Fibrations III - Locally Trivial Maps and Bundles

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1 Locally Trivial Maps

Fix a space B. Recall that if $f : X \to B$ is a space over B and $A \subseteq B$, then we write $X_A = f^{-1}(A) \subseteq X$ and

$$f_A : X_A \to A \tag{1.1}$$

for the space over A obtained by restriction (i.e. pullback).

Definition 1 *Let B be a space and $f : X \to B$ a space over B.*

1) *We say that $f : X \to B$ is **trivial** if there is a space F and a homeomorphism $X \cong B \times F$ over B, where $B \times F$ is the space over B defined by the projection $pr_B : B \times F \to B$.*

2) *We say that $f : X \to B$ is **locally trivial** if each point $b \in B$ is contained in an open set U having the property that X_U is trivial over U.*

*In the case that f is trivial, we call a choice of fiberwise homeomorphism $X \cong B \times F$ a **trivialisation** of f. If f is locally trivial, then we call a homeomorphism $X_U \cong U \times F$ over an open $U \subseteq B$, a **local trivialisation** of f.* $\square$

If $f : X \to B$ is trivial over $U, V \subseteq B$ and $U \cap V \neq \emptyset$, then the fibres of f over any two points of $U \cup V$ are homeomorphic. In particular, if f is locally trivial, then its fibres over any two points in the same connected component of B are homeomorphic. Thus if B is connected, then we can unambiguously talk about the fibre of f. In any case, if f is locally trivial and all its fibres are homeomorphic to F, then we say that f is **locally trivial with fibre** F.

Proposition 1.1 *Let B be a space. Then the full subcategory of Top/B on the (locally) trivial maps is replete and closed under formation of products.*[1]

Proof The first statement is that any map $X \to B$ which is isomorphic in Top/B to a (locally) trivial map is itself locally trivial. The second statement is that if $X \to B$ and $Y \to B$ are (locally) trivial, then so is their product $X \times_B Y$ in Top/B. ∎

Proposition 1.2 *If $\theta : A \to B$ is any map, then the pullback functor $Top/B \xrightarrow{\theta^*} Top/A$ takes trivial maps to trivial maps and locally trivial maps to locally trivial maps.*

Proof We check that $\theta^*(B \times F) \cong A \times F$, and that if $X \to B$ is trivial over $U \subseteq B$, then $\theta^* X$ is trivial over $\theta^{-1}(U) \subseteq A$. ∎

Our interest in locally trivial maps is that they often turn out to be fibrations. In fact, in the following sections we will pin down some precise conditions under which they are, and for this we will need to introduce the idea of *numerability*. When the conditions are met it will open up these examples to study by the tools we developed over the last two lectures to study fibrations.

On the other hand, the theories of fibrations and of locally trivial maps are *not* the same. There are fibrations which are not locally trivial, and locally trivial maps which are not fibrations. Moreover, many of the theorems we proved for fibrations actually have strictly stronger analogues in the world of locally trivial maps. To begin to appreciate the differences between the two theories consider the following simple observation.

Proposition 1.3 *If $f : X \to B$ is a locally trivial map, then it is open, and in particular a quotient map.* ∎

Thus being locally trivial is a very special property of a map. We have already seen examples of fibrations which are either not quotient maps, or which have nonhomeomorphic fibres over the same connected component.

Example 1.1 Let $X \subseteq \mathbb{R}^2$ be the solid triangle with vertices $(0,0), (1,0), (1,0)$ and let $f : X \to I$ be given by projecting onto the first coordinate. Then it is clear that f is a fibration. However f is not locally trivial. All the fibres of f are homotopy equivalent since it is a fibration (cf. Co. 3.6 *Fibrations II*), but they are not all homeomorphic. The fibre of 1 is a point, but the fibre over any other $t \in I$ is an interval. □

[1]Let $\mathcal{C}$ be a category and $\mathcal{A} \subseteq \mathcal{C}$ a subcategory. Then $\mathcal{A}$ is said to be **full** if the morphisms between any two of its objects are all the morphisms between those same objects in $\mathcal{C}$. It is said that $\mathcal{A}$ is **replete** if whenever $x \in ob(\mathcal{C})$ is isomorphic to some $a \in ob(\mathcal{A})$, it follows that $x \in ob(\mathcal{A})$. It is said that $\mathcal{A}$ is closed under products if $\mathcal{C}$ has products, and if whenever $a, b \in ob(\mathcal{A})$, then $a \times b \in ob(\mathcal{A})$.

Example 1.2 In Example 2.4 of *Fibrations I* we gave the example of the fibration $Q \to \mathbb{Q}$, where Q is the rationals in the discrete topology. This map is not locally trivial, since it is not a quotient map. $\square$

Example 1.3 The Long Ray $\mathbb{L}_+$ is constructed and discussed in more detail in the accompanying notes on paracompactness. It is a Hausdorff, locally-Euclidean space modeled on the real line. In fact it satisfies all the requirements for it to be a smooth manifold other than the requirement of second-countability. In particular $\mathbb{L}_+$ is not paracompact.

A locally trivial map over $\mathbb{L}_+$ which is not a fibration was constructed by P. Tulley in her paper [11] (Example #7, pg. 107). While the example is actually not difficult to construct, its strange features rely heavily on the pathological topological properties of the long ray. $\square$

So, now that we have issued a warning to the reader to keep some distance between their ideas of fibrations and of locally trivial maps, let us proceed, as promised, to try to understand when we can conflate the two concepts. Our main results are achieved in the next section after we discuss numerable bundles. Without the assumption of numerability the statements are much less impressive, so the reader should consider the remainder of this section as a warmup for what will follow.

