

Cyclification

*n*Lab materials

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Abstract

For a topological group G , its G -cyclification sends a space A to the homotopy quotient $\mathrm{Map}(G, A) // G$ under translation, equipped with its projection to BG . This is right adjoint to extension, which sends a classifying map to the total space of its principal G -bundle. We discuss the ordinary topological and infinity-categorical adjunctions, and their circle specialization. For $G = S^1$, cyclification is the cyclic loop space construction and the adjunction describes circle dimensional reduction in nonabelian cohomology. We compare double cyclification with toroidification by a homotopy pull-back, identify the integral obstruction to descending a circle-bundle tower, and describe double cyclification as the homotopy quotient of toroidification by the shear subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

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associated *n*Lab entry: <https://ncatlab.org/nlab/show/cyclic+loop+space>

1. Definitions

The categorical conventions are collected in § A: §§ A.1 and A.2 distinguish topological spaces and strict slices from spaces and homotopy slices, while § A.3 recalls nonabelian cohomology. Throughout, spaces and maps are unpointed unless explicitly stated otherwise, and no connectedness assumption is imposed.

1.1. G -Cyclification

Let G be a topological group in the compactly generated setting of Def. A.1; no commutativity or connectedness is assumed. Fix a universal numerable principal right G -bundle $\pi : E = EG \rightarrow B = BG$, as in Def. A.2. Products, pullbacks, quotients, and topological mapping spaces are taken in that setting. The projection to BG is part of a G -cyclification object.

Definition 1.1 (Mapping spaces and translation). For $A \in \text{Top}$, the mapping space $\text{Map}_{\text{Top}}(G, A)$ carries the continuous left G -action

$$(g \cdot \gamma)(t) := \gamma(g^{-1}t), \quad g, t \in G, \quad \gamma \in \text{Map}_{\text{Top}}(G, A). \quad (1)$$

Indeed, $g \cdot (h \cdot \gamma)(t) = \gamma(h^{-1}g^{-1}t) = ((gh) \cdot \gamma)(t)$. Continuity follows from the exponential law (75). Postcomposition by $u : A \rightarrow A'$ is equivariant, so $\text{Map}_{\text{Top}}(G, -)$ takes values in topological spaces with left G -action. Maps $G \rightarrow A$ are unbased; constant maps are fixed points of the action.

Definition 1.2 (Topological G -cyclification). The G -cyclification functor on the ordinary categories of Def. A.4 is

$$\begin{aligned} \text{Top} &\xrightarrow{\text{Cyc}_{\text{Top}}^G} \text{Top}_{/BG} \\ A &\longmapsto (p_A^G : EG \times_G \text{Map}_{\text{Top}}(G, A) \rightarrow BG) \end{aligned} \quad (2)$$

Thus its underlying space is $\text{Map}_{\text{Top}}(G, A) // G$, where the double slash denotes the Borel homotopy quotient for (1). The associated bundle is defined by

$$(eg, \gamma) \sim (e, g \cdot \gamma), \quad p_A^G([e, \gamma]) := \pi(e). \quad (3)$$

On a map $u : A \rightarrow A'$, the functor sends $[e, \gamma]$ to $[e, u \circ \gamma]$, a strict map over BG .

Definition 1.3 (Topological G -extension). The G -extension functor is

$$\begin{aligned} \text{Top}_{/BG} &\xrightarrow{\text{Ext}_{\text{Top}}^G} \text{Top} \\ (p : X \rightarrow BG) &\longmapsto P_p := X \times_{BG} EG \end{aligned} \quad (4)$$

A strict slice morphism $f : (X, p) \rightarrow (Y, q)$ is sent to $(x, e) \mapsto (f(x), e)$. The projection $P_p \rightarrow X$ is the pullback principal G -bundle, with right action $(x, e)g = (x, eg)$. The value of $\text{Ext}_{\text{Top}}^G$ is its total space; the G -action is additional structure on this construction.

These ordinary functors use the fixed universal bundle, actual pullbacks, and strict slice maps, as in Rem. A.5. Their homotopy-theoretic versions are defined separately below.

Lemma 1.4 (Cyclification as a bundle of mapping spaces). *For $z \in BG$, write $E_z := \pi^{-1}(z)$, a right G -torsor. There is a canonical identification*

$$\begin{aligned} (EG \times_G \text{Map}_{\text{Top}}(G, A))_z &\xrightarrow{\cong} \text{Map}_{\text{Top}}(E_z, A) \\ [e, \gamma] &\longmapsto (et \mapsto \gamma(t)) \end{aligned} \quad (5)$$

The topology of $\text{Cyc}_{\text{Top}}^G(A)$ is the topology of this locally trivial bundle of mapping spaces over BG .

Proof. A point $e \in E_z$ identifies G with E_z by $t \mapsto et$. For a function $F : E_z \rightarrow A$, its coordinates are $\gamma_e(t) := F(et)$. Replacing e by eg gives $\gamma_{eg}(t) = \gamma_e(gt) = (g^{-1} \cdot \gamma_e)(t)$, so $[eg, \gamma_{eg}] = [e, \gamma_e]$. This uses the order of multiplication in (1) and does not require G to be abelian. Conversely, this calculation makes the displayed function independent of its representative. On any open set $V \subset BG$ carrying a section $s : V \rightarrow EG$, both descriptions are $V \times \text{Map}_{\text{Top}}(G, A)$, with the same transition maps. The exponential law also gives continuity of fiberwise evaluation. $\square$

Remark 1.5 (The pullback universal G -bundle). The projection p_A^G is a numerable locally trivial bundle with fiber $\text{Map}_{\text{Top}}(G, A)$. Its pullback of $EG \rightarrow BG$ is the principal G -bundle

$$\begin{aligned} EG \times \text{Map}_{\text{Top}}(G, A) &\longrightarrow EG \times_G \text{Map}_{\text{Top}}(G, A) \\ (e, \gamma) &\longmapsto [e, \gamma] \end{aligned} \quad (6)$$

with right action $(e, \gamma)g = (eg, g^{-1} \cdot \gamma)$. Indeed, $(e, \gamma) \mapsto ([e, \gamma], e)$ is a homeomorphism onto $(EG \times_G \text{Map}_{\text{Top}}(G, A)) \times_{BG} EG$. Since EG is contractible, its total space is homotopy equivalent to $\text{Map}_{\text{Top}}(G, A)$.

Definition 1.6 (G -cyclification and extension of homotopy types). Regard G as a group object in $\mathcal{S}$, with pointed delooping BG and basepoint $b : * \rightarrow BG$; thus $\Omega BG \simeq G$. For $A \in \mathcal{S}$, give the derived mapping space $\text{Map}_{\mathcal{S}}(G, A)$ the homotopy coherent translation action of (1). Define

$$\begin{aligned} \mathcal{S} &\xrightarrow{\text{Cyc}^G} \mathcal{S}_{/BG} \\ A &\longmapsto (p_A^G : \text{Map}_{\mathcal{S}}(G, A) // G \rightarrow BG) \end{aligned} \quad (7)$$

The homotopy quotient is the colimit of the action; its projection is induced by the map of G -spaces to the point. Postcomposition gives the action on maps and all higher homotopies. Define also

$$\begin{aligned} \mathcal{S}_{/BG} &\xrightarrow{\text{Ext}^G} \mathcal{S} \\ (p : X \rightarrow BG) &\longmapsto b^*X := X \times_{BG}^h * \end{aligned} \quad (8)$$

Extension acts on morphisms by homotopy pullback; its total space maps to X with homotopy fiber G . Cyclification has the homotopy fiber sequence

$$\text{Map}_{\mathcal{S}}(G, A) \longrightarrow \text{Cyc}^G(A) \xrightarrow{p_A^G} BG. \quad (9)$$

These constructions make sense for every group object in spaces, hence in particular for the homotopy type of every topological group under consideration; see [SS24, § 2.2, (29)–(31)].

Remark 1.7 (Comparison of the topological and homotopy models). The numerable bundle $EG \rightarrow BG$ is a fibration with contractible total space (Def. A.2). Choosing a point above

b therefore identifies $X \times_{BG} EG$ with a model of $X \times_{BG}^h *$: topological extension presents Ext^G .

If the underlying space of G has the homotopy type of a CW complex, then $\text{Map}_{\text{Top}}(G, A)$ computes $\text{Map}_S(G, A)$. Indeed, replace G by a homotopy-equivalent CW complex and use (79); precomposition with a homotopy equivalence induces a homotopy equivalence of mapping spaces. The comparison respects the translation action as a homotopy coherent action. Taking its associated bundle shows that $\text{Cyc}_{\text{Top}}^G$ then presents Cyc^G . Under this hypothesis the ordinary functors preserve weak homotopy equivalences, by the mapping-space comparison and the homotopy fiber sequences over BG , and induce the displayed functors on the localizations of (81).

For a general topological group, the ordinary construction and adjunction still use continuous mapping spaces; the formula in § uses derived mapping spaces and its homotopy quotient there. The CW homotopy-type hypothesis is a sufficient comparison condition, not a restriction on either adjunction. It holds for Lie groups and discrete groups, including S^1 and its finite products.

1.2. S^1 -Cyclification

Definition 1.8 (The circle specialization: free loops and cyclification). Specialize the homotopy-type functors of Def. 1.6 to $G = S^1$, and the ordinary functors of Defs. 1.2 and 1.3 to the model $ES^1 = S^\infty \rightarrow BS^1 = \mathbb{C}P^\infty$ of Def. A.3. The *free loop spaces* are

$$L_{\text{Top}}A := \text{Map}_{\text{Top}}(S^1, A), \quad LA := \text{Map}_S(S^1, A). \quad (10)$$

The translation action specializes to loop rotation,

$$(g \cdot \gamma)(t) = \gamma(g^{-1}t), \quad g, t \in S^1. \quad (11)$$

We omit the group superscript in this case:

$$\text{Cyc} := \text{Cyc}^{S^1}, \quad \text{Ext} := \text{Ext}^{S^1}, \quad p_A := p_A^{S^1},$$

and similarly $\text{Cyc}_{\text{Top}} := \text{Cyc}_{\text{Top}}^{S^1}$ and $\text{Ext}_{\text{Top}} := \text{Ext}_{\text{Top}}^{S^1}$. Explicitly,

$$\begin{aligned} \text{Cyc}_{\text{Top}}(A) &= (ES^1 \times_{S^1} L_{\text{Top}}A \xrightarrow{p_A} BS^1), \\ \text{Cyc}(A) &= (LA // S^1 \xrightarrow{p_A} BS^1) \end{aligned} \quad (12)$$

and

$$\begin{aligned} \text{Ext}_{\text{Top}}(p : X \rightarrow BS^1) &= X \times_{BS^1} ES^1, \\ \text{Ext}(p : X \rightarrow BS^1) &= X \times_{BS^1}^h *. \end{aligned} \quad (13)$$

The underlying space of $\text{Cyc}(A)$ is its *cyclic loop space*. We will use the unindexed notation in the circle-bundle discussion and examples below.

Remark 1.9 (The circle projection and its universal Chern class). Specializing (6) gives the principal circle bundle

$$\begin{aligned} ES^1 \times L_{\text{Top}}A &\longrightarrow ES^1 \times_{S^1} L_{\text{Top}}A \\ (e, \gamma) &\longmapsto [e, \gamma] \end{aligned} \quad (14)$$

If $u := c_1(ES^1 \rightarrow BS^1) \in H^2(BS^1; \mathbb{Z})$, its Chern class is $c_1^{\text{Cyc}} := p_A^* u$. Its total space is homotopy equivalent to $L_{\text{Top}}A$. This explains the customary homotopy-theoretic notation $S^1 \rightarrow LA \rightarrow \text{Cyc}(A)$, while (14) specifies the actual topological principal bundle. The

projection of cyclification itself fits into

$$LA \longrightarrow \text{Cyc}(A) \xrightarrow{p_A} BS^1. \quad (15)$$

Example 1.10 (Cyclification of $B^{n+1}\mathbb{Z}$). Let $n \geq 0$, and write $B^r\mathbb{Z} = K(\mathbb{Z}, r)$, including the discrete space $B^0\mathbb{Z} = \mathbb{Z}$. Identify $B := BS^1 \simeq B^2\mathbb{Z}$, with universal Chern class u as in Rem. 1.9. The cup-product map

$$\mu_n = \cup : B^2\mathbb{Z} \times B^n\mathbb{Z} \longrightarrow B^{n+2}\mathbb{Z}$$

represents $\text{pr}_1^* u \cup \text{pr}_2^* \iota_n$, where $\iota_n \in H^n(B^n\mathbb{Z}; \mathbb{Z})$ is the universal class; for $n = 0$, it is the function $m \mapsto m$. There is an equivalence in $\mathcal{S}_B$,

$$\text{Cyc}(B^{n+1}\mathbb{Z}) \simeq \text{hofib}_0(B^2\mathbb{Z} \times B^n\mathbb{Z} \xrightarrow{\cup} B^{n+2}\mathbb{Z}), \quad (16)$$

where the homotopy fiber is taken at zero and its map to B is induced by the first projection. In particular, both squares in

$$\begin{array}{ccccc} B^n\mathbb{Z} \times B^{n+1}\mathbb{Z} & \longrightarrow & \text{Cyc } B^{n+1}\mathbb{Z} & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ B^n\mathbb{Z} & \xrightarrow{(0, \text{id})} & B^2\mathbb{Z} \times B^n\mathbb{Z} & \xrightarrow{\cup} & B^{n+2}\mathbb{Z}, \end{array} \quad (17)$$

are homotopy pullbacks. The case $n = 2$ is the classifying-space calculation of [BS05, Thm. 2.17].

