\mu \otimes (1 \text{ or } \Omega)$$

$$\Gamma^i = (1 \text{ or } \gamma^5) \otimes \tilde{\gamma}^i$$

Therefore, a Dirac spinor in $4 + N$ dimensions, through the ordinary dimensional reduction is equivalent to a $2^{[N/2]}$ Dirac spinor in 4 dimensions or $2.2^{[N/2]}$ Majorana or Weyl spinors in 4 dimensions. There will be a reduction factor : 1/2 if we start with Majorana or Weyl spinors and 1/4 if we start with Majorana-Weyl spinors when they exist. These results are summarized in Table I giving the number of Majorana or Weyl spinors in 4 dimensions corresponding to a given spinor in D dimensions.

We see from Table I, that starting from a Majorana-Weyl spinor in 10 dimensions and reducing to 9, we should obtain a 4 component-spinor although there are no Majorana or Weyl spinors. This means that in 9 dimensions there exists another kind of spinors (pseudo-Majorana) which have half the number of the components of a Dirac spinor (Van Nieuwenhuizen

TABLE I

D	4	5	6	7	8	9	10	11	12
DIRAC	2	2	4	4	8	8	16	16	32
MAJ.	1	-	-	-	-	-	8	8	16
WEYL	1	-	2	-	4	-	8	-	16
M - W	-	-	-	-	-	-	4	-	-

1980)

4 Supersymmetric Yang-Mills in D = 10, N = 1 and Supersymmetric Yang-Mills in D = 4, N = 4

From the previous table, it can be seen that the maximal dimension in which we can have a supersymmetric Yang-Mills theory (with spin≤ 1) is D = 10. For D$>$10, the dimensional reduction would lead to particles with spin $\geq 3/2$ or equivalently with N$>$4 supersymmetry.

Supersymmetric Yang-Mills in D = 10, N = 1 (Gliozzi et al. 1977 ; Brink et al. 1977). A vector field in 10 dimensions has 8 degrees of freedom as a Majorana-Weyl spinor field. Therefore it seems plausible that there could exist a supersymmetric Yang-Mills theory with only 1 vector field and a Majorana-Weyl spinor field. It is in fact possible and the Lagrangian is given by

$$S = \int d^{10}x \ \mathrm{Tr}\left[-\frac{1}{4} F_{MN}F^{MN} + \frac{i}{2} \bar{\lambda} \Gamma^M D_M \lambda \right]$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M + i g_0 [A_M, A_N]$$

$$D_M \lambda = \partial_M \lambda + i g_0 [A_M, \lambda]$$

A_M and λ belong to the Lie algebra of a group G. The Γ_{11}'s are 32 x 32 matrices. λ is a Majorana-Weyl spinor and satisfies $\lambda = \Gamma_{11}\lambda$

S is invariant under the following supersymmetry transformations

$$\delta A_M = i \bar{\epsilon} \Gamma_M \lambda \quad , \quad \delta\lambda = 2 F_{MN} \Gamma^{MN} \epsilon$$

with $\epsilon = \Gamma_{11} \epsilon$

Let us note that the same action is supersymmetric for D = 6 if λ is a Weyl spinor and D = 4 if λ is a Majorana spinor.

Supersymmetric Yang-Mills in D = 4, N = 4 : It is obtained by dimensional reduction of the supersymmetric Yang-Mills theory in D = 10. Let us define 6 real 4 x 4 independent antisymmetric matrices $(\alpha^i)_{ab}$, $(\beta^i)_{ab}$ (i = 1,2,3) which satisfy the algebra of O(4) $\sim$ SU(2) x SU(2)

$$\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = -2\delta^{ij}$$

$$[\alpha^i, \beta^j] = 0 \ , \ [\alpha^i, \alpha^j] = -2\epsilon^{ijk}\alpha^k \ , \ [\beta^i, \beta^j] = -2\epsilon^{ijk}\beta^k$$

$$\frac{1}{2}\varepsilon^{abcd}(\alpha^i)_{cd} = (\alpha^i)_{ab} \quad , \quad \frac{1}{2}\varepsilon^{abcd}(\beta^i)_{cd} = -(\beta^i)_{ab}$$

We can then write the Γ matrices in 10 dimensions and in the Majorana representation as

$$\Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \qquad \mu = 0,1,2,3$$

$$\Gamma^{3+i} = i I_4 \otimes \begin{pmatrix} 0 & \beta^3\alpha^i \\ \beta^3\alpha^i & 0 \end{pmatrix}$$

$$\Gamma^{6+i} = \gamma_5 \otimes \begin{pmatrix} \beta^i & 0 \\ 0 & \beta^3\beta^i\beta^3 \end{pmatrix}$$

therefore

$$\Gamma^{11} = I_4 \otimes \begin{pmatrix} 0 & \beta^3 \\ -\beta^3 & 0 \end{pmatrix}$$

A Majorana-Weyl spinor is then written as

$$\lambda = (\lambda_a \ , \ -\beta^3_{ab}\lambda_b) \qquad a,b = 1,2,3,4$$

λ_a being a 4 dimensional Majorana spinor.

As has been done previously, we split the 10 components of the vector fields $A_M \longrightarrow (A_\mu, A_i, B_i)$. With the previous decomposition of the spinor field, this immediately gives the reduced Lagrangian

$$S_R = \int d_4x \ \mathrm{Tr}\Big\{ -\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu A_i)^2 + \frac{1}{2}(D_\mu B_i)^2 + \frac{i}{2}\bar{\lambda}_a\gamma^\mu D_\mu \lambda_a + \frac{g}{2}\bar{\lambda}_a[\alpha^i_{ab}A_i + i\gamma_5\beta^i_{ab}B_i, \lambda_b] + \frac{g^2}{4}\left([A_i, A_j]^2 + [B_i, B_j]^2 + 2[A_i, B_j]^2\right) \Big\}$$

A_i and B_i are respectively scalar and pseudoscalar fields. The reduction of the transformation of supersymmetry $\epsilon = \Gamma_{11}\epsilon$ is consistent with $\partial\epsilon/\partial y_i = 0$. Then we get 4 supersymmetries with parameters ϵ_a from

$$\epsilon = (\epsilon_a \ , \ -\beta^3_{ab}\epsilon_b)$$

S_R is then invariant under the N = 4 supersymmetry transformations

$$\delta A_\mu = 2i \ \bar{\epsilon}^a \gamma_\mu \lambda^a$$

$$\delta A_i = -2\,\bar{\epsilon}^a \alpha^i_{ab} \lambda^b \quad , \quad \delta B_i = 2i\,\bar{\epsilon}^a \gamma_5 \beta^i_{ab} \lambda^b$$

$$\begin{aligned}\delta\lambda^a = {} & 2F_{\mu\nu}\gamma^{\mu\nu}\epsilon^a + 4i\,(D_\mu A_i \alpha^i_{ab} + i\gamma_5 D_\mu B_i \beta^i_{ab})\gamma_\mu \epsilon^b \\ & - ig\,(2\epsilon_{ijk}[A^i, A^j]\alpha^k_{ab} + 2\epsilon_{ijk}[B^i, B^j]\beta^k_{ab} \\ & - 2i\gamma_5 [A^i, B^j](\alpha^i\beta^j)_{ab})\,\epsilon^b\end{aligned}$$

It is also invariant as expected under a $O(6) \sim SU(4)$ global symmetry of parameters $\Lambda_{ij} = -\Lambda_{ji}$, $\Lambda'_{ij} = -\Lambda'_{ji}$ and $\tilde{\Lambda}_{ij}$

$$\delta A_\mu = 0$$

$$\delta A_i = \Lambda'_{ij} A_j - \tilde{\Lambda}_{ij} B_j \quad , \quad \delta B_i = \Lambda_{ij} B_j + \tilde{\Lambda}_{\ell i} A_\ell$$

$$\delta\lambda_a = -\tfrac{1}{4}[\varepsilon^{ijk}\beta^k_{ab}\Lambda_{ij} + \varepsilon^{ijk}\alpha^k_{ab}\Lambda'_{ij} + i\gamma_5(\alpha^i\beta^j)_{ab}\tilde{\Lambda}_{ij}]\lambda_b$$

The ordinary formulation of N = 4 Yang-Mills, as it is described by the representation of N = 4 supersymmetry, is made with the scalar fields A_{ab} and B_{ab} respectively self-dual and antiself-dual, defined by

$$A_{ab} = \alpha^i_{ab} A_i \quad , \quad B_{ab} = \beta^i_{ab} B_i$$

Since this multiplet (A_μ, A_i, B_i, λa) is an irreducible representation of N = 4 supersymmetry, we now really have a unified theory of vector, scalar and spinor particles. All relations between coupling constants are dictated by the N = 4 supersymmetry.

This conformal invariant theory has remarkable renormalizability properties. It has been shown that the β function is zero up to three-loop order and that there exists, at the one loop order, a gauge in which the theory is finite (Grisaru 1981). It has been conjectured that $\beta = 0$ to all orders and that the theory is finite. This could be linked with the fact that the N = 4 Yang-Mills supermultiplet is CPT self-conjugate and that the global symmetry is SU(4) and not U(4).

II DIMENSIONAL REDUCTION WITH GRAVITATION

When we consider theories in higher dimensions which include gravity (or equivalently which are invariant under local transformations of coordinates) their interpretation in 4 dimensions leads to the concept

of spontaneous compactification of space-time. The 4-dimensional theories obtained by dimensional reduction have new features. As previously, really unified theories are obtained if we start now with supergravity theories in higher dimensions (for example D = 11).

1 Spontaneous compactification of space-time

Let us start with a theory with gravitation in 4 + N dimensions. As previously, if we want to interpret it in 4 dimensions, we need the background 4 + N space-time to be a product of a compact space of dimension N by an ordinary 4 dimensional space-time. But now this requirement should follow from the field equations (for the metric $g_{\hat{\mu}\hat{\nu}}$ and eventually the other fields) : This is called spontaneous compactification of space-time (Cremmer & Scherk 1976 b, 1977a ; Cremmer et al. 1977 b ; Luciani 1978 ; Palla 1979).

Then the theory is expanded around this particular solution using "harmonic" functions for the invariance group G of this compact space of N dimensions. The problem of showing the stability of such a solution is in general difficult. If there are several possible classical solutions, the choice between them is not obvious since the energy is not well defined. We could even have solutions for various decompositions of the 4 + N space-time : boundary conditions could eventually choose between them. We could also ask that only quantum corrections provide such a spontaneous compactification, but we would need the 4 + N dimensional theory to have a meaning at the quantum level (finite theory ?).

It has been shown that such solutions exist. In particular, in the system Einstein + Yang-Mills, it has been shown that we can require that the 4-dimensional space-time should be flat and the internal space a sphere S_N. If we can make the limit $L \to 0$, ($L \sim$ size of the compact space) and keep some fields with a finite (or zero) 4-dimensional mass, we can truncate the theory in a consistent way and deduce a theory in 4 dimensions with a finite number of fields. This is equivalent to retaining solutions which have a specific property of symmetry.

The most simple dimensional reduction consists in compactifying on a product of torus (this is always consistent with the equations of motion and is, as previously, dictated only by boundary conditions) and letting the size of the torus go to zero. This is equivalent to assuming that the fields do not depend on the extra-coordinates. This is the ordinary dimensional reduction. To 1 degree of freedom in 4 + N dimensions corresponds 1 degree of freedom in 4 dimensions. This should not be the case for other compactified spaces like a sphere for example.

2 Dimensional reduction

As we have already seen, the solutions of 4 + N dimensional equations of motion which satisfy $\partial\phi/\partial y_i = 0$ can be derived from the Lagrangian $\int d_4x \sqrt{g(x_\mu)}\, L(\phi(x_\mu))$ where L is identical with $\mathcal{L}$ up to some rescaling of the coupling constants and the fields in order to give them the canonical dimensions of a theory in 4 dimensions. These rescaling factors disappear after integration over $d^N y$. The world indices M will be split as previously into 4-dimensional world indices and internal indices.

In 4 + N dimensions, the theory has the complete reparametrization invariance under the local coordinates transformations

$$\delta x^M = -\xi^M(x)$$

$$\delta\phi = \xi^M \partial_M \phi$$

$$\delta A_M = \xi^N \partial_N A_M + \partial_M \xi^N A_N$$

In 4 dimensions, after dimensional reduction, the remaining invariances will be those consistent with the condition

$$\partial_i \phi = \partial_i A_N = 0$$

$$* \; \delta\phi = \xi^\mu \partial_\mu \phi + \cancel{\xi^i \partial_i \phi}$$

$$\partial_i(\delta\phi) = 0 \implies \partial_i \xi^\mu = 0$$

$$* \; \delta A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + \partial_\mu \xi^i A_i + \cancel{\xi^i \partial_i A_\mu}$$

$$\partial_i(\delta A_\mu) = 0 \implies \partial_j \partial_\mu \xi^i = 0$$

$$* \; \delta A_i = \xi^\nu \partial_\nu A_i + \cancel{\partial_i \xi^\nu} A_\nu + \cancel{\xi^j \partial_j A_i} + \partial_i \xi^j A_j$$

$$\partial_k(\delta A_i) = 0 \implies \partial_k \partial_i \xi^j = 0$$

The results are summarized in

$$\xi^\mu = \xi^\mu(x_\nu)\;,\; \xi^i = a^i{}_j x^j + \xi^i(x_\nu)\;,\; a^i{}_j = \mathrm{Cste}$$

The rescalings we had to make on the fields restrict to transformations which preserve the volume element $d^N y$ i.e. $a^i{}_i = 0$. These transformations correspond to the following symmetries in 4 dimensions

$$\begin{cases} \delta x^\mu = -\xi^\mu(x) & \text{Reparametrization invariance in 4 dimensions} \\ \delta x^i = -\xi^i(x) & U(1)^N \text{ local invariance} \\ \delta x^i = -a^i{}_j x^j & SL(N,R) \text{ global invariance} \end{cases}$$

3 Dimensional reduction of pure gravitation

(a) Let us apply the previous discussion to the reduction of the pure gravitation (Cho & Freund 1975 ; Cho & Jang 1975 ; Cremmer & Julia 1978, 1979 ; Scherk & Schwarz 1979 b). We start with the Einstein-Cartan formulation of gravitation described by the action (we have chosen K = 1)

$$S = -\frac{1}{4}\int d^{4+N}x \; e\, R(\omega, e)$$

where $e_M{}^A$ is the vielbein field, $e = \det e_M{}^A$, ω_{MAB} is the connection for the local Lorenzt group SO(N + 3, 1)

$$R(\omega,e) = e^{MA} e^{NB}\left(\partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{MAC}\omega_N{}^C{}_B - \omega_{NAC}\omega_M{}^C{}_B\right)$$

ω is an independent field which does not propagate. We can solve its equation of motion and obtain $\omega = \omega(e)$. The invariances are the reparametrization in 4 + N dimensions and the local SO(N + 3,1) Lorentz invariance. Let us now perform the dimensional reduction. Writing $e_M{}^A$ as

$$e_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & e_\mu{}^a \\ e_m{}^\alpha & e_m{}^a \end{pmatrix} \qquad \begin{array}{l} \alpha \text{ indices for } SO(3,1) \\ a \text{ indices for } SO(N) \end{array}$$

Of course, we require $\partial_i e_M{}^A = 0$. Moreover, we break the local SO(3+N,1) invariance into SO(3,1) x SO(N) by imposing the condition : $e_m{}^\alpha = 0$. This condition does not restrict the invariances derived from the reparametrization invariance discussed in the previous section.

$$\delta e_m{}^\alpha = \xi^\mu \partial_\mu e_m{}^\alpha + a_m{}^i e_i{}^\alpha = 0$$

$$\delta e_\mu{}^\alpha = \xi^\nu \partial_\nu e_\mu{}^\alpha + \partial_\mu \xi^\nu e_\nu{}^\alpha$$

$$\delta e_m{}^a = \xi^\nu \partial_\nu e_m{}^a + a_m{}^i e_i{}^a$$

$$\delta e_\mu{}^a = \xi^\nu \partial_\nu e_\mu{}^a + \partial_\mu \xi^i e_i{}^a + \partial_\mu \xi^\nu e_\nu{}^a$$

Defining $B_\mu{}^i = e_a{}^i e_\mu{}^a$ $\quad (e_a{}^i e_j{}^a = \delta^i_j)$, we obtain

$$\delta B_\mu^i = \xi^\nu \partial_\nu B_\mu^i + \partial_\mu \xi^\nu B_\nu^i + \partial_\mu \xi^i - a^i{}_j B_\mu^j$$

Therefore :

- $B_\mu{}^i$ are N vectors, gauge fields for $U(1)^N$ in the representation $\bar{F}$ of SL(N,R)
- $e_m{}^a$ are N^2 scalar fields, in the N representation of SL(N,R) (index m) and in the N representation of SO(N) (index a)
- $e_\mu{}^\alpha$ is the ordinary vierbein