Lemma 1.4 *If $f : X \to B \times [0,1]$ is trivial over $B \times [0,\frac{1}{2}]$ and $B \times [\frac{1}{2},1]$, then f is trivial over $B \times I$.* $\blacksquare$

Proof Since $(B \times [0,1/2]) \cap (B \times [1/2,1]) = B \times \{1/2\} \cong B$ is nonempty, the locally-trivial map has a well-defined fibre F. Write

$$X_0 = X_{B\times[0,\frac{1}{2}]}, \qquad X_1 = X_{B\times[\frac{1}{2},1]}, \qquad X_{01} = X_0 \cap X_1 = X_{B\times\{\frac{1}{2}\}} \tag{1.2}$$

and choose trivialisations

$$\alpha_0 : F \times B \times [0,1/2] \xrightarrow{\cong} X_0, \qquad \alpha_1 : F \times B \times [1/2,1] \xrightarrow{\cong} X_1. \tag{1.3}$$

Here we break our normal conventions and write the fibres on the left. Now we have the transition function defined by these trivialisations

$$\alpha_0^{-1}\alpha_1 : F \times B \times \{1/2\} \xrightarrow{\cong} F \times B \times \{1/2\} \tag{1.4}$$

which is an isomorphism over $B \cong B\times\{1/2\}$. In this way we have a map $\widetilde{\alpha} : B \to Homeo(F)$. Let

$$\beta : F \times B \times [1/2,1] \to F \times B \times [1/2,1] \tag{1.5}$$

be the map

$$\beta(x,a,t) = (\widetilde{\alpha}(a)(x), a, t). \tag{1.6}$$

Then β is an isomorphism over $B \times [1/2,1]$ and satisfies

$$\beta|_{F\times B\times\{1/2\}} = \alpha_0\alpha_1^{-1}|_{F\times B\times\{1/2\}}. \tag{1.7}$$

Now consider the diagram

$$\begin{tikzcd}
F \times B \times \{\frac{1}{2}\} \arrow[rr] \arrow[d] & & F \times B \times [\frac{1}{2}, 1] \arrow[r, "\beta^{-1}"] \arrow[d] & F \times B \times [\frac{1}{2}, 1] \arrow[d, "\alpha_1"] \\
F \times B \times [0, \frac{1}{2}] \arrow[rr] \arrow[dr, "\alpha_0"'] \arrow[urr, phantom, "\lrcorner"] & & F \times B \times [0, 1] \arrow[dr, dashed] & X_1 \arrow[d, "\subseteq"] \\
& X_0 \arrow[rr, "\subseteq"] & & X.
\end{tikzcd} \tag{1.8}$$

The square is a pushout, and the solid part of the diagram commutes by construction. Hence the dotted arrow can be filled in. Since $X = X_0 \cup_{X_{01}} X_1$ we see easily that it is both a map over $B \times I$ and a homeomorphism. ∎

Proposition 1.5 *If $f : X \to I$ is locally trivial, then f is trivial.*

Proof By compactness we can cover I by finitely many open sets over each of which f is trivial. Next we can choose a sufficently large integer n so that $\frac{1}{n}$ is Lebesgue number for this covering. Then f is trivial over each of the closed intervals $[\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, \ldots, n-1$. We take $B = *$ and repeatedly apply Lemma 1.4 to get the statement. ∎

Proposition 1.6 *If $f : X \to I^n$ is locally trivial for some $n \geq 0$, then f is trivial.*

Proof The argument is formally the same as 1.5. We covering I^n be finitely many open sets which trivialise f and choose an integer N large enough to make $\frac{n}{N}$ a Lebesgue number for the covering. Then f is trivial over each product

$$\left[\frac{i_1}{N}, \frac{i_1+1}{N}\right] \times \cdots \times \left[\frac{i_n}{N}, \frac{i_n+1}{N}\right], \qquad i_1, \ldots, i_n = 0, \ldots, N-1 \tag{1.9}$$

and we get the statement with an iterative application of Lemma 1.4. ∎

Corollary 1.7 *Let $f : X \to B$ be a locally trivial map. Then f has the homotopy lifting property with respect to all cubes I^n and discs D^n for all $n \geq 0$.*

Proof It suffices to show the statement for I^n. Assume given a homotopy $G : I^n \times I \to B$ and a map $\varphi : I^n \to X$ satisfying $f\varphi = G_0$. Now take the pullback of f along H and consider the following diagram

$$\begin{tikzcd}
I^n \arrow[dr, dashed, "\widetilde{\varphi}"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "in_0"'] & & \\
& G^*X \arrow[r, "\widetilde{G}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\
& I^n \times I \arrow[r, "G"] & B.
\end{tikzcd} \tag{1.10}$$

Since the outer part of the diagram commutes the dotted arrow $\widetilde{\varphi}$ completes uniquely.

Now, the map p_f in the diagram is locally trivial by 1.2, and hence completely trival by Corollary 1.7. Thus we can find a homeomorphism

$$\alpha : G^*X \cong I^n \times I \times F \tag{1.11}$$

over $I^n \times I$, where F is the fibre of f over the connected component of B which meets $G(I^n \times I)$. The composition $\alpha\widetilde{\varphi}$ is then a map $I^n \to I^n \times I \times F$ of the form

$$x \mapsto (x, 0, \phi(x)) \tag{1.12}$$

where $\phi : I^n \to F$ is some map. Let $H : I^n \times I \to X$ be the map

$$H : I^n \times I \xrightarrow{(x,t)\mapsto(x,t,\phi(x))} I^n \times I \times F \xrightarrow{\alpha^{-1}} G^*X \xrightarrow{\widetilde{G}} X. \tag{1.13}$$

Then $f(H(x,t)) = G(x,t)$ and $H(x,0) = \varphi(x)$, so H is a solution to the initial lifting problem. ∎

Remark Maps which have the homotopy lifting property with respect to all cubes I^n, $n \geq 0$ are called *Serre fibrations*. It turns out that this requirement is equivalent to having the homotopy lifting property with respect to all CW complexes [9], Pr. 3.3.5, pg. 85. Serre fibrations play an imporatant rôle in many problems, but past a few comments will not be of much interest to us. □

1.1 Fibre Bundles

In this brief section we discuss extra structure that can be imposed on locally trivial maps over B. We will not make much use of these objects, so the reader wishing to skip ahead may do so.

