8 Homotopy theory of Fredholm operators

This chapter is about homotopy groups of the sets of Fredholm operators, unitary and self-adjoint Fredholm operators, Fredholm pairs, and other operator classes. Both bounded and unbounded Fredholm operators with various topologies are dealt with, and as an application a characterization of the spectral flow is proved in Section 8.4. Part of the presentation below follows closely the textbook by Booß-Bavnbek and Wojciechowski [32], as well as the excellent lecture notes by Schröder [165] which are unfortunately only available in German. When dealing with unbounded self-adjoint operators equipped with the gap metric, another crucial element of proof is taken from a paper of Joachim [108] that is apparently not particularly well known. Along the way, several homotopy equivalences are proved and this is summarized in Section 8.7. Several fundamental results are needed (in particular, the long exact sequence of homotopy groups of fiber bundles and the stable homotopy groups of the general linear groups as computed by Bott) and are recalled in Appendix A.3 for the convenience of the reader.

8.1 Homotopy groups of essentially gapped unitaries

For the stable general linear group $\mathrm{GL}(\infty,\mathbb{C})$, the homotopy groups are known by Bott's celebrated result, see (A.3) in Appendix A.3. It was then proved by Palais [141] and Shvarts [179] that one can enlarge $\mathrm{GL}(\infty,\mathbb{C})$ to the invertible operators $\mathbb{G}^c(\mathcal{H})$ in the unitization of the compact operators without changing the homotopy groups. More precisely, consider

$$\mathbb{K}(\mathcal{H})^\sim = \{T \in \mathbb{B}(\mathcal{H}) : T - \mathbf{1} \in \mathbb{K}(\mathcal{H})\} = \mathbf{1} + \mathbb{K}(\mathcal{H}),$$

and the subset of invertibles

$$\mathbb{G}^c(\mathcal{H}) = \mathbb{G}(\mathcal{H}) \cap \mathbb{K}(\mathcal{H}),$$

equipped with the norm topology. Then

$$\pi_k\big(\mathbb{G}^c(\mathcal{H})\big) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \tag{8.1}$$

A proof of the latter fact can also be found in [165]. Based on this, one can directly state the homotopy groups of the essentially gapped unitary operators. Indeed, by Proposition 3.7.2, the set $\mathbb{F}\mathbb{U}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : -1 \notin \mathrm{spec}_{\mathrm{ess}}(U)\}$ of essentially gapped unitaries can be retracted to the set $\mathbb{U}^c(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U - \mathbf{1} \in \mathbb{K}(\mathcal{H})\}$. Furthermore, the polar decomposition provides the following:

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Proposition 8.1.1. *With respect to the norm topology,* $\mathbb{U}^c(\mathcal{H})$ *is a deformation retract of* $\mathbb{G}^c(\mathcal{H})$.

Proof. For $T \in \mathbb{G}^c(\mathcal{H})$, clearly, also $|T|^2 = T^*T \in \mathbb{G}^c(\mathcal{H})$. Moreover, for $s \in \mathbb{R}$,

$$|T|^s = \oint_\Gamma \frac{dz}{2\pi\imath} z^{\frac{s}{2}} (z\mathbf{1} - |T|^2)^{-1},$$

for some contour surrounding the (positive) spectrum of $|T|^2$ once in the positive sense. Due to the resolvent identity, one has $|T|^s \in \mathbb{G}^c(\mathcal{H})$. Therefore $U = T|T|^{-1} \in \mathbb{U}^c(\mathcal{H})$ and the path $s \in [0,1] \mapsto U|T|^s$ lies in $\mathbb{G}^c(\mathcal{H})$. Thus the homotopy

$$h : \mathbb{G}^c(\mathcal{H}) \times [0,1] \to \mathbb{G}^c(\mathcal{H}), \quad h(T,t) = T|T|^{-t}$$

is well defined and clearly norm-continuous. Moreover, $h(T,0) = T$ for $T \in \mathbb{G}^c(\mathcal{H})$, $h(T,1) \in \mathbb{U}^c(\mathcal{H})$ for $T \in \mathbb{G}^c(\mathcal{H})$ by the above, and $h(U,t) = U$ for $U \in \mathbb{U}^c(\mathcal{H})$ and $t \in [0,1]$, and therefore h is a deformation retraction of $\mathbb{G}^c(\mathcal{H})$ onto $\mathbb{U}^c(\mathcal{H})$. □