We can define two tensor metrics invariant in the local transformations SO(3,1) x SO(N)

$$g_{\mu\nu} = \eta_{\alpha\beta}\, e_\mu{}^\alpha e_\nu{}^\beta \qquad \eta_{\alpha\beta} = (+---)$$

$$g_{ij} = \eta_{ab}\, e_i{}^a e_j{}^b \qquad \eta_{ab} = (--\ldots-)$$

The complete metric tensor g_{MN} is then written in the Kaluza-Klein parametrization

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + B_\mu^k g_{k\ell} B_\nu^\ell & B_\mu^k g_{ik} \\ B_\nu^k g_{kj} & g_{ij} \end{pmatrix}$$

It has, in particular, the property

$$\det g_{MN} = \det g_{\mu\nu} \det g_{ij} \quad (\equiv g\,\Delta)$$

Because of the $U(1)^N$ gauge invariance, B_μ^k must appear in the reduced Lagrangian only through $G_{\mu\nu}{}^k$. The simplest way to obtain gauge invariant objects in 4 dimensions is to start with flat tensors in 4 + N dimensions which are scalar under reparametrization. Defining the anholonomy coefficients Ω_{ABC} by

$$[\partial_A, \partial_B] = [e^M{}_A \partial_M , e^N{}_B \partial_N] = \Omega_{AB}{}^C \partial_C$$

we can rewrite S, after integration by part, as

$$S = \frac{1}{16}\int d^{4+N}x \sqrt{g}\left[\Omega_{ABC}^2 - 2\Omega_{ABC}\Omega^{CAB} - 4\Omega_{CA}{}^A \Omega^C{}_B{}^B\right]$$

After dimensional reduction in the SO(3 + N,1) gauge $e_m^\alpha = 0$, the only non vanishing coefficients are $\Omega_{\alpha\beta\gamma}$,

$$\Omega_{\beta\alpha c} = e_{ic}\, e^\mu{}_\alpha e^\nu{}_\beta\, G^i_{\mu\nu}$$

$$\Omega_{\alpha bc} = -\Omega_{b\alpha c} = -e^i{}_b\, e^\mu{}_\alpha\, \partial_\mu e_{ic}$$

Then, we immediately get the reduced action in 4 dimensions

$$S = -\frac{1}{4}\int d_4x \sqrt{g}\sqrt{\Delta}\Big\{ R - \frac{1}{4} g_{ij} G^i_{\mu\nu} G^j_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + \frac{1}{4} g^{\rho\sigma}(g^{ik}g^{j\ell} - g^{ij}g^{k\ell})\partial_\rho g_{ik}\partial_\sigma g_{j\ell}\Big\}$$

We can eliminate $\Delta^{1/2} = |\det g_{ij}|^{1/2}$ in front of R by a Weyl rescaling

$$e_{4\mu}{}^\alpha = e_\mu{}^\alpha \Delta^{1/4}$$

We finally obtain

$$S = -\frac{1}{4}\int d_4x \sqrt{g_4}\Big\{ R_4 - \frac{1}{4}\sqrt{\Delta}\, g_{ij}\, g_4^{\mu\rho} g_4^{\nu\sigma} G^i_{\mu\nu} G^j_{\rho\sigma} - \frac{1}{8} g_4^{\mu\nu}\partial_\mu \log\Delta\, \partial_\nu \log\Delta + \frac{1}{4} g_4^{\mu\nu}\partial_\mu g_{ij}\partial_\nu g^{ij}\Big\}$$

This describes 1 graviton, N gauge vector fields and N(N+1)/2 massless scalar fields. The invariancesare the reparametrization in 4 dimensions, local $U(1)^N$ and global SL(N,R) as well as local SO(N) and SO(3,1) Lorentz hidden in the metrics g_{ij} and $g_{\mu\nu}$. As previously, it is not a unified theory of scalar, vector and tensor particles since no symmetry relates $g_{\mu\nu}$, g_{ij} and $B_\mu{}^i$. Let us remark that we can extend the SL(N,R) symmetry to GL(N,R) by adding the following scale transformations which preserve S

$$g_{ij} \to \lambda\, g_{ij} \quad , (\Delta \to \lambda^N \Delta) \quad , B_\mu^i \to \lambda^{-\frac{1}{2}-\frac{N}{4}} B_\mu^i$$

(b) <u>Structure of the scalar fields</u>

Defining $\bar{g}_{ij}$ such that det $\bar{g}_{ij} = 1$ or $g_{ij} = \bar{g}_{ij}\Delta^{1/N}$, the part of the Lagrangian which describes the self interaction of the scalar fields (together with their couplingsto gravity) is written as

$$\mathcal{L}_S = \frac{1}{16}\int d_4x\, V_4\, g_4^{\mu\nu}\left[\left(\frac{1}{2}+\frac{1}{N}\right)\partial_\mu \log\Delta\, \partial_\nu \log\Delta - \partial_\mu \bar{g}_{ij}\partial_\nu \bar{g}^{ij}\right]$$

$\bar g_{ij}$ is a symmetric matrix of determinant equal to 1. It is an element of the coset space SL(N,R)/SO(N). This structure is better seen when we reintroduce a N-bein $\bar e_i{}^a$ such that

$$\bar g_{ij} = \bar e_i{}^a \eta_{ab} \bar e_j{}^b$$

$\bar e_i{}^a$ is an element of SL(N,R) defined only up to a SO(N) local transformation. If we forget for a moment about the local SO(N) transformations, a possible Lagrangian for $\bar e_i{}^a$ invariant under SL(N,R) is

$$L \sim -\partial_\mu \bar e_i{}^a \partial^\mu \bar e^i{}_a \qquad (\ \bar e^i{}_a \text{ inverse matrix of } \bar e_i{}^a\ )$$

$$\sim \mathrm{Tr}[(\bar e^{-1}\partial_\mu \bar e)^2]$$

where $\bar e^{-1}\partial_\mu \bar e$ is now an element of the Lie algebra of SL(N,R)

It is invariant even under SL(N,R) x SL(N,R). Since SL(N,R) is non compact, $\mathrm{Tr}[(\bar e^{-1}\partial_\mu \bar e)^2]$ has both positive and negative terms. Therefore, L cannot be positive definite. This non-positivity problem can be solved as usually by introducing a local gauge invariance : SO(N) in this case. We now start from the gauge invariant Lagrangian

$$L \sim \mathrm{Tr}[(\bar e^{-1} D_\mu \bar e)^2]$$

with $D_\mu \bar e_i{}^a = \partial_\mu \bar e_i{}^a - \bar e_i{}^b \Omega_{\mu a}{}^b$

$\Omega_{\mu a}{}^b$ is a gauge field for the local SO(N) invariance, it belongs to the Lie algebra of SO(N), i.e. is antisymmetric in a and b. However, we do not introduce any kinetic terms for Ω_μ so that the Lagrangian being quadratic in Ω_μ, we can solve its equations of motion

$$L \sim \mathrm{Tr}[(\bar e^{-1}\partial_\mu \bar e - \Omega_\mu)^2]$$

We can decompose $\bar e^{-1}\partial_\mu \bar e$ into two parts parallel and perpendicular to SO(N) with respectto the Killing metric i.e. in antisymmetric and symmetric part in a and b

$$\bar e_a{}^i \partial_\mu \bar e_i{}^b = (\bar e_a{}^i\partial_\mu \bar e_i{}^b)_{\parallel} + (\bar e_a{}^i \partial_\mu \bar e_i{}^b)_\perp$$

The equation for Ω_μ is solved immediately by

$$\Omega_{\mu a}{}^b = (\bar e_a{}^i\partial_\mu \bar e_i{}^b)_\parallel \quad \left(=\tfrac12(\bar e_a{}^i\partial_\mu \bar e_i{}^b - a \leftrightarrow b)\right)$$

so that after insertion in the Lagrangian, it becomes

$$L \sim \mathrm{Tr}[(\bar e^{-1}\partial_\mu \bar e)^2_\perp]$$

Since SO(N) is the maximal compact subgroup of SL(N,R), L is now positive definite. Although Ω_μ has disappeared, L is still gauge invariant. It is invariant under the following transformation on $\bar e$

$$\bar e_i{}^a \longrightarrow S_i{}^j \bar e_j{}^b O_b{}^a(x)$$

$S_i{}^j$ being a constant matrix of SL(N,R) and $O_b{}^a(x)$ a local matrix of SO(N). With a little algebra, we can show that this Lagrangian is in fact only function of the gauge invariant quantity $\bar g_{ij}$ and can be rewritten as

$$L \sim -\mathrm{Tr}(\partial_\mu \bar g\, \partial^\mu \bar g^{-1})$$

If we write $\bar e$ as exp w where w is an element of the Lie algebra of SL(N,R). SL(N,R) and O(N) do not act linearly on w. The maximal subgroup which acts linearly on w is a diagonal subgroup isomorph to SO(N)

$$\bar e \longrightarrow O\, \bar e\, O^{-1}$$

This structure is a particular case of what is called G/H σ-model where now the scalar fields are defined on the coset space G/H, for example, the CP^{N-1} models are SU(N)/U(N-1)σ- models. We can use a non compact group G provided H is the maximal subgroup. Although the gauge invariance H could seem artificial, the N-bein formalism is necessary when we couple the theory to fermions. Moreover, we know from the properties of the 2-dimensional CP^{N-1} model, that it could eventually lead to non-trivial quantum effects : this local symmetry could become dynamical.

4 Dimensional reduction of coupled matter

Let us now consider some matter in 4 + D dimensions coupled to gravity : scalar matter and vector matter. This last one will exhibit some particular problems which will occur in a more complicated way later on in the dimensional reduction of 11-dimensional supergravity.

(a) Scalar matter

Let us start with the scalar Lagrangian

$$L_S = \int d^{4+N}x \sqrt{g}\, g^{MN}\partial_M\phi\,\partial_N\phi$$

Assuming as usual $\partial_i \phi = 0$ and with the previous formula for g_{MN} we get immediately (taking into account the previous Weyl rescaling)

$$L_S^{(4)} = \int d_4x\, V_4\, g_4^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi$$

(b) Vector matter

For simplicity, we shall consider only an abelian gauge field described by the Lagrangian

$$L_V = \int d^{4+N}x \sqrt{g}\, g^{MN} g^{PQ} F_{MP} F_{NQ}$$

L_V is invariant under the following transformations of A_M (together with corresponding transformations of g_{MN})

$$\delta A_M = \xi^N \partial_N A_M + \partial_M \xi^N A_N + \partial_M \Lambda$$

After dimensional reduction with $\partial_i \Lambda = 0$, we get

$$\delta A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + \partial_\mu \xi^i A_i + \partial_\mu \Lambda$$

$$\delta A_i = \xi^\nu \partial_\nu A_i + a_i{}^j A_j$$

As already noticed, A_μ transforms not only under its own gauge group (Λ) but also under the $U(1)^N$ gauge group associated with the vector fields $B_\mu{}^i$. However, we can define a new vector field A'_μ which is invariant under this $U(1)^N$ gauge group. For this, we apply the general method which is to start with a flat tensor in 4 + N dimensions A_A

$$A_A = e_A{}^M A_M$$

and reduce it in 4 dimensions. We get

$$A_\alpha = e_\alpha{}^\mu A_\mu + e^i{}_\alpha A_i = e_\alpha{}^\mu (A_\mu - B_\mu^i A_i) \equiv e_\alpha{}^\mu A'_\mu$$

$$A_a = e_a{}^i A_i + e_a'{}^\mu A_\mu = e^i{}_a A_i$$

The reduction of the Lagrangian is obtained by starting by the flat tensor

$$F_{AB} = e_A{}^M e_B{}^N F_{MN} = e_A{}^M \partial_M A_B - e_B{}^M \partial_M A_A - \Omega_{AB}{}^C A_C$$

for which we get

$$F_{\alpha\beta} = e_\alpha{}^\mu e_\beta{}^\nu (\partial_\mu A'_\nu - \partial_\nu A'_\mu + G_{\mu\nu}^i A_i)$$

$$F_{\alpha a} = e_a{}^i e_\alpha{}^\mu \partial_\mu A_i$$

$$F_{ab} = 0$$

The Lagrangian in 4 dimensions is then immediately obtained after the previously defined Weyl rescaling noting that $\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma}$ is a Weyl rescaling

invariant

$$L_V^{(4)} = \int d_4x\, V_4 \left[\sqrt{\Delta}\, g_4^{\mu\rho} g_4^{\nu\sigma} (F'_{\mu\nu} + G^i_{\mu\nu} A_i)(F'_{\rho\sigma} + G^j_{\rho\sigma} A_j) + g_4^{\mu\nu} g^{ij} \partial_\mu A_i \partial_\nu A_j \right]$$

As previously, the SL(N,R) invariance can be extended to GL(N,R) by adding to the previous scale transformations, the following ones

$$A'_\mu \longrightarrow \lambda^{-N/4} A'_\mu \quad , \quad A_i \longrightarrow \lambda^{1/2} A_i$$

This example has shown us how to define pure gauge fields in 4 dimensions.

5 Supergravity in 11 dimensions (Cremmer et al. 1978 b)

As we have seen, the maximal dimension in which a supersymmetric theory leads, after reduction in 4 dimensions, to $N \leq 8$ supersymmetric theory is D = 11. Therefore, supergravity in 11 dimensions should correspond to maximal N = 8 supergravity in 4 dimensions. The onshell massless states in D dimensions are classified by O(D − 2). A graviton g_{MN} has 1/2 (D −2) (D − 1) − 1 or 44 degrees of freedom in 11 dimensions. A gravitino Ψ_M with a Majorana condition has $1/2\ 2^{[D/2]}$ (D − 3) or 128 degrees of freedom in 11 dimensions. Therefore, 84 bosonic degrees of freedom are missing : it corresponds in fact to the representation A_{ijk} antisymmetric of O(9). We can associate to it the covariant tensor A_{MNP} with the gauge invariance

$$\delta A_{MNP} = 3\, \partial_{[M} \xi_{NP]} \qquad \xi_{NP} = -\xi_{PN}$$

If this guess is correct, only gauge fields appear in this theory and then its construction should be relatively simple. This is true and we can construct a locally supersymmetric Lagrangian function of the 3 fields $e_M{}^A$, Ψ_M and A_{MNP}. Because of gauge invariance A_{MNP} appears mainly through its field strength $F_{MNPQ} = 4\, \partial_{[M} A_{NPQ]}$. The Lagrangian is given by

$$L = \int d^{11}x \Big\{ -\frac{e}{4} R(\omega) - \frac{ie}{2} \bar\Psi_M \Gamma^{MNP} D_N\!\left(\frac{\omega+\hat\omega}{2}\right) \Psi_P - \frac{e}{48} F_{MNPQ} F^{MNPQ} + \frac{2}{(144)^2} \varepsilon^{IJKLMNOPQRS} F_{IJKL} F_{MNOP} A_{QRS} + \frac{e}{192} \left(\bar\Psi_R \Gamma^{RSMNPQ} \Psi_S + 12 \bar\Psi^M \Gamma^{PQ} \Psi^N\right)\left(F_{MNPQ} + \hat F_{MNPQ}\right) \Big\}$$

with $\omega_{MAB} = \hat{\omega}_{MAB} - \frac{i}{4}\bar{\Psi}_P \Gamma_{MAB}{}^{PQ} \Psi_Q$

$$\hat{\omega}_{MAB} = \omega^{o}_{MAB}(e) + \frac{i}{2}\left(\bar{\Psi}_M \Gamma_B \Psi_A - \bar{\Psi}_M \Gamma_A \Psi_B + \bar{\Psi}_B \Gamma_M \Psi_A\right)$$

Replacing $\hat{\omega}$ by $\omega + \ldots$, ω is solution of its own equation of motion when it is considered as an independent variable (lst order formalism)

$$\hat{F}_{MNPQ} = F_{MNPQ} - 3\,\bar{\Psi}_{[M} \Gamma_{NP} \Psi_{Q]}$$

The term εFFA is gauge invariant up to a total derivative. L is invariant under the following supersymmetry transformations

$$\delta e_M{}^A = -i\,\bar{\epsilon}\,\Gamma^A \Psi_M$$

$$\delta A_{MNP} = \frac{3}{2}\,\bar{\epsilon}\,\Gamma_{[MN}\Psi_{P]}$$

$$\delta \Psi_M = \left(D_M + \frac{1}{4}\hat{\omega}_{MAB}\Gamma^{AB} + \frac{i}{144}\left(\Gamma^{NOPQ}{}_M - 8\,\delta^N_M \Gamma^{OPQ}\right)\hat{F}_{NOPQ}\right)\epsilon$$

$\hat{\omega}$ and $\hat{F}$ are supercovariant objects (their variations by supersymmetry do not content $\partial\epsilon$ terms).

The bosonic invariances are :

(1) reparametrization invariance $\xi^M(x)$

(2) local Lorentz invariance SO(10,1) $\Lambda_A{}^B(x)$

(3) local abelian invariance for A_{MNP} $\xi_{NP}(x)$

We can show that the algebra closes on-shell as usual. The geometrical interpretation of A_{MNP} as a gauge field is still unclear : $e_M{}^A$ "gauge" the translations P_M, Ψ_M gauge the supersymmetries , what do A_{MNP} gauge ?