Proof. We work with the derived mapping spaces of Defs. A.6 and A.7. We use the circle adjunction proved independently in § 2.1 below. Let $c : X \rightarrow B$ classify the principal circle bundle $p : P \rightarrow X$. By Rem. A.8 we may use a CW representative of X and the pulled-back universal bundle, so $P \simeq \text{Ext}(c)$.

First we establish the natural homotopy fiber sequence

$$\text{Map}(P, B^{n+1}\mathbb{Z}) \xrightarrow{p_!} \text{Map}(X, B^n\mathbb{Z}) \xrightarrow{c \cup -} \text{Map}(X, B^{n+2}\mathbb{Z}), \quad (18)$$

with fiber at the zero map. Here $(c \cup -)(a) = \mu_n \circ (c, a)$, and on components $p_!$ is integration over the oriented circle fiber. Keeping this statement at the level of mapping spaces will also retain the nullhomotopies needed for the asserted pullback.

Let $L \rightarrow X$ be the associated complex line bundle, oriented so that $c_1(L) = c^*u$ and $S(L) \cong P$. Write $D(L)$ for its disk bundle and $\text{Th}(L) := D(L)/S(L)$ for its pointed Thom space. The inclusion of the sphere bundle is a cofibration. Thus the pointed cofiber sequence and its Puppe continuation are

$$P_+ \longrightarrow D(L)_+ \longrightarrow \text{Th}(L) \longrightarrow \Sigma P_+ \longrightarrow \Sigma D(L)_+,$$

where Y_+ means that a disjoint basepoint has been added. Applying $\text{Map}_*(-, B^{n+2}\mathbb{Z})$, using $D(L) \simeq X$ and $\Omega B^{n+2}\mathbb{Z} \simeq B^{n+1}\mathbb{Z}$, gives a homotopy fiber sequence

$$\text{Map}(P, B^{n+1}\mathbb{Z}) \longrightarrow \text{Map}_*(\text{Th}(L), B^{n+2}\mathbb{Z}) \xrightarrow{s^*} \text{Map}(X, B^{n+2}\mathbb{Z}), \quad (19)$$

where $s : X_+ \rightarrow \text{Th}(L)$ is induced by the zero section.

Let $U \in \widetilde{H}^2(\text{Th}(L); \mathbb{Z})$ be the integral Thom class. Cup product with U , using the Thom diagonal $\text{Th}(L) \rightarrow X_+ \wedge \text{Th}(L)$, defines

$$\Phi_L : \text{Map}(X, B^n\mathbb{Z}) \longrightarrow \text{Map}_*(\text{Th}(L), B^{n+2}\mathbb{Z}). \quad (20)$$

This map is an equivalence: on π_k at zero it is the Thom isomorphism

$$H^{n-k}(X; \mathbb{Z}) \xrightarrow{\cup U} \widetilde{H}^{n+2-k}(\text{Th}(L); \mathbb{Z}), \quad k \geq 0,$$

with negative-degree groups zero. This includes the bijection on components, and additivity gives the same conclusion at every component. The zero-section identity $s^*U = c_1(L) = c^*u$ identifies $s^*\Phi_L$ with $c \cup -$. These are the usual Thom and Euler-class descriptions of the Gysin sequence; see [Hat02, § 4.D, especially Cor. 4D.9 and pp. 442–444]. Choose the Thom class and this identification on the universal line bundle and pull them back along c . The construction is then natural, including its homotopies, as c varies. Substituting (20) into (19) proves (18).

The same Puppe sequence identifies the preceding map as

$$\Omega \text{Map}(X, B^{n+2}\mathbb{Z}) \simeq \text{Map}(X, B^{n+1}\mathbb{Z}) \xrightarrow{p^*} \text{Map}(P, B^{n+1}\mathbb{Z}). \quad (21)$$

Hence, in the homotopy-fiber description of (18), pullback of a class from X is the inclusion of the loop coordinate over zero. We use the suspension and Thom identifications with this convention. This observation retains the data needed for the constant-loop map in (28).

Now set $F_n := \text{hofib}_0(\mu_n)$, with its projection to B . The defining homotopy pullback gives

$$\text{Map}_{\mathcal{S}/B}(c, F_n) \simeq \text{hofib}_0\left(\text{Map}(X, B^n\mathbb{Z}) \xrightarrow{c \cup -} \text{Map}(X, B^{n+2}\mathbb{Z})\right). \quad (22)$$

Indeed, a map over B is a map $a : X \rightarrow B^n\mathbb{Z}$ together with a specified nullhomotopy of $\mu_n(c, a)$. By (18) and the circle adjunction of Cor. 2.5, naturally in c ,

$$\text{Map}_{\mathcal{S}/B}(c, F_n) \simeq \text{Map}(P, B^{n+1}\mathbb{Z}) \simeq \text{Map}_{\mathcal{S}/B}(c, \text{Cyc}(B^{n+1}\mathbb{Z})).$$

Yoneda in $\mathcal{S}/B$ proves (16), including compatibility with the projection to BS^1 . Under this equivalence, the second class of a pair (c, a) is $a = p_!h$, where $h \in H^{n+1}(P; \mathbb{Z})$ is the class corresponding to the map into $\text{Cyc}(B^{n+1}\mathbb{Z})$.

The right square of (17) is now the defining homotopy pullback of F_n . Pull it back along $(0, \text{id}) : B^n\mathbb{Z} \rightarrow B \times B^n\mathbb{Z}$. Cup product with zero has its canonical nullhomotopy, so the resulting space is

$$\text{hofib}_0(0 : B^n\mathbb{Z} \rightarrow B^{n+2}\mathbb{Z}) \simeq B^n\mathbb{Z} \times \Omega B^{n+2}\mathbb{Z} \simeq B^n\mathbb{Z} \times B^{n+1}\mathbb{Z}.$$

This proves the left square as well. Equivalently, it identifies the fiber of cyclification over the basepoint of BS^1 with the familiar splitting $L(B^{n+1}\mathbb{Z}) \simeq B^{n+1}\mathbb{Z} \times B^n\mathbb{Z}$ into evaluation and based loops, with factors ordered as in the diagram. $\square$

Example 1.11 (Cyclification of $BS^1 \simeq B^2\mathbb{Z}$). Of particular importance (for the discussion of iterated cyclification in § 2.2) is the special case $n = 1$ of Ex. 1.10, where we have

$$\text{Cyc}(BS^1) \simeq \text{Cyc}(B^2\mathbb{Z}) \simeq \text{hofib}_0(B^2\mathbb{Z} \times B\mathbb{Z} \xrightarrow{\cup} B^3\mathbb{Z}), \quad (23)$$

hence forming a universal cone whose components we will label as follows:

$$\begin{array}{ccc} \text{Cyc}(BS^1) & \longrightarrow & * \\ \downarrow (f_2^A, f_1) & \swarrow f_2^M & \downarrow \\ B^2\mathbb{Z} \times B\mathbb{Z} & \xrightarrow{\cup} & B^3\mathbb{Z}. \end{array} \quad (24)$$

Here $f_2^A := p_{BS^1}$ is the projection (12), while f_1 is the projection to $B\mathbb{Z}$. The symbol f_2^M denotes the specified nullhomotopy

$$f_2^M : f_2^A \cup f_1 \simeq 0.$$

Accordingly, a map from X to this homotopy fiber consists of maps $f_2^A : X \rightarrow B^2\mathbb{Z}$ and $f_1 : X \rightarrow B\mathbb{Z}$, together with this nullhomotopy; we use the same symbols for the pulled-

back universal data. In particular, f_2^M is part of the homotopy-fiber data, rather than an independent map $X \rightarrow B^2\mathbb{Z}$.

If we are given a map out of a circle extension,

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & B^2\mathbb{Z} \\ \text{fib}(c_1) \downarrow & & \\ X & \searrow c_1 & \\ & & B^2\mathbb{Z}, \end{array}$$

then for the Ext/Cyc-adjunct map

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\alpha}} & \text{Cyc}B^2\mathbb{Z} \\ & \searrow (c_1, f_1(\tilde{\alpha})) & \downarrow (f_2^A, f_1) \\ & & B^2\mathbb{Z} \times B\mathbb{Z} \end{array}$$

we have

$$[f_1(\tilde{\alpha})] = \text{fib}(c_1)_![\alpha]. \quad (25)$$

Below, these data play a role in

- (i) the constant-loop comparison in Def. 1.13,
- (ii) the obstruction map in Prop. 2.10.

1.3. Toroidification

We next specialize G -cyclification to the product of two circles. All mapping spaces and homotopy quotients in this subsection are taken in $\mathcal{S}$ unless the ordinary model is indicated explicitly. The factor order will also fix the order of the circle-bundle towers in § 2.2.

Definition 1.12 (Toroidification). Write $T^2 := S_{\text{in}}^1 \times S_{\text{out}}^1$, with inner coordinate z and outer coordinate w . The *toroidification functor* is the specialization of Def. 1.6 to this group:

$$\begin{aligned} \mathcal{S} & \xrightarrow{\text{Tor} := \text{Cyc}^{T^2}} \mathcal{S}_{/BT^2} \\ A & \mapsto (p_A^{T^2} : \text{Map}(T^2, A) // T^2 \rightarrow BT^2) \end{aligned} \quad (26)$$

Here $BT^2 \simeq BS_{\text{in}}^1 \times BS_{\text{out}}^1$, and the translation action is

$$((s, t) \cdot F)(z, w) = F(s^{-1}z, t^{-1}w).$$

We pronounce Tor as “toroidification” and also use it for the underlying space when its projection is displayed or understood. The ordinary model is $\text{Tor}_{\text{Top}} := \text{Cyc}_{\text{Top}}^{T^2}$; since T^2 is a CW complex it presents (26), by Rem. 1.7. Its adjunction is the case $G = T^2$ of Thms. 2.1 and 2.3. Integral changes of the two torus coordinates act on this family and its base, as made explicit in Prop. 2.15.

Definition 1.13 (The constant-loop comparison on bases). Constant loops define an S_{out}^1 -equivariant map from BS_{in}^1 , with trivial action, to $\text{Map}(S_{\text{out}}^1, BS_{\text{in}}^1)$. Its homotopy quotient is the map

$$\begin{aligned} BT^2 = BS_{\text{in}}^1 \times BS_{\text{out}}^1 & \xrightarrow{\iota} \text{Cyc}(BS^1) \\ (a, [e]) & \mapsto [e, \text{const}_a] \end{aligned} \quad (27)$$

The element formula uses the topological model. In particular, $f_2^A \circ \iota = p_{BS^1} \circ \iota = \text{pr}_{\text{out}}$. The factor order is that of Def. 1.12.

Ex. 1.11 gives a direct description in the homotopy-fiber model (23). Use $B^2\mathbb{Z} \simeq \Omega B^3\mathbb{Z}$. With the canonical nullhomotopy of $b \cup 0$, the map is

$$\begin{aligned} (B^2\mathbb{Z})_{\text{in}} \times (B^2\mathbb{Z})_{\text{out}} &\xrightarrow{\iota} \text{Cyc}(BS^1) \\ (a, b) &\longmapsto (b, 0, \ell_a) \end{aligned} \quad (28)$$

Here the target coordinates are (f_2^A, f_1, f_2^M) , and $\ell_a \in \Omega B^3\mathbb{Z}$ is the loop corresponding to a , regarded as a nullhomotopy of $b \cup 0$ by concatenation with the canonical one. Thus $f_2^A = b$, $f_1 = 0$, and $f_2^M = \ell_a$: the outer class is the first coordinate, while the inner class is retained in the nullhomotopy coordinate. This notation describes maps and families of maps in $\mathcal{S}$, not only points or cohomology classes.

