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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 3

Goal: Aspects of 3-dimensional gauge theory on a point.

Recall: Γ, finite group, we have the identification

$$Rep_{\mathbb{C}}\Gamma = \mathbb{C}\Gamma - mod$$

$$Rep_k\Gamma = k\Gamma - mod$$

This gives a useful way of thinking about representations as modules over an associative algebra. For infinite groups, one way to specify classes of representations it is often convenien to specify group algebra. $E.g., \mathbb{C}_c(G), L^1(G)$.

G, complex affine group (reductive: $GL_n\mathbb{C}, E_8(\mathbb{C})$).

Want to look at group actions on categories. In particular, dg categories or infinity categories. If G is algebraic we want the action to be somehow algebraic.

$$a_g\colon \mathcal{C} \to \mathcal{C}, \quad g \in G$$

Want to say a_g depend algebraically on g.

To get arond this we think about group algebras.

$$\Gamma \to \mathrm{Vect}\Gamma, * \qquad \Gamma \times \Gamma \to \Gamma$$

So we have a monoidal category.

Γ-action on $\mathcal{C}$:= VectΓ-module category.

Two natural candidates for a group algebra. These are the grouup algebras for G valued in quasi-coherent sheaves or $\mathcal{D}$-modules.

$$G \to Q(G) \quad \text{quasi-coherent sheaves on } G$$

$$G \to \mathcal{D}(G) \quad \mathcal{D}-\text{modules on } G$$

These are both monoidal dg categories.

$$M\colon G \times G \to G$$
$$M(F \odot G) =: F \times G$$

We want to define topological field theories where to the point we can assign (four different possibilities):

- $\bullet \mapsto$ VectΓ-modules (3-dimensional TFT)
- "$Q(\mathcal{C})G$-categories" $\{QG - modules\}$ $QG \otimes \mathcal{C} \to \mathcal{C}$ (2-dimensional TFT)
- "smooth G-categories" $\{\mathcal{D} - modules\}$ (1-dimensional TFT)

There is a modified version to give 2-dimensional TFT's.

For finite group Γ:

We have an action of G on a variety X, an action of $Q(G)$ on $Q(X)$, and $\mathcal{D}(G)$ on $\mathcal{D}(X)$.

This is where examples come from.

What kind of structures do these categories of G-categories have?

For $H \subset G$,
$Q(G/H) \in Z_G^Q(\cdot)$ is a natural example. So we look at the endomorphisms of the induced representation. This should be a Hecke algebra, i.e., functions on a double coset.

$$EndQ(G/H) = Q(H\backslash G/H) = \cdot/H \times_{\cdot/G} \cdot/H$$

E.g., (for $H = G$),

$$End\text{Vect} = \cdot/G = Q(BG) = Rep_{\mathbb{C}}G, \otimes$$

1.1 Morita theory

X, finite set.
$Fun(X \times X)$ algebra $= EndFun(X)$ is Morita equivalent to $\mathbb{C} = Fun(\cdot)$

I.e., $Fun(X \times X) - mod \cong \mathbb{C} - mod$

For $X \to Y$ surjective,

$$Fun(X \times_Y X) - mod \cong Fun(Y) - mod = \text{Vect}(Y)$$

$Q(H\backslash G/H) = Q(\cdot/H \times_{\cdot/G} \cdot/H)$

Theorem 1. *(BZ-Francis-Nadler)*
$\forall H$,

$$Q(H\backslash G/H) - mod \cong Q(G) - mod \cong (RepG) - mod$$

Primary Example:

Let G be a reductive group, e.g., $GL_n(\mathbb{C})$, containing $K = B$ Borel.

G/B = flag variety = complete flags in $\mathbb{C}^n$.

Beilinson-Bernstein says tehre is an isomorphism between: actions of $\mathcal{D}G$ on $\mathcal{D}(G/B)$ and $\mathcal{D}G$-actions on $\mathfrak{g} - mod_0$

We have a left action by $\mathcal{D}(G)$ on $\mathcal{D}(G/B)$ and a right action by $\mathcal{D}(B\backslash/B) = \mathcal{H}$ the finte Hecke category (roughly "Category O")

$$B\backslash(G/B) \leftrightarrow \text{Schubert cells} \leftrightarrow W \text{ Weyl group}$$

$$K(\mathcal{H}) = \mathbb{Z}W$$

"Bases" of $\mathcal{H}$ labelled by $w \in W$.

$$\ni T_w \text{ standard } i_{w*}\mathbb{C}_w$$

$\mathcal{H} \to$ version of $\mathbb{Z}B_G$ $[B_{n-1}$ for GL_n

$$T_{s_i} \times T_{s_j} \times T_{s_i} = T_{s_j} \times T_{s_i} \times T_{s_j}$$
$$T_{s_i}^2 \neq Id$$

So if we give an action of $\mathcal{H}$ on some catgory $\mathcal{C}$ gives an action of the braid group on $\mathbb{C}$. This is how many actions of braid group arise in Khovanov homology.

Studying modules for $\mathcal{D}(G)$ is not good.

We want

$$\cdot \mapsto \mathcal{H} - modules \Rightarrow \text{Theorem (BZ-Nadler) 2d extended TFT's}$$

We have the Harish Chandra modules $\mathcal{D}(K\backslash G/B)$ and we can take actions of $\mathcal{H}$ on these.

$Z(S^1)$ [TFT of dim ≥ 2]

FIGURES

For $A \in Z(pt)$, we get two maps

$$Z(S^1) \to EndA$$

$Z(S^1)$ = "Hochschild cohomology/center"

$$EndA \to Z(S^1)$$

$Z(S^1)$ = "Hochschild homology/dimension of Z(pt)"

Center = end of $Id_{Z(\cdot)} = \hom_{\mathcal{A}-\mathcal{A}^{op}}(\mathcal{A}, \mathcal{A})$

$Z(\cdot) = \mathcal{A}$-mod

Dimension = FIGURES TO BE INCLUDED

$$Z(\cdot) = \mathcal{A} - mod \Rightarrow Dim = \mathcal{A} \otimes_{\mathcal{A} \otimes \mathcal{A}^{op}} \mathcal{A}$$

$$\text{Vect}\Gamma - mod = Z(\cdot)$$

$$Center(\text{Vect}\Gamma) = Z(S^1)$$

Here the center is the Drinfel'd center.

Theorem 2. *(BZ-Nadler-Francis) QG has dimension and center equal to* $Q(G_{ad}/G) = Z(S^1)$

Also for a Reimann surface Σ,

$$Z_G^Q(\Sigma) = R\Gamma(\mathcal{M}_G(\Sigma), \mathcal{O})$$

Next time:
Character theory

$$\cdot \mapsto \mathcal{H} - mod$$

$$S^1 \mapsto \text{Luzstig's character sheaves}$$

$$\begin{tikzcd} & G/B \arrow[dl] \arrow[dr] & \\ G_{ad}/G & & B\backslash G/B \end{tikzcd}$$