Universal simplicial bundles and inner automorphism n-groups

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Introduction

In [10] the authors construct the 3-group of inner automorphisms of a 2-group, namely, those automorphisms which are given by conjugating with the lowest dimensional elements:

$$\mathrm{INN}(G_2) \subset \mathrm{AUT}(G_2) = \mathbf{Gray}(\Sigma G_2, \Sigma G_2),$$

where Σ associates to a monoidal groupoid the obvious one-object bigroupoid. This 3-group is equivalent to the trivial 3-group and so can be thought of as being contractible.

In the special case of the 2-group being discrete, i.e. a group G, the construction gives rise to the underlying groupoid of EG. This groupoid has a canonical functor to ΣG, the underlying groupoid of BG. If we consider G as a discrete simplicial group, then $WG = N(\mathrm{INN}(G))$ and $\overline{W}G = N\Sigma G$, with W and $\overline{W}$ the universal bundle and classifying space functors for simplicial groups. The same construction should also work for 2-groups, considered as 3-coskeletal simplicial groups.

We point out that similar work is being undertaken in [2]

1 Simplicial bundles

A simplicial K-bundle on X, for K a simplicial group, is a simplicial set P with a K action (as an object over X), a simplicial map

$$P \to X$$

such that

$$K \times P \to P \times_X P$$

is an isomorphism.

A universal simplicial bundle is one with a contractible total space. There is a canonical model, see for example [8], for the universal K-bundle. However, in the special cases where the simplicial group is a constant simplicial group

$$\cdots\ G \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G \substack{\longrightarrow\\ \longrightarrow} G,$$

or the nerve of a strict 2-group

$$\cdots\ G \ltimes H \ltimes H \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G \ltimes H \substack{\longrightarrow\\ \longrightarrow} G$$

the universal bundle also has the structure of a simplicial group. These simplicial groups have only one and two nontrivial homotopy groups respectively, and with the second example, the group structure on the universal bundle is already fairly complicated. From the canonical definition it is not at all obvious how to continue this structure, so we introduce an isomorphic simplicial object whose group structure is manifest. We then relate this model for the universal bundle to the results in [11] and [9] where the two examples above are considered.

1.1 The traditional classifying complex and universal bundle

Definition 1.1. Let K be a simplicial group. The *classifying complex* $\overline{W}K$ of K has n-simplices given by

$$\overline{W}K_n = K_{n-1} \times \ldots \times K_0, \qquad \overline{W}K_0 = \{*\}$$

and face and degeneracy operators

$$
\begin{aligned}
\bar{\partial}_0(g_n, \ldots, g_0) &= (g_{n-1}, \ldots, g_0), \\
\bar{\partial}_{i+1}(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\
\bar{s}_0(g_{n-1}, \ldots, g_0) &= (id_{K_n}, g_{n-1}, \ldots, g_0), \\
\bar{s}_{i+1}(g_{n-1}, \ldots, g_0) &= (s_i g_{n-1}, \ldots, s_0 g_{n-i}, id_{K_{n-i}}, g_{n-i-1}, \ldots, g_0)
\end{aligned}
$$

for $n > 0$ and

$$
\bar{\partial}_i(g_0) = *, \ i = 1, 2; \qquad \bar{s}_0(*) = (id_{K_0}).
$$

for $n = 0$.

For a group G, this is the nerve of the standard one-object groupoid it defines.

$$
\overline{W}G = N(\Sigma G)
$$

Definition 1.2. Let K be a simplicial group. The universal K-bundle WK has as its set of n-simplices

$$
WK_n = K_n \times \overline{W}K_n = K_n \times \ldots \times K_0
$$

and face and degeneracy operators

$$
\begin{aligned}
\partial_0(g_n, \ldots, g_0) &= (\partial_0 g_n g_{n-1}, g_{n-2}, \ldots, g_0), \\
\partial_i(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\
s_i(g_n, \ldots, g_0) &= (s_i g_n, \ldots, s_0 g_{n-i}, id_{K_{n-i}}, g_{n-i-1}, \ldots, g_0)
\end{aligned}
$$

The simplicial group K includes into WK as the first factor, and there is a Kan fibration

$$
WK \to \overline{W}K
$$

with fibre the underlying simplicial set of K.