Equivalently, both squares in the following diagram are homotopy pullbacks:

$$\begin{array}{ccccc} (B^2\mathbb{Z})_{\text{in}} \times (B^2\mathbb{Z})_{\text{out}} & \xrightarrow{\iota} & \text{Cyc}(BS^1) & \longrightarrow & * \\ \text{pr}_{\text{out}} \downarrow & \lrcorner & \downarrow (f_2^A, f_1) & \lrcorner & \downarrow \\ (B^2\mathbb{Z})_{\text{out}} & \xrightarrow{(\text{id}, 0)} & (B^2\mathbb{Z})_{\text{out}} \times B\mathbb{Z} & \xrightarrow{\cup} & B^3\mathbb{Z}. \end{array} \quad (29)$$

Verification of the homotopy-fiber description. Let $(a, b) : X \rightarrow BT^2$, and let $p : P_b \rightarrow X$ be the outer circle bundle. Constant loops correspond under the circle adjunction to the pulled-back map $a \circ p : P_b \rightarrow B^2\mathbb{Z}$. In the Gysin fiber sequence of Ex. 1.10, the map preceding integration is

$$\text{Map}(X, B^2\mathbb{Z}) \xrightarrow{p^*} \text{Map}(P_b, B^2\mathbb{Z}).$$

As explained in (21), this is the inclusion of the loop coordinate $\Omega \text{Map}(X, B^3\mathbb{Z})$ into the homotopy fiber of $b \cup -$. It sends a to $(0, \ell_a)$ in that fiber. Including the outer class b gives (28), naturally in X . Pulling the cup-product fiber back along $(\text{id}, 0)$ gives

$$(B^2\mathbb{Z})_{\text{out}} \times \Omega B^3\mathbb{Z} \simeq (B^2\mathbb{Z})_{\text{out}} \times (B^2\mathbb{Z})_{\text{in}},$$

with the same map to $\text{Cyc}(BS^1)$. Reordering the two factors proves (29). $\square$

2. Properties

2.1. The Ext/Cyc Adjunction

We first work with the fixed topological group G and universal bundle $E = EG \rightarrow B = BG$ of Def. A.2. The circle case is recorded after the general adjunctions.

Theorem 2.1 (The ordinary G -adjunction). *The functors of Defs. 1.2 and 1.3 form an adjunction*

$$\text{Ext}_{\text{Top}}^G : \text{Top}_{/BG} \rightleftarrows \text{Top} : \text{Cyc}_{\text{Top}}^G, \quad \text{Ext}_{\text{Top}}^G \dashv \text{Cyc}_{\text{Top}}^G.$$

For $p : X \rightarrow BG$ and $A \in \text{Top}$, its natural bijection is

$$\text{Hom}_{\text{Top}}(X \times_{BG} EG, A) \cong \text{Hom}_{\text{Top}_{/BG}}(p, \text{Cyc}_{\text{Top}}^G(A)). \quad (30)$$

No CW-type assumption on G is required for this ordinary adjunction.

Proof. Given $F : X \times_{BG} EG \rightarrow A$, restrict it to the G -torsor over each x and define its transpose by

$$\tilde{F}(x) := [e, \gamma_{x,e}], \quad \gamma_{x,e}(t) := F(x, et), \quad e \in E_{p(x)}, \quad t \in G. \quad (31)$$

Replacing e by eg gives $\gamma_{x,eg}(t) = \gamma_{x,e}(gt) = (g^{-1} \cdot \gamma_{x,e})(t)$, so (3) leaves the class unchanged. Its projection is $p(x)$, making $\tilde{F}$ a strict slice map. For continuity, choose a local section $s : V \rightarrow EG$. On $p^{-1}(V)$ the formula is the exponential transpose of

$$\begin{aligned} p^{-1}(V) \times G &\longrightarrow A \\ (x, t) &\longmapsto F(x, s(p(x))t) \end{aligned}$$

The exponential law (75) gives continuity, and these expressions agree on overlaps.

Conversely, a strict slice map $f : X \rightarrow \text{Cyc}_{\text{Top}}^G(A)$ specifies, by Lem. 1.4, a function $f_x : E_{p(x)} \rightarrow A$. Define

$$\check{f}(x, e') := f_x(e'); \quad f(x) = [e, \gamma], \quad e' = et \implies \check{f}(x, e') = \gamma(t). \quad (32)$$

This is independent of representatives and continuous by local evaluation. Restricting and then evaluating recovers F ; conversely, evaluation and restriction recover the given family. Thus $\check{\tilde{F}} = F$ and $\tilde{\check{f}} = f$. The constructions commute with precomposition by strict slice maps and postcomposition by maps of A , proving naturality.

The bijection is also a natural homeomorphism of topological mapping spaces, with the strict slice topology of (78). For an arbitrary parameter space Z , the identity $(Z \times X) \times_{BG} EG \cong Z \times (X \times_{BG} EG)$ and the exponential law identify continuous Z -families on both sides. These natural bijections on maps from every Z imply the homeomorphism by Yoneda in Top . $\square$

Proposition 2.2 (Unit and counit for G). *For $p : X \rightarrow BG$, the unit is*

$$\begin{aligned} X &\xrightarrow{\eta_p^G} \text{Cyc}_{\text{Top}}^G(P_p) \\ x &\longmapsto [e, t \mapsto (x, et)] \end{aligned} \quad (33)$$

For $A \in \text{Top}$, the counit is fiberwise evaluation,

$$\begin{aligned} (EG \times_G \text{Map}_{\text{Top}}(G, A)) \times_{BG} EG &\xrightarrow{\varepsilon_A^G} A \\ ([e, \gamma], et) &\longmapsto \gamma(t) \end{aligned} \quad (34)$$

Under the homeomorphism in Rem. 1.5, it is $(e, \gamma) \mapsto \gamma(1_G)$, where 1_G is the identity of G .

Proof. These are the transposes of the identity maps of P_p and $\text{Cyc}_{\text{Top}}^G(A)$, by (31) and (32). The first triangle identity evaluates the map $t \mapsto (x, et)$ at 1_G , recovering (x, e) . For the second, the composite at $[e, \gamma]$ gives $t \mapsto \varepsilon_A^G([e, \gamma], et) = \gamma(t)$, hence recovers $[e, \gamma]$. Both identities are strict equalities of continuous maps. $\square$

Theorem 2.3 (The G -adjunction of spaces). *For every group object G in spaces, the functors of Def. 1.6 form an adjunction of ∞ -categories*

$$\text{Ext}^G : \mathcal{S}_{/BG} \rightleftarrows \mathcal{S} : \text{Cyc}^G, \quad \text{Ext}^G \dashv \text{Cyc}^G.$$

For $p : X \rightarrow BG$ and $A \in \mathcal{S}$, this is the natural equivalence of spaces

$$\text{Map}_{\mathcal{S}}(X \times_{BG}^h *, A) \simeq \text{Map}_{\mathcal{S}_{/BG}}(p, \text{Cyc}^G(A)). \quad (35)$$

Equivalently, $\text{Ext}^G = b^*$ and $\text{Cyc}^G \simeq b_*$, the right adjoint of pullback along $b : * \rightarrow BG$.

Proof. Write $B := BG$. The classification of families of spaces gives the straightening equivalence

$$\begin{aligned} \mathcal{S}_{/B} &\xrightarrow{\simeq} \text{Fun}(B, \mathcal{S}) \\ p &\longmapsto F_p \end{aligned} \quad (36)$$

Here $F_p(z) := \text{hofib}_z(p)$, and paths in B act by transport in this family. This is the case of straightening over an ∞ -groupoid; see [Lur09, § 2.2.1]. Pullback b^* is evaluation at b , whose right adjoint is right Kan extension along $b : * \rightarrow B$.

The pointwise right Kan extension formula is

$$(\text{Ran}_b A)(z) \simeq \lim_{\text{Path}_B(z, b)} A \simeq \text{Map}_{\mathcal{S}}(\text{Path}_B(z, b), A). \quad (37)$$

Here $(z \downarrow b)$ is the space of paths from z to b , and the limit of a constant A -diagram over a space K is $\text{Map}_{\mathcal{S}}(K, A)$. For the formula and its universal property, see [Lur09, §§ 4.3.2–4.3.3].

At b , the value is $\text{Map}_{\mathcal{S}}(\Omega B, A) \simeq \text{Map}_{\mathcal{S}}(G, A)$. We check its action, including the order of multiplication. In incoming-path coordinates, transport by g takes $f(h)$ to $f(hg)$. Reversing paths, write $\gamma(t) := f(t^{-1})$; the transported function is

$$f(t^{-1}g) = \gamma(g^{-1}t).$$

This is exactly (1), also for nonabelian G . For a group object these formulas express the corresponding coherently associative operations. Unstraightening therefore gives $\text{Map}_{\mathcal{S}}(G, A) // G \rightarrow BG$, namely $\text{Cyc}^G(A)$.

The universal property of right Kan extension now gives

$$\text{Map}_{\text{Fun}(B, \mathcal{S})}(F_p, \text{Ran}_b A) \simeq \text{Map}_{\mathcal{S}}(F_p(b), A).$$

Since $F_p(b) = X \times_B^h *$, this is (35). Its naturality as an equivalence of mapping spaces includes all higher homotopies required for the adjunction. $\square$

Remark 2.4 (Relation between the two adjunctions). Under the comparison hypothesis in Rem. 1.7, the topological unit and counit of Prop. 2.2 present those of Thm. 2.3. Intrinsically, for every G one has

$$\text{Ext}^G \text{Cyc}^G(A) \simeq \text{Map}_{\mathcal{S}}(G, A),$$

and the counit is evaluation at 1_G . The mapping spaces in (35) are derived; topological representatives require the replacements of Rem. A.8.

Corollary 2.5 (The circle adjunction). *For the circle functors of Def. 1.8, the general adjunctions specialize to*

$$\begin{aligned} \text{Ext}_{\text{Top}} : \text{Top}_{/BS^1} &\rightleftarrows \text{Top} : \text{Cyc}_{\text{Top}}, & \text{Ext}_{\text{Top}} \dashv \text{Cyc}_{\text{Top}}, & \text{Ext} \dashv \text{Cyc}. \\ \text{Ext} : \mathcal{S}_{/BS^1} &\rightleftarrows \mathcal{S} : \text{Cyc}, \end{aligned}$$

In particular,

$$\text{Map}_{\mathcal{S}}(X \times_{BS^1}^h *, A) \simeq \text{Map}_{\mathcal{S}_{/BS^1}}(c, \text{Cyc}(A)), \quad c : X \rightarrow BS^1. \quad (38)$$

The unit and counit are (33) and (34) with $G = S^1$; we suppress their G -superscripts in this specialization.

Proof. Substitute $G = S^1$ in Thms. 2.1 and 2.3 and Prop. 2.2. Since S^1 is a CW complex, the comparison in Rem. 1.7 applies to these topological and homotopy models. $\square$

Remark 2.6 (The adjunction on nonabelian cohomology). For a fixed G , let $c : X \rightarrow BG$ and put $P_c := X \times_{BG}^h *$. Applying π_0 to (35) gives the natural bijection

$$H(P_c; A) \cong H_c(X; p_A^G) \quad (39)$$

in the notation of Defs. A.11 and A.16. A map from the total space of the classified principal G -bundle to A corresponds to $f : X \rightarrow \text{Cyc}^G(A)$ with a specified homotopy $p_A^G \circ f \simeq c$, modulo homotopy in the slice. The chosen classifying data are retained. For pointed connected A , the left side can be written $H^1(P_c; \Omega A)$. The coefficient family on the right has fiber $\text{Map}_s(G, A)$, which need not be connected. For $G = S^1$, this is the circle dimensional-reduction formula, with $p_A^G = p_A$ and fiber LA .

Remark 2.7 (Comparison to the literature). The ordinary adjunction is already stated for a general topological group G , including its unit and counit, in [BSS19, Def. 2.43 and Thm. 2.44]; their Rem. 2.45 also distinguishes it from the homotopy-invariant construction. Our left-translation convention is related to their right-translation coordinates by inversion on G . See also [Sch17, § 4]. The description as pullback and dependent product for an arbitrary group object in an ∞ -topos is developed in [SS24, § 2.2, especially (29)–(31)]. The general proof above specializes that construction to the ∞ -category of spaces, retaining an arbitrary group G .

For the circle, earlier rational versions occur in [FSS17, § 3; FSS18a, § 3; FSS18b, § 2; FSS19, § 8]; see also [SV23, § 2.4; GSS25, § 2.2]. For coefficients $A = BH$ associated to a connected Lie group H , related bundle correspondences appear in [BV05], particularly for $H = E_8$, and in the trivial-circle-bundle setting in [MS04; Sat09; Sat10].

For a principal G -bundle supplied with a trivialization, the mapping-space equivalence reduces to the exponential law

$$\text{Map}_s(X \times G, A) \simeq \text{Map}_s(X, \text{Map}_s(G, A)).$$

In the circle case with $A = BH$, this underlies the bundle-theoretic *caloron correspondence*: compare [GM88; MV10b; HMV12]. These treatments also retain bundle and, in suitable versions, connection data. The statement here is their homotopy-theoretic counterpart at the level of classifying spaces.

2.2. Iterated Cyclification

In this subsection we use the circle specialization of Def. 1.8 and Cor. 2.5. We reserve B for the base of a bundle tower. To make iteration precise, let $U : \mathcal{S}_{/BS^1} \rightarrow \mathcal{S}$ forget the projection. We continue to write $\text{Cyc}(A)$ for the underlying space $U(\text{Cyc}(A))$ whenever the projection is displayed or understood; in particular,

$$\text{Cyc}^2(A) := U(\text{Cyc}(U(\text{Cyc}(A)))).$$

The same convention applies to Cyc_{Top} and its iteration. All relevant structure maps are retained in the slice mapping spaces and diagrams below. Toroidification and its base comparison are defined in § 1.3. The model of $\text{Cyc}(BS^1)$ in Ex. 1.11 will make the obstruction to comparing a tower with a torus bundle explicit.