Let us finally note that for the counting of degrees of freedom we could have used a field A_{MNOPQR} with gauge invariance $\delta A_{MNOPQR} = 6\,\partial_{[M}\xi_{NOPQR]}$. However, although we can construct a free supersymmetric theory with this field, it is impossible to construct an interactive theory (Nicolai et al. 1981)

6 Dimensional reduction of 11-dimensional supergravity in 4 dimensions

We shall perform now the ordinary dimensional reduction of the 11-dimensional supergravity (Cremmer & Julia 1978, 1979). We shall also perform duality transformations which change a pseudovector field into a vector field as well as a two-rank antisymmetric tensor field into a scalar field. This will make the physical content of the theory more similar to the ordinary content of N = 8 supergravity. However, the hunt for the hidden symmetries will be discussed only in the next part.

The field content in 4 dimensions is sumamrized in Table II and has to be compared with the field content of N = 8 supergravity :

$g_{\mu\nu}$	$\Psi_\mu{}^A$	$A_\mu^{[AB]}$	$\chi^{[ABC]}$	$\Phi^{[ABCD]} = \frac{1}{24}\varepsilon^{ABCDEFGH}\bar{\Phi}_{EFGH}$
1	8	28	56	70

TABLE II

11-D Field	4-D Field	Number	d° of freedom	Total d° of freedom
	$g_{\mu\nu}$	1	2	2
$g_{MN} \rightarrow$	$g_{\mu i}$	7	2	14
	g_{ij}	28	1	28
				44
	$A_{\mu\nu\rho}$	1	0	0
	$A_{\mu\nu i}$	7	1	7
$A_{MNP} \rightarrow$	$A_{\mu ij}$	21	2	42
	A_{ijk}	35	1	35
				84
	$\psi_\mu{}^A$	8	2	16
$\Psi_M \rightarrow$	$\psi_i{}^A$	56	2	112
	(A = 1...8)			128

(a) Diagonalization of gauge transformations

In 11 dimensions for A_{MNP} we have the invariance under the following reparametrization and gauge transformations

$$\delta A_{MNP} = 3\,\partial_{[M}\xi^{Q} A_{NP]Q} + \xi^{Q}\partial_{Q} A_{MNP} + 3\,\partial_{[M}\xi_{NP]}$$

Restricting to $\partial_i \xi_{NP} = 0$ and previous conditions on ξ^{M}, we get

$$\delta A_{ijk} = \xi^{\mu}\partial_\mu A_{ijk} + 3\, a_{[i}^{\ \ell} A_{jk]\ell}$$

$$\delta A_{\mu ij} = \xi^{\nu}\partial_\nu A_{\mu ij} + \partial_\mu \xi^{\nu} A_{\nu ij} + \partial_\mu \xi^{k} A_{kij} - 2\, a_{[i}^{\ k} A_{\mu j]k} + \partial_\mu \xi_{ij}$$

As usual, $A_{\mu ij}$, $A_{\mu\nu i}$ and $A_{\mu\nu\rho}$ are invariant under the $U(1)^7$ gauge transformations. We can as previously define invariant tensors starting from $A_{ABC} = e_A^{\ M} e_B^{\ N} e_C^{\ P} A_{MNP}$ or in 4 dimensions

$$A'_{ijk} = A_{ijk}$$

$$A'_{\mu ij} = e_\mu^{\ \alpha} e_i^{\ a} e_j^{\ b} A_{\alpha ab} = A_{\mu ij} - B_\mu^{\ k} A_{kij}$$

$$A'_{\mu\nu i} = e_\mu^{\ \alpha} e_\nu^{\ \beta} e_i^{\ a} A_{\alpha\beta a} = A_{\mu\nu i} + 2\, B_{[\mu}^{\ j} A_{\nu]ji} + B_\mu^{\ i} B_\nu^{\ j} A_{jki}$$

Under the gauge transformations ξ^{i} and ξ_{ij} we have

$$\delta A'_{ijk} = 0 \qquad ; \quad \delta A'_{\mu ij} = \partial_\mu \xi_{ij}$$

$$\delta A'_{\mu\nu i} = 2\, \partial_{[\mu}\xi_{\nu]i} + 2\, B_{[\mu}^{\ j}\partial_{\nu]}\xi_{ji}$$

therefore $A'_{\mu\nu i}$ is no longer invariant under the ξ_{ij} gauge transformations. There is in fact no way to define a $A''_{\mu\nu i}$ which should be a gauge field for $\xi_{\mu i}$ transformations and invariant under the ξ^{i} and ξ_{ij} gauge transformations. The same problem arises also for $A'_{\mu\nu\rho}$. We can nevertheless define completely gauge invariant tensors starting from F_{ABCD}

$$F_{\alpha bcd} = e_\alpha^{\ \mu} e_b^{\ i} e_c^{\ j} e_d^{\ k}\, \partial_\mu A_{ijk}$$

$$F_{\alpha\beta cd} = e_\alpha^{\ \mu} e_\beta^{\ \nu} e_c^{\ i} e_d^{\ j}\, F^{+}_{\mu\nu ij}$$

$$F_{\alpha\beta\gamma d} = e_\alpha^{\ \mu} e_\beta^{\ \nu} e_\gamma^{\ \rho} e_d^{\ i}\, F^{+}_{\mu\nu\rho i}$$

$$F_{\alpha\beta\gamma\delta} = e_\alpha^{\ \mu} e_\beta^{\ \nu} e_\gamma^{\ \rho} e_\delta^{\ \sigma}\, F^{+}_{\mu\nu\rho\sigma}$$

This defines in particular

$$F^{+}_{\mu\nu ij} = \partial_\mu A'_{\nu ij} - \partial_\nu A'_{\mu ij} + G_{\mu\nu}^{\ \ k} A_{ijk}$$

$$F^{+}_{\mu\nu\rho i} = 3\,\partial_{[\mu} A'_{\nu\rho]i} + 3\, G_{[\mu\nu}^{\ \ \ k} A'_{\rho]ik}$$

(b) Duality transformations

They allow to transform a magnetic field into an electric field or an antisymmetric tensor $A_{\mu\nu}$ into a scalar field ϕ. Let us start with the following Lagrangian

$$\mathcal{L} = \mathcal{L}(F_{\mu\nu}, F_{\mu\nu\rho}, F_{\mu\nu\rho\sigma})$$

where $F_{\mu\nu}$, $F_{\mu\nu\rho}$ and $F_{\mu\nu\rho\sigma}$ are, respectively, the field strengths of the fields A_μ, $A_{\mu\nu}$ and $A_{\mu\nu\rho}$ which do not appear explicitly in $\mathcal{L}$. Up to topological subtleties or harmonic solutions, these field strengths are characterized by the constraints

$$\partial_{[\mu} F_{\nu\rho]} = 0 \quad , \quad \partial_{[\mu} F_{\nu\rho\sigma]} = 0 \quad , \qquad \text{no constraints for } F_{\mu\nu\rho\sigma}$$

We can therefore use a first order formalism by implementing these constraints via Lagrange multiplier B_μ and ϕ i.e. by adding to $\mathcal{L}$

$$\Delta\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma}\left(B_\mu \partial_\nu F_{\rho\sigma} + \phi\, \partial_\mu F_{\nu\rho\sigma}\right)$$

$$= -\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu B_\mu F_{\rho\sigma} + \partial_\mu\phi\, F_{\nu\rho\sigma}\right)$$

In $\mathcal{L} + \Delta\mathcal{L}$ we can now integrate on the fields $F_{\mu\nu}$, $F_{\mu\nu\rho}$ and $F_{\mu\nu\rho\sigma}$ which appear algebraically and we get a new 2nd order Lagrangian $\mathcal{L}'$

$$\mathcal{L}' = \mathcal{L}'(G_{\mu\nu}, \partial_\mu\phi)$$

with $\quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

In the present case, we shall make the following transformation

$$A_{\mu ij} \rightarrow B_\mu^{\ ij} \quad , \quad A_{\mu\nu i} \longrightarrow \phi^{i} \quad , \qquad A_{\mu\nu\rho} \quad \text{eliminated.}$$

Because of the previous discussion on the various gauge invariances, $\mathcal{L}$ is function of $F^{+}_{\mu\nu ij}$, $F^{+}_{\mu\nu\rho i}$, $F^{+}_{\mu\nu\rho\sigma}$ and A_{ijk}. Taking into account the definition of F^{+}_{i}), we must add to $\mathcal{L}$ (after integration by part)

$$\Delta\mathcal{L} = -\varepsilon^{\mu\nu\rho\sigma}\Big[\tfrac{1}{12}\partial_\mu \phi^i F^4_{\nu\rho\sigma i} + \tfrac{1}{8}\phi^i G^k_{\nu\rho} F^4_{\mu\sigma ik} + \tfrac{1}{8} G^{ij}_{\mu\nu}\left(F^4_{\rho\sigma ij} - G^k_{\rho\sigma} A_{ijk}\right)\Big]$$

Remarks : 1. At the quantum level ; there is no exact equivalence between $A_{\mu\nu}$ and ϕ. For example, they contribute in a different way to topological counterterms at the one-loop level.

2. Taking into account the harmonic solutions allows to introduce some extra-parameters in the theory. This is easily seen in the case of the elimination of $F_{\mu\nu\rho\sigma}$ (Duff 1981 ; Aurilla et al. 1981). If we start from the Lagrangian $\mathcal{L} = -\frac{e}{48} F^2_{\mu\nu\rho\sigma}$

The variations with respect to $F_{\mu\nu\rho\sigma}$ and $A_{\mu\nu\rho}$ lead respectively to

$$\delta\mathcal{L}/\delta F_{\mu\nu\rho\sigma} = 0 \qquad F_{\mu\nu\rho\sigma} = 0 \tag{I}$$

$$\delta\mathcal{L}/\delta A_{\mu\nu\rho} = 0 \qquad \partial^\mu F_{\mu\nu\rho\sigma} = 0 \quad \text{or} \quad F_{\mu\nu\rho\sigma} = a\,\varepsilon_{\mu\nu\rho\sigma} \tag{II}$$

where a is an arbitrary constant.

It is easy to see that (II) can be derived from the following Lagrangian

$$\mathcal{L}' = -\frac{e}{48} F^2_{\mu\nu\rho\sigma} + a\,\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}$$

when $F_{\mu\nu\rho\sigma}$ is a curl, the extra term is a total derivative. For the solution $F_{\mu\nu\rho\sigma} = a\,\varepsilon_{\mu\nu\rho\sigma}$, $\mathcal{L}$ has the value $e a^2$, it corresponds to a cosmological constant. This mechanism can be implemented in the reduction of 11-dimensional supergravity (Aurilla et al. 1981)

(c) Bosonic Lagrangian

We then get the complete bosonic Lagrangian in terms of the fields $g_{\mu\nu}$, g_{ij}, B^i_μ, B^{ij}_μ, ϕ^i and A_{ijk} ($i = 1\ldots7$) after the Weyl rescaling

$$\mathcal{L}_B = -\frac{e}{4} R + L_S + L_V$$

with

$$L_S = -\frac{e}{16} g^{\mu\nu}\partial_\mu g^{ij}\partial_\nu g_{ij} + \frac{e}{32} g^{\mu\nu}\partial_\mu \log\Delta\,\partial_\nu \log\Delta - \frac{e}{12} g^{\mu\nu}\partial_\mu A_{ijk}\partial_\nu A_{lmn} g^{il} g^{jm} g^{kn}$$
$$- \frac{e}{8\Delta} g_{iq} g^{\mu\nu}\left(\partial_\mu \phi^i - \tfrac{1}{3}\sqrt{\Delta}\,{}^*A^{ijke}\partial_\mu A_{jke}\right)\left(\partial_\nu \phi^q - \tfrac{1}{3}\sqrt{\Delta}\,{}^*A^{qrst}\partial_\nu A_{rst}\right)$$

where ${}^*A^{ijke} = \frac{1}{6\sqrt{\Delta}}\,\varepsilon^{ijkemno} A_{mno}$

$$L_V = \frac{e}{16}\sqrt{\Delta}\, g^{\mu\rho} g^{\nu\sigma} g_{ij}\, G^i_{\mu\nu} G^j_{\rho\sigma} + \frac{1}{48}\varepsilon^{\mu\nu\rho\sigma}\,{}^*A^{ijke} A_{ijp} A_{keq} G^p_{\mu\nu} G^q_{\rho\sigma}$$
$$+ \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma} G^k_{\mu\nu} G^{ij}_{\rho\sigma} A_{ijk} - \frac{e}{8\sqrt{\Delta}} Y^{ij} \mathcal{M}_{ij,pq} Y^{pq}$$

where $Y^{ij}_{\mu\nu} = G^{ij}_{\mu\nu} + \frac{1}{2}\sqrt{\Delta}\,{}^*A^{ijke} A_{kem} G^m_{\mu\nu} + \frac{1}{2}\left(\phi^i G^j_{\mu\nu} - \phi^j G^i_{\mu\nu}\right)$

$$\mathcal{M}_{ij,ke} = \left[\tfrac{1}{2}\left(g^{ik} g^{jl} - g^{il} g^{jk}\right) - j\,{}^*A^{ijke}\right]^{-1}$$

For the indices $\mu\nu$, 1 stands for $1/2\left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right)$ and j stands for $\frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}$

(d) Reduction of the spinor fields

The 11 Γ matrices 32 x32 of the Clifford algebra can be written as

$$\Gamma^\alpha = \gamma^\alpha \otimes 1\!\!1^A{}_B \qquad \alpha = 0,1,2,3$$

$$\Gamma^a = \gamma_5 \otimes (\Gamma^a)^A{}_B \qquad a = 1,\ldots 7$$

$$A = 1\ldots 8$$

The $(\Gamma^a)^A{}_B$ form a Clifford algebra of SO(7) and are real and antisymmetric 8 x 8 matrices.

As for $A_{\mu ij}$ we can define ψ_μ and ψ_i fields which are invariant under $U(1)^7$ gauge transformations. We also make some Weyl rescaling on these spinor fields associated with a rediagonalization of the kinetic term for ψ_μ and χ_a

$$\psi_\mu = e^\alpha_{\mu(4)}\Delta^{-1/8}\left(\psi_M e^M{}_\alpha - \tfrac{1}{2}\gamma_5\gamma_\alpha \Gamma^a \psi_M e^M{}_a\right)$$

$$\chi_a = \Delta^{-1/8}\psi_M e^M{}_a$$

Then, from the kinetic term in 11 dimensions for ψ_M, we get

$$L^{kin}_F = -\frac{ie}{2}\bar\psi_\mu \gamma^{\mu\nu\rho}\partial_\nu \psi_\rho - \frac{ie}{2}\bar\chi^A{}_a\left(\tfrac{1}{2}\Gamma^a\Gamma^b + \eta^{ab}\right)_A{}^B \gamma^\mu \partial_\mu \chi_{bB}$$
+ scalar couplings

A final diagonalization of χ^A_a 's kinetic terms is made by the redefinition $\lambda_{ABC} = \frac{3}{2}\Gamma^a_{[AB}\chi_{aC]}$

which leads to

$$L^{\mathrm{kin}}_{\frac{1}{2}} = \frac{ie}{12}\,\bar\lambda_{ABC}\,\gamma^\mu\partial_\mu\lambda_{ABC}$$

Some further chiral transformations on ψ_μ^A and λ_{ABC} are made.

$$\psi_\mu^A = (i\gamma_5)^{-1/2}\psi^A_{\mu\,\mathrm{new}} \quad , \quad \lambda_{ABC} = (i\gamma_5)^{-1/2}\lambda_{ABC\,\mathrm{new}}$$

(e) Spinor couplings

In order to obtain a simple form for the spinor couplings, a lot of algebraic properties of $(\Gamma^a)_A{}^B$ matrices are required (Cremmer & Julia 1979), for instance

$$\Gamma^{[ab}_{AB}\gamma^{c]}_{B} = -\frac{1}{3\sqrt2}\,\Gamma^{[ab}_{[AB}\Gamma^{c]}_{CD]}\lambda_{BCD}$$

the use of λ_{ABC} is crucial in this tremendous simplification and we get finally

$$\begin{aligned}
L_F = {} & \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu^A\gamma_\nu\gamma_5(\partial_\nu\delta_A^B - Q^\circ_{\nu A}{}^B)\psi_\rho^B \\
& + \frac{ie}{12}\bar\lambda_{ABC}\gamma^\mu(\partial_\mu\delta_A^D - 3Q^\circ_{\mu A}{}^D)\lambda_{DBC} \\
& - \frac{e}{3\sqrt2}\bar\psi_{\mu D}\gamma^\nu\gamma^\mu\bar P^{\circ\,ABCD}_\nu\lambda_{ABC} \\
& + \frac{e}{4\sqrt2}\bar\psi^A_\mu\gamma^\rho\mathcal{F}^\circ_{AB}\gamma^\sigma\psi^B_\nu - \frac{ie}{8}\bar\psi^C_\mu\mathcal{F}^\circ_{AB}\gamma^\mu\lambda_{ABC} \\
& - \frac{e}{288\sqrt2}\,\eta\,\varepsilon^{ABCDEFGH}\bar\lambda_{ABC}\mathcal{F}^\circ_{DE}\lambda_{FGH}
\end{aligned}$$

