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GROUP THEORY FOR FEYNMAN DIAGRAMS IN NON-ABELIAN GAUGE THEORIES*

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ABSTRACT

A simple and systematic method for the calculation of group-theoretic weights associated with Feynman diagrams in non-Abelian gauge theories is presented.

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I. INTRODUCTION

The increased interest in non-Abelian gauge theories has in recent years led to computation of many higher order Feynman diagrams.[1-5] Asymptotic form factor calculations[1-2] are of especial interest, because they suggest that it might be possible to sum up diagrams with arbitrary numbers of soft gluons just as one can sum up soft photon processes in QED. In such a program the analysis of the momentum integrals proceeds by the traditional techniques developed for QED calculations. The new aspect, characteristic of non-Abelian gauge theories, is emergence of a group-theoretic weight (or weight,[6] for short) associated with each Feynman diagram. The dramatic cancellations between various diagrams[2] occur through interplay of their group-theoretic weights and their momentum space integrals. So the study of weights becomes of interest, as it might suggest cancellation patterms needed for summations of soft gluon diagrams.

In this paper we give a general method for computing group-theoretic weights, and give explicit rules for SU(n), SO(n), Sp(n) and G_2 gauge symmetry groups. We restrict ourselves to the models with quarks in fundamental representation and gluons in the adjoint (regular) representation, but the method can be extended to higher representations.

Our evaluation procedure is very simple. First we interpret the weight itself as a Feynman integral over a discrete lattice, and introduce Feynman diagrammatic notation to replace the unwieldly algebraic expressions replete with dummy indices. Then we give two graphical relations ("integration rules"); the first eliminates all three-gluon vertices, and the second eliminates all internal gluon lines. The result is a sum over a unique set of irreducible group-theoretic tensors which form a natural basis for all Lie algebras. All this is accomplished without recourse to any explicit representation of the group generators and structure constants. As a byproduct we learn how to count quickly the number of invariant couplings for arbitrary numbers of quarks and gluons, thus avoiding involved reductions of outer products of representations by Young-tableaux.

The above approach is at variance with the customary procedure of expressing weights in terms of Casimir operators.[2] While it is appealing to express simple diagrams in terms of quadratic Casimir operators (so that the form of the expression is independent of the particular gauge group and the particular representation), for higher order diagrams there is no simple way of relating weights to generalized Casimir operators,[7,8] and such an approach becomes very cumbersome.

In the past the most weight calculations have involved SU(n) and even more specifically SU(3). This has led to development of methods specific to SU(n).[9-16] For the sake of completeness and comparison, we pursue this traditional line for a while and find ourselves in a cul-de-sac.

The organization of the paper is as follows. In Section II we state the Feynman rules and introduce diagrammatic notation. In Section III we derive various relationships true for all Lie groups, while particular groups are defined in Section IV. The completeness relationships, which are the crux of our method, are derived in Section V. The ease of weight evaluation is demonstrated in Section VI. In Section VII we discuss group-theoretic tensor bases, and relations between basis tensors for specific representations. Full Feynman rules are stated in Appendix A. Appendix B is a long discussion of an older method of weight evaluation, specific to SU(n). Appendix C contains a sketchy discussion of the exceptional group F_4.

II. FEYNMAN RULES

For our model we take a Yang-Mills theory[17] with n massive quarks and N massless gluons, defined by the classical Lagrangian density

$$\mathscr{L} = -\frac{1}{4} F_i^{\mu\nu} F_{i\mu\nu} + \bar{\psi}(i\not{D}-m)\psi \quad ,$$

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + g\, C_{ijk} A_j^\mu A_k^\nu \quad , \tag{2.1}$$

$$D_{ab}^\mu = \delta_{ab}\partial^\mu - ig\, A_i^\mu (T_i)_{ab} \quad ,$$

$$a, b = 1, 2, \ldots n \quad , \quad i, j = 1, 2, \ldots N \quad .$$

The N traceless hermitian $[n{\times}n]$ matrices T_i belong to the fundamental representation of a compact, simple Lie group $\mathscr{G}$

$$\left[T_i, T_j\right] = i\, C_{ijk} T_k \quad , \tag{2.2}$$

$$\mathrm{Tr}\, T_i = 0 \quad , \tag{2.3}$$

normalized by

$$\mathrm{Tr}\,(T_i T_j) = a\, \delta_{ij} \quad . \tag{2.4}$$

(We leave a arbitrary throughout this paper.)

The structure constants C_{ijk} are calculable from T_i matrices by tracing:

$$iC_{ijk} = \frac{1}{a}\mathrm{Tr}(T_i T_j T_k - T_k T_j T_i) \quad . \tag{2.5}$$

They obey a Jacobi identity

$$C_{i\ell m} C_{mjk} + C_{j\ell m} C_{imk} + C_{k\ell m} C_{ijm} = 0 \quad , \tag{2.6}$$

which is nothing but the commutator (2.2) for the adjoint (or regular) representation of $\mathscr{G}$, constructed from matrices

$$(C_i)_{jk} = -iC_{ijk} \quad . \tag{2.7}$$

The Lagrangian (2.1) generates the usual Feynman diagrams. There is no mixing between the spacetime and the gauge group $\mathscr{G}$, and the Feynman amplitude associated with a diagram G factorizes into $W_G M_G$, where W_G is the group theoretic weight consisting of various $(T_i)_{ab}$ and C_{ijk}, and M_G arises from the integrals over internal momenta and is similar to QED Feynman amplitudes. Even though M_G will not concern us in this paper, we give the rules for its computation in Appendix A. We note that while in momentum space there are four-gluon vertices, for W_G there exist only 3-gluon couplings, because the group-theoretic factors in a four-gluon vertex have form $C_{ijk}C_{k\ell m}$.

The group-theoretic weight W_G is a product of the following factors (all repeated indices are summed over);

a) for each internal quark line, a factor δ_{ab}; $a, b = 1, 2, \ldots n$,

b) for each internal gluon or ghost line, a factor δ_{ij}; $i, j = 1, 2, \ldots N$,

c) for each quark-quark-gluon vertex, a factor $(T_i)_{ab}$,

d) for each three-gluon or ghost-ghost-gluon vertex a factor $-iC_{ijk}$,

e) for the four-gluon vertex vertex, the following factors

$$
\begin{array}{lll}
- C_{imj}C_{k\ell m} & \text{multiplying} & (g_{\lambda\nu}g_{\mu\xi} - g_{\lambda\xi}g_{\mu\nu}) \\
- C_{ikm}C_{m\ell j} & \text{multiplying} & (g_{\lambda\mu}g_{\nu\xi} - g_{\lambda\xi}g_{\mu\nu}) \\
- C_{im\ell}C_{mkj} & \text{multiplying} & (g_{\lambda\nu}g_{\mu\xi} - g_{\lambda\mu}g_{\xi\nu})
\end{array}
$$

where gluon group and Lorentz indices are paired as (i,λ), (j,μ), (k,ν), (ℓ,ξ) (see Fig. 19).

W_G can be thought of as a sum of all possible paths of the interacting particles over a compact, discrete lattice characterized by the group $\mathscr{G}$. So W_G can itself be drawn as a Feynman diagram, with rules depicted in Fig. 1.

According to (2.4) and (2.5) the "vertices" $(T_i)_{ab}$ and C_{ijk} scale as $\sqrt{a}$, so that the arbitrary normalization $\sqrt{a}$ is a "coupling constant."[18] We shall use powers of $\sqrt{a}$ to count the number of vertices in W_G.

III. LIE ALGEBRA IN DIAGRAMMATIC NOTATION

In this section we shall transcribe the defining Lie algebra relations into weight diagrams W_G, and derive a number of relations true for all Lie groups. We omit all indices; the equivalent points on the paper represent the same index in all terms of a diagrammatic equation.[20]

Diagrammatically, the defining equations (2.2) through (2.6) are given in Fig. 2. Figure 2f is a statement of the skew symmetry of C_{ijk}. Figures 2g and 2h count the numbers of quarks and gluons, respectively; $\delta_a^a = n$, $\delta_i^i = N$. The above definitions already enable us to perform some simple calculations. For example, to calculate the quadratic Casimir operator for the fundamental representation, Fig. 3a, we form a trace (join the external quark lines) and use Figs. 2c,g,h, as outlined in Fig. 4, to obtain

$$C_F = a\,\frac{N}{n} \tag{3.1}$$

Joining gluon indices in commutators Figs. 2a and 2e leads to relations in Figs. 3c and 3d. Similarly, the relation Fig. 3e follows from the commutation relation Fig. 2a.

The antisymmetry of C_{ijk} leads to vanishing of nonplanar diagrams of Fig. 5, as well as all diagrams that contain these as subdiagrams. This follows from the commutation relations of Fig. 2, but it is easily seen as a consequence of the skewness of C_{ijk}, Fig. 2f. For example, interchange of vertices $1 \leftrightarrow 2$ in Fig. 5a and $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $5 \leftrightarrow 6$ in Fig. 5d gives a factor $(-1)^3$ from skewness of C_{ijk}, while the diagrams are mapped into themselves.

The obscure diagram of Fig. 5d is related to the Peterson graph[21] in graph theory, while Fig. 5a is related to the famous nonplanar Kuratowski graph.[22,23]

One quickly runs out of relations achievable by Lie algebra manipulations. For example, at this point we have no clue to the evaluation of the gluon Casimir operator C_A of Fig. 3b, let alone any more complicated diagram, like the one of Fig. 6. For that it is necessary to concentrate on specific groups. In the next section we proceed to define relationships characterizing particular groups.

IV. FUNDAMENTAL REPRESENTATIONS

A. Special Unitary Groups SU(n)

The fundamental representation of SU(n) is a set of all unitary ($U^+U=1$) and unimodular ($\det U=1$) $[n\times n]$ matrix transformations acting on an n-dimensional vector space (the fundamental quark n-tuplet).[26-28] They can be parametrized exponentially by $N=n^2-1$ Gell-Mann's traceless hermitian $[n\times n]$ λ-matrices;[29]

$$U = e^{i\epsilon_i\lambda_i} \qquad i = 1,2,\ldots N$$
$$(\lambda_i)^*_{ab} = (\lambda_i)_{ba} \tag{4.1}$$

They satisfy the Lie algebra of Fig. 2a, where $T_i = \frac{1}{2}\lambda_i$. To obtain an element M_{ab} of SU(n) from an arbitrary $[n\times n]$ hermitian matrix $\mathcal{M}_{ab}$, we use a projection operator[26] $P[SU]$ to impose the tracelessness condition of Fig. 2b.

$$M_{ab} = P[SU]^{cd}_{ab}\,\mathcal{M}_{cd} \tag{4.2}$$

$$P[SU]^{cd}_{ab} \equiv \delta_{ac}\delta_{bd} - \frac{1}{n}\,\delta_{cd}\delta_{ab} \tag{4.3}$$

Diagramatic representation of $P[SU]$ is given in Fig. 7a.

B. Special Orthogonal Groups SO(n)

The fundamental representation of SO(n) is a set of all orthogonal ($R^T R = 1$) and unimodular ($\det R = 1$) $[n \times n]$ matrix transformations over an n-dimensional real vector space. They can be parametrized exponentially by $N = \frac{1}{2} n(n-1)$ antisymmetric hermitian rotation matrices[27,28] T_i

$$R = e^{i\epsilon_i T_i}$$
$$(T_i)_{ab} = -(T_i)_{ba} \tag{4.4}$$

The antisymmetry is diagramatically depicted in Fig. 8a. As its consequence the expression for C_{ijk} of Fig. 2d simplifies to Fig. 8b. The projection operator $P[SO]$ from Fig. 7b imposes the antisymmetry condition on an arbitrary $[n \times n]$ hermitian matrix.

C. Symplectic Groups Sp(n)

The fundamental representation of Sp(n) is a set of all matrix transformations over an n-dimensional (n even) real vector space which preserve a skew symmetric metric[27]

$$g_{ab} = R_{ac} R_{bd} g_{cd}$$
$$g_{ab} = -g_{ba} \tag{4.5}$$

$$g_{ac} g_{cb} = -\delta_{ab} \tag{4.6}$$

They can be parametrized exponentially by $N = \frac{n}{2}(n+1)$ hermitian matrices T_i which satisfy

$$(T_i)_{ca} g_{cb} + g_{ac} (T_i)_{cb} = 0 \tag{4.7}$$

We introduce the diagrammatic notation for g_{ab} in Fig. 9 and use it to construct the projection operator $P[Sp]$ of Fig. 7c. Using relations in Fig. 9 $P[Sp]$ can be rewritten in a number of ways, all equivalent.