Proposition 2.8 (Two successive circle bundles). *Consider a tower of principal circle bun-*

dles with specified classifying data,

$$\begin{array}{ccc}
 X & & \\
 q \downarrow & & \\
 B' & \xrightarrow{c'} & BS^1 \\
 p \downarrow & & \\
 B & \xrightarrow{c} & BS^1.
 \end{array} \tag{40}$$

In $\mathcal{S}$ this means that $B' \simeq \text{Ext}(c)$ over B and $X \simeq \text{Ext}(c')$ over B' , with these identifications included in the data. Let

$$\tilde{c}' := \text{Cyc}(c') \circ \eta_c : B \xrightarrow{\eta_c} \text{Cyc}(B') \xrightarrow{\text{Cyc}(c')} \text{Cyc}(BS^1) \tag{41}$$

be the adjoint transpose of c' , where η_c is the circle unit of Cor. 2.5, presented topologically in Prop. 2.2. For every $A \in \mathcal{S}$ there is a natural equivalence

$$\text{Map}_{\mathcal{S}}(X, A) \simeq \text{Map}_{\mathcal{S}/\text{Cyc}(BS^1)}\left((B, \tilde{c}'), (\text{Cyc}^2(A), \text{Cyc}(p_A))\right). \tag{42}$$

It is natural in A and in maps compatible with the bundle and classifying data.

The correspondence proceeds by two successive transpositions. First,

$$\begin{array}{ccc}
 X \dashrightarrow^f A & \longleftrightarrow & \begin{array}{ccc} B' & \xrightarrow{\tilde{f}} & \text{Cyc}(A) \\ & \searrow c' & \swarrow p_A \\ & BS^1 & \end{array}
 \end{array} \tag{43}$$

and the second transpose $\tilde{\tilde{f}} := \text{Cyc}(\tilde{f}) \circ \eta_c$ fits into

$$\begin{array}{ccc}
 B & \xrightarrow{\tilde{\tilde{f}}} & \text{Cyc Cyc}(A) \\
 c \downarrow & \searrow \tilde{c}' & \swarrow \text{Cyc}(p_A) \\
 BS^1 & \xleftarrow{p_{BS^1}} & \text{Cyc}(BS^1)
 \end{array} \tag{44}$$

Here commuting triangles carry specified homotopies and their higher coherences, as in Def. A.7. The left triangle is part of the transpose $\tilde{c}'$; a map in the slice over $\text{Cyc}(BS^1)$ already supplies the required compatibility over BS^1 .

For the ordinary topological models, with the bundles identified with the chosen universal pullbacks, there is likewise a natural bijection

$$\text{Hom}_{\text{Top}}(X, A) \cong \text{Hom}_{\text{Top}/\text{Cyc}_{\text{Top}}(BS^1)}\left((B, \tilde{c}'), (\text{Cyc}_{\text{Top}}^2(A), \text{Cyc}_{\text{Top}}(p_A))\right). \tag{45}$$

Here Cyc_{Top} and $\text{Cyc}_{\text{Top}}^2$ denote the underlying spaces, as above. The unit, projections, and triangles are taken in the strict topological sense.

Proof. Put $\mathcal{D} := \mathcal{S}/_{BS^1}$. The upper-circle adjunction and the slice mapping-space formula give

$$\begin{aligned}
 \text{Map}_{\mathcal{S}}(X, A) &\simeq \text{Map}_{\mathcal{D}}(c', p_A) \\
 &\simeq \text{hofib}_{c'}\left(\text{Map}(B', \text{Cyc}(A)) \xrightarrow{(p_A)_*} \text{Map}(B', BS^1)\right).
 \end{aligned} \tag{46}$$

By naturality, the lower-circle adjunction identifies the map inside this fiber with

$$\mathrm{Map}_{\mathcal{D}}(c, \mathrm{Cyc}(\mathrm{Cyc}(A))) \xrightarrow{\mathrm{Cyc}(p_A)_*} \mathrm{Map}_{\mathcal{D}}(c, \mathrm{Cyc}(BS^1)),$$

and sends its basepoint c' to $\tilde{c}'$. Taking the homotopy fiber at that point gives

$$\mathrm{Map}_{\mathcal{S}}(X, A) \simeq \mathrm{Map}_{\mathcal{D}/\mathrm{Cyc}(BS^1)}(\tilde{c}', \mathrm{Cyc}(p_A)). \quad (47)$$

Finally, the canonical equivalence of iterated slices

$$(\mathcal{S}/BS^1)_{/(\mathrm{Cyc}(BS^1) \xrightarrow{p_{BS^1}} BS^1)} \simeq \mathcal{S}/\mathrm{Cyc}(BS^1)$$

proves (42). A map into $\mathrm{Cyc}(BS^1)$ already supplies its map to BS^1 by composition with p_{BS^1} , including the coherent compatibility. This explains why the left triangle in (44) imposes no additional condition. Both transposes have the form $g \mapsto \mathrm{Cyc}(g) \circ \eta_c$, with the appropriate classifying object, so the correspondence is the one displayed in the diagrams. Naturality holds throughout at the level of mapping spaces.

For the ordinary statement, apply Thm. 2.1 twice to the chosen universal pullbacks. Replace homotopy fibers by fibers of sets, mapping spaces by hom-sets, and the equivalence of iterated slices by their ordinary isomorphism. This proves (45), with the same transpose formulas and strictly commuting triangles. $\square$

2.2.1. Relation to Toroidification

We now compare this iteration with toroidification from Def. 1.12, using the base map of Def. 1.13. Throughout the remainder of this subsection all mapping spaces, pullbacks, fibers, and quotients are interpreted in $\mathcal{S}$. The circle bundles may be represented by numerable principal bundles over CW representatives, as in Rem. A.8. We retain the inner–outer factor order fixed in § 1.3.

Proposition 2.9 (Toroidification as a pullback of double cyclification). *There is a natural comparison $j_A : \mathrm{Tor}(A) \rightarrow \mathrm{Cyc}^2(A)$, characterized by viewing a principal T^2 -bundle as its ordered tower of circle bundles. With ι described in the cup-product model of Ex. 1.11 and Def. 1.13, the square*

$$\begin{array}{ccc} \mathrm{Tor}(A) & \xrightarrow{j_A} & \mathrm{Cyc}^2(A) \\ p_A^{T^2} \downarrow & \lrcorner & \downarrow \mathrm{Cyc}(p_A) \\ BT^2 & \xrightarrow{\iota} & \mathrm{Cyc}(BS^1) \end{array} \quad (48)$$

is a homotopy pullback, naturally in A . Equivalently,

$$\mathrm{Tor}(A) \simeq BT^2 \times_{\mathrm{Cyc}(BS^1)}^h \mathrm{Cyc}^2(A).$$

Proof. Let $d = (d_{\mathrm{in}}, d_{\mathrm{out}}) : B \rightarrow BT^2$, and let $P_{\mathrm{in}} \rightarrow B$ and $P_{\mathrm{out}} \rightarrow B$ be the classified circle bundles. Their fiber product is the classified principal torus bundle. It fits into the homotopy-cartesian square

$$\begin{array}{ccc} P_{\mathrm{in}} \times_B P_{\mathrm{out}} & \longrightarrow & P_{\mathrm{in}} \\ q \downarrow & \lrcorner & \downarrow \\ P_{\mathrm{out}} & \xrightarrow{p_{\mathrm{out}}} & B. \end{array} \quad (49)$$

The upper bundle of the tower $P_{\text{in}} \times_B P_{\text{out}} \rightarrow P_{\text{out}} \rightarrow B$ is classified by $d_{\text{in}} \circ p_{\text{out}}$. Its transpose under the outer-circle adjunction is

$$\widetilde{d_{\text{in}} \circ p_{\text{out}}} \simeq \iota \circ (d_{\text{in}}, d_{\text{out}}). \quad (50)$$

Indeed, on the outer torsor above $b \in B$, the map $d_{\text{in}} \circ p_{\text{out}}$ is constant with value $d_{\text{in}}(b)$. The transpose formula (31) therefore gives exactly (27). This also specifies the equivalence and its coherences for the derived models.

The torus adjunction and Prop. 2.8 now give, naturally in d and A ,

$$\begin{aligned} \text{Map}_{\mathcal{S}_{/BT^2}}(d, \text{Tor}(A)) &\simeq \text{Map}(P_{\text{in}} \times_B P_{\text{out}}, A) \\ &\simeq \text{Map}_{\mathcal{S}_{/\text{Cyc}(BS^1)}}(\iota \circ d, \text{Cyc}(p_A)). \end{aligned} \quad (51)$$

The last space is the space of maps over BT^2 from d into the asserted pullback. Yoneda in $\mathcal{S}_{/BT^2}$ proves the equivalence and produces j_A . The construction retains the map from the total space into A and all specified classifying data. $\square$

Proposition 2.10 (The universal obstruction class). *The degree-one projection of Ex. 1.11, expressed in its homotopy-fiber coordinates, is*

$$\begin{aligned} \text{Cyc}(BS^1) &\xrightarrow{f_1} B\mathbb{Z} \\ (f_2^A, f_1, f_2^M) &\longmapsto f_1 \end{aligned} \quad (52)$$

Then

$$BT^2 \xrightarrow{\iota} \text{Cyc}(BS^1) \xrightarrow{f_1} B\mathbb{Z} \quad (53)$$

is a homotopy fiber sequence over the zero basepoint, where ι is the constant-loop comparison of Def. 1.13. For a tower $X \xrightarrow{q} B' \xrightarrow{p} B$ with classifying map $\tilde{c}' : B \rightarrow \text{Cyc}(BS^1)$, the composite $f_1 \circ \tilde{c}'$ represents

$$[f_1 \circ \tilde{c}'] = p_! c_1(q) \in H^1(B; \mathbb{Z}), \quad (54)$$

with the standard circle orientation. The map ι presents the universal cover of $\text{Cyc}(BS^1)$.

Proof. The projection to $B\mathbb{Z}$ in (23) has homotopy fiber obtained by setting $f_1 = 0$. The defining cup product becomes zero, so

$$\text{hofib}_0(f_1) \simeq (B^2\mathbb{Z})_{\text{out}} \times \Omega B^3\mathbb{Z} \simeq (B^2\mathbb{Z})_{\text{out}} \times (B^2\mathbb{Z})_{\text{in}}.$$

Its map into the cup-product fiber is $(b, \ell_a) \mapsto (b, 0, \ell_a)$, precisely ι after putting the inner factor first, by (28). Equivalently, (29) is the fiber square, with the outer $B^2\mathbb{Z}$ factor displayed. This proves (53), including its nullhomotopy.

For the tower, Ex. 1.11 identifies the coordinate f_1 of $\tilde{c}'$ with the fiber integral of the class represented by $c' : B' \rightarrow B^2\mathbb{Z}$. That class is $c_1(q)$, so (54) follows. Finally, the displayed fiber BT^2 is connected and simply connected and the base $B\mathbb{Z}$ is a $K(\mathbb{Z}, 1)$. Thus f_1 induces an isomorphism on fundamental groups and ι is the universal cover in homotopy types. $\square$

Corollary 2.11 (Toroidification as a homotopy fiber). *Set $\omega_A := f_1 \circ \text{Cyc}(p_A)$, where f_1 is the degree-one projection in the model of Ex. 1.11, as specified in (52). There is a natural homotopy fiber sequence*

$$\text{Tor}(A) \xrightarrow{j_A} \text{Cyc}^2(A) \xrightarrow{\omega_A} B\mathbb{Z}. \quad (55)$$

For $f : B \rightarrow \text{Cyc}^2(A)$, the space of lifts through j_A , including a specified homotopy to f , is the space of nullhomotopies of $\omega_A \circ f$. Thus a lift exists if and only if $[\omega_A \circ f] = 0$

in $H^1(B; \mathbb{Z})$. When nonempty, this lift space is a torsor for the discrete group $H^0(B; \mathbb{Z})$; equivalently, it is a torsor for $\text{Map}(B, \mathbb{Z})$ as a space.

Proof. Paste the homotopy pullback (48) with the fiber square of (53). This proves (55). Applying $\text{Map}(B, -)$ gives the asserted identification of lift spaces. The target $B\mathbb{Z} = K(\mathbb{Z}, 1)$ represents H^1 , so vanishing of the class is equivalent to existence of a nullhomotopy. After choosing one nullhomotopy, the space of choices is equivalent to

$$\Omega \text{Map}(B, B\mathbb{Z}) \simeq \text{Map}(B, \mathbb{Z}),$$

which is discrete with set of points $H^0(B; \mathbb{Z})$. The choice is used only to identify the torsor with this group. $\square$

Proposition 2.12 (Descent criterion for a tower of circle bundles). *For a tower $X \xrightarrow{q} B' \xrightarrow{p} B$, a compatible principal torus structure means a principal T^2 -bundle over B together with an identification of its ordered circle tower with the given tower. The following conditions are equivalent:*

- (i) *The tower admits a compatible principal torus structure.*
- (ii) *There is a principal circle bundle $r : P \rightarrow B$ and an isomorphism $X \cong B' \times_B P$ as circle bundles over B' (or the corresponding equivalence with classifying data in $\mathcal{S}$).*
- (iii) *$c_1(q)$ belongs to the image of $p^* : H^2(B; \mathbb{Z}) \rightarrow H^2(B'; \mathbb{Z})$.*
- (iv) *$p_! c_1(q) = 0$ in $H^1(B; \mathbb{Z})$.*

A choice in (ii) is illustrated by the homotopy-cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\quad} & P \\ q \downarrow & \lrcorner & \downarrow r \\ B' & \xrightarrow{p} & B. \end{array} \tag{56}$$

Any given map $X \rightarrow A$ is retained under the resulting lift to $\text{Tor}(A)$. There is no further obstruction involving A .