Proposition 1.3. *WK is contractible.*

As already pointed out, for an ordinary group G, the object WG gives us the nerve of the 2-group $G \ltimes G \rightrightarrows G = \mathrm{INN}(G)$, and as such is a simplicial group. However, the group structure on WG_n for $n > 1$ becomes increasingly complicated, being the semidirect product

$$
(\ldots(G \ltimes G) \ltimes G \ldots) \ltimes G.
$$

However, there is a groupoid isomorphic to $\mathrm{INN}(G)$, namely $codisc(G)$, whose arrows have a less awkward group structure. For $(g_i, h_i) \in Mor(codisc(G))$,

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

and the trend continues up the nerve, with multiplication given componentwise. The isomorphism

$$\begin{array}{rcl} \Phi : \mathrm{Inn}(G) & \to & codisc(G) \\ (g,h) & \mapsto & (g, hg) \end{array}$$

is more than an isomorphism of groupoids: it is an isomorphism of strict 2-groups. Its inverse is

$$\begin{array}{rcl} \Phi^{-1} : codisc(G) & \to & \mathrm{Inn}(G) \\ (g,h) & \mapsto & (g, hg^{-1}) \end{array}$$

Both $\mathrm{Inn}(G)$ and $codisc(G)$ are equipped with functors to ΣG:

$$\begin{array}{rccl} \pi^{\mathrm{Inn}} : & \mathrm{Inn}(G) & \to \Sigma G, & \pi^{\mathrm{Inn}}(g,h) = h \\ \pi^{codisc} : & codisc(G) & \to \Sigma G, & \pi^{codisc}(g,h) = hg^{-1}, \end{array}$$

giving us a commuting diagram

$$\begin{array}{ccc} \mathrm{Inn}(G) & \xrightarrow{\sim \quad \Phi} & codisc(G) \\ & {\scriptstyle \pi^{\mathrm{Inn}}} \searrow \quad \swarrow {\scriptstyle \pi^{codisc}} & \\ & \Sigma G & \end{array}$$

The difference between the two approaches is precisely the difference between Milnor's construction of the universal bundle, and the universal bundle as described in [11] (section 3), for example. Using the realisation of the nerve of $codisc(G)$ as a model for EG, one can instantly see that EG is a topological group, and moreover Segal [11] has proved that G is a closed subgroup of EG, and is normal iff G is abelian.

In this context, we have a pair of exact sequences

$$\begin{array}{ccccc} disc(G) & \xrightarrow{i} & \mathrm{Inn}(G) & \xrightarrow{\pi^{\mathrm{Inn}}} & \Sigma G \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle =} \\ disc(G) & \xrightarrow{j} & codisc(G) & \xrightarrow{\pi^{codisc}} & \Sigma G \end{array}$$

with

$$i(g) = (g, 1), \qquad j(g) = (g, g),$$

and ΣG is only a 2-group if G is abelian.

1.2 The "codiscrete" version of the universal bundle

We will now construct the analogue of *codisc*(G) for a general simplicial group K, which we will call $\mathcal{E}K$. Define the simplices as

$$\mathcal{E}K_n = K_n \times \ldots \times K_0.$$

The face and degeneracy maps are

$$\begin{aligned}
\partial_0(g_n, \ldots, g_0) &= (g_{n-1}, g_{n-2}, \ldots, g_0), \\
\partial_i(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, g_{n-i-1}, \ldots, g_0), \\
s_i(g_n, \ldots, g_0) &= (s_i g_n, \ldots, s_0 g_{n-i}, g_{n-i}, g_{n-i-1}, \ldots, g_0).
\end{aligned}$$