+ quartic fermionic terms

where η is a self duality coefficient ($\eta = \pm 1$) depending on the explicit representation of the Γ's matrix of SO(7) defined by

$$\Gamma^a_{[AB}\Gamma^b_{CD]} = \frac{\eta}{4!}\varepsilon_{ABCDEFGH}\Gamma^a_{[EF}\Gamma^b_{GH]}$$

$Q^\circ_{\nu A}{}^B$, $\bar P^{\circ\,ABCD}_\nu$ and $\mathcal{F}^\circ_{AB}$ are given in terms of $e^i{}_a$, ϕ^i, A_{ijk}, $G^i_{\mu\nu}$ and $G^{ij}_{\mu\nu}$ by

$$\begin{aligned}
Q^\circ_{\nu A}{}^B = -\frac{1}{4}\Big[& e^i{}_a\partial_\nu e_{ib}\Gamma^{ab} + e_{ia}\Big(\frac{\partial_\nu\phi^i}{\sqrt\Delta} - \frac{1}{3}{}^*A^{ijk\ell}\partial_\nu A_{jk\ell}\Big)\Gamma^a \\
& - \frac{i}{3}\gamma_5\, e^i{}_a e^j{}_b e^k{}_c\partial_\nu A_{ijk}\Gamma^{abc}\Big]_A{}^B
\end{aligned}$$

$$\begin{aligned}
\bar P^{\circ\,ABCD}_\nu = \frac{1}{8}\Big\{ & e^i{}_a\partial_\nu e_{ib}\Gamma^{ac}_{[AB}(\Gamma^b{}_c)_{CD]} + e_{ia}\Big(\frac{\partial_\nu\phi^i}{\sqrt\Delta} - \frac{1}{3}{}^*A^{ijk\ell}\partial_\nu A_{jk\ell}\Big)\Gamma^{ac}_{[AB}(\Gamma_c)_{CD]} \\
& + 2i\gamma_5\, e^i{}_a e^j{}_b e^k{}_c\partial_\nu A_{ijk}\Gamma^{ab}_{[AB}\Gamma^c_{CD]}\Big\}
\end{aligned}$$

$P^\circ_{\nu ABCD}$ satisfies the self-duality relation. ($\bar P$ designs the complex conjugate)

$$\bar P^{\circ\,ABCD}_\nu = \frac{\eta}{4!}\varepsilon^{ABCDEFGH}P^\circ_{\nu EFGH}$$

$$\begin{aligned}
\mathcal{F}^\circ_{AB} = \frac{1}{2\sqrt2}\gamma^{\rho\sigma}\Big\{ & -G^i_{\rho\sigma}e_{ia}\Gamma^a\Delta^{3/4} + \Delta^{-1/4}\bar{\mathcal{M}}_{ij,pq}e^i_a e^j_b\Gamma^{ab} \\
& \times\Big(G^{pq}_{\rho\sigma} + \phi^p G^q_{\rho\sigma} + \frac{1}{8}\sqrt\Delta\,{}^*A^{pqmn}A_{mn\ell}G^\ell_{\rho\sigma}\Big)\Big\}_{AB}
\end{aligned}$$

We can note at this stage that the Noether coupling to $\bar\psi_\mu\lambda$: $P^\circ_{\nu\,ABCD}$ is nothing else than the square root of the scalar "kinetic term" L_S as it is usually in supergravity.

(f) Supersymmetry transformation laws

In the reduction process, we assume that ϵ does not depend on the extra-coordinates : that is necessary for the compatibility with the no-dependence of the fields on these coordinates. The 32 components then split into 8 ϵ^A 4-component spinors in 4 dimensions.

In order to determine the supersymmetry transformation laws in 4 dimensions, we have to keep in mind all transformations on the fields we have made during the reduction and also redefine ϵ^A by chiral and Weyl transformations. In order to preserve the gauge condition $e_m{}^\alpha = 0$, it is necessary to add a compensating SO(10,1) gauge transformation $\Omega^\alpha{}_a$. We can always redefine δ_S by combining with local SO(3,1) or local SO(7) transformations. This will simplify δ_S and allow to keep the canonical form for $\delta e_\mu{}^\alpha$

$$\delta e_\mu{}^\alpha = -i\,\bar\epsilon^A\gamma^\alpha\psi_{\mu A}$$

When we perform duality transformations, the new fields are not function of the original fields (except on-shell and in a non local way) we derive their transformation laws such that the first order Lagrangian is supersymmetric following Van Nieuwenhuizen's method. Let us show in a simple example how this works. Let us start from a symmetric theory with a Lagrangian $\mathcal{L}(F_{\mu\nu\rho}, \psi)$ where $F_{\mu\nu\rho} = 3\,\partial_{[\mu}A_{\nu\rho]}$. It is invariant under some transformations $\delta A_{\mu\nu}$ and $\delta\psi$. Let us now consider the first order Lagrangian

$$\mathcal{L}' = \mathcal{L}(F_{\mu\nu\rho}, \psi) + \varepsilon^{\mu\nu\rho\sigma}\partial_\sigma F_{\mu\nu\rho}\,\phi$$

Under $\delta\psi$ and $\delta F_{\mu\nu\rho} = 3\,\partial_{[\mu}\delta A_{\nu\rho]}$, $\delta\mathcal{L}$ is zero only if $F_{\mu\nu\rho}$ is a curl, therefore it can be written as

$$\delta\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma}\partial_\sigma F_{\mu\nu\rho}\, S$$

Then it is obvious from

$$\delta\mathcal{L}' = \varepsilon^{\mu\nu\rho\sigma}\partial_\sigma F_{\mu\nu\rho}\, S + \varepsilon^{\mu\nu\rho\sigma}\partial_\sigma F_{\mu\nu\rho}\,\delta\phi$$

$$(\partial_{[\sigma}\delta F_{\mu\nu\rho]} = 0)$$

that by choosing $\delta\phi = -S$ the Lagrangian $\mathcal{L}'$ is now invariant under $\delta\psi$, $\delta F_{\mu\nu\rho} = 3\,\partial_{[\mu}\delta A_{\nu\rho]}$, and $\delta\phi = -S$ and we can proceed to the elimination of $F_{\mu\nu\rho}$. Although this method is simple in principle, it can be very complicated in practice to determine S.

The next step for this theory will be the hunt for hidden symmetries which will be discussed in the next part.

7 Concluding remarks on dimensional reduction

(a) The dimensional reduction suggests hidden symmetries. One example which will be developped in length is of course the N = 8 supergravity. But previously already, the possibility of reducing the 10 dimensional supergravity has suggested that there could exist a formulation of N = 4 supergravity with an SU(4)$\sim$SO(6) invariance of the Lagrangian, instead of only an invariance of the equation of motion. This theory has been constructed directly and shown to be equivalent to the other formulation of N = 4 supergravity as far as the equations of motion are concerned. This has allowed, in the same time, the discovery of the first hidden symmetry in supergravity : SU(1,1) for N = 4 supergravity (Cremmer et al. 1978 a).

(b) There exists a method called "dimensional reduction by Legendre transform" which can partially solve the problem of finding the auxiliary fields (Sohnius et al. 1981 ; West 1981).

(c) There exists a modified and more general version of dimensional reduction (Cho & Freund 1975 ; Scherk & Schwarz 1979 b). It still associates to 1 degree of freedom in 4 + N dimensions, 1 degree of freedom in 4 dimensions. The gravitational sector instead of generating $U(1)^N$ gauge group, generates a non abelian gauge group with N generators.

If we require the absence of cosmological constant and the positivity of scalar potential, we obtain what has been called "flat group". Starting from a massless theory, it allows to introduce mass parameters and to generate a potential term for the scalar fields. In the case of a supersymmetric theory (without gravitation), it ends up with a theory which has soft supersymmetry breaking in general. In the case of a supergravity theory (local supersymmetry) (Scherk & Schwarz 1979 ; Cremmer et al.1979) we end up with a theory invariant under some new local transformations like supersymmetry but the local algebra has changed and in general it is not possible to extract a global algebra from it.

III HIDDEN SYMMETRIES OF N = 8 SUPERGRAVITY IN 4 DIMENSIONS

From dimensional reduction of the 11-dimensional supergravity we have obtained a N = 8 supergravity theory described in terms of the fields $g_{\mu\nu}$, ψ_μ^A, B_μ^i, B_μ^{ij}, λ_{ABC} (or χ^A_a), g^{ij}, ϕ^i which form representations of SO(7). The natural question arises : is it the same as the N = 8 supergravity expected and described in terms of irreducible representation of SO(8) ? A first remark is that the N = 7 supergravity is expected to be the same as the N = 8, the fields being described by irreducible representation of SO(7) in particular the scalar fields which are in the irreducible 35 representation of SO(7), but the scalar fields coming from the reduction g^{ij}, Φ^i are in 28 + 7 representation of SO(7). This problem will be solved by the discovery of the hidden symmetries of N = 8 supergravity and by a special property of SO(8) : the triality which allows two different embeddings of SO(7) in SO(8).

1 General ideas for the hunt for hidden symmetries

The main idea is to try to generalize what we have learned from dimensional reduction. The scalar fields coming from the tensor metric are described by a coset space SL(7,R)/SO(7). Therefore we would like to describe all scalars and pseudoscalars by a coset space G/H. It describes, as we have seen, an element $\mathcal{V} \in G$ defined up to a local transformation of H (Coleman et al. 1969 ; Callan et al. 1965). The theory will be invariant under the change $\mathcal{V} \to g\,\mathcal{V}\,h(x)$ $g \in G$, $h(x) \in H$ together with some transformations on the other fields. Writing $\mathcal{V} = \exp w$ $w \in \mathcal{L}ie\,(G)$

the maximal linear subgroup acting on w will be H_d defined by

$$\mathcal{V} \to h^{-1}\,\mathcal{V}\,h \qquad h \in H$$

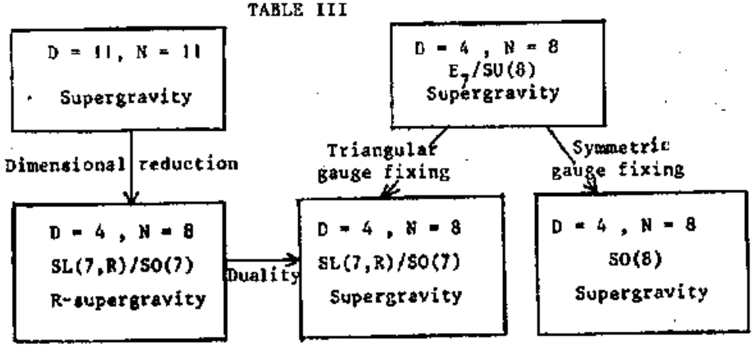
For the N = 8 supergravity, the maximal linear group acting on the 70 scalar fields ϕ_{ABCD} is expected to be SU(8). Therefore, we should have dim G = dim SU(8) + 70 = 133. This is consistent with G = E_7. G must be non compact since it contains the non compact group SL(7,R) derived from dimensional reduction. In fact, there exists a non compact version $E_{7(+7)}$ (the normal form) whose maximum compact subgroup is SU(8). This will ensure the positivity of the Lagrangian for the scalar fields.

From previous results for $N \leq 4$ (Ferrara et al. 1977 ; Cremmer et al. 1977 a ; Cremmer & Scherk 1977 b) we know that H_D is realized only on the equations of motion when vector fields are present. In fact, for vector fields G and H_D exchange the equations of motion $\partial^\mu H_{\mu\nu} = 0$ and the Bianchi identities $\partial^\mu \tilde{G}_{\mu\nu} = 0$ (see also Gaillard & Zumino 1981). Therefore, if there are N vector fields, 2 N should be an irreducible representation of H_D and therefore of G. In fact, the fundamental representation of E_7 has dimension 56 consistent with 28 vector fields.

From dimensional reduction we have learned also that vector fields are singlet for the gauge group SO(7) and that the spinor fields are singlet for the global group SL(7,R). We then expect to describe the supergravity by : 1 graviton singlet for E_7 and SU(8) ; 8 gravitinos singlet for E_7 and in representation 8 for SU(8) ; 28 vector fields whose field strengths $G_{\mu\nu}$ and $\tilde{H}_{\mu\nu} \sim \delta\mathcal{L}/\delta G_{\mu\nu}$ are singlet for SU(8) and in the 56 representation of E_7, 56 spin 1/2 fields singlet for E_7 and in the 56 representation of SU(8), the 70 scalar fields being described by a 56 x 56 matrix of E_7, which transforms as 56 for E_7 and 28 complex for SU(8).

Therefore we shall show that there exists a E_7/SU(8) formulation of N = 8 supergravity described by the pattern shown in Table III.

TABLE III



The R-supergravity is the one obtained directly from 11 dimensions where some of the scalar fields are described by antisymmetric tensors. The SO(8) supergravity is the ordinary formulation of N = 8 supergravity on which partial results were obtained (de Wit & Freedman 1977 ; de Wit 1979). When we fix the gauge of SU(8), then the group E_7 acts on all the fields, in particular on the spinor fields via transformations of SU(8) which are function of the scalar fields.

2 Restoration of SL(8,R) symmetry

Before defining E_7 and showing that it is a symmetry of the theory, we shall perform an intermediate step at the bosonic level and exhibit the SL(8,R) symmetry and local SO(8) symmetry which unify B_μ^i with B_μ^{ij}, and g^{ij} with ϕ^i.

In the absence of pseudoscalar A_{ijk} the bosonic Lagrangian is written

$$\mathcal{L}_B^{(+)} = -\frac{e}{16}\partial_\mu g^{ij}\partial^\mu g_{ij} - \frac{e}{8\Delta}\partial_\mu\phi^i\partial^\mu\phi^j g_{ij} + \frac{e}{32}\frac{\partial_\mu\Delta\partial^\mu\Delta}{\Delta^2}$$
$$+\frac{e}{16}\sqrt{\Delta}\, g_{ij} G^i_{\mu\nu} G^{\mu\nu j} - \frac{e}{8\sqrt{\Delta}} g_{ip} g_{jq}\left[G^{ij}_{\mu\nu} + \tfrac{1}{2}(\phi^i G^j_{\mu\nu} - \phi^j G^i_{\mu\nu})\right]\left[G^{pq}_{\rho\sigma} + \tfrac{1}{2}(\phi^p G^q_{\rho\sigma} - \phi^q G^p_{\rho\sigma})\right] g^{\mu\rho} g^{\nu\sigma}$$

Defining $S^{i'j'} = \Delta^{-3/4}\begin{pmatrix} \Delta g^{ij} - \phi^i\phi^j \,, & \phi^j \\ \phi^i & , -1 \end{pmatrix}$ with det S = 1

$$G^{i'j'}_{\mu\nu} = \left(G^{ij}_{\mu\nu} \,, -\tfrac{1}{2} G^i_{\mu\nu}\right) \,, \quad \left(G^{i'8}_{\mu\nu} = -\tfrac{1}{2} G^i_{\mu\nu}\right)$$

$\mathcal{L}_B^{(+)}$ takes now the very simple form

$$\mathcal{L}_B^{(+)} = -\frac{e}{16}\partial_\mu S^{i'j'}\partial^\mu S_{i'j'} - \frac{e}{16} S_{i'p'} S_{j'q'} G^{i'j'}_{\mu\nu} G^{p'q'}_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

As discussed previously, we can reintroduce a 8-bein $\mathcal{V}_{i'}{}^{a'}$ defined up to local SO(8) transformations $S_{i'j'} = \mathcal{V}_{i'}{}^{a'}\mathcal{V}_{j'}{}^{b'}\eta_{a'b'}$

A specific choice of $\mathcal{V}_{i'}{}^{a'}$ which breaks SO(8) local invariance down to SO(7) is

$$\mathcal{V}_{i'}{}^{a'} = \Delta^{-1/8}\begin{pmatrix} e_i{}^a & , & 0 \\ \phi^j e_j{}^a & , & \sqrt{\Delta} \end{pmatrix} \qquad (\det\mathcal{V} = 1)$$

Defining $A_{abc} = e^i{}_a e^j{}_b e^k{}_c A_{ijk}$ and ${}^*A^{abcd} = \frac{1}{6}\varepsilon^{abcdefg} A_{efg}$, we can rewrite the vector part of the Lagrangian as

$$L_V = -\frac{1}{8} e\, \mathcal{V}_{i'}{}^{a'}\mathcal{V}_{j'}{}^{b'}\mathcal{V}_{k'}{}^{c'}\mathcal{V}_{l'}{}^{d'} G^{i'j'}_{\mu\nu} G^{k'l'}_{\rho\sigma} \mathcal{N}^{\mu\nu,\rho\sigma}_{a'b',c'd'}$$

$\mathcal{N}$ is a 28 x 28 matrix function of A_{abc} only, that we shall not write explicitly (see Cremmer & Julia 1979). The complete bosonic Lagrangian will simplify when we shall take into account the E_7 symmetry.