Proof. A principal product-group bundle is the fiber product of its two quotient circle bundles. Conversely, a pair of circle bundles over B gives a principal T^2 -bundle by their fiber product, with componentwise action. These constructions prove the equivalence of (i) and (ii), with the outer quotient identified with B' . Classification of circle bundles by their integral first Chern classes proves the equivalence of (ii) and (iii).

Write $e := c_1(p)$. The integral Gysin sequence contains the exact segment

$$\begin{array}{ccccc} H^0(B; \mathbb{Z}) & \xrightarrow{e \cup -} & H^2(B; \mathbb{Z}) & \xrightarrow{p^*} & H^2(B'; \mathbb{Z}) \\ & & \xrightarrow{p_!} & H^1(B; \mathbb{Z}) & \xrightarrow{e \cup -} & H^3(B; \mathbb{Z}). \end{array} \tag{57}$$

See [Hat02, § 4.D], or the mapping-space proof in Ex. 1.10. Exactness gives (iii) if and only if (iv). The final assertion follows either by transporting $X \rightarrow A$ across the equivalence in (ii), or directly from (48). $\square$

Remark 2.13 (Existence and the data of a reduction). Vanishing of the class in (54) asserts existence; a lift also includes a descent identification, or equivalently the nullhomotopy in Cor. 2.11. If B is connected, possible descended Chern classes differ by integer multiples

of $e = c_1(p)$, by (57). This describes their image in $H^2(B; \mathbb{Z})$, whereas the full space of reductions of a fixed tower is the $\mathbb{Z}$ -torsor of Cor. 2.11. The shift by a multiple of e is the Chern-class action of the shear subgroup described in Prop. 2.16, with the sign determined by its integer parametrization. Distinct reductions can have the same descended Chern class, for example when $e = 0$. In particular, if $H^1(B; \mathbb{Z}) = 0$, every such tower admits a compatible torus structure, with a choice of reduction still required.

Lemma 2.14 (A loop-group model for double cyclification). *Let $LS^1 := \text{Map}_{\text{Top}}(S^1, S^1)$, with pointwise multiplication, and let outer rotations act by*

$$(\rho_b g)(w) := g(b^{-1}w).$$

Define $W := LS^1 \rtimes_{\rho} S^1$, with product

$$(g, b)(h, d) = (g(h \circ b^{-1}), bd).$$

It acts on T^2 by the bundle automorphisms

$$\Phi_{g,b}(z, w) := (g(bw)z, bw), \quad (58)$$

and hence on $\text{Map}(T^2, A)$ by inverse precomposition. There is a natural equivalence

$$\text{Cyc}^2(A) \simeq \text{Map}(T^2, A) // W. \quad (59)$$

Under this equivalence, $\text{Cyc}(BS^1) \simeq BW$, and $\text{Cyc}(p_A)$ is the projection to BW . Thus BW is a group model for the cup-product homotopy fiber in Ex. 1.11 and (23).

Proof. For an S^1 -space Y , apply $\text{Map}(S^1, -)$ to the bundle $Y // S^1 \rightarrow BS^1$. Its homotopy fiber over the constant map is $\text{Map}(S^1, Y)$, since mapping out of S^1 preserves homotopy pullbacks. The base $\text{Map}(S^1, BS^1)$ is connected, because $H^2(S^1; \mathbb{Z}) = 0$, and its loop group at that map is

$$\Omega \text{Map}(S^1, BS^1) \simeq \text{Map}(S^1, \Omega BS^1) \simeq LS^1.$$

Thus $\text{Map}(S^1, BS^1) \simeq BLS^1$. Transport on the fiber is the pointwise gauge action. The classification of families of spaces, (36), now gives

$$\text{Map}(S^1, Y // S^1) \simeq \text{Map}(S^1, Y) // LS^1. \quad (60)$$

This argument identifies the whole family over BLS^1 , so it includes parameter-dependent bundles and their trivializations. It does not require a simultaneous choice of trivialization in every family.

The constant basepoint above is rotation-invariant, and rotation conjugates the pointwise gauge action by ρ . Hence (60) respects outer rotations; taking their homotopy quotient replaces LS^1 by W . For $Y = \text{Map}(S^1, A)$ with inner translation, the exponential law identifies $\text{Map}(S^1, Y)$ with $\text{Map}(T^2, A)$. The element (g, b) acts as

$$F(z, w) \mapsto F(g(w)^{-1}z, b^{-1}w).$$

This is inverse precomposition by (58). Applying the same construction to $Y \rightarrow *$ identifies the base with BW and the structure map with the projection, proving all the assertions. This is the homotopy-type version of the caloron correspondence; compare [MV10a, Prop. 2.5] for the smooth bundle correspondence. $\square$

2.2.2. Relation to Modular Transformations

Proposition 2.15 (Modular action on toroidification). *The discrete group $\text{GL}_2(\mathbb{Z})$ acts naturally on $\text{Tor}(A)$, compatibly with its action on BT^2 . In the ordered coordinates of Def. 1.12,*

a matrix and its torus automorphism are related by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \phi_M(z, w) := (z^a w^b, z^c w^d). \quad (61)$$

The action on mapping spaces is

$$\Phi_M(F) := F \circ \phi_{M^{-1}},$$

and the induced map Θ_M on toroidification fits into

$$\begin{array}{ccc} \mathrm{Tor}(A) & \xrightarrow{\Theta_M} & \mathrm{Tor}(A) \\ p_A^{T^2} \downarrow & & \downarrow p_A^{T^2} \\ BT^2 & \xrightarrow{B\phi_M} & BT^2. \end{array} \quad (62)$$

Thus this is an action on the coefficient family together with its base. If $c_{\mathrm{in}}, c_{\mathrm{out}}$ are the two universal Chern classes pulled back along $p_A^{T^2}$, then

$$\Theta_M^* \begin{pmatrix} c_{\mathrm{in}} \\ c_{\mathrm{out}} \end{pmatrix} = M \begin{pmatrix} c_{\mathrm{in}} \\ c_{\mathrm{out}} \end{pmatrix}. \quad (63)$$

Proof. The monomial formulas give $\phi_M \phi_N = \phi_{MN}$ and hence $\Phi_M \Phi_N = \Phi_{MN}$. For the translation action of $t \in T^2$,

$$\Phi_M(t \cdot F) = \phi_M(t) \cdot \Phi_M(F).$$

Applying the homotopy-quotient construction to this compatibility gives Θ_M and (62). Functoriality supplies the group-action coherences, and postcomposition in A commutes with every map used. The two components of ϕ_M tensor the circle bundles with exponents (a, b) and (c, d) , respectively; additivity of the first Chern class proves (63). $\square$

Proposition 2.16 (Double cyclification as a quotient by the shear subgroup). *Consider the subgroup*

$$\mathrm{Sh}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subset \mathrm{GL}_2(\mathbb{Z}). \quad (64)$$

It is isomorphic to $\mathbb{Z}$ by the displayed parameter n . Under (61), it acts by

$$\sigma_n(z, w) := (zw^n, w). \quad (65)$$

This is precisely the subgroup acting as the identity both on $S_{\mathrm{in}}^1 \subset T^2$ and on the quotient $T^2/S_{\mathrm{in}}^1 \cong S_{\mathrm{out}}^1$. Restrict the action of Prop. 2.15 to this subgroup. There are natural equivalences in the slice ∞ -category $\mathcal{S}_{/B\mathbb{Z}}$:

$$\begin{aligned} \mathrm{Cyc}^2(A) &\simeq \mathrm{Map}(T^2, A) // (T^2 \rtimes \mathrm{Sh}_2(\mathbb{Z})) \\ &\simeq \mathrm{Tor}(A) // \mathrm{Sh}_2(\mathbb{Z}) \simeq \mathrm{Tor}(A) // \mathbb{Z}. \end{aligned} \quad (66)$$

On the left, the structure map is precisely the map of (55), namely

$$\omega_A := f_1 \circ \mathrm{Cyc}(p_A) : \mathrm{Cyc}^2(A) \xrightarrow{\mathrm{Cyc}(p_A)} \mathrm{Cyc}(BS^1) \xrightarrow{f_1} B\mathbb{Z}.$$

Here f_1 is the degree-one projection of Ex. 1.11 and (52). On the quotient expressions, use the canonical maps to $B\mathrm{Sh}_2(\mathbb{Z})$, induced respectively by the semidirect-product projection and the homotopy-quotient projection. Identify $B\mathrm{Sh}_2(\mathbb{Z}) \simeq B\mathbb{Z}$ with the generator sign for which the base projection corresponds to f_1 , as justified below. This choice is made once, independently of A ; the final $\mathbb{Z}$ -action in (66) is transported along the same group

identification. Writing q_A and $q_A^{\mathbb{Z}}$ for the resulting quotient projections, the equivalences fit into the homotopy-commutative diagram

$$\begin{array}{ccccc} \mathrm{Cyc}^2(A) & \xrightarrow{\simeq} & \mathrm{Tor}(A) // \mathrm{Sh}_2(\mathbb{Z}) & \xrightarrow{\simeq} & \mathrm{Tor}(A) // \mathbb{Z} \\ & \searrow \omega_A & \downarrow q_A & \swarrow q_A^{\mathbb{Z}} & \\ & & B\mathbb{Z} & & \end{array}$$

Thus these are equivalences over $B\mathbb{Z}$ with the indicated homotopies, not merely equivalences of underlying spaces. The action is the restriction of the full integral change-of-coordinates action. On Chern classes a shear sends

$$(c_{\mathrm{in}}, c_{\mathrm{out}}) \mapsto (c_{\mathrm{in}} + nc_{\mathrm{out}}, c_{\mathrm{out}}).$$

Under these equivalences the quotient map is j_A , identified with the homotopy-fiber map over $0 \in B\mathbb{Z}$ in (55). In particular, j_A is a principal $\mathbb{Z}$ -cover in homotopy types.

Proof. In the group W of Lem. 2.14, consider the subgroup H whose gauge transformations are monomials $g(w) = aw^n$. It contains all torus translations $\tau_{a,b}(z, w) = (az, bw)$ and the transformations σ_n . Since

$$\sigma_n \tau_{a,b} \sigma_n^{-1} = \tau_{ab^n, b}, \quad (67)$$

every element of H has a unique expression $\tau_{a,b} \sigma_n$, and $H \cong T^2 \rtimes \mathrm{Sh}_2(\mathbb{Z}) \cong T^2 \rtimes_{\sigma} \mathbb{Z}$. These identifications describe its action on T^2 . In particular, the induced shear action on $\mathrm{Tor}(A)$ is exactly the restriction constructed in Prop. 2.15.

The inclusion $H \rightarrow W$ is a weak equivalence of topological groups. To see this directly, every loop $g : S^1 \rightarrow S^1$ of winding number n has a unique expression

$$g(w) = aw^n \exp(2\pi i f(w)), \quad a = g(1), \quad f : S^1 \rightarrow \mathbb{R}, \quad f(1) = 0.$$

The based real-valued function f exists by the lifting property of $\mathbb{R} \rightarrow S^1$ and varies continuously with the based nullhomotopic loop. Scaling f to zero contracts the remaining factor. Thus the monomial circle aw^n is a deformation retract of each winding-number component. With the outer S^1 factor, this proves that $H \rightarrow W$ is a weak equivalence; the deformation need not be a group homomorphism.

An integral matrix is the identity on the inner circle and the outer quotient exactly when it has the form in (64); this proves the subgroup description. Restriction along a weak equivalence of these group models induces an equivalence of homotopy quotients. Apply this to (59). Finally, the normal subgroup $T^2 \subset H$ gives the iterated-quotient equivalence

$$\mathrm{Map}(T^2, A) // H \simeq (\mathrm{Map}(T^2, A) // T^2) // \mathbb{Z}.$$

The Chern-class formula is the specialization of (63) to the shear matrix. The subgroup T^2 consists precisely of constant gauge transformations and outer rotations. Its inclusion classifies the product-bundle tower (49); hence its induced map of quotients is j_A , as characterized in Prop. 2.9.

