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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 2

2-dimensional TQFT (extended)

$$G\text{finite group } \to Z_G$$
$$Z_G(\cdot) = Rep_{\mathbb{C}}G$$

The action map was a reflection of the fact that the class functions were the center, since class functions act as endomorphisms of any given representation.

$$act : \mathbb{C}G_{ad}/G \leftrightarrow End_G A : char$$

$$act\colon \mathbb{C}G_{ad}/G = Z(\mathbb{C}G) \quad (Rep_{\mathbb{C}}G = Mod_{\mathbb{C}}G)$$

$$char\colon End_G A \ni Id_A \mapsto \chi_A \in \mathbb{C}G_{ad}/G$$

Example
$G \supset K$ subgroup

$$A = V_{G,K} = \mathbb{C}[G/K = Ind_K^G\mathbb{C}$$

The Hecke algebra $\mathcal{H}_{G,K} = End_G V_{G,K}$

$V_{G,K}$ represents $RepG \to \text{Vect}$ by $A \mapsto A^K$.

$$Hom_G(V_{G,K}, A) = A^K \text{action by } \mathcal{H}_{G,k} = End()^K$$

$$\mathcal{H}_{G,K} = \hom_G(V_{G,K}, V_{G,K}) = \mathbb{C}[G/K]^K = \mathbb{C}[K\backslash G/K]$$

$\mathbb{C}[K\backslash G/K] \subset \mathbb{C}[G]$

$$K\backslash G \times_K G/K \to K\backslash G$$

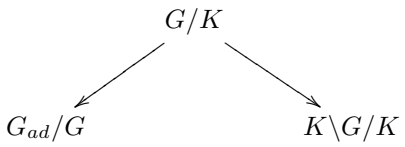
G gauge theory

$\mathcal{M}_G(interval) = \cdot/G$ If we mark the endpoints with K and the interval with G we get $K\backslash G/G$

If $K' = \{1\}$, then $\mathcal{M}(interval) = G$.

This has to do with convolution of the Hecke algebra. I missed how.

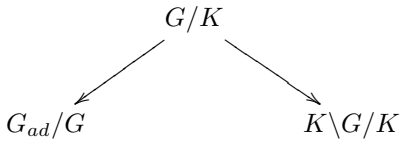
Then we get a correspondence which is $\mathcal{M}$ of the cylinder over the circle and the intercal. So this gives



Pulling and pushing gives

$$\mathbb{C}G_{ad}/G \to EndV_{G,K}$$

TH character of a reduced representation $CharV_{G,K}$ comes from



considering K/K sitting inside G/K.

Now we switch to complex Lie groups.

$G \to$ complex reductive Lie group (or affine algebraic groups/$\mathbb{C}$)

The notion of fields on a manifold will still be

$$\mathcal{M}_G(M) = \{G \text{ local systems}\} = \{\pi_1(M) \to G\}/\sim$$

$\mathcal{M}_G(S^1) = G_{ad}/G \supset H^{reg}/W$

$\mathcal{M}_G(\Sigma_g) = \left\{A_1, \ldots, A_g, B_1, \ldots, B_G \in G \mid \prod[A,B] = 1\right\}$ an affine algebraic variety

$\mathbb{F}_q \to \mathcal{M}_{G(\mathbb{F}_q)}(\Sigma_g) \quad TFT(q)$ Havsel-Rodriguez-Villegas

FILL IN BLANKS

2-dimensional extended TFT → 3-dimensional extended TFT

$$\#$$

Σ	$\#$	Vect
S^1	Vect	Cat
$\circ$	Cat	$2-$Cat

Geometric function theory $\mathbb{C}G-Mod$

X variety, stack $\rightarrow$ "categorified space of functions"

- quasicoherent sheaves, $Q(X)$ is the dg category of quasicoherent sheaves
- $\mathcal{D}$-modules

- basic objects: algebraic vector bundles (SOMETHING HERE)
generate others by taking kernels, cokernels, and complexes.