3 Definition of E_7 (+7)

We shall define E_7 by the infinitesimal transformations in the 56 dimensional fundamental representation. This 56-space is spanned by 2 antisymmetric tensors x^{ij} and y_{ij} ($i, j = 1 \ldots 8$)

$$\delta x^{ij} = \Lambda^i{}_k x^{kj} + \Lambda^j{}_k x^{ik} + \frac{1}{24} \varepsilon^{ijklmnop} \Sigma_{mnop} y_{kl}$$

$$\delta y_{ij} = \Lambda_i{}^k y_{kj} + \Lambda_j{}^k y_{ik} + \Sigma_{ijkl} x^{kl}$$

with $\Lambda_i{}^k = -\Lambda^i{}_k$; $\Lambda^i{}_i = 0$, Σ_{ijkl} totally antisymmetric.

$\Sigma = 0$ corresponds to the subgroup SL(8,R)

There are two invariants for E_7, one bilinear invariant which shows that E_7 is a subgroup of Sp(56) with the symplectic metric Ω

$$I = x_1^{ij} y_{2ij} - x_2^{ij} y_{1ij}$$

Another one is quartic and characterizes E_7

$$J = x^{ij} y_{jk} x^{kl} y_{li} - \frac{1}{4} x^{ij} y_{ij} x^{kl} y_{kl} + \frac{1}{96} \left[\varepsilon^{ijklmnop} y_{ij} y_{kl} y_{mn} y_{op} + \varepsilon_{ijklmnop} x^{ij} x^{kl} x^{mn} x^{op} \right]$$

An element X of E_7 being also an element of Sp(56) satisfies

$$X^t \Omega X = \Omega \qquad X^{-1} = -\Omega X^t \Omega$$

$$\Omega^2 = -\mathbb{1} \qquad \Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

The subgroup SU(8) of E_7 is characterized by

$$[X, \Omega] = 0 \quad \text{or} \quad X^t X = \mathbb{1}$$

4 E_7 formulation for scalars and vectors

Forgetting for the present time the fermions, the vectors' equations of motion can be written as $\partial_\mu (e \tilde{H}^{\mu\nu}_{i'j'}) = 0$ where $e \tilde{H}^{\mu\nu}_{i'j'} = \partial L / \partial G^{i'j'}_{\mu\nu}$ (since $B^{i'j'}_\mu$ appears only through $G^{i'j'}_{\mu\nu}$). The Bianchi identity is written as $\partial_\mu (e \tilde{G}^{\mu\nu}_{i'j'}) = 0$

We therefore expect that $\mathcal{F}_{\mu\nu} = (G^{i'j'}_{\mu\nu}, H_{\mu\nu i'j'})$ could be a vector of the

56 representation of E_7. Since $H_{\mu\nu i'j'}$ is a function of $G^{i'j'}_{\mu\nu}$ and of the scalar fields, there must exist a relation between $\mathcal{F}$ and the matrix $\mathcal{V}$ describing the scalar fields (or the metric $\mathcal{R} = \mathcal{V}^t \mathcal{V}$ invariant under the expected local SU(8)). The simplest covariant relation is $\Omega \tilde{\mathcal{F}} = \mathcal{R} \mathcal{F}$. Since we know $\mathcal{F}$, this defines the matrix $\mathcal{R}$ in terms of the original scalar fields $S^{i'j'}$ and A_{abc}. We find

$$\mathcal{R} = \mathcal{V}_+^t \mathcal{P}_- \mathcal{V}_+$$

where

$$\mathcal{V}_+ = \begin{pmatrix} \mathcal{V}^{[a'}{}_{i'} \mathcal{V}^{b']}{}_{j'} , & 0 \\ 0 & , \mathcal{V}_{[a'}{}^{i'} \mathcal{V}_{b']}{}^{j'} \end{pmatrix}$$

and $\mathcal{P}_-$ is function of A_{abc} only and is computed from $\mathcal{N}_{a'b'c'd'}$

Defining

$$A_{a'b'c'd'} = 4 A_{[a'b'c'} \delta^8_{d']}$$

$$^*A^{a'b'c'd'} = \frac{1}{24} \varepsilon^{a'b'c'd'e'f'g'h'} A_{e'f'g'h'}$$

($^*A$ is zero if one of its indices is equal to 8) and making the convention that all contractions are made on antisymmetric pairs of indices to define the product of 28 x 28 matrices, we find

$$\mathcal{P}_- = \mathcal{P}_-^t = \begin{pmatrix} X , & Z \\ Z^t , & Y \end{pmatrix}$$

where $X = X^t$, $Y = Y^t$ and Z are 28 x 28 matrices defined by

$$X = \mathbb{1} + A^2 + \frac{1}{2} {}^*A A + \frac{1}{2} A {}^*A + \frac{1}{4} A ({}^*A)^2 A$$

$$Z = A + {}^*A + \frac{1}{2} A^2 {}^*A + \frac{1}{2} A ({}^*A)^2 + \frac{1}{6} A {}^*A A - \frac{1}{12} A {}^*A A^2 {}^*A$$

$$Y = \mathbb{1} + ({}^*A)^2 + \frac{1}{2} A {}^*A + \frac{1}{2} {}^*A A + \frac{1}{4} {}^*A A^2 {}^*A$$

We must prove that $\mathcal{R}$ is a matrix of E_7. Since $\mathcal{V}_+^t$ and $\mathcal{V}_+$ are in E_7, it is sufficient that $\mathcal{P}_-$ should be in E_7. For this we show that $\mathcal{P} = \mathcal{V}_-^t \mathcal{V}_-$ with

$$\mathcal{V}_- = \exp \begin{pmatrix} 0 & , {}^*A^{a'b'c'd'} \\ A_{a'b'c'd'} , & 0 \end{pmatrix} \equiv \exp V$$

In this form it is obvious that $\mathcal{V}_- \in E_7$. To prove $\mathcal{P}_- = \mathcal{V}_-^t \mathcal{V}_-$ it is necessary to notice that A and $^*A$ as defined previously satisfy $^*A A {}^*A = 0$ so that V is nilpotent $V^4 = 0$. Therefore the expression of the exponential has only a finite number of terms and is polynomial in A.

Defining $\mathcal{V} = \mathcal{V}_- \mathcal{V}_+$ $\mathcal{F}_{\mu\nu}$ now satisfies

$$\mathcal{V}\mathcal{F}_{\mu\nu} = \Omega\,\mathcal{V}\tilde{\mathcal{F}}_{\mu\nu}$$

If this matrix $\mathcal{R}$ describes the scalar field, we must have

$$L_S = -\gamma e\,\mathrm{Tr}(\partial_\mu \mathcal{R}\,\partial_\nu \mathcal{R}^{-1})\,g^{\mu\nu}$$

Computing this explicitly in terms of $\mathcal{V}_{\omega}{}^{\bullet}$, A_{ijk} we check that it is true and find $\gamma = 1/192$. This can be written in a form which exhibits the local SU(8) gauge invariance using the vielbein $\mathcal{V}$

$$L_S = \frac{e}{48}\,\mathrm{Tr}([D_\mu \mathcal{V}\,\mathcal{V}^{-1}]^2)$$

The complete bosonic Lagrangian is now written in the simple form

$$L_B = -\frac{e}{4}R - \frac{e}{192}g^{\mu\nu}\mathrm{Tr}(\partial_\mu \mathcal{R}\,\partial_\nu \mathcal{R}^{-1}) + \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma}G^{ij}_{\mu\nu}H_{\rho\sigma ij}$$

where $H_{\rho\sigma ij}$ is defined in terms of $G^{ij}_{\mu\nu}$ and $\mathcal{R}$ by the relation

$$\Omega\tilde{\mathcal{F}} = \mathcal{R}\mathcal{F} \quad \text{and} \quad \mathcal{F} = \begin{pmatrix} G^{ij}_{\mu\nu} \\ H_{\mu\nu ij}\end{pmatrix}$$

$$\tilde{H}^{\mu\nu}_{ij} = \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}H_{\rho\sigma ij} = \frac{1}{e}\,\partial\mathcal{L}_B/\partial G^{ij}_{\mu\nu}$$

$\mathcal{R}$ being a symmetric matrix 56 x 56 of E_7.

Let us now prove the complete invariance of the equations of motion derived from this Lagrangian which has the general form

$$L_B = L_{E_7} + \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma}G^{ij}_{\mu\nu}H_{\rho\sigma ij}(B^{ij}_\mu, g_{\mu\nu}, \mathcal{R})$$

where L_{E_7} is invariant under E_7.

(1) The equations of motion of B^{ij}_μ together with the Bianchi identities for $G_{\mu\nu}$ are

$$\partial_\mu(e\tilde{H}^{\mu\nu}_{ij}) = 0 \quad , \quad \partial_\mu(e\tilde{G}^{\mu\nu ij}) = 0$$

which can be written $\partial_\mu(e\tilde{\mathcal{F}}^{\mu\nu}) = 0$ covariant for E_7.

(2) The other fields belong to precise representation of E_7. Let us call them ϕ

$$\frac{\delta\mathcal{L}}{\delta\phi} = \frac{\delta L_{E_7}}{\delta\phi} + \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma}G^{ij}_{\mu\nu}\frac{\delta H_{\rho\sigma ij}}{\delta\phi}$$

$$= \frac{\delta L_{E_7}}{\delta\phi} + \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma}\left(G^{ij}_{\mu\nu}\frac{\delta H_{\rho\sigma ij}}{\delta\phi} - \frac{\delta G^{ij}_{\mu\nu}}{\delta\phi}H_{\rho\sigma ij}\right)$$

The second term has the form $\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^t_{\mu\nu}\,\Omega(\delta\mathcal{F}_{\rho\sigma}/\delta\phi)$ and consequently transforms under E_7 in the inverse way of ϕ since the invariant of E_7 is

written $\mathcal{F}^t\Omega\,\mathcal{G}$. Therefore

$$\delta\mathcal{L}/\delta\phi = 0 \quad \text{is covariant under } E_7$$

This completely proves the invariance of this bosonic theory under E_7, the local SU(8) (as the local Lorentz invariance SO(3,1)) being hidden in the metric $\mathcal{R}$ (as $g_{\mu\nu}$).

5 SO(7), SO(8), SU(8), $E_7(+7)$

Before discussing the coupling to fermions, it is necessary to reformulate E_7 in a basis which features the subgroup SU(8). This basis linked to the basic space of the Clifford algebra of SO(7) will show also the two ways of embedding SO(7) into SO(8).

(a) Clifford algebra of SO(7) and Lie algebra of SO(8)

Let us start with the 7 $(\Gamma^a)_A{}^B$ matrices (a = 1...7, A = 1...8) which form the Clifford algebra of SO(7)

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1}$$

The $(\Gamma^a)_A{}^B$ are real and antisymmetric. From them we define the antisymmetrized product of Γ matrices. Γ^a, Γ^{ab} are antisymmetric matrices and Γ^{abc} are symmetric matrices. They form together a complete basis for the 8 x 8 matrices. The other matrices are related to them via the "duality" relation

$$\Gamma^{abcdefg} = \varepsilon^{abcdefg}\mathbb{1} \quad , \quad \Gamma^{abcd} = -\frac{1}{6}\varepsilon^{abcdefg}\Gamma_{efg}$$

The Γ^a and Γ^{ab} satisfy the commutation relations

$$\begin{cases} [\Gamma^{ab}, \Gamma^{cd}] = -8\,\eta^{[a[c}\Gamma^{b]d]} \\ [\Gamma^{ab}, \Gamma^{c}] = -2(\eta^{ac}\Gamma^b - \eta^{bc}\Gamma^a) \\ [\Gamma^{a}, \Gamma^{b}] = 2\Gamma^{ab} \end{cases}$$

Therefore, defining $\Gamma^{a8} = -\Gamma^a$ $(\eta^{88} = -1)$ and noting $a' = (a, 8)$ we get

$$[\Gamma^{a'b'}, \Gamma^{c'd'}] = -8\,\eta^{[a'[c'}\Gamma^{b']d']}$$

This is the Lie algebra of SO(8) : a' labels the 8 vector representation of SO(8) and A labels the 8 spinor representation of SO(8). To the transformation $\delta x^{a'} = \Lambda^{a'}{}_{b'}x^{b'}$ ($\Lambda^{a'}{}_{b'}$ is antisymmetric) of SO(8) corresponds in the spinor representation

$$\delta x^A = \frac{1}{4}(\Gamma^{a'b'})_A{}^B\Lambda_{a'b'}x^B \equiv \Lambda^A{}_B x^B$$

$\Lambda^A{}_B$ is again an antisymmetric 8 x 8 matrix (but different from $\Lambda_{a'b'}$). However, the two 8 representation of SO(8) are not equivalent : there is no transformation $x^A = Z^A{}_{a'} x^{b'}$ which would imply the relation between $\Lambda^{a'b'}$ and $\Lambda_A{}^B$. But for the 28 antisymmetric representation of SO(8) spanned by $x^{a'b'} = -x^{b'a'}$ the SO(8) transformation $\Lambda_{a'}{}^{b'}$ on $x^{a'b'}$ generates the SO(8) "vector" representation of parameter $\Lambda_A{}^B$ on $x^{AB} = \frac{1}{4}(\Gamma^{a'b'})_A{}^B x^{a'b'}$.

(b) SU(8) basis for E_7

The basis which features the SU(8) subgroup is defined from x^{ij} and y_{ij} by $z_{AB} = \frac{1}{4}(\Gamma^{ij})_A{}^B (x^{ij} + i y_{ij})/\sqrt{2} = -z_{BA}$

Using a lot of properties of Γ matrices we can show that E_7 is defined in this basis by the infinitesimal transformations

$$\delta z_{AB} = \Lambda_A{}^C z_{CB} + \Lambda_B{}^C z_{AC} + \Sigma_{ABCD}\, \bar{z}^{CD}$$

where $\bar{z}^{CD}$ is the complex conjugate of z_{CD} ; $\Lambda_A{}^B$ is a traceless antihermitian 8 x 8 matrix, Σ_{ABCD} is totally antisymmetric and satisfies the self-duality condition

$$\Sigma_{ABCD} = \frac{\eta}{4!}\varepsilon_{ABCDEFGH}\,\bar{\Sigma}^{EFGH} \qquad (\eta = \pm 1)$$

The bilinear invariant is now written in term of the symplectic matrix Ω'

$$z_{1AB}\bar{z}_2^{AB} - z_{2AB}\bar{z}_1^{AB} \qquad \Omega' = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}$$

The subgroup SU(8) corresponds to $\Sigma = 0$ or to matrices of the group E_7 which satisfy $[X', \Omega'] = 0$ or equivalently since $X'^{-1} = -\Omega' X'^{\dagger}\Omega'$ for $X' \in E_7$, to $X'X'^{\dagger} = 1$

(c) Reformulation of the previous results in this basis

We make the corresponding change on the matrix describing the scalar fields $\mathcal{V}'$

$$\mathcal{V}' = \begin{pmatrix} u_{AB}{}^{MN} & , v_{ABMN} \\ \bar{v}^{AB\,MN} & , \bar{u}^{AB}{}_{MN} \end{pmatrix}$$

$\partial_\mu \mathcal{V}'\mathcal{V}'^{-1}$ is an element of the Lie algebra of E_7 and can therefore be written

$$\partial_\mu \mathcal{V}'\mathcal{V}'^{-1} = \begin{pmatrix} 2\, Q_{\mu[A}{}^{[C}\delta_{B]}{}^{D]} & , P_{\mu ABCD} \\ \bar{P}_\mu{}^{ABCD} & , 2\,\bar{Q}_\mu{}^{[A}{}_{[C}\delta^{B]}{}_{D]} \end{pmatrix}$$

with $P_{\mu ABCD} = \frac{\eta}{24}\varepsilon_{ABCDEFGH}\,\bar{P}_\mu{}^{EFGH}$

The splitting between $(\partial_\mu \mathcal{V}'\mathcal{V}'^{-1})_{\parallel}$ and $(\partial_\mu \mathcal{V}'\mathcal{V}'^{-1})_{\perp}$ with respect to SU(8) is now obvious

$$(\partial_\mu \mathcal{V}'\mathcal{V}'^{-1})_{\parallel} = 2\begin{pmatrix} Q_{\mu[A}{}^{[C}\delta_{B]}{}^{D]} & , 0 \\ 0 & , \bar{Q}_\mu{}^{[A}{}_{[C}\delta^{B]}{}_{D]} \end{pmatrix}$$

$$(\partial_\mu \mathcal{V}'\mathcal{V}'^{-1})_{\perp} = \begin{pmatrix} 0 & , P_{\mu ABCD} \\ \bar{P}_\mu{}^{ABCD} & , 0 \end{pmatrix} = D_\mu \mathcal{V}'\mathcal{V}'^{-1}$$

Under the transformations

$$\mathcal{V}' \to S(x)\mathcal{V}'G \qquad S(x) \in SU(8)\,,\ G \in E_7$$

$Q_{\mu A}{}^B$ and $P_{\mu ABCD}$ are invariant under E_7. $Q_{\mu A}{}^B$ transforms as a gauge field for the local SU(8) and $P_{\mu ABCD}$ transforms as a covariant tensor for this local SU(8).