Let $f : X \to B$ be a locally trivial map. Then it is always possible to cover B with open sets, over each of which f is trivial. An *atlas* for f is a particular choice of covering. More formally:

Definition 2 *An* ***atlas*** *for a locally trivial map $f : X \to B$ is a family $\mathcal{A} = \{(U_i, \alpha_i)\}_{i\in\mathcal{I}}$ where the $U_i \subseteq B$ are open sets which cover B, and each $\alpha_i : U_i \times F \xrightarrow{\cong}_{U_i} X_{U_i}$ is local trivialisation of f. We call the pairs (U_i, α_i)* ***charts*** *for f.* □

Suppose given two charts $(U_i, \alpha_i), (U_j, \alpha_j)$ for f such that the intersection $U_{ij} = U_i \cap U_j$ is nonempty. We let

$$\alpha_j^i = \alpha_j^{-1}|_{U_{ij}} \circ \alpha_i|_{U_{ij}} : U_{ij} \times F \xrightarrow{\cong} U_{ij} \times F. \tag{1.14}$$

This is a fibrewise automorphism of the trivial overspace $U_{ij} \times F$ given by the top line in the following diagram

$$\begin{tikzcd} U_{ij} \times F \arrow[r, "\alpha_i|", "\cong"'] \arrow[dr] & X_{U_{ij}} \arrow[r, "\alpha_j|^{-1}", "\cong"'] \arrow[d] & U_{ij} \times F \arrow[dl] \\ & U_{ij} & \end{tikzcd} \tag{1.15}$$

We call the α^i_j the **transition functions** associated to the atlas $\mathcal{A}$. Each α^i_j has the form $(b, x) \mapsto (b, \widetilde{\alpha}^i_j(b, x))$ for some map $\widetilde{\alpha}^i_j : U_{ij} \times F \to F$, and we check easily that for fixed $b \in U_{ij}$, the map $F \to F$, $x \mapsto \widetilde{\alpha}^i_j(b, x)$ is a homeomorphism. Thus the adjoint of $\widetilde{\alpha}^i_j$ is a function $U_{ij} \to Homeo(F)$, which is continuous if F is locally compact. If F is not locally compact then we may need to alter the topology on the homeomorphism group to achieve this. If each of the maps $U_{ij} \to Homeo(F)$ takes values in a given subgroup $G \subseteq Homeo(F)$, then we say that the atlas defines a G**-structure** on $f : X \to B$, and call G the **structure group** of $\mathcal{A}$.

Definition 3 *A* ***locally trivial fibre bundle*** *over B with fibre F is a pair of a locally trivial map $X \to B$ with fiber F and a particular choice of atlas $\mathcal{A}$ for it.* $\square$

Some of the most frequently examples of G-structures can be found amongst the following.

1) $F = G$ is a locally compact topological group and the inclusion $G \subseteq Homeo(G)$ is that sending $g \in G$ to the left translation map $l_g : h \mapsto gh$.

2) $F = \mathbb{K}^n$ is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{K}\}$ and $G = Gl_n(\mathbb{K}) \subseteq Homeo(\mathbb{K}^n)$.

3) $F = S^n$, and $G = Homeo_+(S^n) \subseteq Homeo(S^n)$ is the subgroup of orientation preserving self-homeomorphisms.

4) $F = M$ is a closed C^∞-manifold and $G = Diff(M) \subseteq Homeo(M)$ is the subgroup of self-diffeomorphisms of M.

2 Numerably Trivial Maps

Let B be a fixed base space.

Definition 4 *A locally trivial map $f : X \to B$ is said to be* ***numberably trivial*** *if it is trivial over each member of a numerable covering of B.* $\square$

If $f : X \to B$ is locally trivial, then we say that an atlas $\mathcal{A} = \{(U_i, \alpha_i)\}_{\mathcal{I}}$ for f is **numerable** if $\{U_i\}_{\mathcal{I}}$ is numerable covering of B. Thus f is numerably trivial if and only if it admits a numerable atlas. We call a fibre bundle $(f, \mathcal{A})$ **numerable** if its atlas $\mathcal{A}$ is numerable. According to Milnor's Theorem (*Partitions of Unity* Theorem 4.1, see 3.1 below) any bundle admits a countable atlas.

Proposition 2.1 *If $f : X \to B$ is numerably trivial, then it is trivial over the members of a countable numerable covering $\mathcal{U} = \{U_n \subseteq B\}_{n \geq 1}$.* ∎

Without some assumptions on B not every locally trivial map will be numerably trivial. Example 1.3 discusses one such map. However, from the characterising property of paracompactness we have the following.

Lemma 2.2 *Any locally trivial map $f : X \to B$ over a paracompact space B is numerably trivial. All bundles over a paracompact space B are numerable.* ∎

Recall that all CW complexes and manifolds are paracompact. More generally all metric spaces are paracompact. Thus in each of these cases the theory of locally trivial maps outlined above is equivalent to the better behaved theory of numerably trivial maps.

We feel that we should stress, however, that there are numerably trivial maps over any base space. For instance any trivial map is numerably trivial. Thus it is not useful to push paracompactness to the forefront. For while paracompactness is not homotopy invariant, there is a homotopy invariant way to transfer numerable structures between spaces. This is explained more fully in the next section, and is a consequence of the fact, proven now, that pullback respects numerable triviability.

Lemma 2.3 *If $\mathcal{U} = \{U_i\}_{\mathcal{I}}$ is a numerable covering of B and $\theta : A \to B$ is any map, then $\theta^*\mathcal{U} = \{\theta^{-1}(U_i)\}_{\mathcal{I}}$ is a numerable covering of A.*

Proof The family $\theta^*\mathcal{U}$ is certainly an open covering of A. If $\{\xi_i : B \to I\}_{\mathcal{I}}$ is a numeration of $\mathcal{U}$, then we claim the that the family

$$\theta^*\xi_i = \xi_i\theta : A \to I, \qquad i \in \mathcal{I} \tag{2.1}$$

is a numeration of $\mathcal{U}$. Indeed, for $a \in A$ we have

$$\sum_{\mathcal{I}}(\theta^*\xi_i)(a) = \sum_{\mathcal{I}}\xi_i(\theta_i(a)) = 1 \tag{2.2}$$

so the $\theta^*\xi_i$ form a partition of unity on A. To see that the family $\{\theta^*\xi_i\}_{\mathcal{I}}$ is locally-finite observe that

$$Supp(\theta^*\xi_i) = \overline{\theta^{-1}\xi_i^{-1}(0,1]} \subseteq \theta^{-1}\big(\overline{\xi^{-1}(0,1]}\big) = \theta^{-1}(Supp(\xi_i)). \tag{2.3}$$