D. Exceptional Group G_2

We have defined the classical groups by their geometric properties, such as length preservation for SO(n). For exceptional groups it is much harder to find such interpretations. The original Cartan's proof of their existence does not lead to any geometric intuition—it is only recently that a unified interpretation of all exceptional groups as algebras over octonians has emerged.[30] In this framework G_2 is the automorphism group of octonians, i.e., it is a set of all [7×7] real matrices G_{ab} such that the transformation

$$e'_a = G_{ab} e_b \qquad a, b = 1, 2, \ldots 7 \tag{4.8}$$

preserves the octonic multiplication rule[31]

$$e_a e_b = -\delta_{ab} + f_{abc} e_c \quad , \tag{4.9}$$

where f_{abc} is a fully antisymmetric tensor. f_{abc} are given explicitly in Ref. 31; for our purposes it is sufficient to note that octonians satisfy the alternativity condition: if

$$[xyz] \equiv (xy)z - x(yz) \quad , \tag{4.10}$$

then

$$[xyz] = [zxy] = [yzs] = -[yxz] \quad , \tag{4.11}$$

where x, y, z are arbitrary octonians. In Fig. 10 we introduce diagrammatic notation for the tensor f_{abc}. Then the important relation Fig. 10c follows from the multiplication rule (4.9) and the alternativity condition (4.11).[32]

To preserve the octonic multiplication rule (4.9), matrices G_{ab} must satisfy

$$G_{ac} G_{bd} \delta_{cd} = \delta_{ab} \tag{4.12}$$

$$G_{ad} G_{be} G_{cf} f_{def} = f_{abc} \tag{4.13}$$

They can be parametrized exponentially by N=14 matrices $(T_i)_{ab}$ satisfying

$$(T_i)_{ab} = -(T_i)_{ba} \qquad i=1,2\ldots 14 \tag{4.14}$$

and

$$(T_i)_{ad} f_{dbc} + (T_i)_{bd} f_{adc} + (T_i)_{cd} f_{abd} = 0 \tag{4.15}$$

Equation (4.14) states that G_2 is a subgroup of SO(7). Equation (4.15) is reminiscent of a gauge invariance condition (Fig. 10e). That N=14 is well known; however, in our approach we shall eventually be able to compute N from Eq. (3.1). In Fig. 11 we list various derived relations.

This time we show how to constrict the projection operator step by step, in Fig. 12. (By the same procedure SO(3) is the isomorphism group of quaternions. For quaternions the associator (4.10) is trivially zero, and relation from Fig. 10c is replaced by the familiar identity for ϵ_{ijk} tensors, Eq. (B.8).)

V. COMPLETENESS RELATIONS

The projection operators defined in Fig. 7 will now enable us to reduce a group-theoretic weight W_G to a sum of lower order weights, and thus evaluate W_G without reference to any explicit matrix representation. We achieve this by deriving a completeness relationship for each group.[33]

A projection $P[\mathscr{G}]\mathscr{M}$ of an arbitrary hermitian matrix $\mathscr{M}$ is an element of the group $\mathscr{G}$, and can be expanded in terms of a complete set of basis matrices $(T_i)_{ab}$, as drawn in Fig. 13b. The expansion coefficients are evaluated by tracing this equation with T_j matrix, leading to the relationship Fig. 13c. As this is true for arbitrary $\mathscr{M}$, it can be removed from the equation, and we obtain a general completeness relation of Fig. 13d. (Here we have used $P[\mathscr{G}]T=T$, i.e., projection operator leaves the elements of the algebra unchanged.)

In Fig. 14 we write out a completeness relation for each of the groups considered. Now its significance is clear; it enables us to reduce a W_G with an internal gluon line to a sum of lower order weight diagrams, multiplied by the square of the coupling constant $\sqrt{a}$.[34]

VI. EVALUATION OF GROUP-THEORETIC WEIGHTS

Evaluation of any W_G is now trivial. It proceeds in two steps:

1) Eliminate all three-gluon vertices C_{ijk} by Fig. 2d (or by Fig. 8b, if the group is SO(n), Sp(n) or G_2).

2) Eliminate all internal gluon lines by the appropriate completeness relation of Fig. 14.

As an example, we evaluate the SO(n) quadratic Casimir operator for the adjoint representation (gluons) in Fig. 15. We find

$$C_A = a(n-2) \qquad (6.1)$$

Other such results are tabulated in Fig. 16.

Also note that a completeness relation fully characterizes the group. We can start with a completeness relation from Fig. 14 and the definition Fig. 2d and derive all the general results of Section III. Such computations provide useful checks of the correctness of our completeness relations. For example, the reader can check Eq. (3.1) for each group, and thus verify the expressions for N, the number of gluons.

VII. KOLO BASES AND RELATIONS BETWEEN BASIS TENSORS

The procedure outlined in the previous sections always leads us to a unique set of tensors; $(T_i)_{ab}$ and traces over T_i matrices. In other words, we are expressing all W_G in terms of the fundamental representation. Let us illustrate this by writing all irreducible tensor invariants for a process with r external gluons and no external quarks; the set of all distinct traces over r T_i matrices (Fig. 17).

We name such basis kolo bases, because they are reminiscent of r people dancing a kolo (a Yugoslav folk dance; "kolo" translates as "wheel"). β_r, the number of all distinct tensors of rank r, is the number of ways in which r people can form kolos by holding hands, with a restriction that nobody dances alone—i.e., tracelessness.

β_r can be calculated in a number of painful ways, such as by Young tableaux,[12,35] or by the method of Appendix B. However, it turns out that β_r has already been calculated in 1708,[36-40] and is known as a number of derangements, or subfactorial

$$\beta_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots (-1)^n \frac{1}{n!}\right) \tag{7.1}$$

Not all kolos (tensor bases) thus enumerated are necessarily independent. Relations between them arise in two ways; from the group structure, and from the dimensionality of the fundamental representation.

β_r was calculated from a single condition, tracelessness. Thus kolos are natural bases for all Lie groups, and SU(n) in particular. For SO(n), Sp(n), G_2 the antisymmetry relations of Figs. 8 and 9 relate the clockwise and anticlockwise directions of loops in Fig. 17, and the number of independent bases is reduced

$$\text{SO(n)}: \quad \beta_3 = 1, \ \ \beta_4 = 6, \ \text{etc.} \tag{7.2}$$

Relations dependent on the dimensionality of the fundamental representation arise from the characteristic equations for [n×n] matrices A.[41] Characteristic polynomial[42] (where x is any [n×n] matrix) is defined as

$$P(x) = \det |A - Ix| = \sum_{k=0}^{n} (-x)^{n-k} \frac{1}{k!} \delta^{a_1 a_2 \ldots a_k}_{b_1 b_2 \ldots b_k} A_{a_1 b_1} \ldots A_{a_k b_k}$$

where

$$\delta^{ab\ldots f}_{pq\ldots u} = \det \begin{vmatrix} \delta_{ap} & \delta_{bp} & \cdots & \delta_{fp} \\ \delta_{aq} & \delta_{bq} & & \\ \vdots & & & \\ \delta_{au} & & \cdots & \delta_{fu} \end{vmatrix}$$

is the generalized Kronecker delta. Identity P(A)=0 yields the characteristic equation for A;

$$0 = \sum_{k=0}^{n} A^{n-k} \frac{(-1)^k}{k!} \delta^{a_1 a_2 \ldots a_k}_{b_1 b_2 \ldots b_k} A_{a_1 b_1} A_{a_2 b_2} \ldots A_{a_k b_k}$$

Now if we substitute $A = a_i T_i$, where T_i are generators of the gauge group $\mathscr{G}$, for each n we obtain various relations between tensor invariants. As an example, we work out n=4 case diagrammatically in Fig. 18a. The indices are symmetrized because the whole expression is multiplied by a symmetric factor $a_i a_j a_k a_\ell$, summed over all i, j, k and ℓ. A more familiar relationship is worked explicitly for SU(2) and SU(3) in Figs. 18c and 18d. The SU(3) relationship can be rewritten in terms of d_{ijk} tensors, the form in which it has been originally derived by Macfarlane et al.[11] Higher SU(n) relationships have been worked out in Ref. 13. All such relations are really of little interest to us, because they do not affect the correctness of our general procedure for weight evaluation.

APPENDIX A

COMPLETE FEYNMAN RULES FOR $M_G W_G$

With the definition of the group theoretic weight W_G given in Section II, the rules for M_G are easily constructed by consulting some standard reference, such as Abers and Lee.[17] In this appendix we state the full rules for (unbroken) non-Abelian gauge theories as an extension of the rules for constructing Feynman-parametric integrals given previously.[43] Factors of the rule 5, of Ref. 43 are now replaced by the factors of Fig. 19. Additionally M_G gets a factor -1 for each quark or ghost loop. We refrain from enumerating various renormalization factors, as the on-the-mass-shell renormalized amplitude suffers from serious infrared divergences.

APPENDIX B

EVALUATION OF SU(N) WEIGHTS USING f- AND d-TENSOR BASES

In this appendix we extend the SU(3) method of Dittner[12] to SU(n). Generalized Gell-Mann's $[n\times n]$ λ matrices together with I, iI and iλ span all complex matrices,[29] so we can write a multiplication law for λ matrices as

$$\mathrm{SU(n)}: \quad \lambda_i\lambda_j = \left(\frac{4a}{n} + ib\right) \delta_{ij} I + \left(d_{ijk} + if_{ijk}\right) \lambda_k \quad . \tag{B.1}$$

This relation, which has no obvious analogues for other simple groups, is the departure point for most of the earlier attempts at weight evaluation.[10-16] The tensors δ_{ij}, d_{ijk} and f_{ijk} are numerically invariant in the sense that they are the same for all equivalent representations $\lambda_i \rightarrow u^+\lambda_i u$, $u^+u=1$. They are real by definition. b=0 because of the hermiticity of λ_i, while a is the arbitrary normalization of Eq. (2.4).

Dittner's approach proceeds in three steps. First, by the repeated application of λ-multiplication formula (depicted in Fig. 20b) all products of form $\lambda_i\lambda_j \ldots \lambda_m$ are reduced to combinations of the three basic tensors δ_{ij}, d_{ijk} and if_{ijk}. Second, bases of β_r independent tensors for processes with r external gluons are constructed. Third, the weight W_G is expanded in terms of the appropriate basis, and the expansion coefficients are calculated by solving a set of β_r linear equations.

The first step results in diagrams of general form of Fig. 20e, where the blobs involve only δ_{ij}, d_{ijk} and f_{ijk}. Generally we do not distinguish between quark colors, so one traces over the quark lines, and weight calculation is reduced to the problem of evaluating "vacuum blobs" consisting purely of gluons. Such a "blob" is evaluated by looking at its subdiagrams with r=2, 3, ... external gluon legs, and rewriting them in terms of some basis set of tensors.

The simplest set of tensors for each r is easily constructed (see Fig. 21). To enumerate them, we start a systematic construction by drawing all Catalan's[37,40,44] trees in Fig. 22, whose number is Catalan's number (the number of ways in which a product of n numbers can be evaluated)

$$a_{r-1} = \frac{(2r-4)!}{(r-1)!\,(r-2)!}\ ; \qquad a_0 \equiv 0 \tag{B.2}$$

By (r-1)! permutations of all branches, and factor two for each crotch (f- or d-tensor), we obtain the number of all distinct connected tensors

$$\bar{\alpha}_r = 2^{r-2}(2r-5)!!\ , \qquad \bar{\alpha}_1 \equiv 0,\ \bar{\alpha}_2 \equiv 1 \tag{B.3}$$

where (2n-1)!! is the product of first n odd integers; $7!! = 7.5.3.1$. To relate $\bar{\alpha}_r$ to the α_r, the number of all distinct tensors (connected and unconnected) we introduce generating functions

$$A(t) \equiv \sum_{r=0}^{\infty} \frac{\alpha_r}{r!}\, t^r \tag{B.4}$$

$$\bar{A}(t) \equiv \sum_{r=0}^{\infty} \frac{\bar{\alpha}_r}{r!}\, t^r = \frac{1}{12}\left(-1+6t+(1-4t)^{3/2}\right) \tag{B.5}$$

The numbers of connected and disconnected graphs are related in the usual fashion

$$A(t) = e^{\bar{A}(t)} \tag{B.6}$$

By differentiation with respect to t, this can be restated as

$$\alpha_r = \sum_{r=0}^{r-1} \binom{r-1}{k}\, \bar{\alpha}_{r-k}\alpha_k \tag{B.7}$$

which enables us to calculate recursively α_r listed in Fig. 21.

However, tensors so constructed are redundant, and if we attempted to use them to expand an arbitrary tensor with r external gluons, we would not be able to calculate the expansion coefficients, because the determinant of the system of α_r equations vanishes for r>3.

So, our next task is to find all the relations between α_r tensors. These stem from the associativity of T_i matrices. For example, $Tr(T_iT_jT_kT_\ell)$ can be evaluated in two ways, by pairing matrices either as $Tr(T_iT_j)(T_kT_\ell)$ or $Tr(T_jT_k)(T_\ell T_i)$, and then using Fig. 20b. The two evaluations give the relationship of Fig. 23a. There are (4-1)!=6 distinct connected kolos (Fig. 17) with four dancers each, giving us $\bar{\gamma}_4$=6 relationships. We cast those in the form familiar from literature;[10-12] three equations for the real parts (Fig. 23b) and three for the imaginary parts (Fig. 23c). The two relations in Fig. 23b which involve disconnected parts are SU(n) generalizations[10,11] of SU(2) relationship

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{j\ell}\delta_{im} \tag{B.8}$$

Number of such associativity relations for arbitrary r is again related to Catalan's number, which is nothing but the number of associativity patterns

$$\bar{\gamma}_r = (r-1)!\ (a_{r-1} - 1) \qquad r\geq 2 \tag{B.9}$$

For each r there are

$$\beta_r = \bar{\alpha}_r - \bar{\gamma}_r = (r-1)! \qquad r\geq 2 \tag{B.10}$$

independent connected tensors. The total number of independent tensors β_r is given by

$$B(t) \equiv \sum_{r=0}^{\infty} \frac{\beta_r}{r!} t^r \tag{B.11}$$

$$\bar{B}(t) = \sum_{r=2}^{\infty} \frac{\bar{\beta}_r}{r} t^r = -t-\ell n(1-t) \tag{B.12}$$

$$B(t) = e^{\bar{B}(t)} = \frac{e^{-t}}{1-t} \tag{B.13}$$

But B(t) is precisely the generating functional for subfactorials, so we have rederived the simple counting of (7.1) in a complicated way.

