Universal simplicial bundles and inner automorphism $n$-groups

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July 31, 2007

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Introduction

In [10] the authors construct the 3-group of inner automorphisms of a 2-group, namely, those automorphisms which are given by conjugating with the lowest dimensional elements:

$$\text{INN}(G_2) \subset \text{AUT}(G_2) = \text{Gray} (\Sigma G_2, \Sigma G_2),$$

where $\Sigma$ associates to a monoidal groupoid the obvious one-object bigroupoid. This 3-group is equivalent to the trivial 3-group and so can be thought of as being contractible.

In the special case of the 2-group being discrete, i.e. a group $G$, the construction gives rise to the underlying groupoid of $EG$. This groupoid has a canonical functor to $\Sigma G$, the underlying groupoid of $BG$. If we consider $G$ as a discrete simplicial group, then $WG = N(\text{INN}(G))$ and $\overline{WG} = N \Sigma G$, with $W$ and $\overline{W}$ the universal bundle and classifying space functors for simplicial
groups. The same construction should also work for 2-groups, considered as 3-coskeletal simplicial groups.

We point out that similar work is being undertaken in [2]

1 Simplicial bundles

A simplicial $K$-bundle on $X$, for $K$ a simplicial group, is a simplicial set $P$ with a $K$ action (as an object over $X$), a simplicial map

$$P \to X$$

such that

$$K \times P \to P \times_X P$$

is an isomorphism.

A universal simplicial bundle is one with a contractible total space. There is a canonical model, see for example [8], for the universal $K$-bundle. However, in the special cases where the simplicial group is a constant simplicial group

$$\cdots \xrightarrow{G} G \xrightarrow{G}$$

or the nerve of a strict 2-group

$$\cdots \xrightarrow{G \times H} G \times H \xrightarrow{G}$$

the universal bundle also has the structure of a simplicial group. These simplicial groups have only one and two nontrivial homotopy groups respectively, and with the second example, the group structure on the universal bundle is already fairly complicated. From the canonical definition it is not at all obvious how to continue this structure, so we introduce an isomorphic simplicial object whose group structure is manifest. We then relate this model for the universal bundle to the results in [11] and [9] where the two examples above are considered.

1.1 The traditional classifying complex and universal bundle

**Definition 1.1.** Let $K$ be a simplicial group. The **classifying complex** $WK$ of $K$ has $n$-simplices given by

$$WK_n = K_{n-1} \times \ldots \times K_0, \quad WK_0 = \{\ast\}$$
and face and degeneracy operators

\[
\begin{align*}
\bar{\partial}_0(g_n, \ldots, g_0) &= (g_{n-1}, \ldots, g_0), \\
\bar{\partial}_{i+1}(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_i g_{n-i+1}, \partial_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\
\bar{s}_0(g_{n-1}, \ldots, g_0) &= (\text{id}_{K_n}, g_{n-1}, \ldots, g_0), \\
\bar{s}_{i+1}(g_{n-1}, \ldots, g_0) &= (s_i g_{n-1}, \ldots, s_0 g_{n-i}, \text{id}_{K_{n-i}}, g_{n-i-1}, \ldots, g_0)
\end{align*}
\]

for \( n > 0 \) and

\[
\bar{\partial}_i(g_0) = *, \quad i = 1, 2; \quad \bar{s}_0(*) = (\text{id}_{K_0}).
\]

for \( n = 0 \).

For a group \( G \), this is the nerve of the standard one-object groupoid it defines.

\[
\overline{W}G = N(\Sigma G)
\]

**Definition 1.2.** Let \( K \) be a simplicial group. The universal \( K \)-bundle \( WK \) has as its set of \( n \)-simplices

\[
WK_n = K_n \times \overline{WK}_n = K_n \times \ldots \times K_0
\]

and face and degeneracy operators

\[
\begin{align*}
\partial_0(g_n, \ldots, g_0) &= (\partial_0 g_n g_{n-1}, g_{n-2}, \ldots, g_0), \\
\partial_i(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_i g_{n-i+1}, \partial_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\
s_i(g_n, \ldots, g_0) &= (s_i g_n, \ldots, s_0 g_{n-i}, \text{id}_{K_{n-i}}, g_{n-i-1}, \ldots, g_0)
\end{align*}
\]

The simplicial group \( K \) includes into \( WK \) as the first factor, and there is a Kan fibration

\[
WK \to \overline{WK}
\]

with fibre the underlying simplicial set of \( K \).