Then if $a \in A$ is any point, choose an open neighbourhood $W \subseteq B$ of $\theta(a)$ such that W intersects only finintely many of the sets $Supp(\xi_i)$. Then $a \in \theta^{-1}(W)$ and

$$\theta^{-1}(W) \cap Supp(\theta^*\xi_i) \subseteq \theta^{-1}(W) \cap \theta^{-1}(Supp(\xi_i)) = \theta^{-1}(W \cap Supp(\xi_i)) \tag{2.4}$$

is nonempty for only finitely many indices $i \in \mathcal{I}$. ∎

Given an overspace $f : X \to B$ and a map $\theta : A \to B$ we can form the pullback θ^*X as in the square

$$\begin{tikzcd} \theta^*X \arrow[r] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ A \arrow[r, "\theta"] & B. \end{tikzcd} \tag{2.5}$$

We know from Proposition 1.2 that if f is trivial over $U \subseteq B$, then p_f is trivial over $\theta^{-1}(U) \subseteq A$. Thus from the lemma we have the following.

Corollary 2.4 *If $\theta : A \to B$ is any map, then the pullback functor $A/Top \xrightarrow{\theta^*} B/Top$ sends numerably trivial maps to numerably trivial maps.* ∎

Recall that the product in Top/B of two maps $f : X \to B$ and $g : Y \to B$ is the fibred product

$$X \times_B Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \tag{2.6}$$

equipped with the obvious map $f \times_B g$ to B. According to Proposition 1.2, if both f, g are locally trivial, then so is $f \times_B g$. More is true, however.

Lemma 2.5 *Let $\{\pi_i : B \to I\}_{\mathcal{I}}$ and $\{\rho_j : B \to I\}_{\mathcal{J}}$ be locally-finite partitions of unity on a space B. Then the family*

$$\{\pi_i \cdot \rho_j \mid i \in \mathcal{I},\ j \in \mathcal{J}\} \tag{2.7}$$

is a locally-finite partition of unity on B.

Proof Fix a point $x \in X$. Due to the assumptions of local-finiteness there exist neighbourhoods $U, V \subseteq X$ of x and finite subsets $E \subseteq \mathcal{I}$, $F \subseteq \mathcal{J}$ such that

1) $\pi_k(y) = 0$ if $y \in U$ and $k \notin E$

2) $\rho_l(y) = 0$ if $y \in U$ and $l \notin F$.

Then the existence of E, F implies that the sum converges to unity as required

$$\begin{aligned}
\sum_{\mathcal{I}\times\mathcal{J}} \pi_i(x) \cdot \rho_j(x) &= \sum_{(i,j)\in E\times F} \pi_i(x) \cdot \rho_j(x) \\
&= \left(\sum_{i\in E} \pi_i(x)\right)\left(\sum_{j\in F} \rho_j(x)\right) \\
&= 1 \cdot 1 \\
&= 1.
\end{aligned} \tag{2.8}$$

Moreover it is clear that $U \cap V$ is neighbourhood of x on which only finitely many of the $\pi_i \cdot \rho_j$ are non-zero. Thus we get the claim. ∎

Proposition 2.6 *The product in Top/B of two numerably trivial maps is numerably trivial.*

Proof Assume that $f : X \to B$ is trivial over the members of a numerable open cover $\mathcal{U} = \{U_i \subseteq B\}_{\mathcal{I}}$ and that $g : Y \to B$ is trivial over the members of the numerable open cover $\mathcal{V} = \{V_j \subseteq B\}_{\mathcal{J}}$. Set

$$\mathcal{W} = \{W_{ij} = U_i \cap V_j \mid i \in \mathcal{I},\ j \in \mathcal{J}\}. \tag{2.9}$$

Then both f and g are trivial over the members of $\mathcal{W}$. Let $\{\pi_i\}_{\mathcal{I}}$ be a numeration of $\mathcal{U}$ and $\{\rho_j\}_{\mathcal{J}}$ a numeration of $\mathcal{V}$ and put

$$\sigma_{ij} = \pi_i \cdot \rho_j, \qquad i \in \mathcal{I},\ j \in \mathcal{J}. \tag{2.10}$$

Then by Lemma 2.5 the family $\{\sigma_{ij} \mid i \in \mathcal{I},\ j \in \mathcal{J}\}$ is a locally-finite partition of unity on B. For each $i \in \mathcal{I}$ and $j \in \mathcal{J}$ we have

$$(\pi_i \cdot \rho_j)^{-1}(0, 1] = \left(\pi_i^{-1}(0, 1]\right) \cap \left(\rho_j^{-1}(0, 1]\right) \tag{2.11}$$

and since these cozero sets form a locally-finite family we can take closures (cf. *Partitions of Unity* Lemma 1.1) to get

$$Supp(\pi_i \cdot \rho_j) = Supp(\pi_i) \cap Supp(\rho_j) \subseteq U_i \cap V_j = W_{ij}. \tag{2.12}$$

In particular $\{\sigma_{ij}\}_{\mathcal{I}\times\mathcal{J}}$ is a numeration of $\mathcal{W}$. ∎

3 The Homotopy Theorem for Locally Trivial Maps

In this section we will prove the Homotopy Theorem for Locally Trivial maps. The reader should compare this to Theorem 3.5 in *Fibrations II* which is the the corresponding result for fibrations.