If we let the product on $\mathcal{E}K_n$ be componentwise, these face and degeneracy maps are homomorphisms, because those of K are. $\mathcal{E}K$ is then a simplicial group and its realisation is a topological group. There is again an isomorphism between WK_n and $\mathcal{E}K_n$

$$\begin{aligned}
\Phi_n : WK_n &\longrightarrow \mathcal{E}K_n \\
(g_n, \ldots, g_0) &\mapsto (k_n, k_{n-1}, \ldots, k_0)
\end{aligned}$$

where the k_j are defined recursively as

$$k_n = g_n, \qquad k_j = \partial_0 k_{j+1} g_j \quad (j \leq n-1).$$

One can see that the maps Φ_n define a map Φ of simplicial sets by the use of the standard identities for the boundary and degeneracy maps for K. The inverse map is

$$\begin{aligned}
\Phi^{-1} : \mathcal{E}K &\longrightarrow WK \\
(h_n, \ldots, h_0) &\mapsto (h_n, \partial_0 h_n^{-1} h_{n-1}, \ldots, \partial_0 h_1^{-1} h_0)
\end{aligned}$$

We use the isomorphism Φ to see how K includes into $\mathcal{E}K$:

$$\begin{aligned}
K \hookrightarrow WK &\xrightarrow{\Phi} \mathcal{E}K \\
k \mapsto (k, 1, \ldots, 1) &\mapsto (k, \partial_0 k, \partial_0 \partial_0 k, \ldots, \partial_0^n k)
\end{aligned}$$

The action of K on $\mathcal{E}K$ can now clearly be seen. For example, the left action is

$$((g_n, \ldots, g_0), k) \mapsto (kg_n, \partial_0 k g_{n-1}, \ldots, \partial_0^n k g_0).$$

There is a map

$$\begin{array}{rcl} \pi^{\mathcal{E}} : \mathcal{E}K & \to & \overline{W}K \\ (h_n, \ldots, h_0) & \mapsto & (\partial_0 h_n^{-1} h_{n-1}, \partial_0 h_{n-1}^{-1} h_{n-2}, \ldots, \partial_0 h_1^{-1} h_0), \end{array}$$

which is the quotient by the left action above, and the triangle

$$\begin{array}{ccc} WK & \xrightarrow{\sim \quad \Phi} & \mathcal{E}K \\ & \searrow \quad \swarrow_{\pi^{\mathcal{E}}} & \\ & \overline{W}K & \end{array}$$

commutes. By proposition 1.3, $\mathcal{E}K$ is also contractible.

The construction of $\mathcal{E}K$ is clearly functorial, so we have an endofunctor

$$\mathcal{E} : s\mathbf{Grp} \to s\mathbf{Grp},$$

and a natural transformation

$$\iota : \mathrm{Id}_{s\mathbf{Grp}} \to \mathcal{E}$$

whose component at K is given by the inclusion $K \hookrightarrow \mathcal{E}K$. Notice that

$$\begin{array}{rcl} (\mathcal{E}\mathcal{E}K)_n & = & \mathcal{E}K_n \times \mathcal{E}K_{n-1} \times \ldots \mathcal{E}K_0 \\ & = & (K_n \times \ldots \times K_0) \times (K_{n-1} \times \ldots \times K_0) \times \ldots \times (K_0). \end{array}$$

If $\mathrm{pr}_1 : \mathcal{E}K_j \to K_j$ denotes projection on the first factor, define the maps

$$(\mu_K)_n = \mathrm{pr}_1 \times \ldots \times \mathrm{pr}_1 : \mathcal{E}\mathcal{E}K_n \to K_n \times \ldots \times K_0 = \mathcal{E}K_n,$$

which clearly assemble into a map of simplicial groups

$$\mu_K : \mathcal{E}\mathcal{E}K \to \mathcal{E}K,$$

and these form the components of a natural transformation

$$\mu : \mathcal{E}\mathcal{E} \to \mathcal{E}.$$

Proposition 1.4. *The data above, $(\mathcal{E}, \iota, \mu)$, forms a monad on s**Grp**.*