Once a set of β_r independent tensors has been constructed, the tensor to be simplified is expanded in this basis. By contracting all its indices with each basis tensor, a set of β_r linear equations is obtained. Now it is necessary to solve these equations—for the details, we refer the reader to Dittner's paper.[12] To illustrate the form of the results, we give the reduction of a gluon "box" diagram in two (of many possible) choices of f-, d-bases (Fig. 24a). For comparison with the method of Section VI, we also evaluate the same diagram in kolo basis, Fig. 24b.

To summarize, f-, d-bases reduction of weights is a feasible method for SU(n). However, it suffers from various drawbacks. It does not work for other groups. It is a method of evaluation of weights in terms of adjoint, rather than fundamental representation, and it tends to be messier. It introduces a tensor d_{ijk} that does not appear in the original interaction Lagrangian, and leads to arbitrariness in the choice of tensor bases (note that the kolo bases are unique). Finally, it involves solving large sets of linear equations; already for r=4 we found it convenient to do the algebra on a computer.[45] By contrast, kolo method requires only a sharp pencil and some paper.

APPENDIX C

EXCEPTIONAL GROUP F_4

We hope to treat all exceptional groups (G_2, F_4, E_6, E_7 and E_8) in an expanded version of this paper. Here we merely sketch a treatment of F_4 based on Schafer.[46] F_4 is the isomorphism group of the exceptional simple Jordan algebra of traceless hermitian [3×3] matrices $\underset{\sim}{x}$ with octonian matrix elements. The nonassociative multiplication rule for elements $\underset{\sim}{x}$ can be written as

$$\underset{\sim}{x} \equiv x_a e_a \qquad a=1,2,\ldots 26$$

$$\mathrm{Tr}\, e_a = 0, \qquad e_a \text{ is a } [3\times 3] \text{ basis matrix}$$

$$e_a e_b = e_b e_a = \frac{\delta_{ab}}{3}\,\mathbb{1} + d_{abc} e_c$$

$$\mathrm{Tr}\,\mathbb{1} = 3, \qquad \mathbb{1} \text{ is } [3\times 3] \text{ unit matrix.}$$

Transformations of F_4 preserve the quadratic form $\mathrm{Tr}\,(x^2)$ (the length in 26 dimensional space, so F_4 is a subgroup of SO(26)), as well as a fully symmetric cubic form

$$\mathrm{Tr}\,(xyz) = \mathrm{Tr}\,(yxz) = \mathrm{Tr}\,(yzx)$$
$$= d_{abc} x_a y_b z_c$$

where d_{abc} have some formal similarities to $\frac{1}{2} d_{ijk}$ coefficients of SU(3). The fundamental representation of F_4 algebra consists of N=52 independent [26×26] skew symmetric matrices $(T_i)_{ab}$ (i=1, 2, . . . , 52) which satisfy the condition of Fig. 25e.[47] The definition of Jordan algebra

$$(xy)x^2 = x(yx^2)$$

gives a relation between contractions of three d_{abc} tensors, Fig. 25a. The characteristic equation for traceless [3×3] matrices

$$x^3 - \frac{1}{2}\,\mathrm{Tr}\,(x^2)x - \frac{1}{3}\,\mathrm{Tr}\,(x^2)\,\mathbb{1} = 0$$

(see Fig. 18) gives a relationship between contractions of pairs of d_{abc}, drawn in Fig. 25b. From these identities follow various relationships (Fig. 26), which enable us to construct the projection operator $P\left[F_4\right]$ given in Fig. 26h.

Acknowledgements

I would like to thank the Aspen Center for Physics for the hospitality extended to me, R. Person for the help with Young tableaux calculations and H. Harari, P. Freund, P. Ramond, C. Sachrajda, and G. Tiktopoulos for stimulating discussions.

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18. In the language of Cartan's diagrams, $\sqrt{a}$ sets the length scale for root vectors. In literature there are many conventions for fixing a. For example, to obtain $T_i = \frac{1}{2}\sigma_i$ for SU(2) and $T_i = \frac{1}{2}\lambda_i$ for SU(3), Gell-Mann[29] choses $a = \frac{1}{2}$. If SO(n) is represented by rotation matrices with only two nonzero elements ±1, then a=2. Also, instead of fundamental representation, some other representation could be used to set the scale. Reference 19 uses the quadratic Casimir operator of the adjoint representation as the arbitrary normalization scale.

19. S. L. Glashow and M. Gell-Mann, Ann. Phys. 15, 437 (1961).

20. Similar diagrammatic notation has also been developed by R. Penrose.[21]

21. R. Penrose, in Combinatorial Mathematics and its Applications, edited by D. J. A. Welsh (Academic Press, New York, 1971), pp. 221-244.

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23. Group-theoretic weights have an amusing graph-theoretic interpretation for SO(3). If we consider a planar "vacuum bubble" (no external lines), then W_G is proportional to the number of ways of coloring the lines of the graph with three colors[21] (red, white, blue?). This in turn is related to the chromatic polynomials,[24] Heawood's conjecture[25] and even the four-color problem.[22]

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33. The completeness relation for SU(n) has been given by Macfarlane et al.[11] The derivation of a general completeness relations has been suggested by the SO(n) completeness relation of T. P. Cheng, E. Eichten and Ling-Fong Li, Phys. Rev. D 9, 2259 (1974).

34. A similar observation for the case of SU(n) has been made by Barnes and Delbourgo.[14]

35. We thank R. Pearson for carrying out a Young tableaux calculation to check our numbers. β_r is the number of times singlet appears in the decomposition of a product of r adjoint representations: $\underline{N} \times \underline{N} \times \ldots \times \underline{N} = \beta_r \underline{1} + \ldots$.

36. This number is related to the probability that if r drunk gentlemen randomly pick up their hats upon leaving a party none of them will end up with his own hat. (Not wearing one's own hat is equivalent of tracelessness for matrices.)

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41. This was noted by Macfarlane *et al.*[11] for SU(3) and generalized to SU(n) by Rashid and Saifuddin.[13] For SU(n) group such relationships can be stated as relationships between d_{ijk} coefficients.

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47. Mathematicians call this algebra the derivation algebra of F_4.

FIGURE CAPTIONS

1. (a) quark propagator

   (b) gluon or ghost propagator

   (c) quark-quark-gluon vertex. The arrow denotes the direction of multiplication of T^i matrices. Whenever omitted, it is assumed to be pointing to the left for quarks going through the diagram, or anticlockwise for closed quark loops.

   (d) three-gluon or ghost-ghost-gluon vertex. Indices circle the vertex anticlockwise.

2. (a) Lie commutator for the fundamental representation

   (b) tracelessness condition ("color conservation")

   (c) normalization convention

   (d) $-iC_{ijk}$ in terms of the fundamental representation

   (e) Jacobi identity (or Lie commutator) for the adjoint representation

   (f) skew-symmetry of C_{ijk}

   (g) quark number

   (h) gluon number.

3. (a) quadratic Casimir operator for the fundamental representation

   (b) quadratic Casimir operator for the adjoint representation

   The remaining figures are examples of the reduction of

   (c) a quark-quark-gluon vertex

   (d) a three-gluon vertex, and

   (e) another quark-quark-gluon vertex.

4. A diagramatic computation of the quadratic Casimir operator for the fundamental representation.

5. Some diagrams that vanish because of the skew-symmetry of C_{ijk}.

6. A sixth-order quark-quark-gluon vertex graph.

7. Projection operators for the fundamental representations of (a) SU(n), (b) SO(n), (c) Sp(n), and (d) G_2.

8. (a) skew symmetry of the fundamental representations of SO(n) and G_2

   (b) $-iC_{ijk}$ in terms of the fundamental representation for SO(n), Sp(n) and G_2.

9. Diagrammatic notation for the skew-symmetric metric tensor g_{ab} for the symplectic group Sp(n).

10. (a) diagrammatic notation for the tensor f_{abc} for the exceptional group G_2

    It is

    (b) fully antisymmetric, and

    (c) contractions of several f_{abc} are reducible.

    The fundamental representation of G_2 is

    (d) a subgroup of SO(7) with the additional condition (e).

11. Some derived relations between f_{abc} tensors useful in the computations of weights for G_2.

12. A projection operator for G_2 is constructed by

    (a) expansion in all available fundamental tensors

    (b) their reduction by relation Fig. 10c, and

    (c) imposition of the antisymmetry requirement.

    Finally the coefficients A' and D' are fixed by

    (d) the definition of a projection operator.

    The result is given in Fig. 7d.

13. (a) an arbitrary [n×n] hermitian matrix $\mathcal{M}_{ab}$

(b) expansion of an arbitrary element of the group $\mathcal{G}$ in terms of a basis set

(c) evaluation of the expansion coefficient

(d) a general completeness relation.

14. Completeness relation for (a) SU(n), (b) SO(n), (c) Sp(n), (d) G_2 and (e) F_4.

15. A sample diagrammatic computation: quadratic Casimir operator for the adjoint representation of SO(n).

(a) C_{ijk} are replaced by the fundamental representation

(b) one gluon is eliminated by the completeness relation

(c) the remaining gluon is eliminated by the completeness relation.

16. A tabulation of some simple weight evaluations.

17. Kolo bases for processes with r=2, 3, ... external gluons and no external quarks. These are also the complete and independent bases for SU(n) tensors as long as $n \geq r$.

18. (a) a characteristic equation for [4×4] matrices

(b) symmetrization symbol

(c) characteristic equation for SU(2); there are no d_{ijk} coefficients (see Fig. 20d)

(d) Macfarlane et al. relation for SU(3).

19. Factors for the group-theoretic weights W_G and Feynman momentum integrals M_G.

20. (a) notation for the (fully symmetric) numerical tensor d_{ijk}

(b) multiplication rule for SU(n) matrices $T_i \equiv \frac{1}{2} \lambda_i$

(c) decomposition of three-external gluon quark-loop into real and imaginary parts

(d) d_{ijk} as its real part

(e) reduction of products of T_i matrices to d_{ijk} and f_{ijk} tensors

(f) elimination of quark lines by tracing.

21. Construction of all simple d- and f-tensors with r external gluons.

22. Catalan' trees.

23. (a) associativity of T_i matrices leads to relations between various d- and f-tensors.

All relations between (b) real and (c) imaginary parts of simple tensors with four external gluons.

24. Gluon "box" diagram evaluated in (a) two different f- and d-bases and (b) kolo basis.

25. (a) diagrammatic notation for the tensor d_{abc} of the exceptional group F_4

(b) d_{abc} is fully symmetric

(c) Jordan identity relates contractions of three d_{abc}'s

(d) characteristic equation for traceless hermitian [3×3] matrices of the exceptional simple Jordan algebra relates contractions of two d_{abc}'s

(e) a condition on generators $(T_i)_{ab}$ which follows from the invariance of Tr (xyz).