For the vector fields we redefine $B_\mu^{MN} = \frac{1}{4}(\Gamma^{ij})_M{}^N B_\mu^{ij}$ and correspondingly $G_{\mu\nu}^{MN}$ and $H_{\mu\nu MN}$. $\mathcal{F}$ will become

$$\mathcal{F}'_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} G_{\mu\nu}^{MN} + i H_{\mu\nu MN} \\ G_{\mu\nu}^{MN} - i H_{\mu\nu MN} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{\mu\nu MN} \\ \bar{\mathcal{F}}_{\mu\nu}^{MN} \end{pmatrix}$$

The condition on $\mathcal{F}'$ $\quad \mathcal{V}'\mathcal{F}'_{\mu\nu} = \Omega'\mathcal{V}'\tilde{\mathcal{F}}'_{\mu\nu}$ is now written if we define $\mathcal{V}'\mathcal{F}' = \begin{pmatrix} \mathcal{F}_{AB} \\ \bar{\mathcal{F}}^{AB} \end{pmatrix}$ as

$$\mathcal{F}_{\mu\nu AB} = i\,\tilde{\mathcal{F}}_{\mu\nu AB}$$

The Lagrangian for the bosonic fields is then written

$$L_B = -\frac{e}{4}R + \frac{e}{24}P_{\mu ABCD}\bar{P}_\nu{}^{ABCD}g^{\mu\nu} + \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma}G_{\mu\nu}^{MN}H_{\rho\sigma MN}$$

6 Couplings to the fermions

(a) Let us define the transformations of SU(8) on Majorana spinors : For $\Lambda_A{}^B = \Lambda'_A{}^B + i\,\Lambda''_A{}^B \in \mathcal{L}ie\,(SU(8))$
real antisymmetric real symmetric

$$\delta\lambda_A = (\Lambda'_A{}^B + i\gamma_5\Lambda''_A{}^B)\lambda_B$$

This preserves the Majorana property of λ^A. It is equivalent to using Weyl spinors

$$\lambda_A^{(R)} = \frac{1+\gamma_5}{2}\lambda_A \quad , \quad \lambda^A_{(L)} = \frac{1-\gamma_5}{2}\lambda_A$$

$$\delta\lambda_A^{(R)} = \Lambda_A{}^B \lambda_B^{(R)} \quad , \quad \delta\lambda^A_{(L)} = \Lambda^A{}_B \lambda^B_{(L)} \quad , \quad \Lambda^A{}_B = \overline{\Lambda_A{}^B}$$

(b) We have seen in the previous part that from dimensional reduction the fermionic couplings can be written in terms of the 3 objects $Q^\circ_{\mu A}{}^B$, $P^\circ_{\mu ABCD}$ and $\mathcal{F}^\circ_{AB}$ which are function of the fields g^{ij}, ϕ^i, A_{ijk}, B_μ^{ij} and $B_\mu{}^i$. Let us now compute $Q_{\mu A}{}^B$, $P_{\mu ABCD}$ defined in section 5, using the scalar matrix $\mathcal{V}$ (or $\mathcal{V}'$) defined in section 4. We find (up to the replacement : i by $i\gamma_5$)

$$Q_{\nu A}{}^B = Q^\circ_{\nu A}{}^B \quad , \quad P_{\mu ABCD} = P^\circ_{\mu ABCD}$$

This shows the SU(8) local invariance for the fermionic terms (Note the factor 3 between $\psi_{\mu A}$ and λ_{ABC} for the gauge coupling to $Q_{\nu A}{}^B$).

The coupling to the vector fields appears only through

$$\mathcal{F}^\circ_{AB} = \gamma^{\mu\nu} \mathcal{F}^\circ_{\mu\nu AB}$$

(with always the replacement of $i \to i\gamma_5$). Since $\gamma_{\mu\nu}$ satisfies

$$\tilde{\gamma}_{\mu\nu} = -i\gamma_5\gamma_{\mu\nu}$$

only the part of $\mathcal{F}^{\circ\mu\nu}_{AB}$ which satisfies $\mathcal{F}^{\circ\mu\nu}_{AB} = i\tilde{\mathcal{F}}^{\circ\mu\nu}_{AB}$ is coupled to fermions. In fact computing $\mathcal{F}_{AB\mu\nu}$ defined in section 5 with $\mathcal{V}'(\mathcal{V})$ defined in section 4, we find

$$\mathcal{F}_{\mu\nu AB} = \mathcal{F}^\circ_{\mu\nu AB}$$

(c) We have to take into account that now the real vector for E_7 is no longer $\mathcal{F}$ but $\mathcal{F}^F_{\mu\nu} = (G^{MN}_{\mu\nu}, H^F_{\mu\nu MN})$, where $H^F_{\mu\nu MN}$ contains now fermionic terms. $\mathcal{F}^F_{\mu\nu AB}$ no longer satisfies the constraint $\mathcal{F}^F_{\mu\nu AB} = i\tilde{\mathcal{F}}^F_{\mu\nu AB}$ but the modified one $\hat{\mathcal{F}}_{\mu\nu AB} = i\tilde{\hat{\mathcal{F}}}_{\mu\nu AB}$ with

$$\hat{\mathcal{F}}_{\mu\nu AB} = \mathcal{F}^F_{\mu\nu AB} + 2\sqrt{2}\Big\{ \overline{\psi}^{(L)}_{[\mu\,[A} \psi^{(R)}_{\nu]\,B]} - \frac{i}{\sqrt{2}} \overline{\psi}^{(R)C}_{[\mu} \gamma_{\nu]} \lambda^{(R)}_{ABC} + \frac{\nu}{288} \varepsilon_{ABCDEFGH} \overline{\lambda}^{CDE}_{(R)} \gamma_{\mu\nu} \lambda^{FGH}_{(L)} \Big\}$$

(d) We have not worked completely all quartic terms although they are completely determined by dimensional reduction. We have conjectured that they can be reabsorbed in the Lagrangian by the minimal replacement

$$P_{\mu ABCD} \longrightarrow \frac{1}{2}(P_{\mu ABCD} + \hat{P}_{\mu ABCD}) \quad , \quad \mathcal{F}_{\mu\nu AB} \longrightarrow \frac{1}{2}(\mathcal{F}_{\mu\nu AB} + \hat{\mathcal{F}}_{\mu\nu AB})$$

$\omega_{\mu\alpha\beta}$ being given by its equation of motion.

We have checked that all the $\psi_\mu{}^4$ terms are correct, the dependence of the quartic terms in scalar fields is correct and that the reduction from N = 8 to N = 4 gives the correct quartic terms.

This Lagrangian having the same structure as the general one discussed in section 4, the proof of the E_7 invariance of the theory is exactly the same.

7 <u>The E_7/SU(8) supergravity</u>

We can now write the complete Lagrangian

$$\begin{aligned}
\mathcal{L} = & -\frac{e}{4} R(\omega, e) + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \overline{\psi}_{\mu A} \gamma_\sigma \gamma_5 (\delta_A^B D_\nu(\omega) - Q_{\nu A}{}^B) \psi_{\rho B} \\
& + \frac{e}{8} G^{MN}_{\mu\nu} \tilde{H}^{\mu\nu(F)}_{MN}(B, \mathcal{V}, \psi, \lambda) + \frac{ie}{12} \overline{\lambda}^{ABC} \gamma^\mu (\delta_A^D D_\mu(\omega) - 3 Q_{\mu A}{}^D) \lambda_{BCD} \\
& + \frac{e}{24} P_{\mu ABCD} \overline{P}_\nu{}^{ABCD} g^{\mu\nu} + \frac{e}{6\sqrt{2}} \overline{\psi}_{\mu A} \gamma^\nu \gamma^\mu (\overline{P}_\nu{}^{ABCD} + \hat{\overline{P}}_\nu{}^{ABCD}) \lambda_{BCD} \\
& + \frac{e}{8\sqrt{2}} \Big\{ \overline{\psi}_{\mu A} \gamma^\mu \hat{\mathcal{F}}_{AB} \gamma^\nu \psi_{\nu B} - \frac{i}{\sqrt{2}} \overline{\psi}_{\mu C} \hat{\mathcal{F}}_{AB} \gamma^\mu \lambda_{ABC} - \frac{\nu}{72} \varepsilon^{ABCDEFGH} \overline{\lambda}_{ABC} \hat{\mathcal{F}}_{DE} \lambda_{FGH} \Big\}
\end{aligned}$$

with

$$\hat{P}_{\mu ABCD} = P_{\mu ABCD} + 2\sqrt{2}\Big[\overline{\psi}^{(L)}_{\mu[A} \lambda^{(R)}_{BCD]} + \frac{2}{24} \varepsilon_{ABCDEFGH} \overline{\psi}^E_{\mu(R)} \lambda^{FGH}_{(L)} \Big]$$

$$\hat{\mathcal{F}}_{AB} = \gamma^{\mu\nu} \hat{\mathcal{F}}_{\mu\nu AB}$$

$$\hat{\mathcal{F}}_{\mu\nu AB} = \mathcal{F}^F_{\mu\nu AB} + \sqrt{2}\Big\{ \overline{\psi}^{(R)}_{[\mu\,[A} \psi^{(L)}_{\nu]\,B]} - \frac{i}{\sqrt{2}} \overline{\psi}^{(L)C}_{[\mu} \gamma_{\nu]} \lambda^{(R)}_{ABC} + \frac{\nu}{288} \varepsilon_{ABCDEFGH} \overline{\lambda}^{CDE}_{(L)} \gamma_{\mu\nu} \lambda^{FGH}_{(R)} \Big\}$$

$\hat{\mathcal{F}}_{\mu\nu AB}$ (and therefore $\mathcal{F}^F_{\mu\nu AB}$) is defined by the constraints

$$\hat{\mathcal{F}}_{\mu\nu AB} = i\,\tilde{\hat{\mathcal{F}}}_{\mu\nu AB}$$

We note that in $\mathcal{L}$, one half of the coupling to $\mathcal{F}_{\mu\nu AB}$ has been absorbed in the term $G\tilde{H}^{(F)}$ so that $H^{(F)}$ satisfies as it should

$$e H^{(F)}_{\mu\nu MN} = -\partial\mathcal{L}/\partial G_{\mu\nu MN}$$

Let us summarize the invariances of the theory.

(a) It is invariant under reparametrization in 4 dimensions and local Lorentz SO(3,1) transformations.

(b) It is invariant under local SU(8) transformations acting on spinor fields and on scalar fields.

(c) The equations of motion are invariant under global $E_{7(+7)}$ transformations acting on field strengths of the vector fields and on scalar fields.

(d) It is invariant under the following supersymmetry transformations

$$\delta_s e_\mu^{\ a} = -i\,\bar{\epsilon}_A \gamma^a \psi_{\mu A}$$

$$\delta_s \mathcal{V}'\mathcal{V}'^{-1} = -2\sqrt{2}\begin{pmatrix} 0 & , X_{ABCD} \\ \bar{X}^{ABCD} & 0 \end{pmatrix} \qquad \text{with}$$

$$X_{ABCD} = \bar{\epsilon}^{(L)}_{[A}\lambda^{(R)}_{BCD]} + \frac{1}{24}\varepsilon_{ABCDEFGH}\,\bar{\epsilon}^{E}_{(R)}\lambda^{FGH}_{(L)}$$

$\delta_s B_\mu^{MN}$ is derived from

$$\delta_s\begin{pmatrix} B_\mu^{MN} + i C_{\mu MN} \\ B_\mu^{MN} - i C_{\mu MN}\end{pmatrix} = -2\sqrt{2}\,\mathcal{V}'^{-1}\begin{pmatrix} \bar{\epsilon}^{(L)}_{[A}\psi^{(R)}_{\mu B]} - \frac{i\sqrt{2}}{4}\bar{\epsilon}^{C}_{(R)}\gamma_\mu\lambda^{(R)}_{ABC} \\ \bar{\epsilon}^{[A}_{(R)}\psi^{B]}_{\mu(L)} - \frac{i\sqrt{2}}{4}\bar{\epsilon}^{(L)}_{C}\gamma_\mu\lambda^{ABC}_{(L)}\end{pmatrix}$$

$C_{\mu MN}$ is the dual potential defined only on-shell

$$\delta_s \psi^{(R)}_{\mu A} = \left(D_\mu(\omega)\delta_A^{\ B} - \varphi_{\mu A}^{\ \ B}\right)\epsilon^{(R)}_B - \frac{i}{4\sqrt{2}}\hat{\mathcal{F}}_{AB}\gamma_\mu\epsilon^B_{(L)} + \frac{i}{4}\bar{\lambda}^{(L)}_{ABC}\gamma^\nu\lambda^{DBC}_{(L)}\gamma_\nu\gamma_\mu\epsilon^{(R)}_D - \frac{1}{\sqrt{2}}\bar{\psi}^{B}_{\mu(R)}\gamma^\nu\lambda^{(R)}_{ABC}\epsilon^C_{(L)}$$

$$\delta_s \lambda^{(R)}_{ABC} = -i\sqrt{2}\,\hat{P}_{\mu ABCD}\gamma^\mu\epsilon^D_{(L)} + \frac{3}{4}\hat{\mathcal{F}}_{[AB}\epsilon^{(R)}_{C]}$$

We check directly that $\hat{P}_{\mu ABCD}$, $\hat{\mathcal{F}}_{AB}$ and $\varphi_{\mu A}^{\ \ B}$ are supercovariant.

In fact, having guessed the symmetries of the theory (SU(8) local x E_7 global) we could have written down the Lagrangian and the supersymmetry transformations up to a few numerical coefficients which could have been determined by checking directly the supersymmetry. We shall show how this works for N = 8 supergravity in 5 dimensions in the next part.

8 The symmetric gauge

The gauge (or equivalently the parametrization) which puts the Lagrangian in the SO(8) symmetric form usually used in supergravity is the so-called symmetric gauge. Moreover, it is the only gauge in which there are SU(8) linear transformations acting on the parameters of the vielbein $\mathcal{V}'$ describing the scalar fields (and not only the metric $\mathcal{R}$). By an SU(8) gauge transformation we can impose the condition

$$\mathcal{V}' = \mathcal{V}'^{\dagger}$$

In this gauge, $\mathcal{V}'$ is generated by the part of $E_{7(+7)}$ perpendicular to SU(8) $\quad \mathcal{V}' = \exp X$

$$X = \begin{pmatrix} 0 & w_{ABCD} \\ \bar{w}^{ABCD} & 0 \end{pmatrix} \qquad w_{ABCD} = \frac{1}{24}\varepsilon_{ABCDEFGH}\bar{w}^{EFGH}$$

$\mathcal{V}'$ is then written

$$\mathcal{V}' = \begin{pmatrix} \mathrm{ch}\sqrt{w\bar{w}} & , w\,\dfrac{\mathrm{sh}\sqrt{\bar{w}w}}{\sqrt{\bar{w}w}} \\ \bar{w}\,\dfrac{\mathrm{sh}\sqrt{w\bar{w}}}{\sqrt{w\bar{w}}} & , \mathrm{ch}\sqrt{\bar{w}w}\end{pmatrix}$$

(w and $\bar{w}$ are considered as 28 x 28 matrices)

At this point, it is useful to introduce the so-called inhomogeneous coordinates of E7/SU(8) y. (This discussion is roughly independent of the coset space considered G/H ; we only need H to be a maximal compact subgroup of G or equivalently G/H to be a symmetric space)

$$y_{AB,CD} = \left(w\,\frac{\mathrm{th}\sqrt{\bar{w}w}}{\sqrt{\bar{w}w}}\right)_{AB,CD}$$

This variable y is bounded by $\|y\|<1$. Then we get

$$\mathcal{V}' = \begin{pmatrix} \dfrac{1}{\sqrt{1-y\bar{y}}} & , y\dfrac{1}{\sqrt{1-\bar{y}y}} \\ \bar{y}\dfrac{1}{\sqrt{1-y\bar{y}}} & , \dfrac{1}{\sqrt{1-\bar{y}y}}\end{pmatrix}$$

The action of $E_{7(+7)}$ on y is simple and gives the y 's their names..

$$\mathcal{V}(y)\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} U(y) & 0 \\ 0 & \bar{U}(y)\end{pmatrix}\mathcal{V}(y')$$

constant matrix of E_7 $\qquad$ U matrix of SU(8)

with $\quad y' = (A + yC)^{-1}(B + yD)$

The SU(8) local transformation U is determined by the condition that $\mathcal{V}(y')$ is symmetric and is reabsorbed by gauge transformations.