Proof. The transformation ι satisfies the unit conditions almost by definition, and proving the associativity axiom is an exercise in indices. □

1.3 The Moore complex

There is a functor M from simplicial groups to normal complexes called the Moore complex, which measures homotopical information. In particular, the homology groups of MK are the homotopy groups of K. In fact, more is true:

Proposition 1.5 ([3]). *Let K be a simplicial group. If $MK_n = 0$ for $n > p$, the canonical arrow $K \to Cosk_p(K)$ is an isomorphism.*

We recall that if $K \xrightarrow{\sim} Cosk_p(K)$, K is called p-coskeletal, and is determined completely by its p-truncation. [As an example, the nerve of a category is 2-coskeletal, as it is determined by the sets of objects, arrows and composable pairs.]

The n^{th} group in the Moore complex for a simplicial group K is given by

$$MK_n = \bigcap_{i=1}^{n} \ker \partial_i$$

and the differential is the restriction of ∂_0. If the simplicial group is abelian, that is, an object of sAb, then the Moore complex is the same as the normalised chain complex defined in the usual manner. It is possible to reconstruct the simplicial group from its Moore complex, and there is an equivalence of categories [3]

$$\{p - \text{coskeletal simplicial groups}\} \xrightarrow{\sim} \{\text{length } p \text{ Moore complexes}\}.$$

We would like to see what effect the functor $\mathcal{E}$ has on the Moore complex of a simplicial group K, even though we know $\mathcal{E}K$ is contractible.

To begin with, $\ker \partial_1 = \{(g_n, g_{n-1}, 1, \ldots, 1) \in (\mathcal{E}K)_n | g_n \in \ker \partial_1 \subset K_n\}$. It follows quickly that $(M\mathcal{E}K)_n = MK_n \times MK_{n-1}$.

We also calculate the Moore complex of $\overline{W}K$:

$$M\overline{W}K_n = MK_n \times MK^0_{n-1} := \{(g, h) \in MK_n \times MK_{n-1} | h \in \ker \partial_0\}$$

Proposition 1.6. *If K is an n-coskeletal simplicial group, $\mathcal{E}K$ is an $n+1$-coskeletal simplicial group.*

Proof. This follows from the conjecture and proposition 1.5. □

Consider now the case that $K = NG_2$ for G_2 a strict 2-group, which we consider as a groupoid internal to **Grp**. By the above corollary, WNG_2 is coskeletal, but more interestingly, has a Moore complex of length 2, which has the structure of a 2-crossed module [3]. 2-crossed modules are known to be equivalent to Gray-monoids, which are 2-categories with a slightly weak monoidal structure. In [9] the authors constructed a contractible Gray-monoid associated to G_2, which contained G_2 as a sub-Gray-monoid.

Question 1.7. *Is the nerve of a Gray-monoid a simplicial group?*

A Strict 2-groups as simplicial groups

[NB : section is old, just a record of the calculation below.]

Consider a strict 2-group G_2 as a groupoid internal to groups. Then its nerve NG_2 is the simplicial set with elements of the set of p-simplices p-tuples of composable arrows. This is automatically a simplicial group, and so we can apply W.[1] Take G_2 as the strict 2-group arising from the crossed module $t : H \to G$, with the action of G on H denoted ${}^g h$. Let $\Gamma = NG_2$, which has as n-simplices

$$\Gamma_n = G \times H^n,$$

and the usual face operators. In particular $\partial_0(g, h) = g$ and $\partial_1(g, h) = t(h)g$. The universal Γ-bundle is then given by

$$W\Gamma_n = \prod_{j=0}^{n} G \times H^{n-j}.$$

It is our result that this is the nerve of a 3-group, considered as a monoidal 2-groupoid. We can apply the Moore complex functor M to $W\Gamma$, giving in the terminology of Conduché a precrossed nonabelian complex [3]. This is a normal complex[2] of groups $\ldots \to G_2 \to G_1 \to G_0$, not necessarily abelian, on which G_0 acts by conjugation. A crossed module is such an object of length one: $G_1 \to G_0$.