26. Various relationships for F_4, derived from the definitions of Fig. 25.

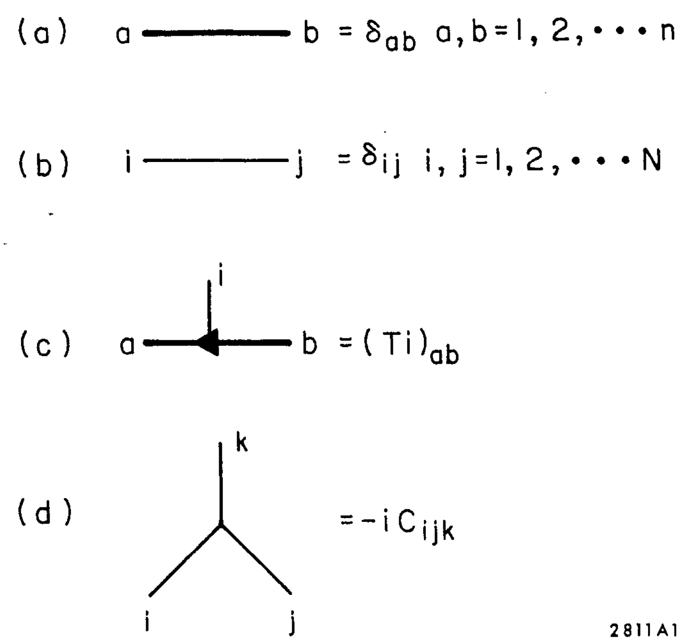


Fig. 1

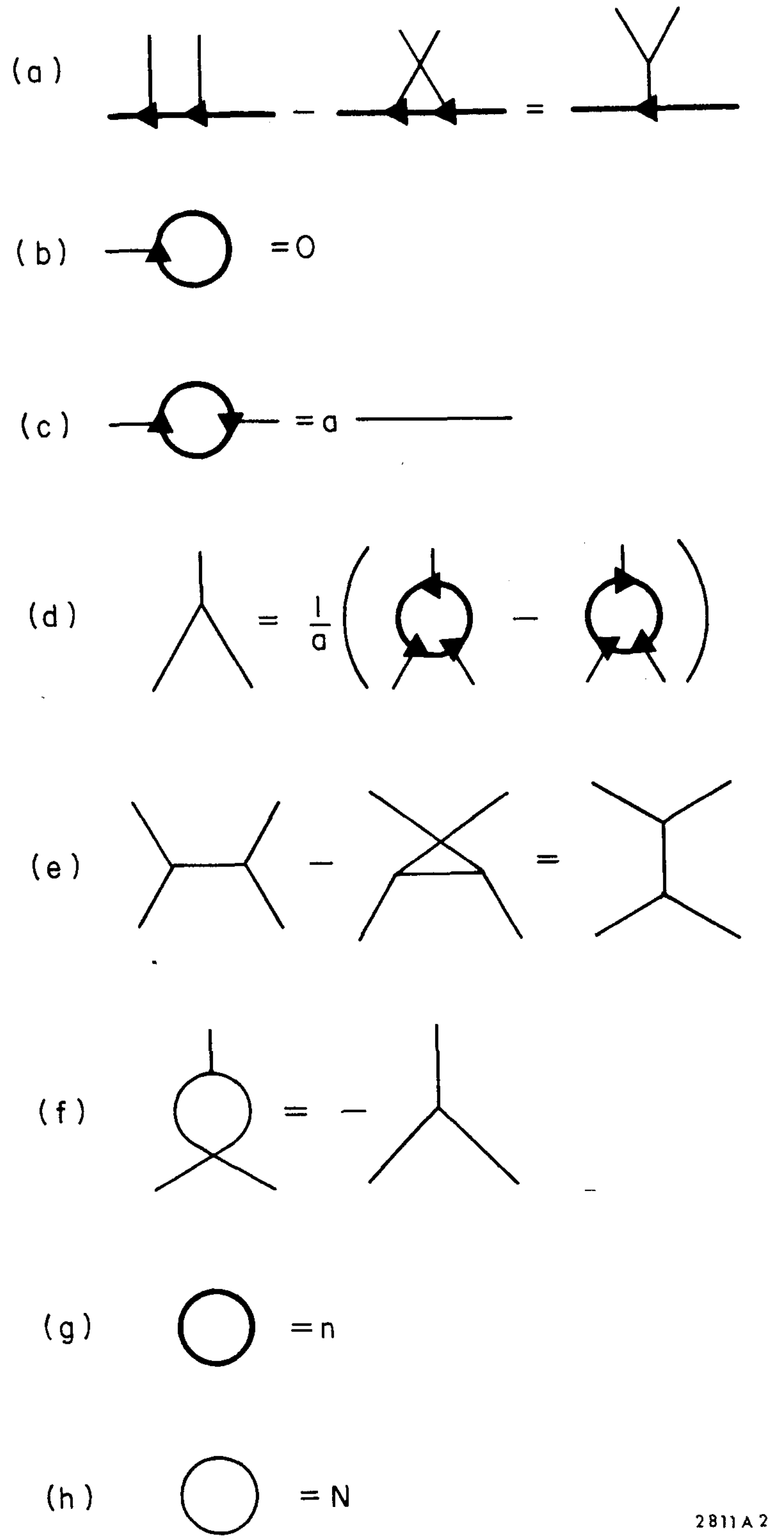


Fig. 2

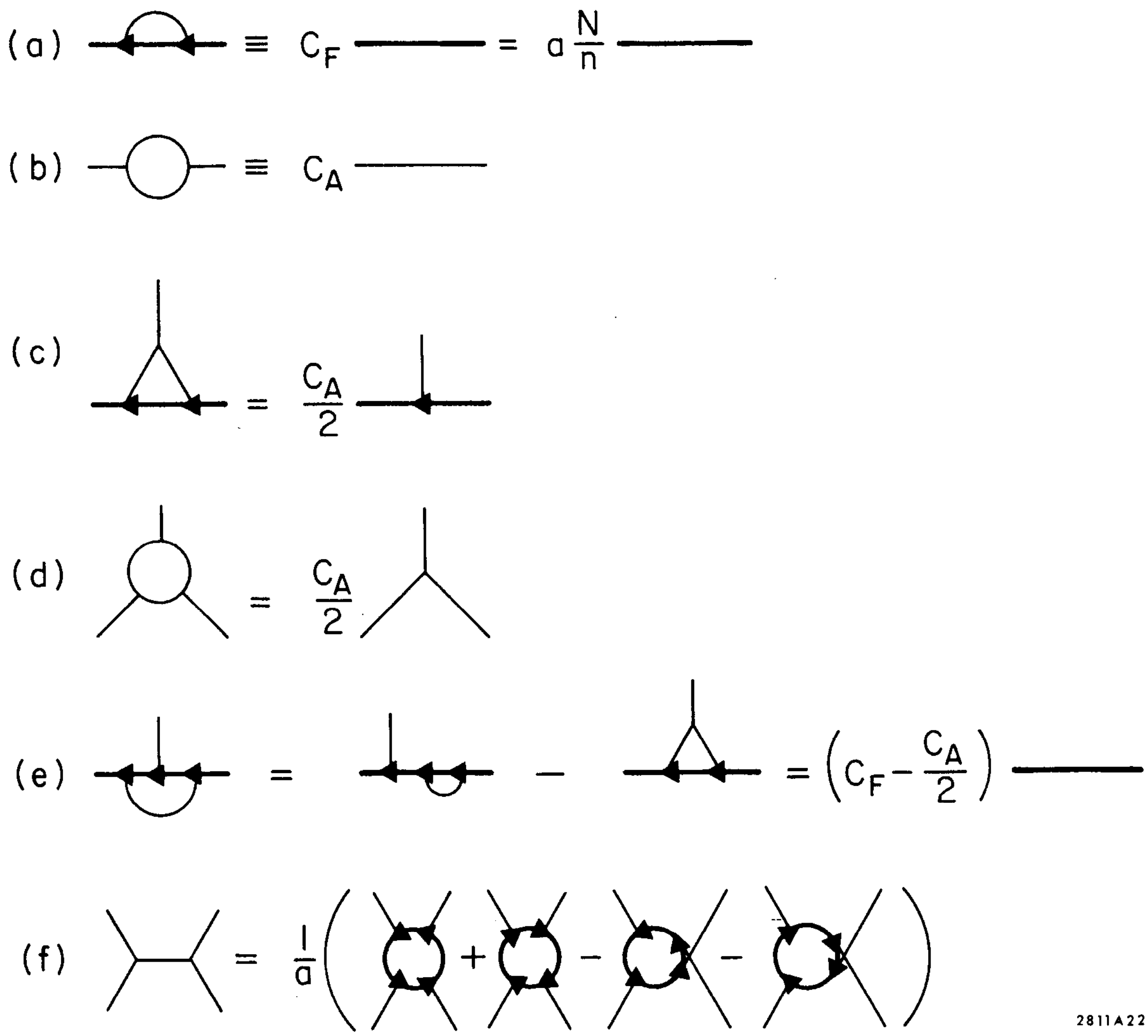

(a)
≡ C_F
= $a\frac{N}{n}$
(b)
≡ C_A
(c)
= $\frac{C_A}{2}$
(d)
= $\frac{C_A}{2}$
(e)
=
−
= $\left(C_F - \frac{C_A}{2}\right)$
(f)
= $\frac{1}{a}$
+
−
−
2811A22

