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KUIPER'S THEOREM FOR HILBERT MODULES

Dedicated to

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in Honor of their Sixtieth Birthdays

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Let A be a C*-algebra with a countable approximate identity, let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable Hilbert space, and denote by $\mathcal{M}(\mathcal{K} \otimes A)$ the multiplier algebra of $\mathcal{K} \otimes A$. The purpose of this note is to prove that the unitary group of $\mathcal{M}(\mathcal{K} \otimes A)$ is a contractible topological space. The result is due to Mingo [7] in the case that A is unital, although the proof given here is different and simpler. The first theorem of this type is due to Kuiper [5], who showed that the unitary group of a separable, infinite dimensional Hilbert space is contractible: the relationship between this result and ours is that $\mathcal{M}(\mathcal{K} \otimes A)$ can be viewed as the algebra of adjoinable operators on the "standard" Hilbert A-module ℓ^2A (see [4]). As with Kuiper's theorem, the result on $\mathcal{M}(\mathcal{K} \otimes A)$ has implications for the representability of K-theory functors in terms of Fredholm families -- see [1] and [7].

The proof rests on one or two facts concerning projections in $\mathcal{M}(\mathcal{K} \otimes A)$. Let us call a projection P in a unital C*-algebra *proper* if both P and $1 - P$ are equivalent to the identity in the sense of Murray and von Neumann. It is useful to introduce another notion

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of equivalence: let us say that two projections P and Q in a unital C*-algebra are *strongly equivalent*, denoted $P \approx Q$, if there exists a unitary W, connected to the identity, such that $WPW^* = Q$. This is the same as requiring that P and Q be connected by a path of projections.

LEMMA 1. *Let* P *and* Q *be equivalent projections in a unital* C*-*algebra. If* $\|PQ\| < 1$ *then* P *and* Q *are strongly equivalent.*

PROOF. Suppose first that $PQ = 0$. Let V be a partial isometry such that $V^*V = P$ and $VV^* = Q$. If R denotes the projection $1 - P - Q$ then the element $W = V + V^* + R$ is a self-adjoint unitary such that $WPW^* = Q$. In the general case, write Q as a matrix

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

with respect to the decomposition of the identity $1 = P + (1 - P)$. Since Q is a projection, a and c are positive elements with norm less than or equal to one, and the relations

$$a = a^2 + bb^* \tag{1}$$

$$b = ab + bc \tag{2}$$

$$c = c^2 + b^*b \tag{3}$$

are satisfied. (These conditions are in fact necessary and sufficient for

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

to be a projection.) The second relation may be written as $(1 - a)b = bc$, and by approximating with polynomials we get that $f(1 - a)b = bf(c)$ for any continuous function $f\colon [0,1] \to \mathbb{C}$. The hypothesis $\|PQ\| < 1$ implies that $\|a\| < 1$, say $\|a\| < 1 - \epsilon$, where $\epsilon > 0$. Let f_t ($t \in [0,1]$), be a norm continuous family of continuous functions on $[0,1]$, with values in $[0,1]$ such that:

(4) $f_0(x) = x$ for all $x \in [0,1]$,

(5) $f_t(x) \leq x$ for all $t \in [0,1]$ and all near 0, and
$f_t(x) \geq x$ for all $t \in [0,1]$ and all x near 1,

(6) $f_1(x) = 1$ for all $x \in [\epsilon,1]$.

Now, let:

$$a_t = 1 - f_t(1 - a)$$

$$b_t = b\left[\frac{f_t(c) - f_t(c)^2}{c - c^2}\right]^{1/2}$$

$$c_t = f_t(c)$$

(condition (5) ensures that b_t is well defined). Then

$$\begin{bmatrix} a_t & b_t \\ b_t^* & c_t \end{bmatrix}$$

is a projection because the relations (1)-(3) are satisfied. Since $\|a\| < 1 - \epsilon$, from (6) we have that $f_1(1 - a) = 1$. In particular, $f_1(1 - a)^2 = f_1(1 - a)$, so that

$$0 = (f_1(1 - a) - f_1(1 - a)^2)b = b(f_1(c) - f_1(c)^2),$$

and thus b_1 is equal to zero. Therefore

$$Q = \begin{bmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{bmatrix} \approx \begin{bmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix},$$

and since

$$P\begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix} = 0,$$

we conclude that

$$P \approx \begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix} \approx Q. \qquad \square$$

LEMMA 2. (i) *If* P *is a proper projection in* $M(K \otimes A) \otimes C(X)$ (*where* X *is a compact space*) *and* $U \in M(K \otimes A) \otimes C(X)$ *is a unitary such that* PUP = P, *then* U *is connected to the identity by a path of unitaries.*

(ii) *If* $Q \in M(K \otimes A) \otimes C(X)$ *is a projection such that* Q *and* 1−Q *contain subprojections equivalent to the identity, then* Q *is proper.*

PROOF. (i) This follows from the fact that $K_1(\mathcal{M}(\mathcal{K} \otimes A) \otimes C(X)) = 0$ (cf. [3], Lemma 3.2). Indeed, U is path connected to the identity in $M_n(\mathcal{M}(\mathcal{K} \otimes A) \otimes C(X))$ for some n, and since U is of the form

$$\begin{pmatrix} U' & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{M}(\mathcal{K} \otimes A) \otimes C(X)$ in the matrix decomposition with respect to P, we may transfer this path to one in $\mathcal{M}(\mathcal{K} \otimes A) \otimes C(X)$ by conjugating with an appropriate isometry.

