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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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**DRAFT VERSION ONLY**

# 1 Lecture 1

**Gauge theory and Representation theory**

Gauge theory captures structures from representation theory.

- Toy model: 2-dimensional gauge theory Representations of finite groups
- 3-dimensional gauge theory Real and complex Lie groups
- 4-dimensional gauge theory geometric Langlands program

What is gauge theory?

A physical theory (Quantum field theory) in which fundamental fields are principal $G$-bundles with connection on a spacetime manifold $M$. (with $M$ in dimensions $2, 3, 4$)

- Classical:
  look for connections satisfying equations (PDE's), flat connections, Yang-Mill's equation)
- Quantum:
  look at all connections by attaching $\mathbb{C}$-linear data (with a weighting by the Yang-Mills action)
- Topological:
  study "coarse" features depending only on topology of spacetime $M$

**Dijkgraaf-Witten theory**

$G$ a finite group.
We assign $Z_G$, a 2-dimensional field theory,
$M$, a manifold of dimension $2, 1$ or $0$ (oriented).

The space we are going to study is the space of gauge fields

$$\mathcal{M}_G(M) = \{\text{G-bundles on M}\} = \{\text{G-Galois covers of M}\} = \{\pi_1(M) \to G\}/\sim$$

We think of this as a finite orbifold: keep track of symmetries (automorphisms).

$$\mathcal{M}_G(\bullet) = BG$$

$$\mathcal{M}_G(S^1) = G_{ad}/G$$

$$\mathcal{M}_G(\Sigma_g) = \left\{A_1, \ldots, A_g, B_1, \ldots, B_g \in G, \prod [A_i, B_i] = 1\right\}/G$$

Quantum field theory tells us to calculate the partition function. $M = \Sigma$, a 2-manifold

$$Z_G(M) = \text{“}\int_{\text{fields on } M} e^{-S(\varphi)} d\varphi\text{”}$$

$$Z_G(\Sigma) = \#\mathcal{M}_G(\Sigma) = \text{weighted \# of G-bundles on } \Sigma = \sum_{\mathcal{P} \in \mathcal{M}_G(M)} \frac{1}{|\text{Aut}\mathcal{P}|}$$

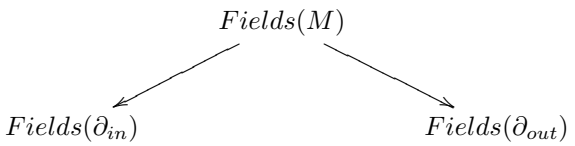
**Locality**

$$Z(M)(\varphi_0) = \int_{\varphi|_{\partial M} = \varphi_0} e^{-S(\varphi)} d\varphi$$

$Z(M) \in \text{Fun}(\text{Fields}(\partial M)) =: Z(\partial M)$

$$2 - \text{manifolds} \to \mathbb{C}$$

$$1 - \text{manifolds} \to \text{Vect}$$




$$Z(\partial_{in}) \to Z(\partial_{out})$$

$$Z(\partial_{in}) \in f \mapsto \{\varphi_{out} \mapsto \in_{\varphi|_{\partial_{out}} = \varphi_{out}} f(\varphi_{in}) e^{-S(\varphi)} d\varphi\}$$

$$= \pi_{out*}(\pi_{in}^* f \cdot e^{-S})$$

$$Z_G(S^1) = \mathbb{C}G_{ad}/G = \mathbb{C}[G]^G = \text{ class functions}$$

figure of Z of caps cups and more...

$$Z(cap) =: 1 = \gamma_1 \in \mathbb{C}[G]^G (\text{skyscraper})$$

$$Z(cup) =: eval_1 \colon \mathbb{C}[G]^G \xrightarrow{tr} \mathbb{C}$$

$$Z(pants) \colon \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \to \mathbb{C}[G]^G (\text{convolution})$$

figure of pants with G-labelled legs going to product on waist

$$G \times G \to G$$

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G]$$

($G$ finite: Fun($G$) = Measures($G$))

This tells us that functions on a group are an associative algebra $\mathbb{C}[G]$ and class funtions $\mathbb{C}[G]^G \subset \mathbb{C}[G]$ are a subgroup.