**Proposition 1.3.** \( WK \) is contractible.

As already pointed out, for an ordinary group \( G \), the object \( WG \) gives us the nerve of the 2-group \( G \ltimes G \Rightarrow G = \text{INN}(G) \), and as such is a simplicial group. However, the group structure on \( WG_n \) for \( n > 1 \) becomes increasingly complicated, being the semidirect product

\[
(\ldots (G \ltimes G) \ltimes G \ldots) \ltimes G.
\]
However, there is a groupoid isomorphic to $\text{INN}(G)$, namely $\text{codisc}(G)$, whose arrows have a less awkward group structure. For $(g_i, h_i) \in \text{Mor}(\text{codisc}(G))$,

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2)$$

and the trend continues up the nerve, with multiplication given component-wise. The isomorphism

$$\Phi : \text{Inn}(G) \to \text{codisc}(G)$$

$$(g, h) \mapsto (g, hg)$$

is more than an isomorphism of groupoids: it is an isomorphism of strict 2-groups. Its inverse is

$$\Phi^{-1} : \text{codisc}(G) \to \text{Inn}(G)$$

$$(g, h) \mapsto (g, hg^{-1})$$

Both $\text{Inn}(G)$ and $\text{codisc}(G)$ are equipped with functors to $\Sigma G$:

$$\pi^{\text{Inn}} : \text{Inn}(G) \to \Sigma G, \quad \pi^{\text{Inn}}(g, h) = h$$

$$\pi^{\text{codisc}} : \text{codisc}(G) \to \Sigma G, \quad \pi^{\text{codisc}}(g, h) = hg^{-1},$$

giving us a commuting diagram

\[
\begin{array}{ccc}
\text{Inn}(G) & \xrightarrow{\sim} & \text{codisc}(G) \\
\pi^{\text{Inn}} \downarrow & & \pi^{\text{codisc}} \downarrow \\
\Sigma G & & \Sigma G
\end{array}
\]

The difference between the two approaches is precisely the difference between Milnor’s construction of the universal bundle, and the universal bundle as described in [11] (section 3), for example. Using the realisation of the nerve of $\text{codisc}(G)$ as a model for $EG$, one can instantly see that $EG$ is a topological group, and moreover Segal [11] has proved that $G$ is a closed subgroup of $EG$, and is normal iff $G$ is abelian.

In this context, we have a pair of exact sequences

\[
\begin{array}{ccc}
disc(G) & \xrightarrow{i} & \text{Inn}(G) & \xrightarrow{\pi^{\text{Inn}}} & \Sigma G \\
\downarrow = & & \Phi \downarrow & & \downarrow = \\
disc(G) & \xrightarrow{j} & \text{codisc}(G) & \xrightarrow{\pi^{\text{codisc}}} & \Sigma G
\end{array}
\]
with
\[ i(g) = (g, 1), \quad j(g) = (g, g), \]
and \( \Sigma G \) is only a 2-group if \( G \) is abelian.

### 1.2 The “codiscrete” version of the universal bundle

We will now construct the analogue of \( \text{codisc}(G) \) for a general simplicial group \( K \), which we will call \( \mathcal{E}K \). Define the simplices as
\[ \mathcal{E}K_n = K_n \times \ldots \times K_0. \]
The face and degeneracy maps are
\[
\partial_0(g_n, \ldots, g_0) = (g_{n-1}, g_{n-2}, \ldots, g_0), \\
\partial_i(g_n, \ldots, g_0) = (\partial_i g_n, \ldots, \partial_i g_{n-i+1}, g_{n-i-1}, \ldots, g_0), \\
s_i(g_n, \ldots, g_0) = (s_i g_n, \ldots, s_0 g_{n-i}, g_{n-i}, g_{n-i-1}, \ldots, g_0).
\]
If we let the product on \( \mathcal{E}K_n \) be componentwise, these face and degeneracy maps are homomorphisms, because those of \( K \) are. \( \mathcal{E}K \) is then a simplicial group and its realisation is a topological group. There is again an isomorphism between \( WK_n \) and \( \mathcal{E}K_n \)
\[
\Phi_n : WK_n \longrightarrow \mathcal{E}K_n \\
(g_n, \ldots, g_0) \mapsto (k_n, k_{n-1}, \ldots, k_0)
\]
where the \( k_j \) are defined recursively as
\[ k_n = g_n, \quad k_j = \partial_0 k_{j+1} g_j \quad (j \leq n - 1). \]
One can see that the maps \( \Phi_n \) define a map \( \Phi \) of simplicial sets by the use of the standard identities for the boundary and degeneracy maps for \( K \). The inverse map is
\[
\Phi^{-1} : \mathcal{E}K \longrightarrow WK \\
(h_n, \ldots, h_0) \mapsto (h_n, \partial_0 h_n^{-1} h_{n-1}, \ldots, \partial_0 h_1^{-1} h_0)
\]
We use the isomorphism \( \Phi \) to see how \( K \) includes into \( \mathcal{E}K \):
\[
K \hookrightarrow WK \xrightarrow{\Phi} \mathcal{E}K \\
k \mapsto (k, 1, \ldots, 1) \mapsto (k, \partial_0 k, \partial_0 \partial_0 k, \ldots, \partial_0^n k)
\]
The action of $K$ on $\mathcal{E}K$ can now clearly be seen. For example, the left action is