Let us note that apart from its definition, there is no simple way to characterize y_{ABCD}, in particular it is not antisymmetric in ABCD. It is easy to compute $\partial_\mu\mathcal{V}'\mathcal{V}'^{-1}$ and get $P_{\mu ABCD}$ and $Q_{\mu A}^{\ \ B}$. In particular

$$P_{\mu\,ABCD} = \left(\frac{1}{\sqrt{1-y\bar{y}}}\,\partial_\mu y\,\frac{1}{\sqrt{1-\bar{y}y}}\right)_{AB,CD}$$

so that the kinetic term of the scalar fields is written in the simple way

$$P_{\mu ABCD}\bar{P}^{\mu ABCD} = \mathrm{Tr}\left(\frac{1}{1-y\bar{y}}\partial_\mu y\,\frac{1}{1-\bar{y}y}\partial_\mu\bar{y}\right)$$

In this gauge the constraint $\bar{\mathcal{G}}_{\mu\nu AB} = i\,\tilde{\mathcal{F}}_{\mu\nu AB}$ is solved by

$H_{\mu\nu AB} = \mathcal{N}_{AB,CD}\,\tilde{G}^{CD}_{\mu\nu}$ $\qquad$ (the notation is now covariant only for SO(8) and no longer for SU(8)) with

$$\mathcal{N}_{AB,CD} = \left(\frac{\mathbb{1} + y}{\mathbb{1} - y}\right)_{AB,CD} \quad \text{where} \quad \mathbb{1}_{AB,CD} = \frac{1}{2}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$$

(we recall that $\mathbb{1}$ stands for $\frac{1}{2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$ and i for $\frac{1}{2e} \varepsilon_{\mu\nu\rho\sigma}$)

Then we get $\quad \mathcal{F}_{\mu\nu\, AB} = \sqrt{2}\left(\sqrt{1-\bar{y}y}\ \frac{1}{1-y}\right)_{AB,CD} G^{CD}_{\mu\nu}$

The supersymmetry transformations have to be modified because the specific gauge transformations we have made, which put V' in the symmetric gauge, depend on the scalar fields.

IV HIDDEN SYMMETRIES FOR SUPERGRAVITIES IN D DIMENSIONS WITH N SPINORIAL CHARGES

We have found that for N = 8 supergravity in 4 dimensions, there exists a non compact global invariance realized non linearly and a hidden local invariance of the theory. These properties are true for all supergravities. Let us give the general idea to find these symmetries. We shall assume that the scalars are always described by a coset space G/H. H which is isomorph to the maximal group linearly realized on the physical states is the group of invariance of the algebra of supersymmetry : for instance SU(N) or U(N) in 4 dimensions, USp(2N) in 5 dimensions. The dimension of G is just equal to the sum of the dimension of H and the number of scalar fields. In order for the scalar to have the right signature, H must be the maximal compact subgroup of G, this fixes the signature of G. We expect that except the scalar fields, all bosonic fields are singlet for the local group, G being realized eventually on-shell, the fermionic fields are singlet for the non compact group G, the scalar fields transform under both groups and can be considered as "vielbein". All these counting arguments fix quite uniquely the group G. We shall have also a guide line by decreasing N or increasing D.

1 Extended supergravities in 4 dimensions

We start from N = 8 and make some consistent truncation which leads to N<8 supergravity theories. We have to take care of CPT invariance. In this way we recover in particular the global invariance SU(1,1) of N = 4 supergravity. It can be formulated as SU(1,1) x SU(4) global invariance of the equations of motion and U(4) local invariance of the Lagrangian.

For N $\leqslant$ 3, since there is no scalar fields the local invariance and the global invariance are both isomorphic to U(N) and can be reduced to the known global U(N) invariance by a field redefinition.

TABLE IV

N	8	7	6	5	4
Spin 2	1	1	1	1	1
Spin 3/2	8	7+1	6	5	4
Spin 1	28	21+7	15+1	10	6
Spin 1/2	56	35+21	20+6	10+1	4
Spin 0	70	35+35	15+15	5+5	1+1
Global group rank	$E_7(+7)$ 7	$E_7(+7)$ 7	$SO^*(12)$ 6	SU(5,1) 5	SU(4) x SU(1,1) 4
Local group rank	SU(8) 7	SU(8) 7	U(6) 6	U(5) 5	U(4) 4

2 Maximal extended supergravities in D dimensions

The maximal supergravities in D dimensions (D<11) are all obtained by dimensional reduction of D = 11, N = 1 supergravity. In order to exhibit the maximal symmetries (we have at least SL(11 - D,R) global x SO(11 - D) local) we have to perform duality transformations on tensor fields, for instance

$$\begin{array}{lll} D = 7 & A_{\mu\nu\rho} \longrightarrow B_{\mu\nu} & \\ D = 6 & A_{\mu\nu\rho} \longrightarrow B_{\mu} & \\ D = 5 & A_{\mu\nu\rho} \longrightarrow \Phi & , A_{\mu\nu} \longrightarrow B_{\mu} \\ D = 4 & A_{\mu\nu\rho} \longrightarrow \text{NOTHING} & , A_{\mu\nu} \longrightarrow \Phi \end{array}$$

The global transformations G_D can eventually be realized on field strength of tensor fields and not on the fields themselves. In this case G_D is a symmetry of the equations of motion and not of the Lagrangian for instance D = 8 for $A_{\mu\nu\rho}$, D = 6 for $A_{\mu\nu}$, D = 4 for A_μ . If D is odd, G_D is always a symmetry of the Lagrangian. The content of the maximal supergravities as well as their symmetries are summarized in Table V (Cremmer 1980 ; see also Morel & Thierry-Mieg 1981 ; Schwarz 1980)

TABLE V

D=9 GL(2, R)global ⊗ SO(2)local

1 e_μ^r , 2 Ψ_μ , 1$A_{\mu\nu\rho}$, 2$A_{\mu\nu}$, 3A_μ , 4 χ , 3 scalars

D=8 $E_{3(+3)}$=SL(3,R) x SL(2,R)global ⊗ [SO(3) x SO(2)] local

1 e_μ^r , 2 Ψ_μ , 1$A_{\mu\nu\rho}$, 3$A_{\mu\nu}$, 6A_μ , 6 χ , 7 scalars

D=7 $E_{4(+4)}$=SL(5,R)global ⊗ SO(5)local

1 e_μ^r , 4 Ψ_μ , 5$A_{\mu\nu}$, 10A_μ , 16 χ , 14 scalars

D=6 $E_{5(+5)}$=SO(5, 5)global ⊗ SO(5) x SO(5) local

1 e_μ^r , 4 Ψ_μ , 5$A_{\mu\nu}$, 16A_μ , 20 χ , 25 scalars

D=5 $E_{6(+6)}$global ⊗ USp(8)local

1 e_μ^r , 8 Ψ_μ , 27A_μ , 48 χ , 42 scalars

D=4 $E_{7(+7)}$global ⊗ SU(8)local

1 e_μ^r , 8 Ψ_μ , 28A_μ , 56 χ , 70 scalars

D=3 $E_{8(+8)}$global ⊗ SO(16)local

1 e_μ^r , 16 Ψ_μ , 128 χ , 128 scalars

The global symmetries are realized on field strengths for the underlying fields. Let us note that in 3 dimensions there are no degrees of freedom for the graviton and the gravitino.

In all dimensions, we have of course the same number of degrees of freedom i.e. 128 bosonic states and 128 fermionic states. For D = 3...8, the global group is E_{11-D}. It has been suggested by Julia (1981 a) that there could exist an inverse process of dimensional reduction : the group desintegration. It is based on the observation that in D dimension the local group x little spin group is always a maximal subgroup of SO(16). A similar statement holds for the global invariances

$$E_8 \supset E_{11-D} \otimes SL(D-2) \quad , \quad SO(16) \supset H_D \otimes SO(D-2)$$

The scalars are described by the coset E_{11-D}/H_D and the on-shell graviton is described by SL(D − 2)/SO(D-2).

In two dimensions, we expect that both G_D and H_D will have an infinite number of generators, in particular G_D should be the affine group $E_8^{(1)}$ $(\sim E_9)$ associated to E_8 (Julia 1981 b).

3 Supergravities in 5 dimensions

In 5 dimensions, the symmetries are invariances of the Lagrangian, so that the structure should be simpler. The knowledge of these theories in 5 dimensions, not only allows a better understanding of the 4 dimensional theories, but also leads to new things such as supersymmetry breaking or off shell formulation in 4 dimensions via various dimensional reduction procedure.

(a) Notations

The metric will be (+ − − − −). As has been explained in part I, we choose as Clifford algebra $\{\gamma_r, \gamma_s\} = 2\eta_{rs}$ the one using the 4-dimensional γ matrices, γ_0 , γ_i (i = 1,2,3) pure imaginary and $\gamma_4 = i\gamma_5$ real with the relation $\gamma_{rstuv} = \varepsilon_{rstuv}$. There are no Majorana spinors in 5 dimensions. Instead of using Dirac spinors, we can double the spinors and use new "reality" conditions. We have only 2N supersymmetry in 5 dimensions. They are classified by USp(2N) (as the massive multiplet in 4 dimensions with central charge $\sim 5^{th}$ dimensions). Defining, Ω^{ab} , the antisymmetric symplectic matrix of USp(2N) (a = 1...N), and using it to lower or raise indices, we define the new "reality" conditions

Bosons $\quad A_\mu^{ab} = A^*_{\mu\,ab}$

Fermions $\quad \Psi_\mu^a = \gamma_5 \Psi^*_{\mu\,a}$

The algebra of supersymmetry is then written

$$\{\bar{Q}^a_\alpha, Q^b_\beta\} = \Omega^{ab}\gamma^\mu_{\alpha\beta} P_\mu$$

(b) Physical content of supergravities

The representation of USp(2N) appearing in supergravity are traceless antisymmetric tensors for instance A_μ^{ab} , χ^{abc} , with

$$\Omega_{ab} A_\mu^{ab} = 0 \quad , \quad \Omega_{ab}\chi^{abc} = 0$$

The representations of supergravity are obtained from the lowest spin representation of 2N supersymmetry by multiplication by appropriate angular momentum (Ferrara & Zumino 1979). (SO(3) classifies the massless states in 5 dimensions) and are given in Table VI.

TABLE VI

s	2	3/2	1	1/2	0	group
N=8	1	8	27	48	42	USp(8)
N=6	1	6	14+1	14'+6	14	USp(6)
	$(J=\frac{1}{2})$	$\otimes$ [1	6	14	14']	
N=4	1	4	5+1	4	1	USp(4)
	(J=1)	$\otimes$	[1	4	5]	
N=2	1	2	1			USp(2)
			$(J=\frac{3}{2})$	$\otimes$ [1 ,	2]	

By counting arguments we can conjecture what the global and the local invariances of the Lagrangian should be. This gives the following results

N=8 $E_{6(+6)}$ global $\otimes$ USp(8) local

N=6 $SU^{*}(6)$ global $\otimes$ USp(6) local

N=4 USp(4) x R global $\otimes$ USp(4) local

N=2 USp(2) global $\otimes$ USp(2) local

(c) N = 8 supergravity in 5 dimensions

We shall show how from the knowledge of the global and local symmetries $E_{6(+6)}$ and USp(8) it is possible to construct the Lagrangian for N = 8 supergravity in 5 dimensions directly (Cremmer et al. 1978 ; Cremmer 1981).

Let us briefly describe $E_{6(+6)}$ by its infinitesimal transformations in the fundamental representation of dimension 27. They act on the vector space spanned by $Z^{\alpha\beta} = -Z^{\beta\alpha}$ $(= Z^{*}_{\alpha\beta})$ $\quad \Omega_{\alpha\beta} Z^{\alpha\beta} = 0$ ($\alpha, \beta = 1 \ldots 8$) and are given by

$$\delta Z^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma} Z^{\gamma\beta} + \Lambda^{\beta}{}_{\gamma} Z^{\alpha\gamma} + \Sigma^{\alpha\beta\gamma\delta} Z_{\gamma\delta}$$

There are 78 generators, 36 compact ones generating the maximum compact subgroup Sp(8) of parameters $\Lambda^{\alpha}{}_{\gamma}$ antihermitian such that $\Lambda_{\alpha\gamma}$ is symmetric, and 42 non compact ones with parameters $\Sigma^{\alpha\beta\gamma\delta}$ antisymmetric and traceless.

E_6 has no bilinear invariant but has a trilinear invariant

$$J = Z^{\alpha\beta} \Omega_{\beta\gamma} Z^{\gamma\delta} \Omega_{\delta\epsilon} Z^{\epsilon\lambda} \Omega_{\lambda\alpha}$$

These properties will be sufficient to write the general structure of the theory. The field content is : 1 graviton $e_{\mu}{}^{r}$, 8 gravitinos $\psi_{\mu}{}^{a}$, 27 vector fields $A_{\mu}{}^{\alpha\beta}$ ($A_{\mu}{}^{\alpha\beta} = -A_{\mu}{}^{\beta\alpha} = (A_{\mu\,\alpha\beta})^{*}$, $\Omega_{\alpha\beta} A_{\mu}{}^{\alpha\beta} = 0$), 48 spin 1/2 fields χ^{abc} (antisymmetric, pseudoreal and traceless) and 42 scalar fields described by a matrix 27 x 27 of E_6 $\mathcal{V}_{\alpha\beta}{}^{ab}$ up to a local transformation of USp(8).

The scalar fields being described by an element $\mathcal{V}_{\alpha\beta}{}^{ab}$ of the coset space E_6/USp(8), their self interaction is described by the associated non-linear σ model with Lagrangian

$$\mathcal{L} \simeq D_{\mu} \mathcal{V}_{\alpha\beta}{}^{ab} \, D^{\mu} (\mathcal{V}^{-1})_{ab}{}^{\alpha\beta} \sim -\mathrm{Tr}(\mathcal{V}^{-1} D_{\mu} \mathcal{V})^2$$

D_{μ} being the covariant derivative with respect to USp(8) using the connection $Q_{\mu a}{}^{b}$ defined by

$$\underbrace{\mathcal{V}_{cd}{}^{\alpha\beta} \partial_{\mu} \mathcal{V}_{\alpha\beta}{}^{ab}}_{\in \text{ Lie } (E_6)} = \underbrace{2\, Q_{\mu[c}{}^{[a} \delta_{d]}{}^{b]}}_{\in \text{ Lie (USp(8))}} + \underbrace{P_{\mu}{}^{ab}{}_{cd}}_{\in \text{ Lie (USp(8))}}$$

The Lagrangian is then written

$$\mathcal{L} \simeq |P_{\mu\, abcd}|^2$$

$P_{\mu\, abcd}$ as well as $Q_{\mu a}{}^{b}$ are invariant under E_6. We can also describe the scalar fields by a metric for this coset space which is covariant for E_6 and invariant for USp(8) $g_{\alpha\beta,\gamma\delta}$

$$g_{\alpha\beta,\gamma\delta} = \mathcal{V}_{\alpha\beta}{}^{ab} \, \Omega_{ac} \, \Omega_{bd} \, \mathcal{V}_{\gamma\delta}{}^{cd}$$

the Lagrangian can be written as

$$\mathcal{L} \sim \partial_{\mu} g_{\alpha\beta,\gamma\delta} \, \partial^{\mu} (g^{-1})^{\alpha\beta,\gamma\delta}$$

However, g cannot be used for describing the couplings to fermions (analogy with $g_{\mu\nu}$ and $e_{\mu}{}^{r}$)

Since there is no quadratic invariant for the vector fields, in order to construct an E_6 invariant kinetic term for the vector fields we must use the metric $g_{\alpha\beta,\gamma\delta}$

$$\mathcal{L}_{V^2} \sim \sqrt{g}\, g_{\alpha\beta,\gamma\delta}\, F_{\mu\nu}^{\alpha\beta} F_{\rho\sigma}^{\gamma\delta} g^{\mu\rho} g^{\nu\sigma}$$

As in 11 dimensions, there exists a trilinear gauge invariant coupling (up to a total derivative). Since there exists a trilinear invariant for E_6 we do not need the scalar metric g (nor the tensor metric $g_{\mu\nu}$)

$$\mathcal{L}_{V^3} \sim \varepsilon^{\mu\nu\rho\sigma\lambda}\, \Omega_{\alpha\beta} F_{\mu\nu}^{\beta\delta}\, \Omega_{\gamma\delta} F_{\rho\sigma}^{\delta\epsilon}\, \Omega_{\epsilon\eta} A_\lambda^{\eta\alpha}$$