The n^{th} group in the Moore complex for a simplcial group K is given by

$$MK_n = \bigcap_{i=0}^{n-1} \ker \partial_i$$

and the differential is the restriction of ∂_n. Usually the Moore functor is denoted N, so we are avoiding any possible clash with the nerve functor. If the simplicial group is abelian, that is, an object of sAb, then the Moore complex is the same as the normalised chain complex defined in the usual manner.

For the case at hand, let $\gamma_n = (\eta_1, \ldots, \eta_n) \in \Gamma_n = N(G_2)_n$, denote the product on the groups Γ_n by $\cdot$ and composition in the groupoid by concatenation. A quick calculation shows that

[1] Simplicial groups are always Kan complexes, and WK is also Kan [8].

[2] A normal complex is a complex such that $\operatorname{im} d$ is normal in $\ker d$ so the homology groups are defined.

$$
\begin{aligned}
\ker\partial_0 &= \{(\gamma_n,\gamma_{n-1},1,\ldots,1)\in W\Gamma_n | \gamma_{n-1}\cdot\partial_0\gamma_n = id_{W\Gamma_{n-1}}\},\\
\ker\partial_0\cap\ker\partial_i &= \{(\gamma_n,\gamma_{n-1},1,\ldots,1)\in W\Gamma_n | \partial_i\gamma_n = id_{W\Gamma_{n-1}} = \partial_i\gamma_{n-1}\},\quad i>0\\
&= \{(\eta_1,\ldots,\eta_n),(\eta'_1,\ldots,\eta'_{n-1}) | (\eta_1,\ldots,\eta_i\eta_{i+1},\ldots,\eta_n) = id_{\Gamma_{n-1}}\\
&\qquad (\eta'_1,\ldots,\eta'_i\eta'_{i-1},\ldots,\eta'_{n+1}) = id_{\Gamma_{n-2}}\}\\
&= \{(1,\ldots,\eta_i,\eta_{i+1},\ldots,1),(1,\ldots,\eta'_i,\eta'_{i+1},\ldots,1) | \eta_i\eta_{i+1} = id_{\Gamma_1} = \eta'_i\eta'_{i+1}\}
\end{aligned}
$$

and hence $\ker\partial_0\cap\ker\partial_i\cap\ker\partial_{i+2} = 1$, whenever this makes sense. So for $n\geq 3$, $MW\Gamma_n = 0$.

The nerve of a category is characterised by the fact the higher simplices are determined by the 1-arrows and composition. One can construct a category from the truncated simplicial object

insert diagram here

There is a functor $tr_n : s\mathbf{Set} \to s\mathbf{Set}_{\leq n}$ which truncates a simplicial object. This has a right adjoint $cosk_n : s\mathbf{Set}_{\leq n} \to s\mathbf{Set}$, described in [5] for the more general case of simplicial objects in a finitely (co?)complete category.

Conduché proves the

Corollary A.1 ([3])**.** *Let K be a simplicial group. If $MK_n = 0$ for $n > p$, the canonical arrow $K \to Cosk_p(K)$ is an isomorphism.*

Since one suggested definition of a weak n-groupoid is an n-coskeletal Kan complex [1][NB: I'm not sure if this is the earliest reference], this is evidence that $W\Gamma$ is the nerve of a bigroupoid. What is this bigroupoid? We shall see this via a construction of Kamps-Porter, associating to a 2-crossed module a Gray-monoid. We first find the 2-crossed module structure on $MW\Gamma$.