$$(g_n, \ldots, g_0, k) \mapsto (kg_n, \partial_0 k g_{n-1}, \ldots, \partial_0^n k g_0).$$

There is a map

$$\pi^\mathcal{E} : \mathcal{E}K \to WK$$

$$(h_n, \ldots, h_0) \mapsto (\partial_0 h_n^{-1} h_{n-1}, \partial_0 h_n^{-1} h_{n-2}, \ldots, \partial_0 h_1^{-1} h_0),$$

which is the quotient by the left action above, and the triangle

$$\begin{array}{ccc}
WK & \xrightarrow{\sim} & \mathcal{E}K \\
\downarrow \Phi & & \downarrow \pi^\mathcal{E} \\
WK & & \\
\end{array}$$

commutes. By proposition 1.3, $\mathcal{E}K$ is also contractible.

The construction of $\mathcal{E}K$ is clearly functorial, so we have an endofunctor

$$\mathcal{E} : sGrp \to sGrp,$$

and a natural transformation

$$\iota : \text{Id}_{sGrp} \to \mathcal{E}$$

whose component at $K$ is given by the inclusion $K \hookrightarrow \mathcal{E}K$. Notice that

$$(\mathcal{E}\mathcal{E}K)_n = \mathcal{E}K_n \times \mathcal{E}K_{n-1} \times \ldots \times \mathcal{E}K_0$$

$$= (K_n \times \ldots \times K_0) \times (K_{n-1} \times \ldots \times K_0) \times \ldots \times (K_0).$$

If $\text{pr}_1 : \mathcal{E}K_j \to K_j$ denotes projection on the first factor, define the maps

$$(\mu_K)_n = \text{pr}_1 \times \ldots \times \text{pr}_1 : \mathcal{E}\mathcal{E}K_n \to K_n \times \ldots \times K_0 = \mathcal{E}K_n,$$

which clearly assemble into a map of simplicial groups

$$\mu_K : \mathcal{E}\mathcal{E}K \to \mathcal{E}K,$$

and these form the components of a natural transformation

$$\mu : \mathcal{E}\mathcal{E} \to \mathcal{E}.$$

**Proposition 1.4.** The data above, $(\mathcal{E}, \iota, \mu)$, forms a monad on $sGrp$.

Proof. The transformation $\iota$ satisfies the unit conditions almost by definition, and proving the associativity axiom is an exercise in indices. $\square$
1.3 The Moore complex

There is a functor $M$ from simplicial groups to normal complexes called the Moore complex, which measures homotopical information. In particular, the homology groups of $MK$ are the homotopy groups of $K$. In fact, more is true:

**Proposition 1.5 ([3]).** Let $K$ be a simplicial group. If $MK_n = 0$ for $n > p$, the canonical arrow $K \to \text{Cosk}_p(K)$ is an isomorphism.

We recall that if $K \sim \text{Cosk}_p(K)$, $K$ is called $p$-coskeletal, and is determined completely by its $p$-truncation. [As an example, the nerve of a category is 2-coskeletal, as it is determined by the sets of objects, arrows and composable pairs.]

The $n^{th}$ group in the Moore complex for a simplicial group $K$ is given by

$$MK_n = \bigcap_{i=1}^{n} \ker \partial_i$$

and the differential is the restriction of $\partial_0$. If the simplicial group is abelian, that is, an object of $sAb$, then the Moore complex is the same as the normalised chain complex defined in the usual manner. It is possible to reconstruct the simplicial group from its Moore complex, and there is an equivalence of categories [3]

$$\{ p - \text{coskeletal simplicial groups} \} \sim \{ \text{length } p \text{ Moore complexes} \}.$$ 

We would like to see what effect the functor $E$ has on the Moore complex of a simplicial group $K$, even though we know $EK$ is contractible.