Now the homotopy groups of $\mathbb{G}^c(\mathcal{H})$ are given by (8.1) by the results of Bott and Palais. Therefore we obtain

Corollary 8.1.2. *With respect to the norm topology, the homotopy groups of the essentially gapped unitary operators are*

$$\pi_k(\mathbb{F}\mathbb{U}(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Corollary 8.1.3. *The spectral flow on closed loops establishes an isomorphism*

$$\mathrm{Sf} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{H})) \to \mathbb{Z}.$$

Proof. Clearly, $\mathrm{Sf} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{H})) = \mathbb{Z} \to \mathbb{Z}$ is a homomorphism. Example 4.5.4 shows that this homomorphism is surjective. It is then a fact that every surjective homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is injective. □

8.2 Homotopy groups of Fredholm operators

It was proved in Theorem 3.3.5 and Corollary 3.3.6 that the connected components $\mathbb{F}_n\mathbb{B}(\mathcal{H})$ of the set of bounded Fredholm operators $\mathbb{F}\mathbb{B}(\mathcal{H})$ with respect to the norm topology are labeled by the index $n \in \mathbb{Z}$, so that

$$\pi_0(\mathbb{F}\mathbb{B}(\mathcal{H})) = \mathbb{Z}.$$

Moreover, one can restate Corollary 3.3.6 as

Corollary 8.2.1. *The index establishes a bijection* $\mathrm{Ind} : \pi_0(\mathbb{FB}(\mathcal{H})) \to \mathbb{Z}$.

As all connected components of $\mathbb{FB}(\mathcal{H})$ are homotopy equivalent by Theorem 3.3.5, the task left is to determine the homotopy groups of the identity component $\mathbb{F}_0\mathbb{B}(\mathcal{H})$. This is done by applying the tools described Appendix A.3.

Theorem 8.2.2. *The homotopy groups of the identity component of the bounded Fredholm operators on* $\mathcal{H}$ *are given by*

$$\pi_k(\mathbb{F}_0\mathbb{B}(\mathcal{H})) = \begin{cases} 0, & k \textit{ odd}, \\ \mathbb{Z}, & k > 0 \textit{ even}. \end{cases}$$

As all connected components of $\mathbb{FB}(\mathcal{H})$ are homotopy equivalent, this directly implies

Corollary 8.2.3. *For all* $n \in \mathbb{Z}$, *the homotopy groups of the component* $\mathbb{F}_n\mathbb{B}(\mathcal{H})$ *of the bounded Fredholm operators are*

$$\pi_k(\mathbb{F}_n\mathbb{B}(\mathcal{H})) = \begin{cases} 0, & k \textit{ odd}, \\ \mathbb{Z}, & k > 0 \textit{ even}. \end{cases}$$

Strictly speaking, one has

$$\pi_k(\mathbb{F}_n\mathbb{B}(\mathcal{H}), T) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}, \end{cases}$$

for all basepoints $T \in \mathbb{F}_n\mathbb{B}(\mathcal{H})$ where, for any topological space X and $b \in X$, the homotopy group $\pi_k(X, b)$ is made by the homotopy classes of continuous maps $f : \mathbb{S}^k \to X$ mapping some fixed point $a_k \in \mathbb{S}^k$ onto b. As the homotopy groups of a connected space X are independent of the basepoint, this is also written as

$$\pi_k(\mathbb{F}_n\mathbb{B}(\mathcal{H})) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}. \end{cases}$$

The proof of Theorem 8.2.2 will use the Bartle and Graves selection theorem as a crucial element for the construction of fiber bundles [34]. A version that is sufficient for the present purposes can be stated as follows: if $\mathcal{E}$ is a Banach space, $\mathcal{U} \subset \mathcal{E}$ a closed subspace, and $\pi : \mathcal{E} \to \mathcal{E}/\mathcal{U}$ is the quotient map, then there exists a continuous (homogeneous but not necessarily linear) right inverse $\rho : \mathcal{E}/\mathcal{U} \to \mathcal{E}$ of π, namely $\pi \circ \rho = \mathrm{id}_{\mathcal{E}/\mathcal{U}}$. A short proof of this version is given in [165].