Recall that low-dimensional face maps $\partial_i : W\Gamma_k \to W\Gamma_{k-1}$ are

blah

We can thus calculate the kernels of $\partial_0,\partial_1 : MW\Gamma_2 \to MW\Gamma_1$:

$$
\begin{aligned}
\ker\partial_0 &= \{(g^{-1},h,k;g,{}^gh^{-1};1)\in G\ltimes H^2\times G\ltimes H\times G\}\\
\ker\partial_1 &= \{(1,a,a^{-1};b,c;b^{-1})\in G\ltimes H^2\times G\ltimes H\times G\}
\end{aligned}
$$

and so

$$MW\Gamma_2 = \ker\partial_0 \cap \ker\partial_1 = \{(1, h, h^{-1}; 1, h^{-1}; 1) \in G \ltimes H^2 \times G \ltimes H \times G\} \simeq H.$$

$MW\Gamma_1$ is the kernel of $\partial_0 : MW\Gamma_1 \to MW\Gamma_0$, which is

$$\ker\partial_0 = \{(g, h; g^{-1}) \in G \ltimes H \times G\} \simeq G \ltimes H$$

The boundary map $\partial_2 : MW\Gamma_2 \to MW\Gamma_1$ is then the restriction of the face map of the same symbol:

$$\partial_2(h) = (t(h), h^{-1})$$

Clearly $MW\Gamma_0 = W\Gamma_0 = G$, and $\partial_1 : MW\Gamma_1 \to MW\Gamma_0$ is

$$\partial_1(g, h) = gt(h),$$

so $MW\Gamma$ is the complex

$$H \xrightarrow{\partial_2} G \ltimes H \xrightarrow{\partial_1} G.$$

Conduché has shown that a Moore complex of length 3 has the structure of a 2-crossed module [3].

Definition A.2. A **2-crossed module** is a complex of length 2

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

of N-groups (N acting on itself by conjugation) and an N-equivariant function

$$\{\cdot,\cdot\} : M \times M \to L$$

called a Peiffer lifting. Define an action

$$\begin{array}{rcl} M \times L & \to & L \\ (m, l) & \mapsto & {}^{m}l := l\{\partial_2 l^{-1}, m\}, \end{array} \tag{1}$$

such that the following axioms are satisfied:

1. $\partial_2\{m, m'\} = (mm'm^{-1})({}^{\partial_1 m}m')^{-1}$,
2. $\{\partial_2 l, \partial_2 l'\} = [l, l'] := ll'l^{-1}l'^{-1}$

3. (a) $\{m, m'm''\} = \{m, m'\}^{mm'm^{-1}}\{m, m''\}$
   (b) $\{mm', m''\} = \{m, m'm''m'^{-1}\}^{\partial_1 m}\{m', m''\}$,

4. $\{m, \partial_2 l\} = ({}^{m}l)({}^{\partial_1 m}l)^{-1}$

5. ${}^{n}\{m, m'\} = \{{}^{n}m, {}^{n}m'\}$,

where $l, l' \in L,\ m, m', m'' \in M$ and $n \in N$.

It follows from these conditions that $\partial_2 : L \to M$ is a crossed module with the action (1).

References

[1] J. Baez, J. Dolan, Categorification

[2] J. Baez, D. Stevenson, A classifying space for 2-bundles, in preparation.

[3] D. Conduché Module croisés généralisés de longueur 2, JPAA 34 (1984) 155-178

[4] B. Day, R.Street, Monoidal Bicategories and Hopf Algebroids, Adv. Math 129 (1997) 99-157

[5] J. Duskin, Memoirs AMS

[6] K.H. Kamps, T. Porter, 2-Groupoid enrichments in homotopy theory, K-theory

[7] J.-L. Loday, Spaces with finitely many homotopy groups, JPAA

[8] J.P. May, Simplicial objects in algebraic topology

[9] D.M.Roberts, U. Schreiber

[10] U. Schreiber, $\Sigma(\mathrm{Aut}(G_2))$-2-transport. Private notes (2006)

[11] G. Segal, Classifying spaces and spectral sequences, Pub. Math. IHES tome 34 (1968)