The following important results were proved in §4 of the lecture on partitions of unity. You should have worked through their proof there

Theorem 3.1 (Milnor) *Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be a numerable open covering of a space B. Then there exists a countable, numerable open covering $\mathcal{V} = \{V_n\}_{n\geq 1}$ of B such that each V_n is a disjoint union of open sets, each of which is contained in some U_i.* ∎

Proposition 3.2 (Stacking Lemma) *Let B be a space and $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ a numerable open covering of $B \times I$. Then there exists a numerable open covering $\{V_j\}_{j\in\mathcal{J}}$ of B, and a family of non-negative real numbers $\{\epsilon_j \in (0,\infty)\}_{j\in\mathcal{J}}$ with the property that for each $j \in \mathcal{J}$ and all $s < t \in I$ with $t - s < \epsilon_j$, there exists $i \in \mathcal{I}$ such that $V_j \times [s,t] \subseteq U_i$.* ∎

Corollary 3.3 *Let $f : X \to B$ be a numerably trivial map. Then f admits a countable atlas. In particular, any locally trivial map over a paracompact space admits a countable atlas* ∎

Proposition 3.4 *Let B be a space and $f : X \to B \times I$ a numerably trivial map. Then there exists a countable numerable covering $\{U_n\}_{n\in\mathbb{N}}$ of B such that f is trivial over $U_n \times I$ for each $n \in \mathbb{N}$.*

Proof Choose a numerable trivialising covering $\{V_j \subseteq B \times I\}_{\mathcal{J}}$ for f. Then Proposition 3.2 says that there is a numerable covering $\{W_i \subseteq B\}_{\mathcal{I}}$ of B and a family $\{\epsilon_i \in (0,\infty)\}_{\mathcal{I}}$ such that f is trivial over $W_i \times [s,t]$ whenever $t - s < \epsilon_i$. But now Corollary 1.7 implies that f is trivial over $W_i \times I$ for each $i \in \mathcal{I}$. To go from here to the countable numerable covering $\{U_n\}_{\mathbb{N}}$ is now a simple application of Milnor's Theorem 3.1. ∎

Proposition 3.5 *Let B be a space and $f : X \to B \times I$ a numerably trivial map. Define a map $\theta : B \times I \to B \times I$ by $\theta(b,t) = (b,1)$. Then there exists a map $\Theta : X \to X$ which makes the diagram*

$$\begin{tikzcd} X \arrow[r, "\Theta"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ B \times I \arrow[r, "\theta"] & B \times I \end{tikzcd} \tag{3.1}$$

a pullback and which restricts to the identity on $X|_{B\times 1}$.

Proof It suffices to consider the case that B is connected. For if we can prove the statement when f is restricted to any connected component of $B \times I$, then the family of maps so produced will sum together to define Θ. Thus in the following we assume that B is connected and write F for the fibre of f.

To begin use Proposition 3.4 to find a countable numerable covering $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ of B such that f trivial over each $U_n \times I$. Also, for each $n \in \mathbb{N}$, fix once and for all a trivialisation

$$\alpha_n : U_n \times I \times F \cong B|_{U_n\times I}. \tag{3.2}$$

Now let $\{\xi_n : B \to I\}_{n\in\mathbb{N}}$ be a numeration of $\mathcal{U}$ and write $\xi = \sup_{\mathbb{N}} \xi_n$. It was shown in *Partitions of Unity* that ξ is continuous and strictly positive, so for each $n \in \mathbb{N}$

$$\mu_n = \frac{\xi_n}{\xi} : B \to I \tag{3.3}$$

is a continuous function whose support lies inside U_n. Notice that $\sup_{\mathbb{N}} \mu_n = 1$ throughout B.

Now for each $n \in \mathbb{N}$ define a map $\theta_n : B \times I \to B \times I$ by setting

$$\theta_n(b,t) = (b, \max\{t, \mu_n(b)\})\,. \tag{3.4}$$

Then θ_n is the identity outside of $U_n \times I$ and on all of $B \times \{1\}$. Covering θ_n there is a map $\Theta_n : X \to X$ which is the identity outside of $f^{-1}(U_n \times I)$, and is defined here by

$$\Theta_n \alpha_n(b,t,e) = \alpha_n(\theta_n(b,t), e), \qquad (b,t,e) \in U_n \times I \times F. \tag{3.5}$$

Then Θ_n is the identity on all of $X|_{B\times\{1\}}$.

The maps in the last paragraph are constructed to make the following square commute

$$\begin{tikzcd} X \arrow[r, "\Theta_n"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ B \times I \arrow[r, "\theta_n"] & B \times I. \end{tikzcd} \tag{3.6}$$

In fact it is not difficult to see that this square is a pullback. There is an induced map $X \to \theta_n^* X$ into the canonical pullback, and a map in the opposite direction is constructed with the help of the trivialisation α_n.

Now form the composition

$$\theta_{(n)} : B \times I \xrightarrow{\theta_n} B \times I \xrightarrow{\theta_{n-1}} \dots \xrightarrow{\theta_2} B \times I \xrightarrow{\theta_1} B \times I. \tag{3.7}$$

Since the covering $\mathcal{U}$ is locally-finite, for each point $(b,t) \in B \times I$ there is an $N \in \mathbb{N}$ such that $\theta_{(N+k)}(b,t) = \theta_{(N)}(b,t)$ whenever $k \geq 0$. This implies that in the limit as $n \to \infty$ the $\theta_{(n)}$ give a well-defined map $B \times I \to B \times I$, which is continuous since it agrees locally with some θ_N. In fact, since $\sup_{\mathbb{N}} \mu_n = 1$ we see that

$$\lim_{n\to\infty} \theta_{(n)} = \theta \tag{3.8}$$

where θ is as in the theorem statement.

Now form the same constructions with the Θ_n. Define $\Theta_{(n)}$ to be the composition

$$\Theta_{(n)} : X \xrightarrow{\Theta_n} X \xrightarrow{\Theta_{n-1}} \dots \xrightarrow{\Theta_{n-2}} X \xrightarrow{\Theta_1} X. \tag{3.9}$$

and set

$$\Theta = \lim_{n\to\infty} \Theta_{(n)}. \tag{3.10}$$

Then Θ is well-defined and continuous for the same reason that θ is. Moreover Θ makes the square (3.1) commute. Since all the squares (3.6) are pullbacks, we conclude from the definition of Θ that (3.1) is a pullback. The last thing to check is that Θ restricts to the identity over $B \times \{1\}$, and this is true because so does each Θ_n. ∎

Given a numerably trivial map $f : X \to B \times I$ write $X_t = X_{X\times\{t\}}$. We think of each X_t as a space over B and of X as a family of spaces over B parametrised by $t \in I$. If θ is as in 3.5 and we let θ^*X be the canonical pullback in the square

$$\begin{tikzcd} \theta^*X \arrow[r] \arrow[d, "p_f"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ B \times I \arrow[r, "\theta"] & B \times I \end{tikzcd} \tag{3.11}$$

then we check that $\theta^*X \cong X_1 \times I$. Thus comparing (3.1) and (3.11) we get a fibrewise isomorphism

$$X \cong X_1 \times I \tag{3.12}$$

over $B \times I$. On the other hand, if we reverse the orientation on the interval in 3.5, then the same argument gives a second fibrewise homeomorphism

$$X \cong X_0 \times I. \tag{3.13}$$

These arguments prove the following. Now $X_0 \times I \cong X \cong X_1 \times I$ over $B \times I$, so $X_0 \cong X_1$ over B. The following proposition summarises these arguments.