To begin with, $\ker \partial_1 = \{(g_n, g_{n-1}, 1, \ldots, 1) \in (EK)_n | g_n \in \ker \partial_1 \subset K_n \}$. It follows quickly that $(MEK)_n = MK_n \times MK_{n-1}$.

We also calculate the Moore complex of $WK$:

$$M\overline{WK}_n = MK_n \times MK_{n-1}^0 := \{(g, h) \in MK_n \times MK_{n-1} | h \in \ker \partial_0 \}$$

**Proposition 1.6.** If $K$ is an $n$-coskeletal simplicial group, $EK$ is an $n+1$-coskeletal simplicial group.

Proof. This follows from the conjecture and proposition 1.5. \qed
Consider now the case that $K = NG_2$ for $G_2$ a strict 2-group, which we consider as a groupoid internal to $\text{Grp}$. By the above corollary, $WNG_2$ is coskeletal, but more interestingly, has a Moore complex of length 2, which has the structure of a 2-crossed module [3]. 2-crossed modules are known to be equivalent to Gray-monoids, which are 2-categories with a slightly weak monoidal structure. In [9] the authors constructed a contractible Gray-monoid associated to $G_2$, which contained $G_2$ as a sub-Gray-monoid.

**Question 1.7.** Is the nerve of a Gray-monoid a simplicial group?
A Strict 2-groups as simplicial groups

[Le texte de la section est vieux, c'est juste un compte rendu de la calcul suivant.]

Considérons un 2-groupe strict $G_2$ comme un groupoïde interne aux groupes. Alors son nerf $NG_2$ est le complexe simplicial avec des éléments du groupe des $p$-simplices $p$-tuples de flèches composables. Cela est automatiquement un groupe simplicial, et donc nous pouvons appliquer $W$.\(^1\) Prenez $G_2$ comme le 2-groupe strict provenant du croisement $t: H \to G$, avec l'action de $G$ sur $H$ notée $^gh$. Soit $\Gamma = NG_2$, qui a les $n$-simplices $\Gamma_n = G \times H^n$, et les opérateurs de face usuels. En particulier $\partial_0(g, h) = g$ et $\partial_1(g, h) = t(h)g$.

Le faisceau universel $\Gamma$ est alors donné par

$$WT_n = \prod_{j=0}^n G \times H^{n-j}.$$  

Il est notre résultat que c'est le nerf d'un 3-groupe, considéré comme un groupoïde monoidal 2-groupe. Nous pouvons appliquer le foncteur de complexe de Moore $M$ à $WT$ en utilisant la terminologie de Conduché un complexe non abélien de croisements [3]. C'est un complexe normal\(^2\) des groupes $\cdots \to G_2 \to G_1 \to G_0$, non nécessairement abélien, sur lequel $G_0$ agit par conjugaison. Un croisement est un objet de longueur un: $G_1 \to G_0$.

Le $n^{\text{e}}$ groupe dans le complexe de Moore pour un complexe simplicial $K$ est donné par

$$MK_n = \bigcap_{i=0}^{n-1} \ker \partial_i$$

et la différentielle est la restriction de $\partial_n$. Généralement le foncteur de Moore est noté $N$, donc nous évitons un possible clash avec le nerve. Si le groupe simplicial est abélien, c'est-à-dire un objet de $sAb$, alors le complexe de Moore est le même que le complexe normalisé de chaîne défini de manière standard.

Pour le cas présent, soit $\gamma_n = (\eta_1, \ldots, \eta_n) \in \Gamma_n = N(G_2)_n$, dénote le produit sur les groupes $\Gamma_n$ par $\cdot$ et la composition dans le groupoïde par concatenation. Un calcul rapide montre que

\(^{1}\)Les groupes simpliciaux sont toujours complexes de Kan, et $WK$ est aussi de Kan [8].

\(^{2}\)Un complexe normal est un complexe tel que $\text{im} \partial$ est normal dans $\ker \partial$ donc les groupes d'homologie sont définis.
\[
\ker \partial_0 = \{ (\gamma_n, \gamma_{n-1}, 1, \ldots, 1) \in W \Gamma_n \mid \partial_0 \gamma_n = id_{W \Gamma_{n-1}} \}, \\
\ker \partial_0 \cap \ker \partial_i = \{ (\gamma_n, \gamma_{n-1}, 1, \ldots, 1) \in W \Gamma_n \mid \partial_i \gamma_n = id_{W \Gamma_{n-1}} = \partial_i \gamma_{n-1} \}, \quad i > 0
\]
and hence \( \ker \partial_0 \cap \ker \partial_i \cap \ker \partial_{i+2} = 1 \), whenever this makes sense. So for \( n \geq 3 \), \( MW \Gamma_n = 0 \).