Since the fermions are scalar for E_6, the coupling of the bosons to fermions must appear through E_6 invariant bosonic expressions : $P_{\mu\, abcd}$, $Q_{\mu a}{}^{b}$ (in covariant derivative $D_\mu \psi_\nu{}^a$ and $D_\mu \chi_{abc}$) or $F_{\mu\nu}^{ab} = \mathcal{V}_{\alpha\beta}{}^{ab} F_{\mu\nu}^{\alpha\beta}$.
The supersymmetry transformation laws $\delta\phi$ are assumed to be covariant with respect to USp(8) and E_6. At this step $\mathcal{L}$ and $\delta\phi$ are determined up to numerical coefficients and quartic fermionic terms. In particular all the non polynomial structure of the scalar fields is known. Supersymmetry of $\mathcal{L}$ and closure of the algebra are used to get rid of the remaining arbitrariness. We then get the Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} = & -\tfrac{1}{4}R - \tfrac{i}{2}\bar{\psi}_\mu{}^a \gamma^{\mu\nu\rho} D_\nu \psi_{\rho a} - \tfrac{1}{8} g^{\mu\rho} g^{\nu\sigma} g_{\alpha\beta,\gamma\delta} F_{\mu\nu}^{\alpha\beta} F_{\rho\sigma}^{\gamma\delta} \\ & + \tfrac{i}{12}\bar{\chi}^{abc}\gamma^\mu D_\mu \chi_{abc} - \tfrac{1}{24} g^{\mu\nu} D_\mu \mathcal{V}_{\alpha\beta}{}^{ab} D_\nu (\mathcal{V}^{-1})^{\alpha\beta}{}_{ab} \\ & - \tfrac{e^{-1}}{12}\varepsilon^{\mu\nu\rho\sigma\lambda} (F_{\mu\nu})^\alpha{}_\beta (F_{\rho\sigma})^\beta{}_\gamma (A_\lambda)^\gamma{}_\alpha + \tfrac{i}{3\sqrt{2}} P_\rho{}^{abcd} \bar{\psi}_{\mu a}\gamma^\rho\gamma^\mu \chi_{bcd} \\ & + \tfrac{i}{4}\mathcal{V}_{\alpha\beta}{}^{ab} F_{\mu\nu}^{\alpha\beta}\left[\bar{\psi}_a^\rho \gamma_\rho \gamma^{\mu\nu}\gamma_\sigma \psi_b^\sigma + \tfrac{1}{\sqrt{2}}\bar{\psi}_\rho^c \gamma^{\mu\nu}\gamma^\rho \chi_{abc} + \tfrac{1}{2}\bar{\chi}_{acd}\gamma^{\mu\nu}\chi_b{}^{cd}\right] \end{aligned}$$

+ quartic terms

We shall not write the quartic terms (see Cremmer 1981). It is invariant under global $E_{6(+6)}$, local USp(8) and the following supersymmetry transformations (up to trilinear fermionic terms)

$$\delta e_\mu{}^r = -i\,\bar{\varepsilon}^a \gamma^r \psi_{\mu a}$$

$$(\mathcal{V}^{-1})^{\alpha\beta}{}_{cd}\, \delta \mathcal{V}_{\alpha\beta,ab} = -2i\sqrt{2}\,(\bar{\varepsilon}_{[a}\chi_{bcd]} + \tfrac{3}{4}\Omega_{[ab}\bar{\varepsilon}_e \chi^e{}_{cd]})$$

$$\delta A_\mu{}^{\alpha\beta} = 2i\,\mathcal{V}^{\alpha\beta}{}_{ab}\,(\bar{\varepsilon}^a \psi_\mu^b + \tfrac{1}{2}\bar{\varepsilon}_c \gamma_\mu \chi^{abc})$$

Cremmer ; Symmetries in extended supergravity

$$\delta\psi_{\mu a} = (D_\mu(\omega)\delta_a^b + Q_{\mu a}{}^b)\epsilon_b - \tfrac{1}{6}\hat{F}_{\rho\sigma}^{\alpha\beta}\mathcal{V}_{\alpha\beta ab}(\gamma^{\rho\sigma}{}_\mu + 2\gamma^\rho\delta_\mu^\sigma)\epsilon^b + \cdots$$

$$\delta\chi_{abc} = \sqrt{2}\,\hat{P}_{\mu abcd}\gamma^\mu\epsilon^d - \tfrac{3}{2\sqrt{2}}\gamma^{\rho\sigma}\hat{F}_{\rho\sigma}^{\alpha\beta}(\mathcal{V}_{\alpha\beta[ab}\epsilon_{c]} + \tfrac{1}{3}\Omega_{[ab}\mathcal{V}_{\alpha\beta c]d}\epsilon^d) + \cdots$$

(d) <u>N = 2 supergravity in 5 dimensions</u>

By consistent truncation we obtain all supergravities in 5 dimensions and their symmetries, in particular the N = 2 supergravity whose Lagrangian is

$$\begin{aligned} e^{-1}\mathcal{L} = & -\tfrac{1}{4}R(\omega) - \tfrac{i}{2}\bar{\psi}_\mu{}^a\gamma^{\mu\nu\rho} D_\nu\left(\tfrac{\omega+\hat{\omega}}{2}\right)\psi_{\rho a} - \tfrac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \\ & + \tfrac{e^{-1}}{6\sqrt{3}}\varepsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F_{\rho\sigma}A_\lambda - \tfrac{i\sqrt{3}}{16}(F_{\mu\nu}+\hat{F}_{\mu\nu})\bar{\psi}^{\rho c}\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi_c^\sigma \end{aligned}$$

$\omega_{\mu rs}$ is defined by its own equation of motion if $\hat{\omega}_{\mu rs} = \omega_{\mu rs} + \tfrac{i}{4}\bar{\psi}^{\rho a}\gamma_{\mu rs\rho\sigma}\psi_a^\sigma$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \tfrac{\sqrt{3}}{4}\bar{\psi}_\mu{}^c\psi_{\nu c}$$

It is invariant under the following supersymmetry transformations

$$\delta e_\mu{}^r = -i\,\bar{\epsilon}^c\gamma^r\psi_{\mu c}$$

$$\delta\psi_{\mu a} = [D_\mu(\hat{\omega}) + \tfrac{1}{4\sqrt{3}}\hat{F}_{\rho\sigma}(\gamma^{\rho\sigma}\gamma_\mu - 2\gamma^\rho\delta_\mu^\sigma)]\epsilon_a$$

$$\delta A_\mu = -\tfrac{\sqrt{3}}{4}\,\bar{\epsilon}^c\psi_{\mu c}$$

The structure of the theory is completely analogous to the 11-dimensional supergravity from which we can derive all supergravities by dimensional reduction and consistent truncation.

V <u>POSSIBLE IMPLICATIONS OF THE SYMMETRIES : CONJECTURES FOR N = 8 SUPERGRAVITY IN 4 DIMENSIONS</u>

In this part, we shall summarize what we know and what we would like to know about N = 8 supergravity at the classical or at the quantum level. This will be developed in more detail in other contributions to the School or to the Workshop. We shall also state the two conjectures for N = 8 supergravity and their implication about the possible relevance of N = 8 supergravity to particle physics.

1 <u>Other problems for N = 8 supergravity</u>

(a) We have seen that the existence of the global E_7 and the local SU(8) symmetries has allowed a "geometrical" description of the scalar fields as describing a coset space $E_7/SU(8)$. A natural question arises : is there a possible "geometrical" interpretation of the spin 1/2 fields ? Although it has not been yet found, it seems that a natural answer should be yes. This should likely provide a tremendous simplification of the quartic fermionic terms and give more light on the supersymmetric structure and eventually on auxiliary fields.

(b) Off-shell formulation

There exists superspace formulation of N = 8 supergravity consistent with the symmetries E_7 and SU(8) but it is an on-shell formulation in the sense that the Bianchi identities on torsion, curvature... imply the equations of motion. The situation is the same for the N = 1 supergravity in 11 dimensions where all geometrical quantities can be written in terms of a single superfield $F_{rstu}(x,\theta)$ satisfying an equation (which implies the equations of motion) (Cremmer & Ferrara 1980 ; Brink & Howe 1980).

At the linearized level of N = 8 supergravity, some attempts have been made to obtain auxiliary fields using the dimensional reduction by Legendre transform starting with N = 8 linearized supergravity in 5 dimensions (Cremmer et al. 1980). This leads to fields which satisfy differential constraints (for gauge fields) which cannot be solved exactly (at least without introducing ghosts or higher derivative terms in the resulting Lagrangian). These constraints can be imposed by Lagrange multiplier, but we lose the closure of the algebra (on the Lagrange multipliers particularly). Moreover, this formulation even if it can be extended to the non linearized case, will not be consistent with the symmetries E_7 and SU(8) (probably only with E_6 and USp(8)).

(c) Supersymmetry breaking

Using a generalized dimensional reduction of Scherk and Schwarz 1979 b, it is possible starting from N = 8 supergravity in 5 dimensions to derive a N = 8 supergravity in 4 dimensions with 4 mass parameters (Cremmer et al. 1979). This new theory is still invariant under some local like-supersymmetry transformations. These new transformations are spontaneously broken and we cannot extract a global algebra from them. However, at one loop, this theory seems still finite (Sezgin & Van Nieuwenhuizen 1981).

(d) Gauging of O(8)

The possibility of gauging O(N) in N-supergravity in 4 dimensions is suggested by the fact that the vector fields of the supergravity multiplet are in general in the adjoint representation of O(N) (true for

N = 1...5 and 8, for N = 6 there is an extra singlet, N = 7 is identical to N = 8). This has been done completely for N = 1...5 and partially for N = 8 (de Wit & Nicolai 1981). This allows the introduction of a new coupling constant : the dimensionless gauge coupling. This requires a huge cosmological constant or a scalar potential which is unbounded from below. This gauging of O(N) can also be viewed as a gauging of the super De Sitter algebra instead of the super Poincaré algebra, the coupling constant being related to the De Sitter radius. Both have the same local algebra . The introduction of the gauge coupling breaks the global invariance and even its subgroup U(N).

The possibility of gauging O(N) has provided the first attempt to superunification, unifying in a same multiplet the vector gauge fields and the graviton (and gravitinos). But, this attempt has not been successful to reproduce the present low energy phenomenology. Let us sketch the arguments for such a statement. First of all, if we want to avoid particles with spin>2 and/or several spin 2 massless particles, since we have no consistent interacting theories for them, we must have $N \leq 8$. Assuming that we can gauge SO(8) and that we can solve the problems due to unbounded potential from below, we immediately see that $SO(8) \not\supset SU(3)_c \times SU(2) \times U(1)$. Despite this problem Gell Mann (1977) has tried to go on noting that $SO(8) \supset SU(3)_c \times U(1)$ and has made an analysis based on the vector-like description $8 \to 3 + \bar{3} + 1 + 1$ with respective charge $(-1/3, 1/3, 1, 0)$. This has shown that at least the muon and its neutrino, the τ and ν_τ , $W^\pm$ are missing.

2 Quantum corrections

This will be discussed in detail by P. Van Nieuwenhuizen, M. Duff and R. Kallosh in this book. As gravity, supergravity cannot be renormalizable in the ordinary sense because the coupling constant K has a dimension. This implies that the counterterms have not the same structure as the original Lagrangian and prevent the absorption of infinities in renormalization constants. Therefore, such a theory can only be finite (or meaningless in perturbation theory).

(a) 1-loop counterterm on shell

For pure gravity with cosmological constant Λ whose action is

$$S = -\frac{1}{2\kappa^2}\int d_4x\sqrt{g}\,(R-2\Lambda)$$

after use of the equation of motion (or redefinition of the background field) the counterterm can be written

$$\Delta S = -\frac{1}{D-4}\,(A\chi + B\delta)$$

with $$\delta = -\frac{\kappa^2\Lambda}{12\pi^2}\,S$$

$$\chi = \frac{1}{32\pi^2}\int d_4x\,\underbrace{\sqrt{g}\,(R_{\mu\nu\rho\sigma}^{\,2} - 4R_{\mu\nu}^{\,2} + R^2)}_{\text{total divergence}} = \text{integer}$$

A and B are constants A = 106/45 , B = -87/10

Therefore, pure gravity is 1-loop finite if $\Lambda = 0$ (no cosmological constant) and $\chi = 0$ (the topological structure of space-time is trivial). For supergravity, S, δ and χ are extended to supersymmetric invariants. It is found that

. if $N \geqslant 5$, then B = 0 (this implies that $\beta(g) = 0$ if g is the O(N) gauge coupling constant ; this has been checked directly).

. A is an integer for $N \geqslant 3$

. For N = 8 if we use the field description of dimensional reduction (63 scalars $+ 7A_{\mu\nu} + 1A_{\mu\nu\rho}$ then A = 0.

This applies also to N = 4 if we use 1 scalar $+ 1A_{\mu\nu}$. Although these theories have not been constructed for N = 5 and 6 we can find such field descriptions using $A_{\mu\nu}$ and $A_{\mu\nu\rho}$ fields which make A = 0.

. There are no boundary contributions to the 1-loop counterterms for N = 8.

N = 8 supergravity is, up to now, the only theory including gravity which is completely finite at 1-loop.

(b) <u>N-loop counterterms</u>

There exists no supersymmetric extension of the 2-loop counterterms of pure gravity. At the linearized level there exist 3-loop counterterms (which could eventually violate E_7 symmetry for N = 8 supergravity ?). Finally there exists a 8-loop counterterm which respects all the symmetries of N = 8 supergravity.

(c) <u>Conjecture N° 1</u>

. <u>N = 8 supergravity exists</u> ! (it is finite)

Although there exist counterterms, the example of N = 4 supersymmetric Yang-Mills is encouraging. This could be linked to the fact that the multiplet of N = 8 supergravity as well as N = 4 super Yang-Mills are CPT self-conjugate. However let us mention that it has been conjectured that all $N \geqslant 3$ supersymmetric theories should be finite. This conjecture implies that we cannot adjust parameters through the renormalization procedure. K is not really a parameter but only a mass scale.

<u>Then supergravity N = 8 (ungauged) has no parameter.</u>

3 <u>Implications of the local symmetry SU(8)</u>

The existence of a local symmetry whose gauge fields are composite fields as in the CP^N models (D'Adda et al. 1978 ; Witten 1979) (or SU(P + N)/U(P) x SU(N))in 2 dimensions (where they are renormalizable) leads to the following conjecture

<u>Conjecture N°2</u>

<u>At the quantum level, the local symmetry SU(8) becomes dynamical. In particular the SU(8) gauge fields can propagate (acquire a kinetic term) and have a Yang-Mills type of coupling with gauge constant g which should be computable</u> (conjecture N°1).

Since SU(8) is big enough to accommodate a grand unified group (like SU(5) for instance), this eliminates the problem found with the gauged SO(8) supergravity and leads to a completely different concept of superunification.

<u>Supergravity is fundamental. It should be viewed as a preon type of theory whose spectrum we should compute.</u>

By supersymmetry we can conjecture that not only the SU(8) gauge fields become dynamical but also other fields and that they could form together a supersymmetric multiplet.

Some conjectures on this multiplet have been made by Ellis et al. 1980 a,b. Namely they choose it as

$$\left(\tfrac{3}{2}\right)^A,\ (1)^A{}_B,\ \left(\tfrac{1}{2}\right)^A{}_{[BC]}\ \text{-----------------}\ \left(-\tfrac{5}{2}\right)^A \quad + \text{CPT conjugate}$$

Making some drastic simplifications (and far from being justified) they can imagine a scenario leading to a breaking of supersymmetry and of SU(8) at the Planck mass directly into SU(5). The massless states they keep are those of a grand unified theory SU(5), a maximal principle leading to 3 families of fermions $(\bar{5} + 10)_L$. All particles are originally in the same multiplet. Some speculations have also been made concerning the restoration of symmetries i.e. the E_7 symmetries of the level of the bound states. Since E_7 is non compact, the only unitary representation have infinite dimension.

More recently, under more conservative assumptions especially in the spin 1/2 sector, Derendinger et al.(1981) have shown that there exist some combinations of supermultiplets which have an anomaly free sector of spin 1/2 : $\left[\text{real SU(8) or SU(8) families } (8 + \overline{28} + 56)_L\right]$ or [real SU(5) or SU(5) families $(\bar{5} + 10)_L$]. The first choice (GUTSU(8)) is not compatible with the requirement that these multiplets should be composite states of N = 8 supergravity. The second choice (GUTSU(5)) implies that there should be an even number of SU(5) families and requires a large number of multiplets (~ 15).

Although these scenarios are far from being justified, this shows that provided the two conjectures are true, the physical spectrum of N = 8 supergravity is probably rich enough to accommodate present low energy phenomenology ($\lesssim 10^{15}$ GeV). To go further, we should have a better understanding of the structure of the N = 8 supergravity, especially the structure of the "multiplet" which contains the SU(8) gauge fields and the behaviour of the various members of this multiplet under the global E_7 and the local SU(8). We should also have solved at least partially the problem of auxiliary fields which should provide a "linearization" of the fermionic terms of the Lagrangian necessary to begin to study the dynamics in particular the appearance of a kinetic term and Yang-Mills coupling for the SU(8) gauge fields. We are also facing now a new problem : what is a finite theory ? In particular, it should be useful to know if there exist some properties which could replace the concept of asymptotic freedom for renormalizable theories.

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