Proposition 3.6 *Let $f : X \to B \times I$ be a numerably trivial map. Write $X_t = X_{B\times\{t\}}$, considered as a space over B. Then there are fibrewise homeomorphisms*

$$X_0 \cong X_1 \tag{3.14}$$

over B. ∎

With the hard work out the way, we can finally present the first important result of this section. We call the theorem the *homotopy theorem* for numerably trivial maps.

Theorem 3.7 *Let $f : X \to B$ be a numerably trivial map over a space B. Suppose given a space A and homotopic maps $\alpha \simeq \beta : A \to B$. Then the two pullbacks $\alpha^*X \to A$ and $\beta^*X \to A$ are homeomorphic over A.*

Proof Choose a homotopy $G : \alpha \simeq \beta$ and form the pullback $p_G : G^*X \to A \times I$. Then by Corollary 2.4, p_G is numerably trivial so we can apply 3.6. This gives homeomorphisms $(G^*X)_0 \cong (G^*X)_1$ over A. But

$$(G^*X)_0 = G_0^*X = \alpha^*X \tag{3.15}$$

and similarly $(G_1^*X) \cong \beta^*X$. Hence

$$\alpha^*X \cong \beta^*X \tag{3.16}$$

over A. ∎

The first obvious application of the homotopy theorem is the following observation.

Corollary 3.8 *A numerably trivial map over a contractible space is trivial. In particular, any locally trivial map over a contractible paracompact space is trivial.* ∎

Thus we generalise Proposition 1.6 to contractible non-compact base spaces like $\mathbb{R}^n$. Comparing these results gives some insight into what the assumption of numerability does for us. The argument for 1.6 relied heavily upon the compactness of I^n to keep inductive gluing arguments to a finite number of stages. The use of partitions of unity provide more powerful tools that greatly streamline the gluing arguments.

With a little bit of thought we can generalise the last corollary.

Corollary 3.9 *Let B be a space with cat$(B) \leq n$. If $f : X \to B$ is numerably trivial, then B can be covered by $\leq n+1$ open sets, over each of which f is trivial.*

Proof If U is an open subset of B and the inclusion $i : U \hookrightarrow B$ is null homotopic, then 3.7 implies that $X_U = i^*X$ is trivial over U. The conclusion follows immediately. ∎

Example 3.1 Any numerably trivial map over a suspension $B = \Sigma A$ is trivialisable over the two cones $C_\pm A \subseteq B$. Thus an atlas consisting of exactly two sets can be found for such a map. A particular case is that of $B = S^n$. We discuss in detail in the next section the process of clutching, which recovers the isomorphism type of a locally trivial map $X \to \Sigma A$ from a particular choice of atlas. □

Example 3.2 Any locally trivial map over $\mathbb{R}P^n$ of $\mathbb{C}P^n$ can be trivialised over $n+1$ open sets. Clearly the standard atlas of smooth charts will suffice. □

The reader should compare the Homotopy Theorem 3.7 with the corresponding result for fibrations which was proved as Theorem 3.5 in *Fibrations II*. The extra structure provided by the assumption of local triviality leads to improved statements. The comparison beteween locally trivial maps and fibrations is now completed by the following result.

Theorem 3.10 *A numerably trivial map $f : X \to B$ is a fibration.*

Proof Assume given a space A and a lifting problem

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, hook, "in_0"'] & X \arrow[d, "f"] \\ A \times I \arrow[r, "G"'] \arrow[ur, dashed] & B. \end{tikzcd} \tag{3.17}$$

We need to show that the dotted arrow can be completed. The construction is essentially that of Corollary 1.7. Thus we start with the pullback $G^*X \to A \times I$, which as a numerably trivial map over $A \times I$ admits a fibrewise homeomorphism

$$G^*X \cong (G^*X)_0 \times I \cong G_0^*X \times I. \tag{3.18}$$

The map $\varphi : A \to X$ determines a section $\widetilde{\varphi}$ of $G_0^*X \to A$. We let $\widetilde{G}$ be the composite

$$\widetilde{G} : A \times I \xrightarrow{\widetilde{\varphi}\times 1} G_0^*X \times I \xrightarrow{\cong} G^*X \to X \tag{3.19}$$

where the last arrow is the canonical map. Then we check directly that $\widetilde{G}$ solves the homotopy lifting problem (3.17). ∎

Corollary 3.11 *Every locally trivial map with paracompact base is a fibration.* ∎

Notice that the numerability assumption on f was needed in the proof of (3.10) only to ensure that the pullback map $G^*X \to A \times I$ was numerable. Hence a generalisation of Corollary 1.7 is available.

Corollary 3.12 *A locally trivial map $f : X \to B$ has the homotopy lifting property with respect to all paracompact spaces.*

Proof If A is paracompact, then so is $A \times I$, since I is locally compact Hausdorff. Since all locally trivial maps over paracompact base are numerable, the proof of 3.10 now goes through as written. ∎

Remark Regular fibrations are defined and discussed in the exercise section of *Fibrations I*. These are fibrations with the extra property that constant homotopies can be lifting to constant homotopies. The reader can check that during the proof of 3.10, the generic solution (3.19) obtained has the required properties.

Proposition 3.13 *A numerably trivial map $f : X \to B$ is a regular fibration.* ∎

4 Examples

4.1 Covering Spaces

Definition 5 *A locally trivial map $f : X \to B$ is said to be a **covering projection** if all its fibres are discrete. The total space X of a covering projection $f : X \to B$ is said to be a **covering space** of B.* □

Covering projections are local homeomorphisms (compare 1.3) and stable under pullback. The results of the last section show that a covering projection is a fibration if it is numerable and in particular if its base is paracompact. In fact a stronger result is obtainable using direct methods [10], Th. 3, pg. 67.