Proof of Theorem 8.2.2. The proof is split into several steps:

Fact 1. $\mathbb{F}_0\mathbb{B}(\mathcal{H})$ is homotopy equivalent to the identity component $\mathbb{G}_0\mathbb{Q}(\mathcal{H})$ of the invertible elements in the Calkin algebra.

Indeed, let ρ be the right inverse of the Calkin projection $\pi : \mathbb{B}(\mathcal{H}) \to \mathbb{Q}(\mathcal{H})$ as given by the Bartle–Graves selection theorem. Because Fredholm operators with vanishing index are compact perturbations of an invertible operator $\mathbb{F}_0\mathbb{B}(\mathcal{H}) = \pi^{-1}(\mathbb{G}_0\mathbb{Q}(\mathcal{H}))$. Thus $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H})$ can be uniquely decomposed into $T = \rho(\pi(T)) + K$ with some compact operator $K \in \mathbb{K}(\mathcal{H})$. Hence $\mathbb{F}_0\mathbb{B}(\mathcal{H})$ is homeomorphic to $\mathbb{G}_0\mathbb{Q}(\mathcal{H}) \times \mathbb{K}(\mathcal{H})$. The contractibility of the compact operators then implies the first fact.

Fact 2. The restriction of the Calkin projection $\hat{\pi} = \pi|_{\mathbb{G}(\mathcal{H})} : \mathbb{G}(\mathcal{H}) \to \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ is a fiber bundle with fiber $\mathbb{G}^c(\mathcal{H})$.

First note that π indeed maps the bounded invertibles into the identity component of the invertibles of the Calkin algebra, due to the connectedness of $\mathbb{G}(\mathcal{H})$. Moreover, $\hat{\pi} : \mathbb{G}(\mathcal{H}) \to \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ is surjective because for each $\widehat{T} \in \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ there is an operator $S = \rho(\widehat{T}) \in \mathbb{F}\mathbb{B}(\mathcal{H})$ with $\mathrm{Ind}(S) = 0$ so that there exists a compact operator $K \in \mathbb{K}(\mathcal{H})$ such that $T = S + K$ is invertible, and clearly $\hat{\pi}(T) = \widehat{T}$. Now fix an operator $\widehat{T}_0$ with associated invertible lift T_0, set $K_0 = T_0 - \rho(\widehat{T}_0)$ and next consider a neighborhood $\mathcal{U}$ of $\widehat{T}_0$. By choosing $\mathcal{U}$ sufficiently small, there is a continuous injective map $\ell : \mathcal{U} \to \mathbb{G}(\mathcal{H})$ defined by $\ell(\widehat{T}) = \rho(\widehat{T}) + K_0$. Note that the image of ℓ lies indeed in the set $\mathbb{G}(\mathcal{H})$ of invertibles because $\mathbb{G}(\mathcal{H})$ is open in $\mathbb{B}(\mathcal{H})$. Moreover,

$$\hat{\pi}^{-1}(\{\widehat{T}\}) = \{\rho(\widehat{T}) + K \text{ invertible} : K \in \mathbb{K}(\mathcal{H})\} = (\rho(\widehat{T}) + K_0)\mathbb{G}^c(\mathcal{H}),$$

the latter because

$$\rho(\widehat{T}) + K = (\rho(\widehat{T}) + K_0)(\mathbf{1} + (\rho(\widehat{T}) + K_0)^{-1}(K - K_0)).$$

Hence $\hat{\pi}^{-1}(\mathcal{U})$ is homeomorphic to $\mathcal{U} \times \mathbb{G}^c(\mathcal{H})$ as claimed. (Note that the fiber bundle is actually a principal bundle with fiber group $\mathbb{G}^c(\mathcal{H})$.)

Fact 3. The homotopy groups $\pi_k(\mathbb{F}_0\mathbb{B}(\mathcal{H}))$ are as stated.