(ii) This follows from the fact that $K_0(\mathcal{M}(\mathcal{K} \otimes A) \otimes C(X)) = 0$ (cf. [3], Lemma 3.2 once more) in a similar manner. □

For the next two lemmas, we fix an infinite sequence $\{e_n\}_{n=1}^{\infty}$ of pairwise orthogonal, rank one projections in $\mathcal{K}$. It is convenient to assume that $1 - \Sigma_{n=1}^{\infty} e_n$ is an infinite dimensional projection in $\mathcal{B}(\mathcal{H})$.

LEMMA 3. *Let* $\{e_n\}_{n\in K}$ *be a subsequence of* $\{e_n\}_{n=1}^{\infty}$ *and let* $\{u_n\}_{n=1}^{\infty}$ *be an approximate unit for* A *such that* $u_{n+1}u_n = u_n$ *for all* n. *There exists a proper projection* $P \in \mathcal{M}(\mathcal{K} \otimes A)$ *such that* $P \leq \Sigma_{n\in K} e_n \otimes u_n$, *and in fact* $(\Sigma_{n\in K} e_n \otimes u_n)P = P$.

We remark that $\Sigma_{n\in K} e_n \otimes u_n$ is easily seen to converge in the strict topology of $\mathcal{M}(\mathcal{K} \otimes A)$.

PROOF. Let $d_n = (u_{n-1} - u_{n-2})^{1/2}$ (where we set $u_0 = u_{-1} = 0$). Choose partial isometries $v_{mn} \in \mathcal{K}$ such that $v_{mn}^* v_{mn} = e_m$ and the $v_{mn}v_{mn}^*$ are pairwise orthogonal projections, each equal to some e_k for which $k \in K$ and $k \geq n$, and let $V_n = \Sigma_{m=1}^{\infty} v_{mn}$. The series $\Sigma_{n=1}^{\infty} V_n \otimes d_n$ converges in the strict topology (cf. [2], Lemma 2.4) to some element V, which is an isometry because $V^*V = \Sigma\, 1 \otimes d_n^2 = 1$. Since $u_k d_n = d_n$ if $k \geq n$, it follows easily that $(\Sigma_{k\in K} e_k \otimes u_k)V = V$. Thus if $P = VV^*$ then $(\Sigma_{k\in K} e_k \otimes u_k)P = P$. That P is proper follows from part (ii) of Lemma 2, together with the fact that $1 - P$ majorizes the proper projection $1 - \Sigma_{k\in K} e_k \otimes 1$. □

If A is unital and we take u_n to be identically equal to 1 then the above lemma is of course trivial.

LEMMA 4. *Let* A *be a* C^**-algebra with a countable approximate unit* *and let* $P, Q \in M(K \otimes A) \otimes C(X)$ *be proper projections. There exist proper projections* P' *and* Q' *such that* $P \geq P'$, $Q \geq Q'$, $\|P'Q'\| < 1$, *and* $P'Q' \in K \otimes A \otimes C(X)$.

PROOF. Let $E = \Sigma_{n=1}^{\infty} e_n \otimes 1 \otimes 1$; then since any two proper projections are unitarily equivalent, there are unitaries V and W such that $P = VEV^*$ and $Q = WEW^*$. Without loss of generality we may assume that $V = 1$ and so $P = E$. Setting $g_n = e_n \otimes u_n \otimes 1$ and $h_n = Wg_nW^*$ (where u_n is as in the previous lemma) we have that $\lim_{m\to\infty} g_m h_n = 0$ for fixed n, and conjugating with W^*, also $\lim_{n\to\infty} g_m h_n = 0$ for fixed m. Choose subsequences g_{m_i} and h_{n_j} such that $\|g_{m_i} h_{n_j}\| < 2^{-i-j}$. By Lemma 3, there are proper projections

$$P' = P'' \otimes 1 \leq \left(\Sigma\, e_{m_i} \otimes u_{m_i}\right) \otimes 1 \leq P$$

and

$$Q' = W(Q'' \otimes 1)W^* \leq W\left(\left(\Sigma\, e_{n_j} \otimes u_{n_j}\right) \otimes 1\right)W^* \leq Q.$$

Since $P'Q' = P'(\Sigma_i\, g_{m_i})(\Sigma_j\, h_{n_j})Q'$, the lemma follows from the facts that $\Sigma_{ij} g_{m_i} h_{n_j} \in K \otimes A \otimes C(X)$ and $\|\Sigma_{ij} g_{m_i} h_{n_j}\| < 1$. □

COROLLARY. *Let* A *be a C*-algebra with a countable approximate identity and let* P *and* Q *be proper projections in* $M(K \otimes A) \otimes C(X)$. *There exists a unitary* W, *connected to the identity, such that* $WPW^* = Q$.

PROOF. Choose projections P' and Q' as in the preceeding lemma. By Lemma 1,

$$P \approx 1 - P \approx P' \approx Q' \approx 1 - Q \approx Q. \qquad \square$$

THEOREM *If* A *has a countable approximate identity then the unitary group of* $M(K \otimes A)$ *is contractible.*

PROOF. Since the unitary group is homotopy equivalent to an open subset of a normed linear space (namely the group of invertible elements, for example), it is homotopy equivalent to a CW-complex (see [6], Lemma 5.2). Thus it suffices to show that any map from a compact space X into the unitary group is homotopic to the identity, or in other words, that the unitary group of $M(K \otimes A) \otimes C(X)$ is connected. Given any element U of this group, and any proper projection P, by the corollary above, there is a unitary W, connected to the identity, such that $WUPU^*W^* = P$. Thus we may assume that

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

in the matrix decomposition with respect to P. Since P is equivalent to $1 - P$, we have that

$$U \sim \begin{pmatrix} U_1U_2 & 0 \\ 0 & 1 \end{pmatrix},$$

and so the theorem follows from Lemma 2. □

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