Proposition 4.1 *Every covering projection is a fibration.* ∎

Clearly covering projections are regular fibrations. In fact they have the stronger unique lifting property. That is, if $f : X \to B$ is a covering projection and $\alpha, \beta : A \to X$ are maps which *i*) agree on at least one point in each component of A, and *ii*) satisfy $f\alpha = f\beta$, then $\alpha = \beta$. This is essentially true because the fibrewise product $X \times_B X$ is again a covering space of B. Not all maps with the unique lifting property are covering maps. The example in 1.2 is one such map.

The covering spaces of B are best behaved when B is connected, locally path-connected and semi-locally simply connected. Under these conditions B has a so-called *universal cover* $f : X \to B$, characterised by the fact that the total space X is simply connected. Note that all CW complexes and manifolds are are locally path-connected and semi-locally simply connected.

Example 4.1 The exponential $p = \exp(2\pi(-)) : \mathbb{R} \to S^1$ is a covering with fibre $\mathbb{Z}$. Note that it is the universal covering of S^1. Here we view $S^1 \subseteq \mathbb{C}$ and trivialise p over the two subsets

$$U_+ = S^1 \setminus \{1\}, \qquad U_- = S^1 \setminus \{-1\}. \tag{4.1}$$

The map p takes each interval $(n, n+1)$, $n \in \mathbb{Z}$, bijectively onto U_+, and the restriction $\mathbb{R}_{U_+} = \bigsqcup_{n\in\mathbb{Z}}(n, n+1)$. Let $\log_n : U_+ \to (n, n+1)$ be the inverse to $p|_{(n,n+1)}$. Then the map

$$\varphi_+ : U_+ \times \mathbb{Z} \to \mathbb{R}_{U_+}, \qquad (z, n) \mapsto \log_n(z) \tag{4.2}$$

is the required trivialisation.

Similarly, p maps each interval $(\frac{n}{2}, \frac{n+1}{2})$, $n \in \mathbb{N}$, bijectively onto U_-, and $\mathbb{R}_{U_-} = \bigsqcup_{n\in\mathbb{Z}}(\frac{n}{2}, \frac{n+1}{2})$ Let $\log'_n : U_n \to (\frac{n}{2}, \frac{n+1}{2})$ be the inverse to $p|_{(\frac{n}{2}, \frac{n+1}{2})}$ and trivialise p over U_- by setting

$$\theta_- : U_- \times \mathbb{Z} \to \mathbb{R}_{U_-}, \qquad (z, n) \mapsto \log'_n(z). \tag{4.3}$$

□

Example 4.2 For each integer $n \geq 1$, the map $p_n : S^1 \to S^1$, $z \mapsto z^n$, is a covering projection. The fibre of p_n is $\mathbb{Z}_n$. Let $p_0 = p$ be the map from example 4.1. Then it is known that the maps p_n, for $n \geq 0$ are, up to fibrewise homeomorphism, all the covering projections onto S^1 with connected total spaces. □

Example 4.3 If $f : X \to A$ and $g : Y \to B$ are locally trivial (numerably trivial), then so is $f \times g : X \times Y \to A \times B$. Clearly, if f, g are covering spaces, then so is $f \times g$. This is a special case of the fact that the cartesian product of two fibrations is a fibration.

This idea gives a basic way to obtain new covering spaces from given ones. In favourable circumstances it is possible to obtain all the interesting covering spaces in this manner. For example the connected coverings of the torus are, up to fibrewise isomorphism, the maps

$$p_0 \times p_0 : \mathbb{R}^2 \to S^1 \times S^1 \tag{4.4}$$

and for $m, n \geq 1$, the maps

$$p_0 \times p_n : \mathbb{R} \times S^1 \to S^1 \times S^1, \qquad p_m \times p_0 : S^1 \times \mathbb{R} \to S^1 \times S^1 \tag{4.5}$$

and for $m, n \geq 1$ the maps

$$p_m \times p_n : S^1 \times S^1 \to S^1 \times S^1. \tag{4.6}$$

Note that $p_0 \times p_0$ is the universal covering of the torus. □

Example 4.4 Let $f : X \to B$ be covering projection onto a CW complex B. For simplicity we will assume that X, and hence B, are connected. Then X has a canonical CW structure for which f is a cellular map. In detail write $B_n \subseteq B$ for the n-skeleton and set

$$X_n = X_{B_n} = f^{-1}(B_n), \qquad n \geq -1. \tag{4.7}$$

Then $X_n \subseteq X_{n+1}$ for each n and $X = \bigcup_{n\geq -1} X_n$. Here we understand $B_{-1} = \emptyset$ so that also $X_{-1} = \emptyset$.

Now B_0 is a discrete set of points and $X_0 = f^{-1}(0) \cong B_0 \times F$, where F is the discrete fibre of f. Thus X_0 is obtained from X_{-1} by attaching 0-cells. Now assume that for some $n \geq 0$ we have shown that X_n is obtained from X_{n-1} by attaching n-cells and consider the following two diagrams

$$\begin{tikzcd} \bigsqcup_{\mathcal{E}} S^n \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\lrcorner"] & \bigsqcup_{\mathcal{E}} D^{n+1} \arrow[d, "\Phi"] \\ B_n \arrow[r] & B_{n+1} \end{tikzcd} \qquad \begin{tikzcd} \bigsqcup_{\widetilde{\mathcal{E}}} S^n \arrow[r] \arrow[d] & \bigsqcup_{\widetilde{\mathcal{E}}} D^{n+1} \arrow[d] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd} \tag{4.8}$$

The left-hand square here is to be the presentation of B_{n+1} as an adjunction space formed by attaching $(n+1)$-cells to B_n. The set $\mathcal{E}$ indexes the $(n+1)$-cells of X. The right-hand square is formed by pulling back $f : X \to B$ to the spaces in the left-hand square, as we now explain more fully.

Since D^{n+1} is contractible, on the top right of (4.8) we have

$$\Phi^* X \cong \left(\bigsqcup_{\mathcal{E}} D^{n+1} \right) \times F \cong \bigsqcup_{\mathcal{E}\times F} D^n \tag{4.9}$$

since F is discrete. Thus we set $\widetilde{\mathcal{E}} = \mathcal{E} \times F$ and get the square on the right-hand side of 4.8. To see that this square is a pushout observe that the map from the canonical pushout space to X_{n+1} is a bijection over B_n and we can construct an inverse by gluing together maps defined over local trivialisations. The conclusion is that X_{n+1} is obtained X_n by attaching $(n+1)$-cells.