Fig. 3

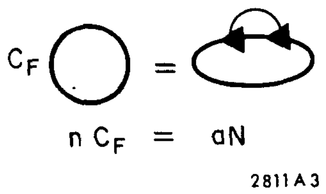


Fig. 4

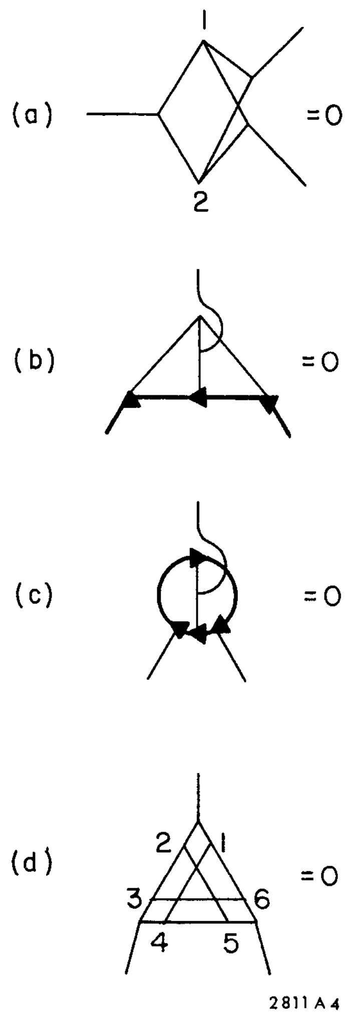


Fig. 5

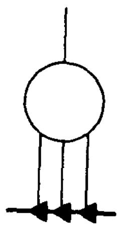


Fig. 6

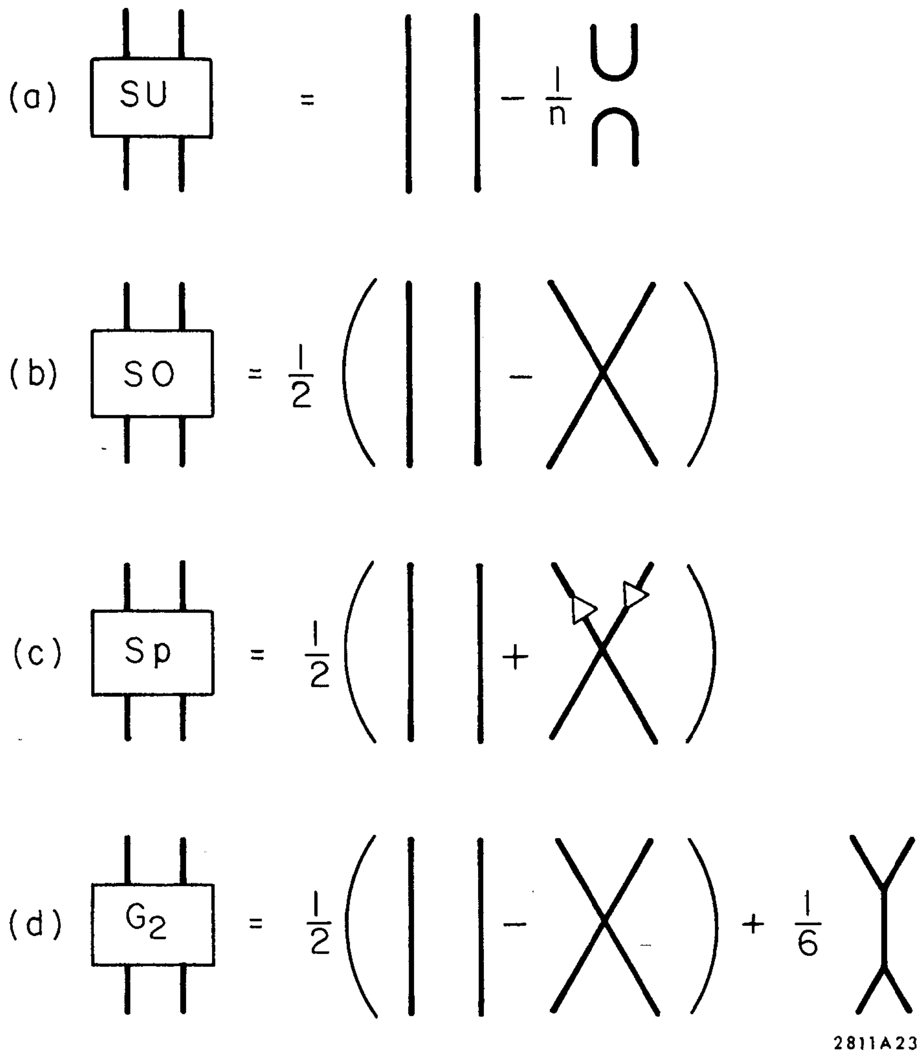


Fig. 7

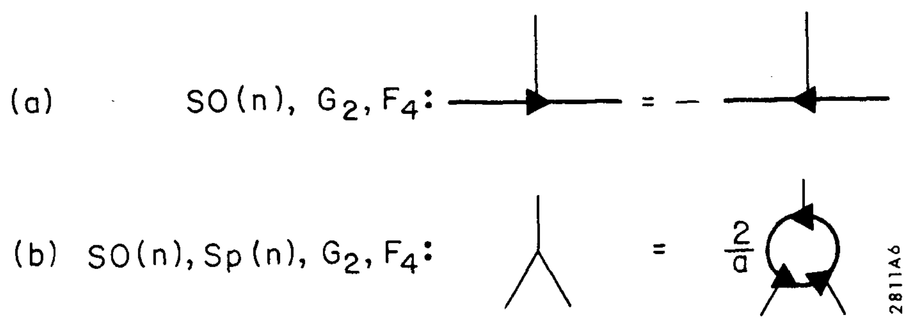


Fig. 8

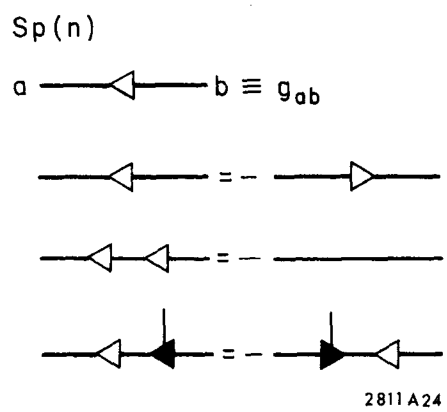


Fig. 9

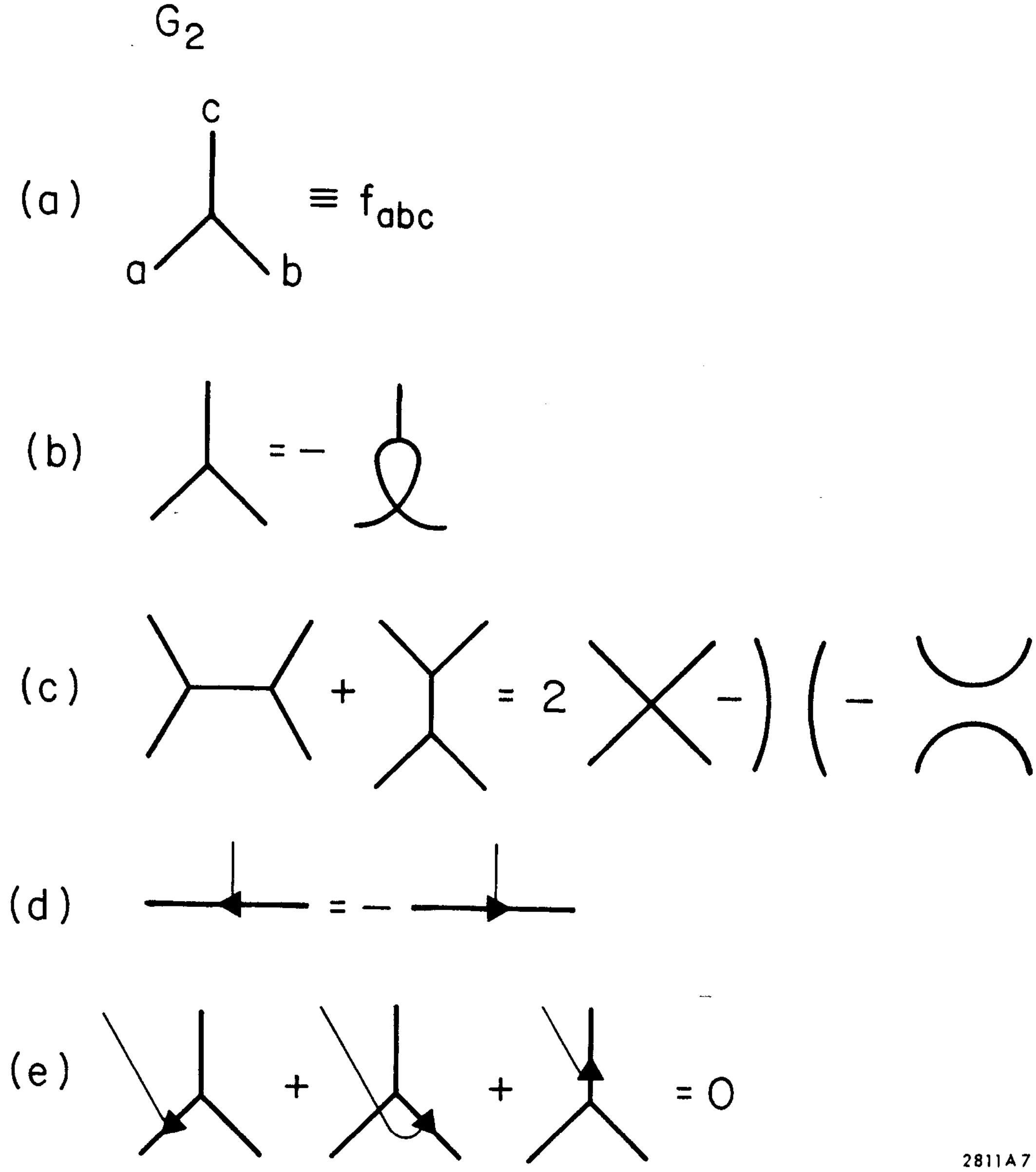


Fig. 10

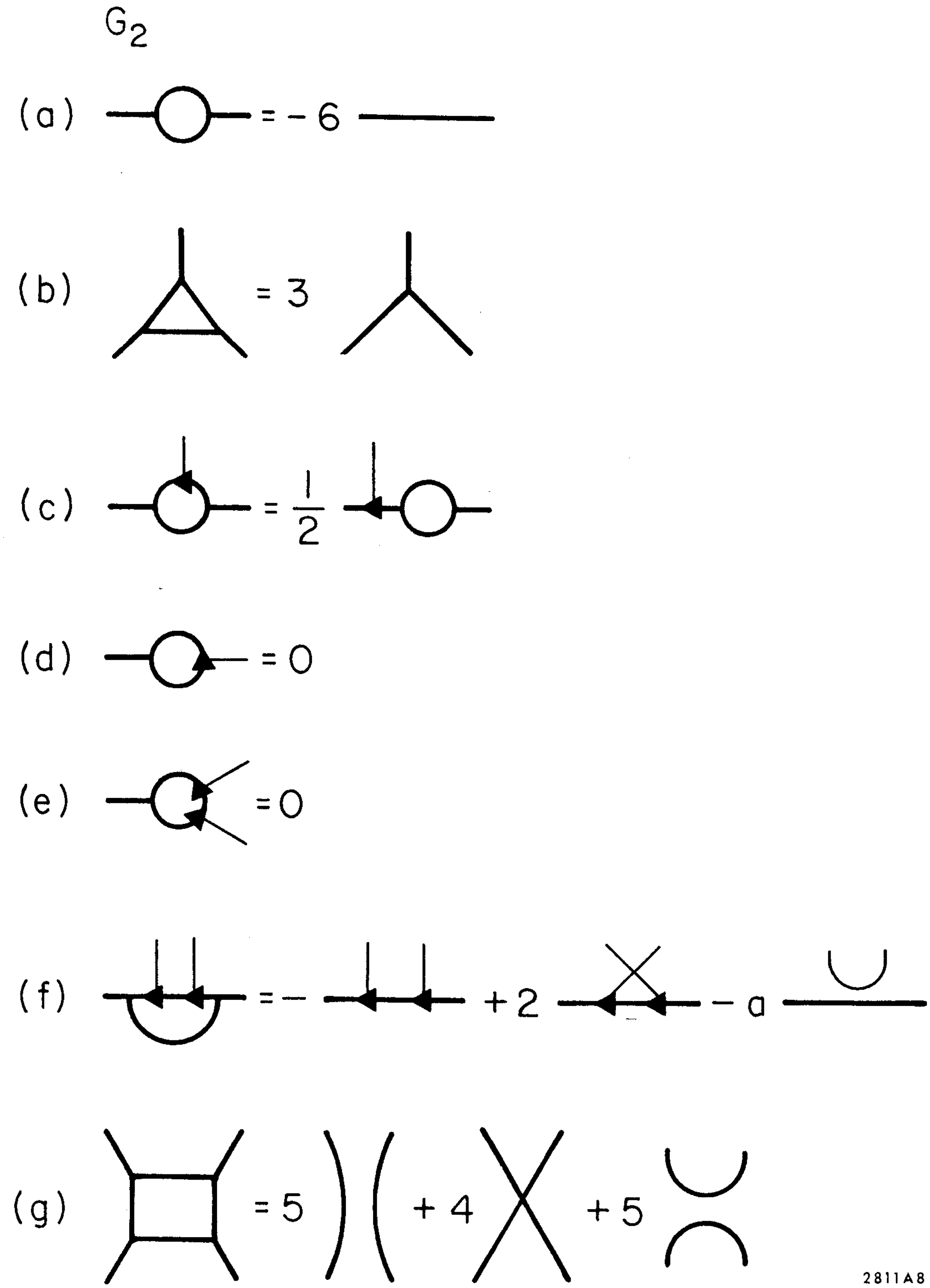