On the bases, this identifies the map ι of (28) with $BT^2 \rightarrow B(T^2 \rtimes \mathrm{Sh}_2(\mathbb{Z}))$. The equivalence

$$B(T^2 \rtimes \mathrm{Sh}_2(\mathbb{Z})) \simeq \mathrm{Cyc}(BS^1)$$

identifies two maps to $B\mathbb{Z}$: the semidirect-product projection, after identifying $\mathrm{Sh}_2(\mathbb{Z})$ with $\mathbb{Z}$, and f_1 . Both induce isomorphisms on fundamental groups, by Prop. 2.10. Choose the sign of the group identification so these isomorphisms agree. Since the base is connected

and $B\mathbb{Z}$ is a $K(\mathbb{Z}, 1)$, the two maps are then homotopic. Fix such a homotopy on the base. The equivalence of (59), restriction from W to H , and the iterated-quotient equivalence all respect their base projections. Pulling back the chosen homotopy therefore identifies all their structure maps with $f_1 \circ \text{Cyc}(p_A) = \omega_A$, naturally in A . This proves (66) in $\mathcal{S}_{/B\mathbb{Z}}$ and supplies the homotopies in the displayed diagram. Taking the fiber over zero recovers j_A and (55). $\square$

Example 2.17 (The point and the universal cover). For $A = *$ the comparison is

$$\text{Tor}(*) = BT^2 \xrightarrow{\iota} \text{Cyc}^2(*) = \text{Cyc}(BS^1) \simeq B(T^2 \rtimes_{\sigma} \mathbb{Z}).$$

Its source has trivial fundamental group, while its target has fundamental group $\mathbb{Z}$, also read directly from the cup-product fiber of Ex. 1.11 and (23). Hence toroidification and double cyclification are not equivalent functors. The monodromy on $\pi_2(BT^2) = \mathbb{Z}^2$ is the shear matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, with the ordered coordinates of Def. 1.12 and a compatible choice of generator. For general A , j_A is a principal $\mathbb{Z}$ -cover; it is a universal cover when $\text{Tor}(A)$ is connected and simply connected.

Example 2.18 (A tower with nonzero obstruction). Let $B = S^1$ and $B' = S^1 \times S^1$, with p the projection to the second factor. Choose the principal circle bundle $q : X_n \rightarrow B'$ whose Chern class is n times the oriented generator of $H^2(B'; \mathbb{Z})$. Fiber integration gives

$$p_! c_1(q) = n \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}.$$

Thus the tower has no compatible principal torus structure for $n \neq 0$. An explicit total space is a mapping torus of the shear σ_n on T^2 , with the sign chosen to match the orientation convention: the inner z -circle action commutes with the gluing, its quotient is the product torus with coordinates (w, t) , and its clutching function along the t -circle is $w \mapsto w^n$. This verifies the Chern number and displays the shear monodromy of the composite $X_n \rightarrow S^1$. For $n = 0$ the construction is the trivial three-torus tower.

Remark 2.19 (Ordered circles, remaining coordinate symmetries, and the literature). The distinction between the full action and the subgroup in (64) is encoded by the sequence

$$1 \longrightarrow S_{\text{in}}^1 \longrightarrow T^2 \longrightarrow S_{\text{out}}^1 \longrightarrow 1.$$

Requiring a torus automorphism to be the identity on its subgroup and quotient forces the matrix to have diagonal entries 1 and lower-left entry 0, and hence gives exactly $\text{Sh}_2(\mathbb{Z})$. Requiring only that it preserve the inner circle as a subgroup gives the larger group

$$N := \left\{ \begin{pmatrix} \varepsilon & n \\ 0 & \delta \end{pmatrix} \mid n \in \mathbb{Z}, \quad \varepsilon, \delta \in \{1, -1\} \right\}. \quad (68)$$

This is the normalizer of $\text{Sh}_2(\mathbb{Z})$ in $\text{GL}_2(\mathbb{Z})$. Indeed, the common fixed lattice of the shear subgroup is $\mathbb{Z}(1, 0)$, so a normalizing matrix must preserve that primitive rank-one sublattice and must have the displayed form. Conversely, direct multiplication gives

$$M \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} M^{-1} = \begin{pmatrix} 1 & \varepsilon \delta k \\ 0 & 1 \end{pmatrix} \quad (M \in N).$$

Consequently the diagonal sign matrices induce a remaining action of

$$N/\text{Sh}_2(\mathbb{Z}) \cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$$

on $\text{Cyc}^2(A) \simeq \text{Tor}(A) // \text{Sh}_2(\mathbb{Z})$. They reverse the two circle coordinates independently. Their action changes the obstruction class $[\omega_A]$ by the factor $\varepsilon \delta$, since this is their conjugation action

on the shear parameter. Thus the shears in (66) are the group being quotiented out; the normalizer describes the coordinate symmetries that preserve that quotient presentation.

A general $M \in \mathrm{GL}_2(\mathbb{Z})$ transports the chosen circle direction and gives an equivalence

$$\mathrm{Tor}(A) // \mathrm{Sh}_2(\mathbb{Z}) \simeq \mathrm{Tor}(A) // (M \mathrm{Sh}_2(\mathbb{Z}) M^{-1}).$$

For example, interchanging z and w conjugates the upper shear subgroup to the lower shear subgroup, giving the presentation for the opposite order of the tower. Further quotienting by all integral coordinate changes is expressed by the natural maps

$$\mathrm{Tor}(A) \xrightarrow{j_A} \mathrm{Cyc}^2(A) \longrightarrow \mathrm{Tor}(A) // \mathrm{GL}_2(\mathbb{Z}), \quad (69)$$

induced by $\{1\} \subset \mathrm{Sh}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Z})$. The last space is equivalently

$$\mathrm{Map}(T^2, A) // (T^2 \rtimes \mathrm{GL}_2(\mathbb{Z})),$$

by the iterated homotopy-quotient construction. These descriptions specify which circle data are retained in the intermediate quotient and which additional coordinate changes are imposed in the last one.

A related comparison in the setting of super-Lie $_{\infty}$ algebras is given in [GSS25, Def. 2.53 and Prop. 2.56]: the toroidal condition removes the fiber-integrated upper extension class. The results here establish the integral comparison in spaces, retain the nullhomotopy needed for a reduction, and identify its shear subgroup inside the full integral coordinate-change group. Their proofs use the general adjunction, the integral Gysin sequence, and the explicit loop-group model above.

3. Examples

Example 3.1 (Integral topological T-duality via cyclification). The case $n = 2$ of Ex. 1.10 identifies

$$\mathrm{Cyc}(B^3\mathbb{Z}) \simeq \mathrm{hofib}_0(B^2\mathbb{Z} \times B^2\mathbb{Z} \xrightarrow{\cup} B^4\mathbb{Z}). \quad (70)$$

A map $B \rightarrow \mathrm{Cyc}(B^3\mathbb{Z})$ classifies a principal circle bundle $p : P \rightarrow B$ with a degree-three integral twist $h : P \rightarrow B^3\mathbb{Z}$, retaining its classifying data. The two degree-two classes are $c = c_1(p)$ and $c^{\vee} = p_! [h]$. Interchanging the factors of $B^2\mathbb{Z} \times B^2\mathbb{Z}$ and using the commutativity homotopy of cup product gives a self-equivalence $\tau : \mathrm{Cyc}(B^3\mathbb{Z}) \rightarrow \mathrm{Cyc}(B^3\mathbb{Z})$. It exchanges c and $c^{\vee}$; consequently it changes the projection to BS^1 in general.

Choose the universal circle T-duality triple over $\mathrm{Cyc}(B^3\mathbb{Z})$: it consists of the two circle bundles $P, P^{\vee} \rightarrow \mathrm{Cyc}(B^3\mathbb{Z})$, twists $h, h^{\vee}$, and an identification of their pullbacks to $P \times_{\mathrm{Cyc}(B^3\mathbb{Z})} P^{\vee}$ satisfying the fiberwise Poincaré condition. Such a triple exists by [BRS06, Thm. 2.7]; its first pair identifies its classifying space with $\mathrm{Cyc}(B^3\mathbb{Z})$, by [BRS06, Thm. 2.13], since the additional covering in that theorem is trivial for circle bundles. We take τ to be induced by exchanging the two pairs. Exchanging twice recovers the original triple, so $\tau^2 \simeq \mathrm{id}$. This fixes a lift of the factor exchange in (70). Our orientations use $c^{\vee} = p_! [h]$ and $c = p_!^{\vee} [h^{\vee}]$; a convention with opposite fiber integration changes these signs.

To state the K-theory comparison, write

$$\mathrm{KU}_j := \Omega^{\infty} \Sigma^j \mathrm{KU}, \quad \mathrm{KU}_0 \simeq \mathbb{Z} \times BU, \quad \mathrm{KU}_1 \simeq U.$$

Indices are taken modulo two using Bott periodicity. Tensoring with complex line bundles defines a homotopy coherent $BU(1)$ -action on the K-theory spectrum and its representing

spaces. In degree j , the quotient projection is

$$h_3^K : \mathrm{KU}_j // \mathrm{BU}(1) \longrightarrow B^2\mathrm{U}(1) \simeq B^3\mathbb{Z}. \quad (71)$$

These are the degree-three twisted K-theory coefficient families; compare Exs. A.13 and A.18 and [AS04]. The domain specifies the degree of h_3^K in each occurrence. With the universal triple and the circle K-orientations fixed, there is a diagram whose horizontal arrows are equivalences:

$$\begin{array}{ccc} \mathrm{Cyc}((\mathrm{KU}_0 // \mathrm{BU}(1))_{\mathrm{rk}=0}) & \xleftarrow{\sim} & \mathrm{Cyc}((\mathrm{KU}_1 // \mathrm{BU}(1))_{\mathrm{wd}=0}) \\ \downarrow & & \downarrow \\ \mathrm{Cyc}(\mathrm{KU}_0 // \mathrm{BU}(1)) & \xleftarrow{\sim} & \mathrm{Cyc}(\mathrm{KU}_1 // \mathrm{BU}(1)) \\ \downarrow \mathrm{Cyc} h_3^K & & \downarrow \mathrm{Cyc} h_3^K \\ \mathrm{Cyc} B^3\mathbb{Z} & \xleftarrow[\tau]{\sim} & \mathrm{Cyc} B^3\mathbb{Z}. \end{array} \quad (72)$$

Here $\mathrm{rk} = 0$ selects the rank-zero component $(\mathrm{KU}_0 // \mathrm{BU}(1))_{\mathrm{rk}=0} \simeq \mathrm{BU} // \mathrm{BU}(1)$. The integer wd is the winding component of the odd loop, under

$$\begin{aligned} \pi_0 \mathrm{Cyc}(\mathrm{KU}_1 // \mathrm{BU}(1)) &\cong \pi_1(\mathrm{KU}_1 // \mathrm{BU}(1)) \\ &\cong \pi_1(U) \cong \mathbb{Z}. \end{aligned}$$

Proof. We give an integral argument for (72). Pull the universal triple back to a parameter space B , and write its correspondence as

$$\begin{array}{ccc} & P \times_B P^\vee & \\ r \swarrow & & \searrow r^\vee \\ P & & P^\vee \\ p \searrow & & \swarrow p^\vee \\ & B. & \end{array} \quad (73)$$

Its twist identification $u : (r^\vee)^* h^\vee \simeq r^* h$ induces the middle arrow of the pull–identify–push transformation

$$\begin{aligned} \mathcal{T}_B : \mathrm{KU}^{j,h}(P) &\xrightarrow{r^*} \mathrm{KU}^{j,r^*h}(P \times_B P^\vee) \\ &\xrightarrow{u^*} \mathrm{KU}^{j,(r^\vee)^*h^\vee}(P \times_B P^\vee) \xrightarrow{(r^\vee)_!} \mathrm{KU}^{j-1,h^\vee}(P^\vee). \end{aligned} \quad (74)$$

This is the standard topological T-duality transformation [BRS06, Def. 2.18]. The last map integrates along the framed circle fiber: a principal circle bundle has the vertical tangent framing supplied by the positive generator of its circle action.

We require the transformation on families of section *spectra*, not just on cohomology groups. The pullback and twist identification already have this form; the twisted Pontryagin–Thom construction supplies the pushforward, naturally under base change [ABG18, §§ 4.3–4.4]. Apply it to the universal correspondence over $\mathrm{Cyc}(B^3\mathbb{Z})$. A family of K-theory module spectra over $\mathrm{Cyc}(B^3\mathbb{Z})$ is a functor from the ∞ -groupoid $\mathrm{Cyc}(B^3\mathbb{Z})$ to KU-module spectra. Hence the resulting transformation is an equivalence if it is so over every point of $\mathrm{Cyc}(B^3\mathbb{Z})$.

Over a point, both circle bundles are S^1 and their twists are trivial. The Poincaré condition says that u^* is tensoring with a line bundle on $S^1 \times S^1$ whose first Chern class is a generator. Thus the fiber map is the circle Fourier–Mukai transform. For completeness, $\mathrm{KU}^*(S^1)$ is free over $\mathrm{KU}^*(*)$ on 1 and an odd generator α . If $\alpha^\vee$ is the generator on the dual circle and

$b \in \mathrm{KU}^{-2}(*)$ is the Bott unit, the Poincaré line has class

$$1 \pm b\alpha\alpha^\vee \quad \text{in } \mathrm{KU}^0(S^1 \times S^1).$$

Indeed, the reduced even group of the torus is $\mathbb{Z}$, detected integrally by the first Chern class. Integration over the first circle sends α to 1 and 1 to 0. The projection formula therefore gives

$$\mathcal{T}(1) = \pm b\alpha^\vee, \quad \mathcal{T}(\alpha) = 1.$$

The sign depends on the chosen twist and orientation conventions and does not affect invertibility. Since b is a unit, this is an isomorphism in every degree, hence an equivalence of the corresponding K-theory module spectra. A map of families of spectra that is an equivalence on every fiber is an equivalence. This proves the universal assertion over $\mathrm{Cyc}(B^3\mathbb{Z})$; taking sections after any base change gives (74). This fiberwise argument also avoids a finite-dimensional restriction on B .