Now we can get a commutative Frobenius algebra $\mathbb{C}[G]^G$.

First, this is commutative. This is because our operation is defined topologically on a surface. Also, this algebra carries a canonical trace: this is given by an "elbow": the trace is non-degenerate.

The claim is this amount of data, the class functions with the unit and inner product $(\mathbb{C}[G]^G, 1, eval_1)$, we can figure out what to assign to any surface.

In physics terms, we say we can "solve this system": $\mathbb{C}[G]^G \to \mathbb{C}[G]^G$

We want to

$$\text{diagonalize simultaneously} \leftrightarrow \text{joint spectrum}$$

$$= Spec\mathbb{C}[G]^G = \{\mathbb{C}[G]^G \to \mathbb{C}\} = \hat{G} = \text{irreducible characters of } G$$

$$Z(\Sigma_g) = \sum_{\chi \in \text{irred chars of } G} \left(\frac{|\, G \,|}{dim\chi}\right)^{2g-2}$$

$$2 - manifold \to Z(M) = \int_{\mathcal{P}} e^{-S} d\varphi$$

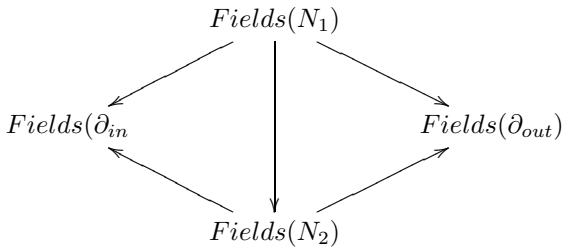
$$1 - manifold \to Z(N) \in \text{Vect} = \text{functions on boundary fields}$$

We fix boundary values and have a "closed 1-manifold" and therefore have a vector space:

$$Z(N) \colon \{\text{fields on } \partial N\} \to \text{Vect}$$

$Z(N)$ is a vector bundle or sheaf on space of fields $Fields(\partial N)$.

$$\partial N(0 - \text{manifold}) \to \{\text{vector bundles on Fields } \partial N\}(\text{a linear category})$$




$$Z\colon \mathrm{Vect}(Fields(\partial_{in})) \overset{Z(N)}{\to} \mathrm{Vect}(Fields(\partial_{out}))$$
$$V \mapsto \pi_{out*}(\pi_{in}^* V \otimes \text{“}e^{-S}\text{”})$$

Where do representations come from?
What do we assign to a point?

$$Z_G(\bullet) = \mathrm{Vect}(Fields(\bullet)) = \mathrm{Vect}(\bullet\text{with symm group } G) = Rep_{\mathbb{C}}[G]$$

The above has another equality to $\mathbb{C}[G] - mod$. The basis for the group algebra is given by delta functions for $g \in G$ and these act on rep???

The 1-dimensional cup has $A, B \in RepG$ labelling the boundaries.

The triangle with sides labelled $\hom(A,B)$ and $\hom(B,C)$ is the natural transformation to the third side $\hom(A,C)$.

From here we get $Z(cup) = Hom_{RepG}(A,B)$

The 2-dimensional cup has a mirror dual version giving

$$Id_A \in End_G A \overset{tr}{\to} \mathbb{C} \in dim A$$

We have maps both ways from $EndA$ to and from $\mathbb{C}[G]^G$

$$\mathbb{C}[G]^G \to End_G X$$
$$\mathbb{C}[G]^G \overset{\sim}{\to} EndId_{RepG}$$
$$\mathbb{C}[G]^G \cong \text{Center or Hochschild cohomology} = Z(\mathbb{C}[G])$$
$$HH^* \to \mathbb{C}[G](a*b = b*a)$$

The dual picture

$$End_G A \to \mathbb{C}[G]^G$$

encodes the fact that

$$\mathbb{C}[G]^G = \text{cocenter or Hochschild homology of } \mathbb{C}[G]$$

$$\mathbb{C}[G] \overset{tr}{\to} HH_*$$
$$tr(a \times b) = tr(b * a)$$