Fig. 11

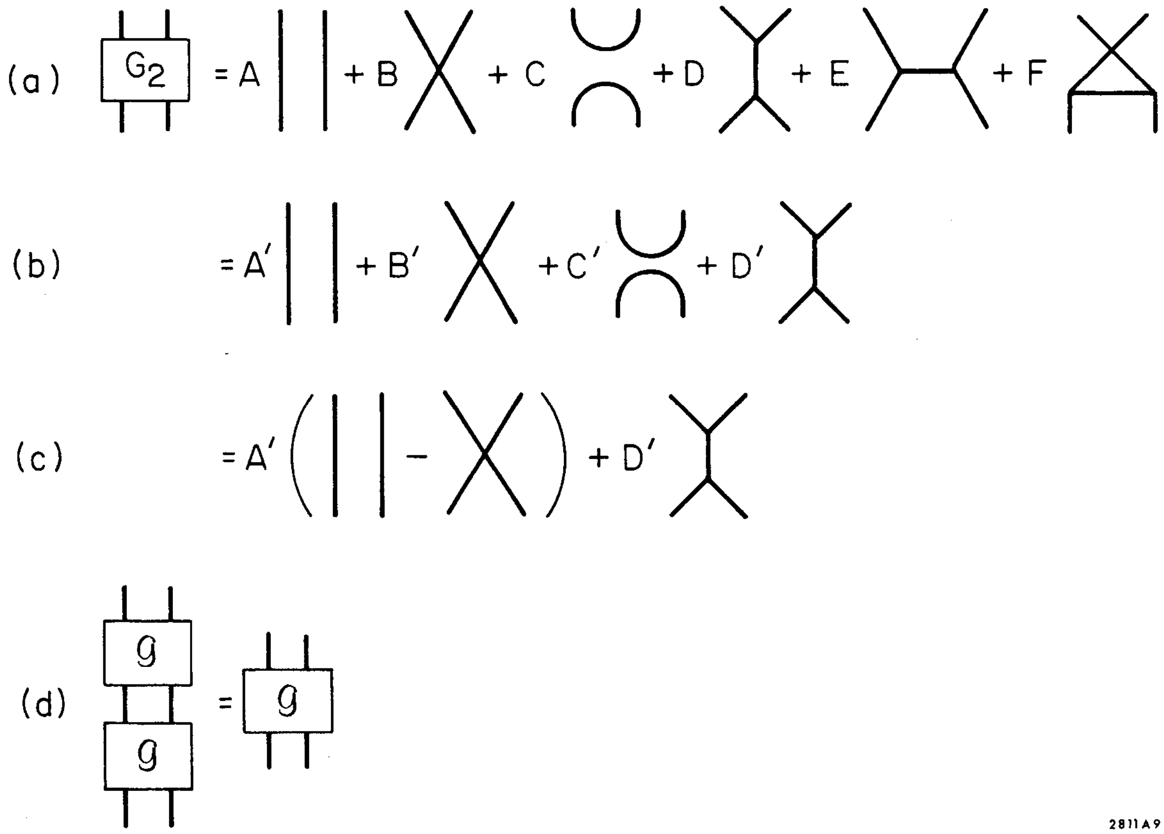


Fig. 12

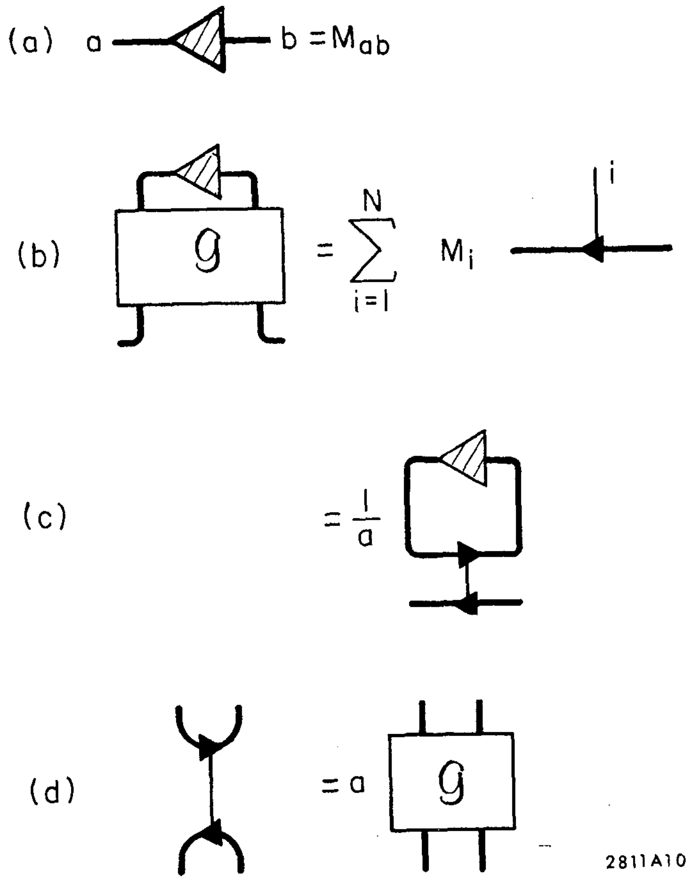


Fig. 13

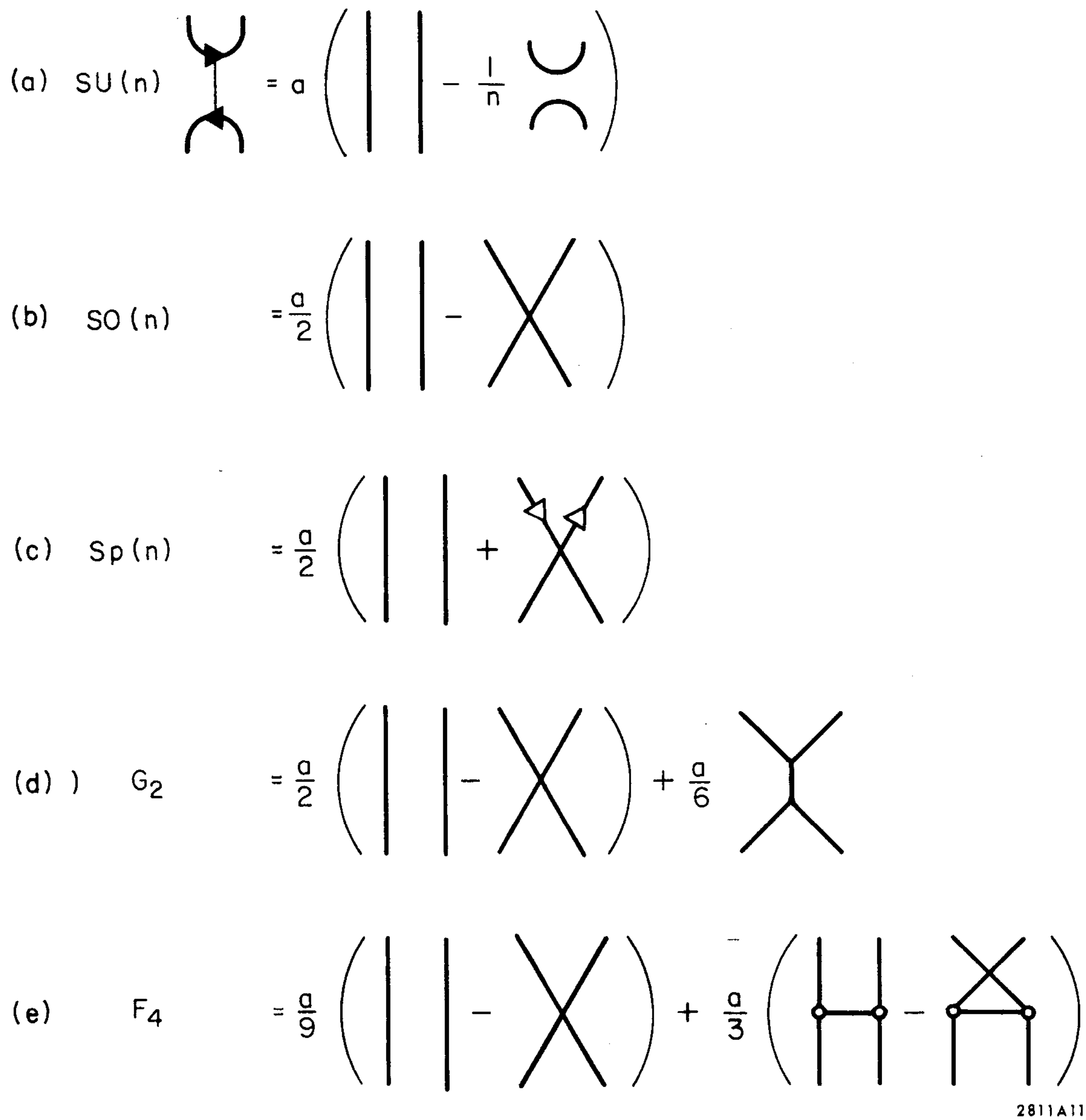
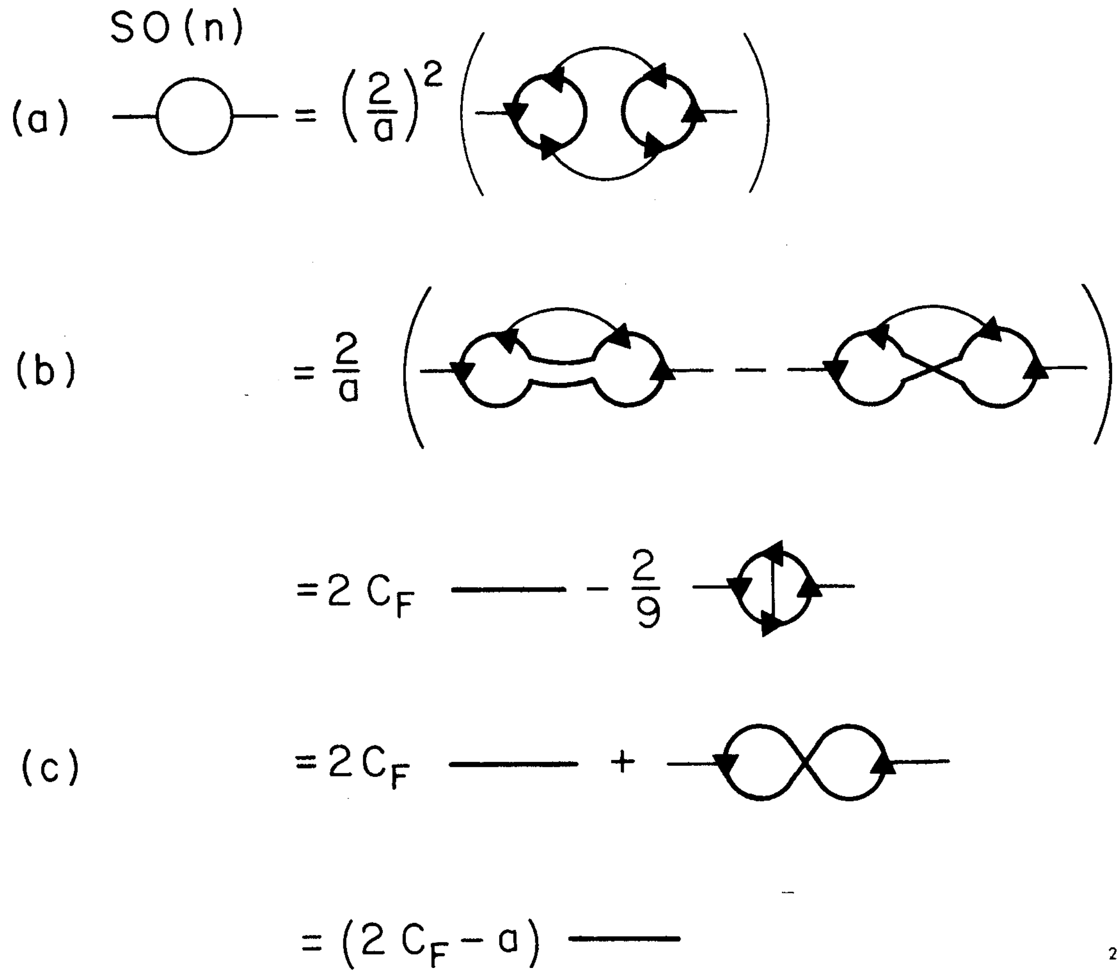


Fig. 14



2811A12

Fig. 15

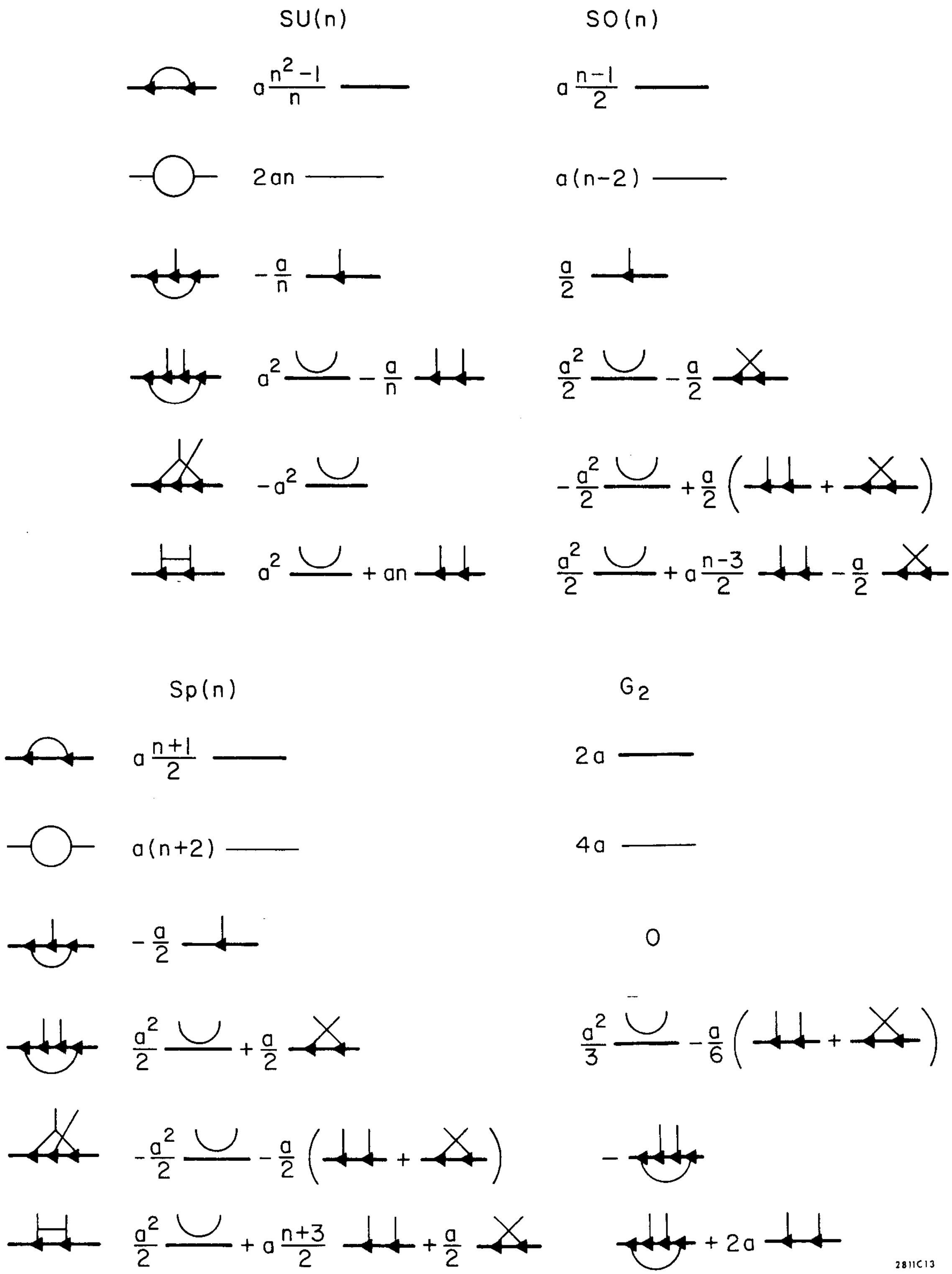
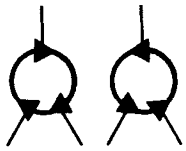
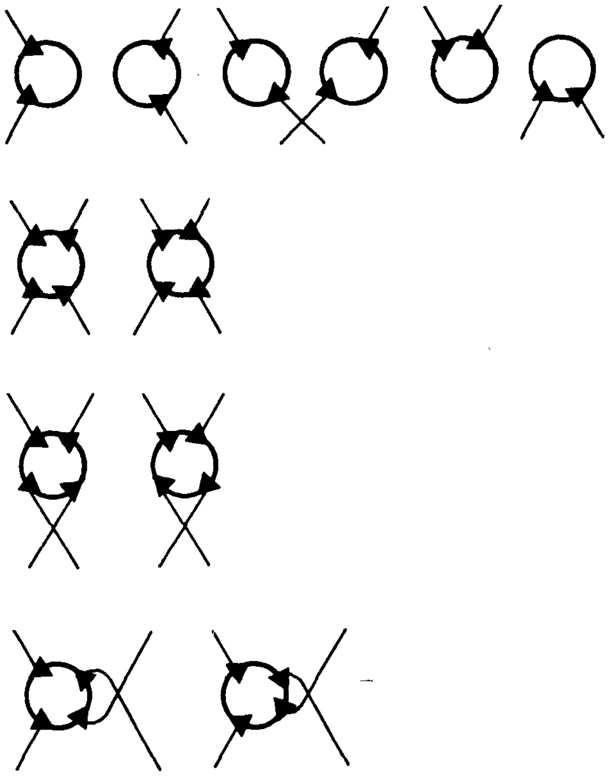
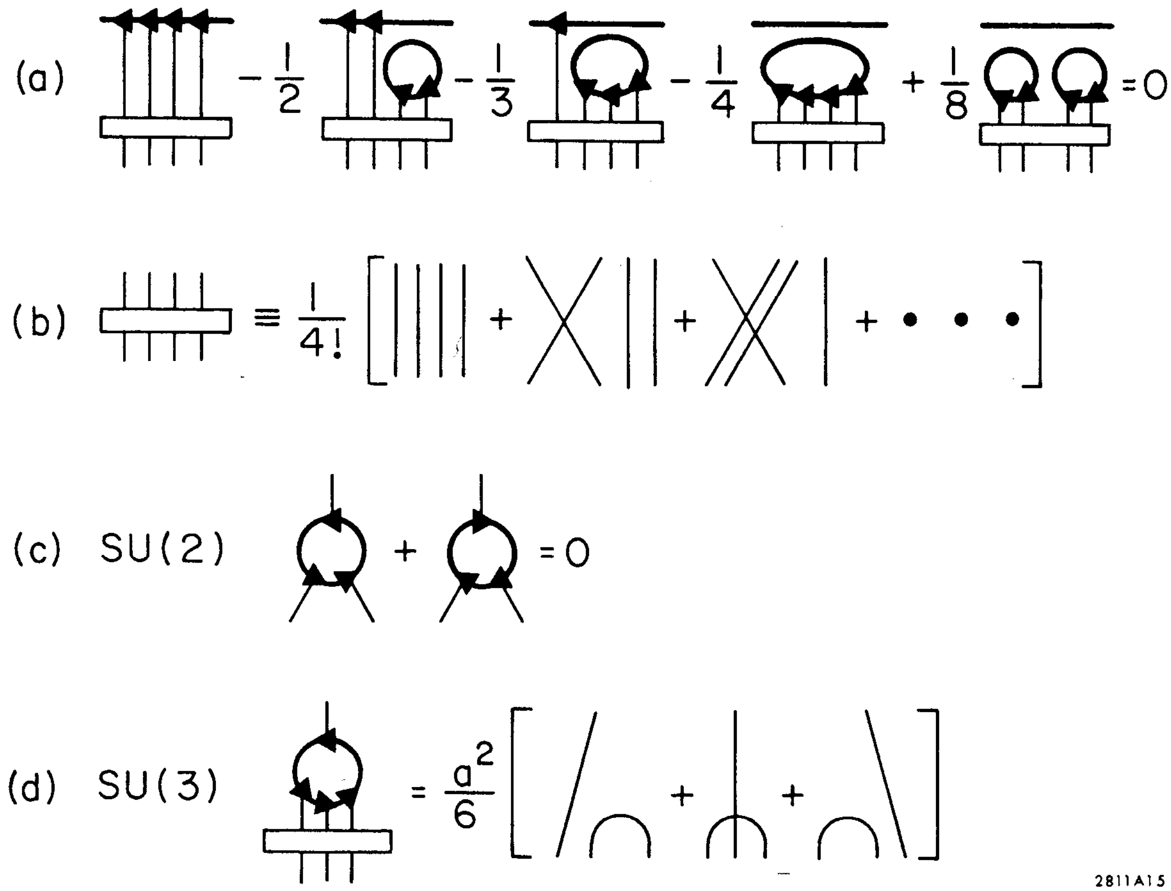