Finally, let $f : B \rightarrow \mathrm{Cyc}(B^3\mathbb{Z})$ classify $(p : P \rightarrow B, h)$. Naturality of the $\mathrm{Ext}/\mathrm{Cyc}$ adjunction, followed by taking homotopy fibers, gives

$$\begin{aligned} \mathrm{Map}_{\mathcal{S}/\mathrm{Cyc}(B^3\mathbb{Z})}(f, \mathrm{Cyc}(h_3^K)) \\ \simeq \mathrm{Map}_{\mathcal{S}/B^3\mathbb{Z}}(h, h_3^K). \end{aligned}$$

The right side is the space of degree- j twisted K-theory sections on P (Prop. A.17). Thus the projection

$$\mathrm{Cyc}(\mathrm{KU}_j//\mathrm{BU}(1)) \xrightarrow{\mathrm{Cyc}(h_3^K)} \mathrm{Cyc}(B^3\mathbb{Z})$$

is precisely the family of these section spaces. Applying Ω^∞ to the equivalence of section spectra and using Bott periodicity yields the middle row of (72) over τ .

On components the fiber computation sends rank to winding number, up to sign. The even coefficient space has components indexed by rank, each simply connected, whereas $\mathrm{KU}_1//\mathrm{BU}(1)$ is connected with $\pi_1(\mathrm{KU}_1//\mathrm{BU}(1)) \cong \mathbb{Z}$. Free loops therefore give precisely the rank and winding components displayed in the diagram. The equivalence restricts to their zero components, proving the top row. $\square$

The construction depends on the specified T-duality triple and its twist identification, and is natural under their pullback. Merely interchanging the two Chern classes does not specify the transformation on twisted K-theory. For the classical integral isomorphism on groups see [BS05, Thm. 3.13; BRS06, Thm. 6.2]; the proof above retains the families needed for the equivalence of coefficient spaces. Related rational models appear in [FSS18b; GSS25, § 3.2].

A. Background

We distinguish constructions on topological spaces from their homotopy-invariant counterparts. The former take place in an ordinary category with actual continuous maps as morphisms; the latter take place in the ∞ -category of spaces and retain homotopies and all higher homotopies. Throughout, spaces and maps are unpointed unless explicitly stated otherwise, and no connectedness assumption is imposed.

A.1. Topological Spaces and Slice Categories

Definition A.1 (The category of topological spaces). Write $\mathbf{Top}$ for the ordinary category of compactly generated weak Hausdorff spaces and continuous maps, with their usual composition. Here a topological space X is:

- (i) *compactly generated* if a subset $A \subset X$ is closed whenever $u^{-1}(A)$ is closed for every continuous map $u : K \rightarrow X$ from a compact Hausdorff space K ;
- (ii) *weak Hausdorff* if $u(K)$ is closed in X for every such map u .

The *compact generation* kX of a space X is its underlying set equipped with the topology whose closed subsets satisfy the test in (i). Products and pullbacks in $\mathbf{Top}$ are formed by compactly generating the corresponding ordinary topological constructions. We write

$$\mathrm{Map}_{\mathbf{Top}}(X, Y) := kC_{\mathrm{co}}(X, Y)$$

for the continuous mapping space; its underlying set is $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$. Here $C_{\mathrm{co}}(X, Y)$ carries the topology generated by

$$\{f : X \rightarrow Y \mid f(u(K)) \subset V\},$$

where $u : K \rightarrow X$ is continuous, K is compact Hausdorff, and $V \subset Y$ is open, followed by compact generation. This compact-Hausdorff test-map formulation of the compact-open topology also covers weak Hausdorff spaces that are not Hausdorff; see [Str, Def. 2.8].

This choice contains CW complexes and manifolds and supplies a Cartesian closed category: the usual exponential correspondence is a natural homeomorphism

$$\mathrm{Map}_{\mathbf{Top}}(Z \times X, Y) \cong \mathrm{Map}_{\mathbf{Top}}(Z, \mathrm{Map}_{\mathbf{Top}}(X, Y)). \quad (75)$$

See [Ste67; Str, §§ 1–2, especially Prop. 2.12]. The subscript $\mathbf{Top}$ distinguishes this topological mapping space from the derived mapping space introduced below.

Definition A.2 (Principal G -bundles and universal bundles). Let G be a topological group in $\mathbf{Top}$. A *principal G -bundle* over X is a map $p : P \rightarrow X$ with a continuous right G -action and local equivariant trivializations

$$p^{-1}(V) \cong V \times G$$

over an open cover of X , with G acting by right multiplication on the second factor. It is *numerable* if it admits a trivializing cover with a subordinate locally finite partition of unity. An isomorphism over X is an equivariant homeomorphism commuting with the projections.

Fix a universal numerable principal right G -bundle

$$\pi : EG \longrightarrow BG \quad (76)$$

with contractible total space, and a point $e_0 \in EG$ over $b_0 \in BG$. Write $b : * \rightarrow BG$ for this basepoint. Such models exist for arbitrary topological groups in our setting, by the Milnor construction; see [Mil56; Hus+08, Chs. 7–8]. One may compactly generate the construction in the category of Def. A.1.

Pullback of (76) identifies ordinary homotopy classes of continuous maps $X \rightarrow BG$ with isomorphism classes of numerable principal G -bundles over X ; see [Hus+08, Ch. 7, Thm. 3.3; May99, § 23.8]. Numerability is automatic over paracompact Hausdorff bases. Numerable locally trivial bundles are fibrations [May99, Ch. 7]; consequently $G \simeq \Omega BG$ as homotopy types, compatibly with the group structures. The ordinary homotopy classes in this classification should be distinguished from derived mapping-space components for arbitrary representatives, as in (79).

Definition A.3 (The universal circle bundle). For the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, with its usual multiplication, we choose

$$\pi : ES^1 = S^\infty \longrightarrow BS^1 = \mathbb{C}P^\infty. \quad (77)$$

This is the principal bundle obtained from scalar multiplication on $S^\infty = \operatorname{colim}_{m \geq 0} S^{2m+1}$ with the CW topology. Its total space is contractible. We choose $e_0 = (1, 0, \dots)$ above $b_0 = [1 : 0 : \dots]$. Thus $BS^1 \simeq K(\mathbb{Z}, 2) = B^2\mathbb{Z}$, and the general classification theorem specializes to the classification of numerable principal circle bundles.

Definition A.4 (The ordinary slice category). For $B \in \mathbf{Top}$, the *slice category* $\mathbf{Top}_{/B}$ has:

- (i) objects given by continuous maps $p : X \rightarrow B$;
- (ii) morphisms from $p : X \rightarrow B$ to $q : Y \rightarrow B$ given by continuous maps $f : X \rightarrow Y$ for which $q \circ f = p$ as an equality of maps.

Thus a morphism is a strictly commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & B & \end{array}$$

and composition and identities are inherited from $\mathbf{Top}$. Its topological mapping space is the strict fiber

$$\operatorname{Map}_{\mathbf{Top}_{/B}}(p, q) := \operatorname{Map}_{\mathbf{Top}}(X, Y)_{\operatorname{Map}_{\mathbf{Top}}(X, B)} \times \{p\}, \quad (78)$$

where the map from the first factor is postcomposition with q . The ordinary morphism set is the underlying set of (78), not its set of path components.

For a fixed topological group G , the ordinary categories relevant to the adjunction are $\mathbf{Top}_{/BG}$ and $\mathbf{Top}$. An object on the slice side is a map to the fixed model of BG ; it need not be a fibration, and no section is specified. The circle case uses $BG = BS^1$.

Remark A.5 (Classifying maps and their bundles). A slice object $p : X \rightarrow BG$ determines the principal bundle

$$\begin{array}{ccc} P_p := X \times_{BG} EG & \longrightarrow & EG \\ \downarrow & \lrcorner & \downarrow \pi \\ X & \xrightarrow{p} & BG. \end{array}$$

A strict slice morphism $f : p \rightarrow q$ induces the equivariant bundle map $(x, e) \mapsto (f(x), e)$ covering f . The slice remembers the chosen classifying map: replacing it by an isomorphism

class of bundles would discard data needed for a statement about ordinary categories. Also, $p : X \rightarrow BG$ is the classifying map, whereas $P_p \rightarrow X$ is the principal G -bundle it classifies. These are different maps. For $G = S^1$, this is the setting used in [BSS19, Def. 2.36].

A.2. Spaces and Homotopy Slice Categories

Definition A.6 (The ∞ -category of spaces). A map $f : X \rightarrow Y$ is a *weak homotopy equivalence* if it induces a bijection on path components and isomorphisms $\pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ for every $x \in X$ and every $n \geq 1$. Let W denote these maps. The ∞ -category of spaces is the ∞ -categorical localization

$$\mathcal{S} := N(\text{Top})[W^{-1}],$$

where N denotes the ordinary nerve. Equivalently, it is modeled by the homotopy coherent nerve of the simplicial category of Kan complexes. An object is a homotopy type (an ∞ -groupoid); the mapping object $\text{Map}_{\mathcal{S}}(X, Y)$ is itself a homotopy type, retaining maps, homotopies between maps, and their higher homotopies.

For these models see [Lur09, Def. 1.2.16.1 and Rem. 1.2.16.3; Hir19]. For topological representatives, choose a CW approximation $QX \rightarrow X$, that is, a weak homotopy equivalence with QX a CW complex. Then a model for the derived mapping space is

$$\text{Map}_{\mathcal{S}}(X, Y) \simeq \text{Sing Map}_{\text{Top}}(QX, Y). \quad (79)$$

Here Sing denotes the singular simplicial set. For a CW complex X , including S^1 , we may take $QX = X$. The ordinary homotopy category $h\mathcal{S}$ has morphism sets

$$\text{Hom}_{h\mathcal{S}}(X, Y) = \pi_0 \text{Map}_{\mathcal{S}}(X, Y).$$

When unambiguous, we use the same symbol for a topological space and its image in $\mathcal{S}$.

Definition A.7 (The slice ∞ -category of spaces). For $B \in \mathcal{S}$, the *slice ∞ -category* $\mathcal{S}_{/B}$ is the ∞ -category of objects of $\mathcal{S}$ over B [Lur09, § 1.2.9]. Its objects are maps $p : X \rightarrow B$. For $p : X \rightarrow B$ and $q : Y \rightarrow B$, its mapping space is

$$\text{Map}_{\mathcal{S}_{/B}}(p, q) \simeq \text{Map}_{\mathcal{S}}(X, Y) \times_{\text{Map}_{\mathcal{S}}(X, B)}^h \{p\}. \quad (80)$$

Here the first map is $q \circ (-)$, and $\times^h$ denotes a homotopy pullback. In particular, a morphism is represented by a map $f : X \rightarrow Y$ together with a *specified* homotopy $H : q \circ f \simeq p$. The mapping space also retains homotopies between these data and all higher coherences. It is not enough merely to require that such a homotopy exist.

For a group object G in spaces, the two ∞ -categories relevant to the adjunction are $\mathcal{S}_{/BG}$ and $\mathcal{S}$. The circle case is $G = S^1$. A morphism in $\mathcal{S}_{/B}$ is an equivalence exactly when its underlying map in $\mathcal{S}$ is an equivalence.

For (80), see [Lur09, Lem. 5.5.5.12 and Rem. 5.5.6.10]. Composing $(f, H) : p \rightarrow q$ and $(g, K) : q \rightarrow r$ gives $g \circ f$ with the concatenated homotopy $r \circ g \circ f \simeq q \circ f \simeq p$. The slice construction supplies its higher coherence.

Remark A.8 (Comparison with topological models). For a topological representative B , let W_B be the strict slice maps whose underlying maps are weak homotopy equivalences. There is an equivalence

$$N(\text{Top}_{/B})[W_B^{-1}] \simeq \mathcal{S}_{/B}. \quad (81)$$

To compute mapping spaces, choose a CW approximation of the source over B and replace the target by a weakly equivalent Serre fibration over B . The singular complex of their strict mapping space (78) then computes (80); see [Hir19; Lur09, Lem. 6.1.3.13].

The category $h(\mathcal{S}_{/B})$ must be distinguished from $(h\mathcal{S})_{/B}$: the latter only asks for a triangle to commute after passing to homotopy classes and does not remember the specified homotopy.

Example A.9 (Homotopies in a slice mapping space). For the object $b : * \rightarrow BS^1$, the strict topological mapping space from b to itself is a point. In the homotopy slice, however, (80) gives

$$\mathrm{Map}_{\mathcal{S}_{/BS^1}}(b, b) \simeq * \times_{BS^1}^h * \simeq \Omega BS^1 \simeq S^1.$$

Indeed, a self-map of the point is unique, while its specified homotopy over BS^1 is a loop at b_0 .