$$X = SpecR \rightarrow \mathbb{C}(R-Mod) = \mathbb{C}(R)\text{up to quasi-isomorphisms}$$

General X: define $Q(X)$ by gluing $Q(U)$ for open cover of X (descent)
II. $X \rightarrow \mathcal{D}(X)$ dg category of $\mathcal{D}$-modules

-basic objects: vector bundles with flat connection
$Q(X) \ni \mathbb{F} + \nabla\colon F \rightarrow F \otimes \Omega^1$

$$T \otimes F \rightarrow F \text{ extends to } \mathcal{D} \otimes F \rightarrow F$$

$\mathcal{D}$-modules are close analogs of functions/distributions on X.
$f \in \mathbb{C}^\infty(X) \quad \delta \in C^\infty(X)$
E.g., $e^{\lambda x} on \mathbb{C}$
E.g., $\delta on \mathbb{C}$

$$\mathcal{D} \cdot f \subset \mathbb{C}^\infty(X) \quad \mathcal{D} \cdot \delta \subset \mathbb{C}^\infty(X)$$

$$\mathcal{D}e^{\lambda X} = \mathcal{D}/\mathcal{D}(\partial - \lambda) \quad (\partial - \lambda)e^{\lambda x} = 0$$
$$\mathcal{D} \cdot \delta_\lambda = \mathcal{D}/\mathcal{D}(\lambda - \lambda)$$

We can do **algebra**:

$Q(X)\otimes$ and $\mathcal{D}(X),\otimes$ are commutative algebra objects of dgcat over $\mathbb{C}$.

symmetric monoidal $(\infty,1)$- category (Lurie DAG II and III)

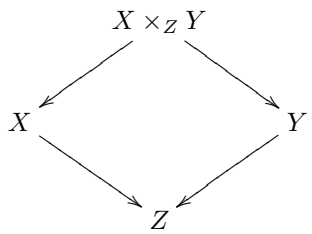
Pullback and pushforward

$\pi\colon X\to Y$

$$\pi^*\colon Q(Y)\to Q(X) \quad \pi^*\colon \mathcal{D}(Y)\to\mathcal{D}(X)$$
$$\pi_*\colon Q(X)\to Q(Y) \quad \pi_*\colon \mathcal{D}(X)\to\mathcal{D}(Y)$$

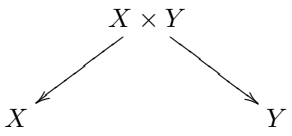
Finite sets

X,Y finite sets



$$Fun(X\times_Z Y)=Fun(X)\otimes_{Fun(Z)}Fun(Y)=Y\times_Z X \text{ matrices}$$
$$=\hom_{Fun(Z)}(Fun(X),Fun(Y))$$

Integral transform



$K\in\mathcal{D}(X\times Y)$

$$K\colon \mathcal{D}(X)\to\mathcal{D}(Y)$$
$$F\mapsto k\times F$$

$$\int_{\pi_Y} F(x)k(x,y)dx=\pi_{Y*}(\pi_X^*F\otimes K)$$

Theorem 1.

- $Q(X)\otimes_{Q(Z)}Q(Y)\cong Q(X\times_Z Y)=$ *integral kernels* $=Fun_{Q(Z)}(Q(X),Q(Y))$
- *Toen for $Z=$point, X,Y schemes*
- *BZ-Francis-Nadler, perfect stacks*

- $\mathcal{D}(X) \otimes_{\mathcal{D}(Z)} \mathcal{D}(Y) = \mathcal{D}(X \times_Z Y) = Fun_{\mathcal{D}(Z)}(\mathcal{D}(X), \mathcal{D}(Y))$

We are going to talk about TFT's where

$$\cdot \mapsto \textit{modules over group algebras } Q(G), \mathcal{D}(G)$$

Beilinson-Bernstein

reps of Lie algebra come from studying $\mathcal{D}$-modules on the flag manifold.