This now uses the long exact sequence of homotopy theory associated to the fiber bundle of Fact 2. It reduces to isomorphisms $\pi_k(\mathbb{G}_0\mathbb{Q}(\mathcal{H})) \cong \pi_{k-1}(\mathbb{G}^c(\mathcal{H}))$ because $\mathbb{G}(\mathcal{H})$ is contractible by Kuiper's theorem and hence has vanishing homotopy groups. Using Fact 1, one deduces $\pi_k(\mathbb{F}_0\mathbb{B}(\mathcal{H})) \cong \pi_{k-1}(\mathbb{G}^c(\mathcal{H}))$ and therefore (8.1) concludes the proof. □

The next aim is to consider the set of unbounded Fredholm operators $\mathbb{F}(\mathcal{H})$ as defined in Definition 6.2.1. They form a subset of the densely defined closed operators $\mathbb{L}(\mathcal{H})$ on which Section 6.1 studied two natural topologies, namely the Riesz and gap topologies. The definition of the Riesz topology is tightly linked to the bounded transform $\mathfrak{F}(T) = T(\mathbf{1} + T^*T)^{-\frac{1}{2}}$ and this leads to Proposition 6.2.18 which states that the spaces $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}\mathbb{B}(\mathcal{H}), \mathcal{O}_N)$ are homotopy equivalent. This directly implies the following main result on the set of unbounded Fredholm operators.

Theorem 8.2.4. *The homotopy groups of* $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ *are the same as those of* $(\mathbb{FB}(\mathcal{H}), \mathcal{O}_N)$ *as given by Corollaries* 8.2.1 *and* 8.2.3.

Let us briefly comment on the space $(\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$. By the bounded transform, it is homeomorphic to $(\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_E)$, which in turn can be shown to be homeomorphic to $(\mathbb{FB}_1(\mathcal{H}), \mathcal{O}_{SE})$ by adapting the argument in the proof of Proposition 6.4.7 (note that d_E is, however, only a pseudometric on $\mathbb{FB}_1(\mathcal{H})$). In [154] it is shown that the identity provides a homotopy equivalence $I : (\mathbb{F}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$. Therefore the homotopy groups of $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$ coincide and are given by Theorem 8.2.4.

8.3 Homotopy groups of bounded self-adjoint Fredholm operators

Recall from Section 3.6 that the set $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ of bounded self-adjoint Fredholm operators equipped with the norm topology has three connected components $\mathbb{FB}_{\mathrm{sa}}^{+}(\mathcal{H})$, $\mathbb{FB}_{\mathrm{sa}}^{-}(\mathcal{H})$, and $\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})$, consisting respectively of those self-adjoint Fredholm operators having only positive essential spectrum, only negative essential spectrum, and having both positive and negative essential spectrum. The components $\mathbb{FB}_{\mathrm{sa}}^{+}(\mathcal{H})$ and $\mathbb{FB}_{\mathrm{sa}}^{-}(\mathcal{H})$ are contractible so that the main task here is to determine the homotopy groups of $\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})$.

Theorem 8.3.1. *With respect to the norm topology, the homotopy groups of* $\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})$ *are*

$$\pi_k(\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Corollary 8.3.2. *The spectral flow on closed loops establishes an isomorphism*

$$\mathrm{Sf} : \pi_1(\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})) \to \mathbb{Z}.$$

Proof. Clearly, $\mathrm{Sf} : \pi_1(\mathbb{FB}_{\mathrm{sa}}^{*}(\mathcal{H})) = \mathbb{Z} \to \mathbb{Z}$ is a homomorphism. By Example 8.3.4 further down, this homomorphism is surjective. As every surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is injective, this implies the claim. ☐

The proof of Theorem 8.3.1 parallels that of Theorem 8.2.2, but there is an extra element stated first.

Lemma 8.3.3. *Let* $Q_0 \in \mathbb{U}_{\mathrm{sa}}^{*}(\mathcal{H})$ *be a proper symmetry with neighborhood*

$$\mathcal{U} = \{Q \in \mathbb{U}_{\mathrm{sa}}^{*}(\mathcal{H}) : \|Q - Q_0\| < 2\}.$$