Fig. 16

r	β_r	
1	0	none (tracelessness)
2	1	
3	2	
4	9	
5	44	
6	265	
7	1854	
8	14833	

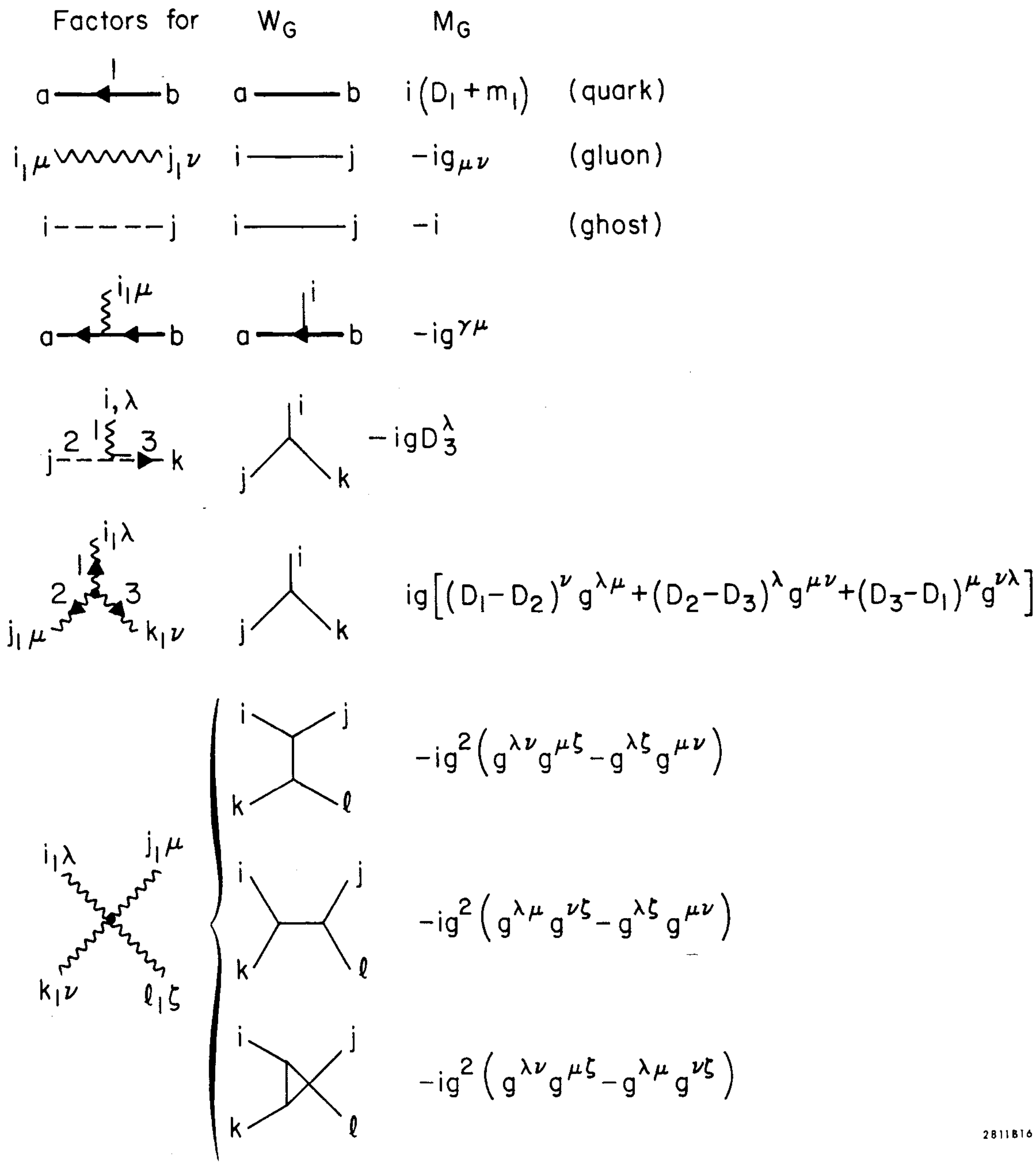
2811A14

Fig. 17



(a)
$-\frac{1}{2}$
$-\frac{1}{3}$
$-\frac{1}{4}$
$+\frac{1}{8}$
$=0$
(b)
$\equiv \frac{1}{4!}$
$+ \cdot \cdot \cdot$
(c) SU(2)
$=0$
(d) SU(3)
$= \frac{a^2}{6}$
2811A15

Fig. 18



Factors for W_G M_G
$i(D_1 + m_1)$ (quark)
$-ig_{\mu\nu}$ (gluon)
$-i$ (ghost)
$-ig\gamma^{\mu}$
$-igD_3^{\lambda}$
$ig\left[(D_1 - D_2)^{\nu} g^{\lambda\mu} + (D_2 - D_3)^{\lambda} g^{\mu\nu} + (D_3 - D_1)^{\mu} g^{\nu\lambda}\right]$
$-ig^2\left(g^{\lambda\nu} g^{\mu\zeta} - g^{\lambda\zeta} g^{\mu\nu}\right)$
$-ig^2\left(g^{\lambda\mu} g^{\nu\zeta} - g^{\lambda\zeta} g^{\mu\nu}\right)$
$-ig^2\left(g^{\lambda\nu} g^{\mu\zeta} - g^{\lambda\mu} g^{\nu\zeta}\right)$
2811B16

Fig. 19

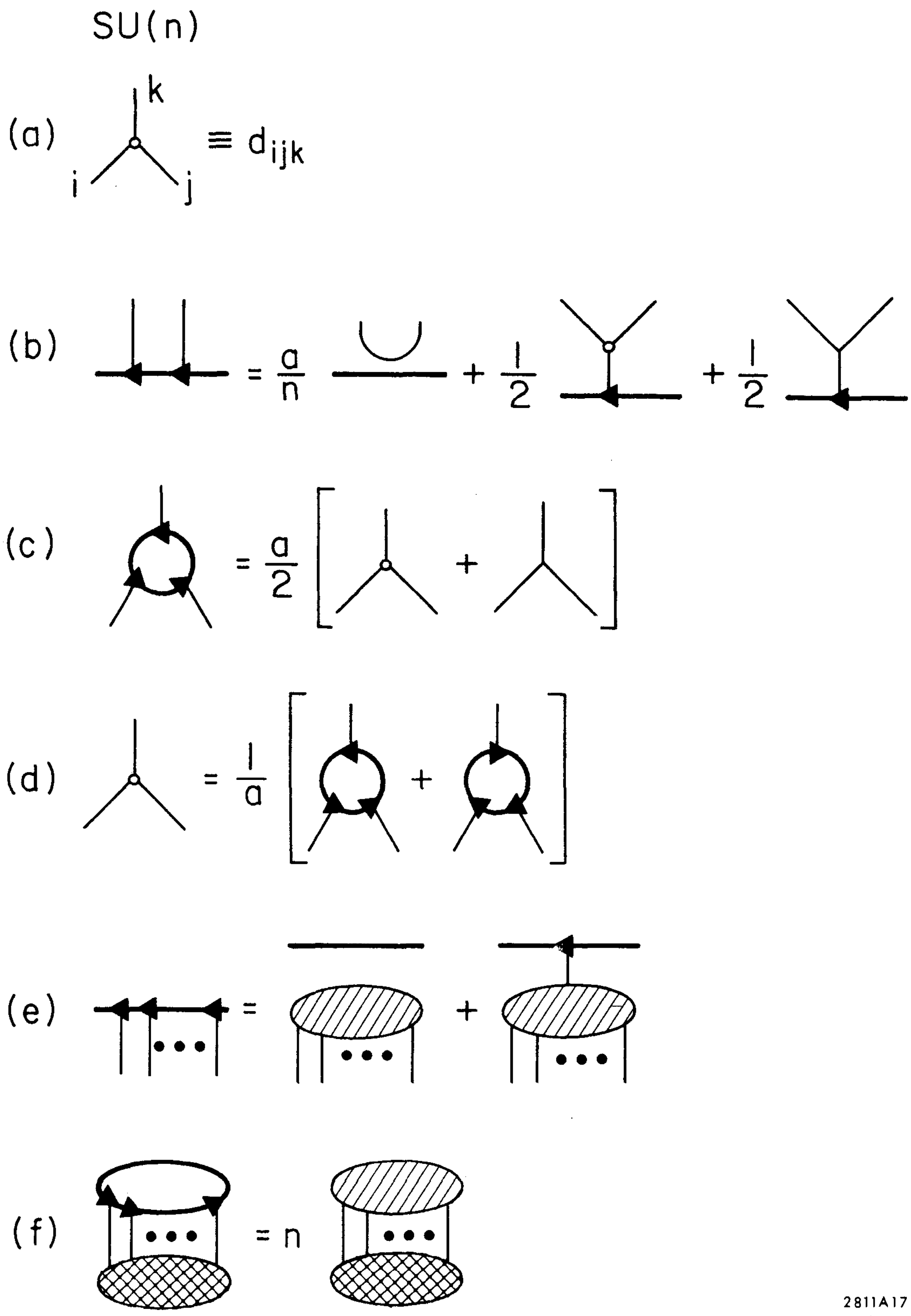
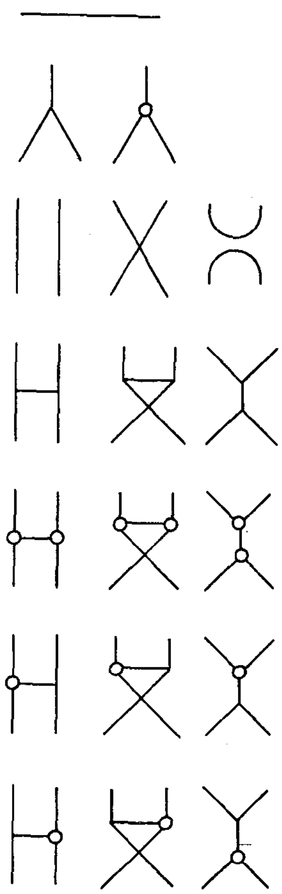