It remains to show that X has the week topology with respect to the cells constructed in the last paragraph. But this follows easily from the fact that f is a quotient map (cf. Pr. 1.3) and the fact that the inverse image of the closed cells of B are disjoint unions of closed cells in X.

In a bit more detail we summarise the above discussion with the following.

Proposition 4.2 *Let B be a connected CW complex. Assume that $f : X \to B$ is a connected covering space with typical fibre F. Then X admits a CW structure with the following properties.*

1) *For each $n \geq 0$, the projection f maps each open n-cell of X onto an open n-cell of B.*

2) *For each $n \geq 0$, each covering transformation $\alpha : X \to X$ of f permutes the set of n-cells of X.*

3) *If $\mathcal{E}_n$ denotes the set of n-cells of B, and $\widetilde{\mathcal{E}}_n$ the set of n-cells of X, then there is a bijection $\widetilde{\mathcal{E}}_n \cong \mathcal{E}_n \times F$.* ∎

Since the proposition neatly describes the cells of the covering X it has a direct and intuitive consequence.

Corollary 4.3 *Let $f : X \to B$ be a connected covering of a connected CW complex B. Assume that X carries the CW structure granted by the Proposition 4.2. If B is finite and X is k-sheeted, then X is finite and the Euler characteristics have the following relation*

$$\chi(X) = k \cdot \chi(B). \quad \blacksquare \tag{4.10}$$

4.2 Clutching

Example 4.5 We take the base space S^n and show how to construct locally trivial maps over S^n with a given fibre F. Let $D^n_+, D^n_- \subseteq S^n$ be the upper and lower hemispheres. Technically it is necessary to work with open sets whose intersection has the form $S^{n-1} \times (-\epsilon, \epsilon)$, but for ease below we will work with closed discs with $D^n_+ \cap D^n_- = S^{n-1}$.

The input data in the construction will be a map

$$\widetilde{\alpha} : S^{n-1} \to Homeo(F) \tag{4.11}$$

which we call a **clutching function**. For simplicity we will assume that F is locally compact and that $Homeo(F)$ is equipped with the compact-open topology. More generally we could equip $Homeo(F)$ with any cosplitting topology. Now form the pushout

$$\begin{tikzcd} S^{n-1} \times F \arrow[r, hook] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner"] & D^n_+ \times F \arrow[d] \\ D^n_- \times F \arrow[r] & (D^n_+ \times F) \cup_\alpha (D^n_- \times F) \end{tikzcd} \tag{4.12}$$

where the top map is the obvious inclusion and α is the map

$$\alpha(z, e) = (z, \widetilde{\alpha}(z)(e)), \qquad (z, e) \in S^{n-1} \times F. \tag{4.13}$$

Then the projection onto the first factor turns

$$X_\alpha = (D^n_+ \times F) \cup_\alpha (D^n_- \times F) \tag{4.14}$$

into a locally trivial space over S^n. We say that X_α is formed by **clutching**. We show below that any locally trivial map over S^n is of this form for some choice of $\widetilde{\alpha}$. Before this we will be interested in the classification of the X_α. We show that up to fibrewise homeomorphism X_α depends on α only through its homotopy class.

To see this we replace the clutching function with a homotopy $\widetilde{\alpha}_t : S^{n-1} \times I \to Homeo(F)$ and run the same construction as before. In detail we form the pushout

$$\begin{tikzcd} S^{n-1} \times I \times F \arrow[r, hook] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\lrcorner"] & D^n_+ \times I \times F \arrow[d] \\ D^n_- \times I \times F \arrow[r] & (D^n_+ \times I \times F) \cup_{\alpha'} (D^n_- \times I \times F) \end{tikzcd} \tag{4.15}$$

where now

$$\alpha'(z, t, e) = (z, t, \widetilde{\alpha}_t(z)(e)). \tag{4.16}$$

The pushout space here is a locally trivial space over $S^n \times I$ and we apply Proposition 3.6 to get that

$$X_{\alpha_0} \cong X_{\alpha_1} \tag{4.17}$$

over S^n.

To see the other claim assume given a locally trivial map $f : X \to S^n$ whose fibre is F and choose trivialisations

$$\alpha_+ : D^n_+ \times F \cong X_{D^n_+}, \qquad \alpha_- : D^n_- \times F \cong X_-. \tag{4.18}$$

Both these maps induce trivialisations of $X_{S^{n-1}}$ and the composite $\alpha_-|^{-1}\alpha_+|_{D^n_+ \times F}$ defines a map

$$\widetilde{\alpha} : S^{n-1} \to Homeo(F). \tag{4.19}$$

We form the space X_α as in (4.12) and use the following pushout diagram to induce a fibrewise homeomorphism between X and X_α

$$\begin{tikzcd} S^{n-1} \times F \arrow[r, hook] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner"] & D^n_+ \times F \arrow[d] \arrow[ddr, bend left, "\alpha_+"] & \\ D^n_- \times F \arrow[r] \arrow[drr, bend right, "\alpha_-"'] & X_\alpha \arrow[dr, dashed] & \\ & & X \end{tikzcd} \tag{4.20}$$

□

Example 4.6 More generally than Example 4.5 we could consider locally trivial maps over the (unreduced) suspension $\widetilde{\Sigma}A$ of a space A. To guarantee numerability we should place some assumption on A. Paracompactness will do, since if A is paracompact, then so is $\widetilde{\Sigma}A$.

Now in (4.12) we replace the northern and southern hemispheres with the upper and lower (open) cones $\widetilde{C}_\pm \subseteq \widetilde{\Sigma}A$ whose intersection is A. Then given a suitable space F to act as fibre, the clutching function takes the form

$$\widetilde{\alpha} : A \to Homeo(F). \tag{4.21}$$

The remainder of the construction is now formally identical to when $A = S^n$. What results is a locally trivial map

$$X_\alpha \to \widetilde{\Sigma}A \tag{4.22}$$

with fibre F. □

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