Notation A.10 (Ambient categories for extension and cyclification). For a fixed topological group G , the ordinary and homotopy-invariant constructions have types

$$\begin{aligned} \mathrm{Ext}_{\mathrm{Top}}^G : \mathrm{Top}_{/BG} &\longrightarrow \mathrm{Top}, & \mathrm{Cyc}_{\mathrm{Top}}^G : \mathrm{Top} &\longrightarrow \mathrm{Top}_{/BG}, \\ \mathrm{Ext}^G : \mathcal{S}_{/BG} &\longrightarrow \mathcal{S}, & \mathrm{Cyc}^G : \mathcal{S} &\longrightarrow \mathcal{S}_{/BG}. \end{aligned}$$

Their definitions are Defs. 1.2, 1.3 and 1.6, and their adjunctions are Thms. 2.1 and 2.3. The comparison conditions are in Rem. 1.7. For $G = S^1$ we suppress the group superscript, as in Def. 1.8 and Cor. 2.5. On the slice side, the projection to BG is part of the object even when its underlying space is also called its cyclification; compare [SS24, § 2.2].

A.3. Nonabelian Cohomology

We use the categories of spaces of §§ A.1 and A.2. All mapping spaces below are derived, and $[X, A] := \pi_0 \mathrm{Map}_{\mathcal{S}}(X, A)$ denotes morphisms in $h\mathcal{S}$ (Def. A.6). For CW complexes these are ordinary homotopy classes; for arbitrary topological representatives use (79) and Rem. A.8. Maps are unpointed even when the coefficient space has a basepoint.

Definition A.11 (Nonabelian cohomology). A *coefficient space* $A \in \mathcal{S}$ determines the non-abelian cohomology functor

$$\begin{aligned} (h\mathcal{S})^{\mathrm{op}} &\xrightarrow{H(-;A)} \mathrm{Set} \\ X &\longmapsto H(X; A) := [X, A] \end{aligned}$$

with pullback along $f : X' \rightarrow X$ given by precomposition. No connectedness assumption on X or A is required, and the resulting sets need not carry a natural group structure.

If A is pointed and connected, its loop space $G := \Omega A$ is a group object in spaces, with delooping $BG \simeq A$. In this case one also writes

$$H^1(X; G) := H(X; BG), \quad H^1(X; \Omega A) \simeq [X, A]. \quad (82)$$

This extends the classical degree-one notation for principal bundles, recalled in Rem. A.12. For the general homotopy-theoretic formulation see [FSS23, § 2; Toë02].

Remark A.12 (Classical degree-one cohomology and principal bundles). The notation $H^1(X; G)$ comes from nonabelian Čech cohomology in degree one. To make the comparison

explicit, let G be a compact Lie group and let X be a CW complex. Write $\mathcal{C}_G$ for the sheaf $U \mapsto C^0(U, G)$ of continuous G -valued maps, with pointwise multiplication. Local sections s_i of a principal right G -bundle on an open cover (U_i) determine transition functions by $s_j = s_i g_{ij}$. They satisfy

$$g_{ij} : U_i \cap U_j \longrightarrow G, \quad g_{ij} g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k. \quad (83)$$

These are Čech 1-cocycles with coefficients in $\mathcal{C}_G$. Changing sections to $s_i h_i$, where $h_i : U_i \rightarrow G$, replaces them by $g'_{ij} = h_i^{-1} g_{ij} h_j$. Taking classes under these changes and refinement of covers gives $\check{H}^1(X; \mathcal{C}_G)$.

Conversely, a cocycle glues the trivial bundles $U_i \times G$ by $(x, g)_j \sim (x, g_{ij}(x)g)_i$. The cocycle condition is exactly the consistency condition for this gluing, while a change by the h_i gives an isomorphic bundle. Together with the classifying-space theorem, this gives

$$\check{H}^1(X; \mathcal{C}_G) \cong \{\text{principal } G\text{-bundles on } X\} / \cong \cong [X, BG] = H^1(X; G). \quad (84)$$

The degree 1 thus records transition data on double overlaps; the delooping description recovers this classical notation. It is generally a pointed set, with the trivial bundle as distinguished element. The coefficient sheaf consists of continuous maps to G , rather than the constant sheaf on the underlying abstract group.

An early systematic source is Grothendieck's 1955 Kansas report, [Gro55, Ch. V]; see also Frenkel's 1957 treatment [Fre57]. For the topological bundle classification theorem see [Hus94]. The CW hypothesis here ensures agreement with the derived mapping-space convention of Def. A.6; the cocycle gluing description itself applies more generally to locally trivial bundles.

Example A.13 (Ordinary and generalized cohomology). Let E be a spectrum represented by an Ω -spectrum $(E_n)_{n \in \mathbb{Z}}$, with $E_n \simeq \Omega E_{n+1}$. Its unreduced cohomology on unpointed spaces is

$$E^n(X) \simeq H(X; E_n) = \pi_0 \text{Map}_s(X, E_n). \quad (85)$$

The infinite loop space structure of E_n supplies the abelian group structure on these sets. See [Ada74; AGP02]. In particular:

- (i) Integral cohomology is represented by Eilenberg–Mac Lane spaces:

$$H^m(X; \mathbb{Z}) \simeq H(X; K(\mathbb{Z}, m)) \quad (m \geq 0).$$

Writing $B^m \mathbb{Z} := K(\mathbb{Z}, m)$, for $m \geq 1$ this becomes $H^m(X; \mathbb{Z}) \simeq H^1(X; B^{m-1} \mathbb{Z})$.

- (ii) Complex K-theory has representing spaces $\mathbb{Z} \times BU$ in even degree and U in odd degree. Here $U = \text{colim}_n U(n)$ and BU is its classifying space. Thus

$$KU^0(X) \simeq H(X; \mathbb{Z} \times BU), \quad KU^1(X) \simeq H(X; U).$$

The subgroup of classes of rank zero on every connected component is

$$KU_{\text{rk}=0}^0(X) \simeq H(X; BU) \simeq H^1(X; \Omega BU).$$

For finite CW complexes the rank is the virtual rank of a difference of vector bundles; see [Ati67; FSS23, Ex. 2.11].

Example A.14 (Cohomotopy and stable cohomotopy). For $n \geq 1$, the sphere S^n represents *unpointed cohomotopy*:

$$\pi^n(X) := H(X; S^n) \simeq H^1(X; \Omega S^n) = [X, S^n]. \quad (86)$$

This is a set in general; no group structure is implied by the notation. See [Bor36; Pon38; Spa49; Hu59, § VII].

The sphere spectrum $\mathbb{S}$ represents stable cohomotopy. For $n \in \mathbb{Z}$,

$$\mathbb{S}^n(X) := [\Sigma^\infty X_+, \Sigma^n \mathbb{S}]_{\text{Sp}} \simeq H(X; \Omega^\infty \Sigma^n \mathbb{S}). \quad (87)$$

Here X_+ is obtained by adjoining a disjoint basepoint, and the brackets with subscript Sp denote morphisms in the homotopy category of spectra. For $n \geq 1$, the stabilization map $S^n \rightarrow \Omega^\infty \Sigma^\infty S^n$ induces $\pi^n(X) \rightarrow \mathbb{S}^n(X)$; compare [Ada74].

Proposition A.15 (Cohomology operations). *A cohomology operation from $H(-; A_1)$ to $H(-; A_2)$ is a natural transformation of set-valued functors on $(h\mathcal{S})^{\text{op}}$:*

$$\Phi_* : H(-; A_1) \longrightarrow H(-; A_2). \quad (88)$$

Postcomposition induces a bijection

$$[A_1, A_2] \cong \text{Nat}(H(-; A_1), H(-; A_2)). \quad (89)$$

Thus operations are classified by $H(A_1; A_2)$.

Proof. A map $\Phi : A_1 \rightarrow A_2$ acts by $[a] \mapsto [\Phi \circ a]$. Conversely, evaluate an operation η on $[\text{id}_{A_1}]$ to obtain $[\Phi] \in [A_1, A_2]$. Naturality with respect to $[a] : X \rightarrow A_1$ gives $\eta_X([a]) = [\Phi \circ a]$. These constructions are inverse. This is the Yoneda lemma applied to $h\mathcal{S}$; compare [FSS23, Def. 2.3]. $\square$

Definition A.16 (Twisted nonabelian cohomology). Fix a space B and a *coefficient family* $p : P \rightarrow B$, regarded as an object of the homotopy slice $\mathcal{S}_{/B}$ of Def. A.7. A *twist* on X is an object $\tau : X \rightarrow B$ of the same slice. The τ -twisted cohomology with coefficients in p is

$$H_\tau(X; p) := \pi_0 \text{Map}_{\mathcal{S}_{/B}}(\tau, p). \quad (90)$$

This defines a contravariant set-valued functor on $h(\mathcal{S}_{/B})$.

By (80), a cocycle is a map $a : X \rightarrow P$ together with a specified homotopy $p \circ a \simeq \tau$, and its class is a connected component of this space of data. Requiring only $[p \circ a] = [\tau]$ in $h\mathcal{S}$ would lose part of the cocycle data; see Rem. A.8 and Ex. A.9.

If a pointed connected space A is identified with the homotopy fiber of p over a chosen point of B , one may abbreviate (90) as $H_\tau^1(X; \Omega A)$ when the coefficient family p is understood. The homotopy type of the fiber A alone does not determine the twisted theory.

See [FSS23, § 3]. The notation $H_\tau(X; p)$ also applies when the fibers are disconnected.

Proposition A.17 (Description by sections). *Form the homotopy pullback*

$$\begin{array}{ccc} \tau^* P & \longrightarrow & P \\ q \downarrow & \lrcorner & \downarrow p \\ X & \xrightarrow{\tau} & B. \end{array}$$

There is a natural equivalence

$$\text{Map}_{\mathcal{S}_{/B}}(\tau, p) \simeq \Gamma_X(\tau^* P), \quad \Gamma_X(\tau^* P) := \text{Map}_{\mathcal{S}_{/X}}(\text{id}_X, q).$$

Consequently $H_\tau(X; p)$ is the set of connected components of the space of homotopy sections of q .

Proof. The universal property of the homotopy pullback identifies a section over X with a map $a : X \rightarrow P$ and a specified homotopy $p \circ a \simeq \tau$, including homotopies between such data and their higher coherences. $\square$

If X is a CW complex and a topological representative of p is a Serre fibration, the ordinary pullback and its space of continuous sections compute this description; compare Def. A.4 and Rem. A.8. For the constant family $\text{pr}_B : B \times A \rightarrow B$, the section space is $\text{Map}_S(X, A)$, so $H_\tau(X; \text{pr}_B) \simeq H(X; A)$.

Example A.18 (Twisted complex K-theory at rank zero). Tensoring virtual vector bundles with line bundles gives a homotopy coherent action of $BU(1)$ on $\mathbb{Z} \times BU$, preserving the rank-zero component BU . Its homotopy quotient defines the coefficient family

$$p_K : BU // BU(1) \longrightarrow B^2U(1) \simeq K(\mathbb{Z}, 3).$$

For a CW complex X and a twist $h_3 : X \rightarrow K(\mathbb{Z}, 3)$, rank-zero twisted K-theory is

$$KU_{\text{rk}=0}^{0, h_3}(X) \simeq H_{h_3}(X; p_K). \quad (91)$$

One may also write $H_{h_3}^1(X; \Omega BU)$ with p_K understood; see [AS04; FSS23, Prop. 3.5].

Proposition A.19 (Twisted cohomology operations over a fixed base). *Let $p_i : P_i \rightarrow B$ be coefficient families. A morphism $\Phi : p_1 \rightarrow p_2$ in $\mathcal{S}_B$ induces operations*

$$\Phi_* : H_\tau(X; p_1) \longrightarrow H_\tau(X; p_2), \quad (92)$$

natural in the object $\tau : X \rightarrow B$ of $h(\mathcal{S}_B)$. Every such natural operation is induced by a unique homotopy class of slice morphisms:

$$\text{Nat}\left(H_{(-)}(-; p_1), H_{(-)}(-; p_2)\right) \cong \pi_0 \text{Map}_{\mathcal{S}_B}(p_1, p_2) = H_{p_1}(P_1; p_2).$$

Proof. Apply the Yoneda lemma in $h(\mathcal{S}_B)$ to the two represented functors. Explicitly, evaluate an operation at the identity of p_1 ; naturality then recovers its value on every slice morphism $\tau \rightarrow p_1$ by postcomposition, just as in Prop. A.15. $\square$

Remark A.20 (Changing the twist). An operation over a fixed base leaves the twist τ unchanged. More generally, a map $\psi : B_1 \rightarrow B_2$ and a map $\Phi : P_1 \rightarrow P_2$, together with a specified homotopy $p_2 \circ \Phi \simeq \psi \circ p_1$, induce

$$H_\tau(X; p_1) \longrightarrow H_{\psi \circ \tau}(X; p_2).$$

The new twist is determined by the map of bases ψ ; see [FSS23, Def. 3.6].

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