Fig. 20

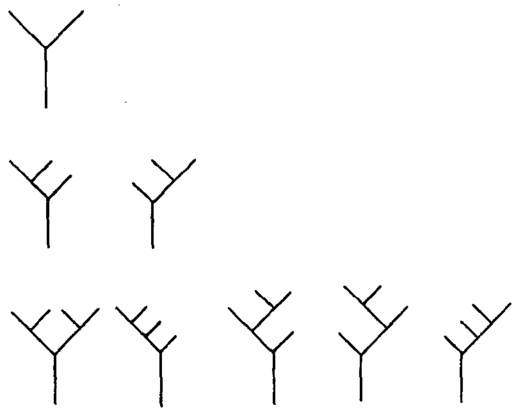
r	α_r
0	1
1	0
2	1
3	2
4	15
5	140
6	1915



2811A18

Fig. 21

r	a_{r-1}	$\bar{\alpha}_r$
3	1	2
4	2	12
5	5	120
6	14	1680



2811A19

Fig. 22

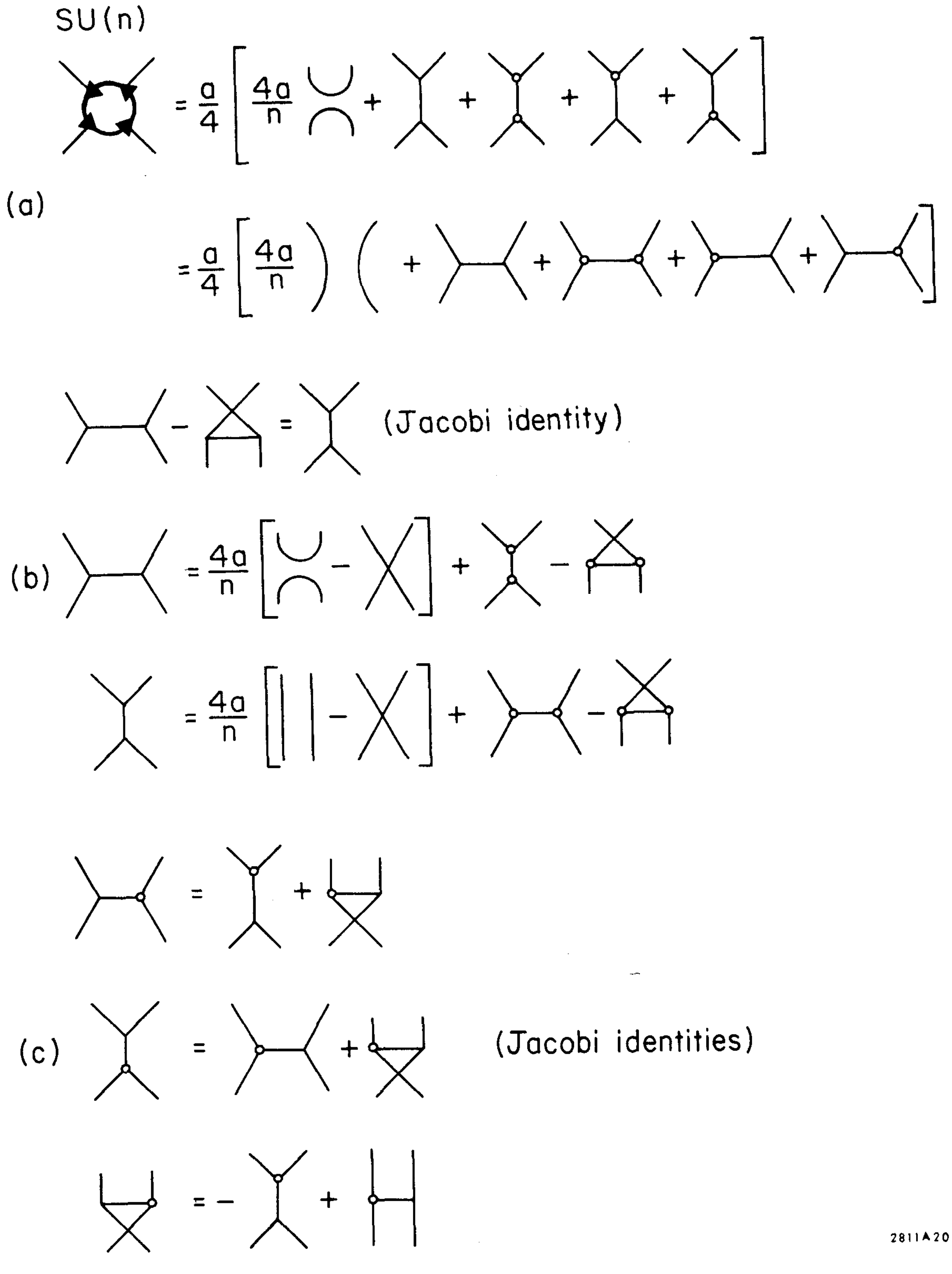


Fig. 23

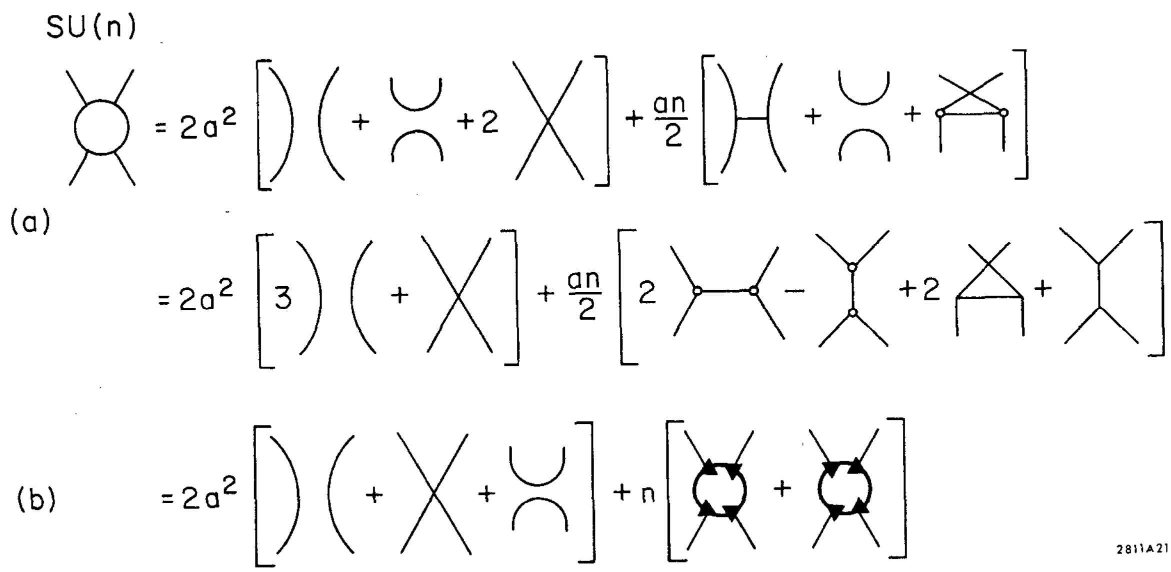


Fig. 24

F_4

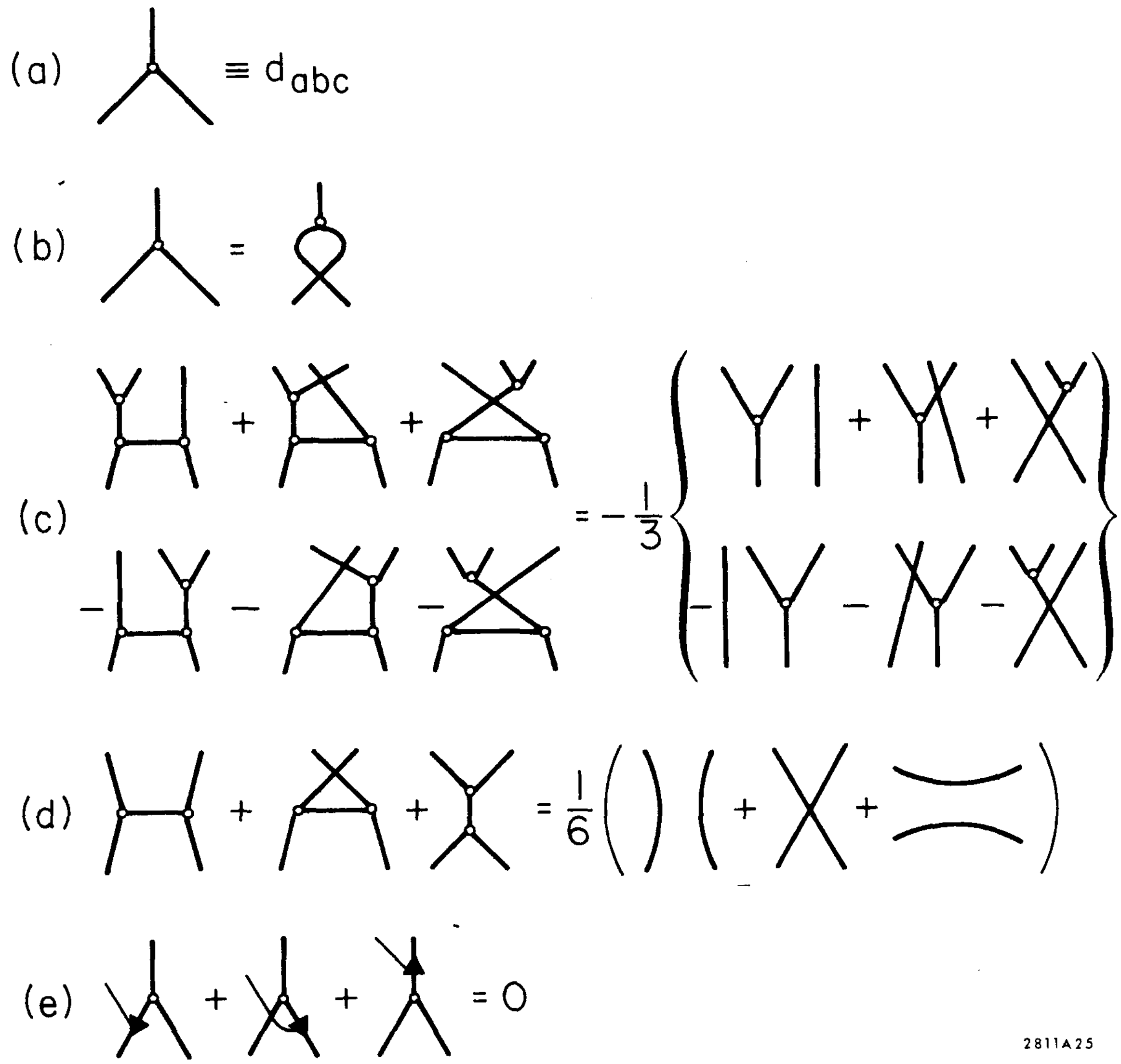


Fig. 25

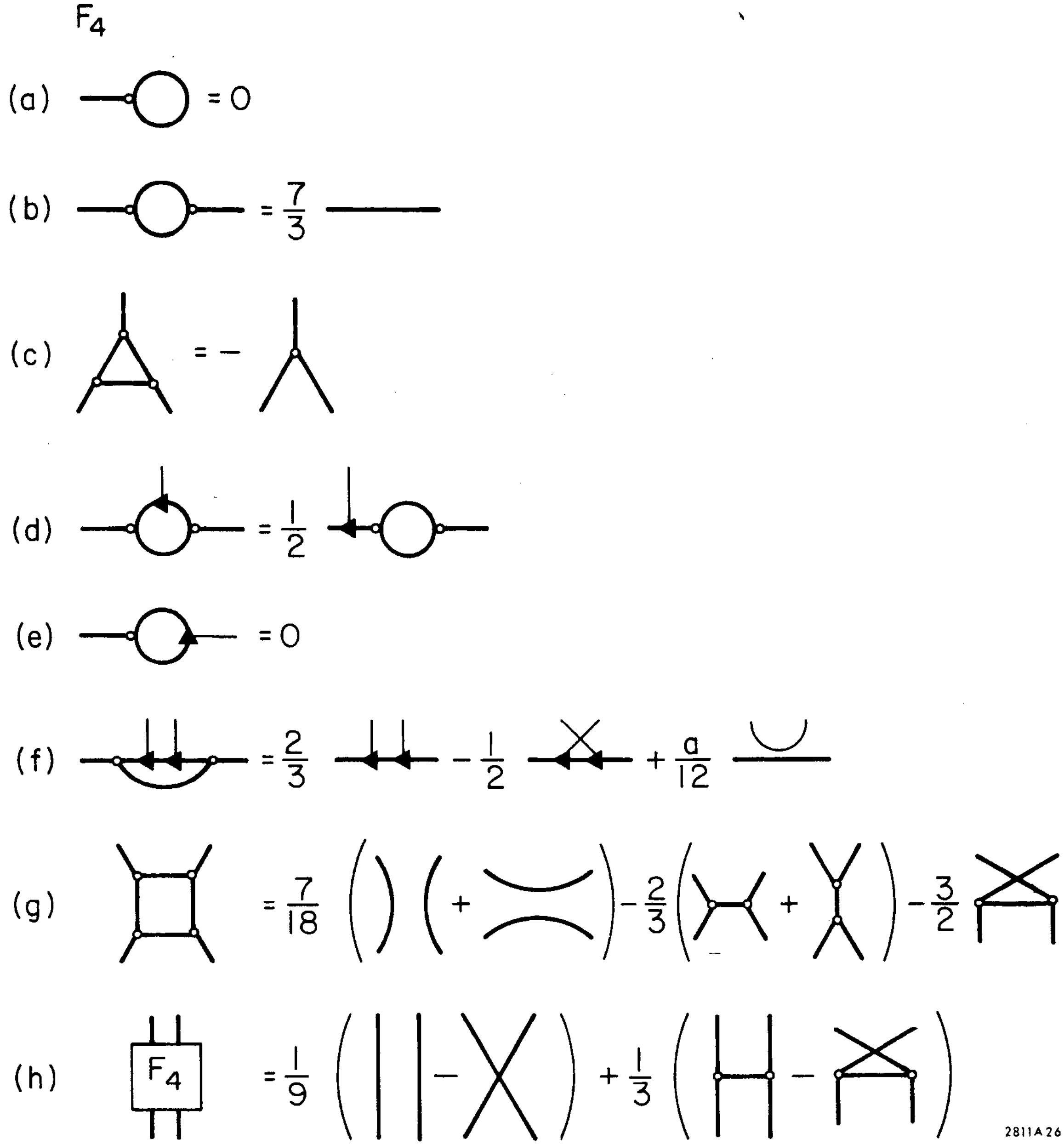


Fig. 26