The nerve of a category is characterised by the fact the higher simplices are determined by the 1-arrows and composition. One can construct a category from the truncated simplicial object

\[\text{insert diagram here}\]

There is a functor \( tr_n : s\text{Set} \to s\text{Set}_{\leq n} \) which truncates a simplicial object. This has a right adjoint \( \text{cosk}_n : s\text{Set}_{\leq n} \to s\text{Set} \), described in [5] for the more general case of simplicial objects in a finitely (co?)complete category.

Conduché proves the

**Corollary A.1** ([3]). Let \( K \) be a simplicial group. If \( MK_n = 0 \) for \( n > p \), the canonical arrow \( K \to \text{Cosk}_p(K) \) is an isomorphism.

Since one suggested definition of a weak \( n \)-groupoid is an \( n \)-coskeletal Kan complex [1][NB: I'm not sure if this is the earliest reference], this is evidence that \( WT \) is the nerve of a bigroupoid. What is this bigroupoid? We shall see this via a construction of Kamps-Porter, associating to a 2-crossed module a Gray-monoid. We first find the 2-crossed module structure on \( MW \Gamma \).

Recall that low-dimensional face maps \( \partial_i : W \Gamma_k \to W \Gamma_{k-1} \) are

\[\text{blah}\]

We can thus calculate the kernels of \( \partial_0, \partial_1 : MW \Gamma_2 \to MW \Gamma_1 \):

\[
\ker \partial_0 = \{ (g^{-1}, h, k; g, g h^{-1}; 1) \in G \times H^2 \times G \times H \times G \} \\
\ker \partial_1 = \{ (1, a, a^{-1}; b, c, b^{-1}) \in G \times H^2 \times G \times H \times G \}
\]
and so

\[ MW_{\Gamma_2} = \ker \partial_0 \cap \ker \partial_1 = \{(1, h, h^{-1}; 1, h^{-1}; 1) \in G \ltimes H^2 \times G \ltimes H \times G\} \simeq H. \]

\[ MW_{\Gamma_1} \text{ is the kernel of } \partial_0 : MW_{\Gamma_1} \to MW_{\Gamma_0}, \text{ which is} \]

\[ \ker \partial_0 = \{(g, h; g^{-1}) \in G \ltimes H \times G\} \simeq G \ltimes H \]

The boundary map \( \partial_2 : MW_{\Gamma_2} \to MW_{\Gamma_1} \) is then the restriction of the face map of the same symbol:

\[ \partial_2(h) = (t(h), h^{-1}) \]

Clearly \( MW_{\Gamma_0} = WT_0 = G \), and \( \partial_1 : MW_{\Gamma_1} \to MW_{\Gamma_0} \) is

\[ \partial_1(g, h) = gt(h), \]

so \( MW \) is the complex

\[ H \xrightarrow{\partial_2} G \ltimes H \xrightarrow{\partial_1} G. \]

Conduché has shown that a Moore complex of length 3 has the structure of a 2-crossed module \([3]\).

**Definition A.2.** A 2-crossed module is a complex of length 2

\[ L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N \]

of \( N \)-groups (\( N \) acting on itself by conjugation) and an \( N \)-equivariant function

\[ \{\cdot, \cdot\} : M \times M \to L \]

called a Peiffer lifting. Define an action

\[ M \times L \to L \]

\[ (m, l) \mapsto m_l := l\{\partial_2 l^{-1}, m\} , \tag{1} \]

such that the following axioms are satisfied:

1. \( \partial_2 \{m, m'\} = (mm'm^{-1})^{(\partial_1 m \partial_1 m')}^{-1} \)

2. \( \{\partial_2 l, \partial_2 l'\} = [l, l'] := l' l^{-1} l'^{-1} \)
3. (a) \( \{m, m'm''\} = \{m, m'\} mm'm^{-1}\{m, m''\} \)
(b) \( \{mm', m''\} = \{m, m'm''m'^{-1}\} \partial_1 m\{m', m''\} \)

4. \( \{m, \partial_2 l\} = (m\cdot)(\partial_1 m\cdot)^{-1} \)

5. \( \cdot m, m'\} = \{m', m'\} \)

where \( l, l' \in L, m, m', m'' \in M \) and \( n \in N \).

It follows from these conditions that \( \partial_2 : L \to M \) is a crossed module
with the action (1).

References

[8] J.P. May, Simplicial objects in algebraic topology