Then there is a continuous map $Q \in \mathcal{U} \mapsto U \in \mathbb{U}(\mathcal{H})$ *such that*

$$Q = UQ_0U^*.$$

Proof. (Note that this is essentially the same argument as in the proof of Proposition 5.3.20.) The unitary will be explicitly constructed, using the orthogonal projections $P = \frac{1}{2}(\mathbf{1} - Q)$ and $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$. Consider the operator

$$M = \mathbf{1} + (P - P_0)(2P_0 - \mathbf{1}).$$

By assumption, $\|(P - P_0)(2P_0 - \mathbf{1})\| < 1$ so that M is invertible. One readily checks

$$PM = PP_0 = MP_0.$$

Hence also $M^*P = P_0M^*$. Therefore $P = MP_0M^{-1}$ and $M^*P(M^*)^{-1} = P_0$ so that upon replacing also

$$P = (MM^*)P(MM^*)^{-1}.$$

This implies $P = (MM^*)^{-\frac{1}{2}}P(MM^*)^{\frac{1}{2}}$. Now set

$$U = (MM^*)^{-\frac{1}{2}}M.$$

This is indeed unitary and satisfies the claim. ☐

Proof of Theorem 8.3.1.

Fact 1. $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is homotopy equivalent to the set $\mathbb{GQ}^*_{\mathrm{sa}}(\mathcal{H})$ of self-adjoint invertible elements in the Calkin algebra having both positive and negative spectrum, which in turn can be retracted to the set $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ of proper symmetries in the Calkin algebra

$$\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H}) = \{\widehat{Q} = \widehat{Q}^* \in \mathbb{Q}(\mathcal{H}) : \mathrm{spec}(\widehat{Q}) = \{-1, 1\}\}.$$

For the proof of this fact, let $\pi_{\mathrm{sa}} : \mathbb{B}_{\mathrm{sa}}(\mathcal{H}) \to \mathbb{Q}_{\mathrm{sa}}(\mathcal{H})$ be the restriction of the Calkin projection π to the self-adjoint bounded operators. Then $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}) = \pi^{-1}_{\mathrm{sa}}(\mathbb{GQ}^*_{\mathrm{sa}}(\mathcal{H}))$. A continuous right inverse ρ_{sa} to π_{sa} is given in terms of the right inverse ρ of π by setting $\rho_{\mathrm{sa}}(\widehat{H}) = \frac{1}{2}(\rho(\widehat{H}) + \rho(\widehat{H})^*)$. Then $H \in \mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ can be uniquely decomposed into $H = \rho_{\mathrm{sa}}(\pi_{\mathrm{sa}}(H)) + K$ with some $K \in \mathbb{K}_{\mathrm{sa}}(\mathcal{H})$, the set of self-adjoint compact operators. Hence $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is homeomorphic to $\mathbb{GQ}^*_{\mathrm{sa}}(\mathcal{H}) \times \mathbb{K}_{\mathrm{sa}}(\mathcal{H})$. The contractibility of the self-adjoint compact operators then implies the first claim. The retraction to $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ can then be done by spectral calculus.

Fact 2. Let $\mathbb{UQ}(\mathcal{H})$ denote the unitary elements in the Calkin algebra and fix some proper symmetry $\widehat{Q}_0 \in \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$. Then the map $\pi_0 : \mathbb{UQ}(\mathcal{H}) \to \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ defined by

$$\pi_0(\widehat{U}) = \widehat{U}\widehat{Q}_0\widehat{U}^*$$

is the base projection of a principal bundle with connected base space and fiber group given by the stabilizer group of $\widehat{Q}_0$,

$$G_0 = \{\widehat{U} \in \mathbb{UQ}(\mathcal{H}) : \widehat{U}\widehat{Q}_0\widehat{U}^* = \widehat{Q}_0\}.$$

For the justification, let us first note that for every $\widehat{Q} \in \mathbb{UQ}^*_{\text{sa}}(\mathcal{H})$ there is a self-adjoint lift $T = \rho_{\text{sa}}(\widehat{Q}) \in \mathbb{B}_{\text{sa}}(\mathcal{H})$, namely $\pi_{\text{sa}}(T) = \pi(T) = \widehat{Q}$. Its essential spectrum is $\text{spec}_{\text{ess}}(T) = \{-1, 1\}$. Hence there is a gap $\Delta \subset (-1, 1)$ somewhere in the spectrum of T, and one can choose an increasing continuous function $f : \mathbb{R} \to [-1, 1]$ with $f(-1) = -1$, $f(1) = 1$, and $\text{supp}(f') \subset \Delta$. Then $Q = f(T)$ is a symmetry on $\mathcal{H}$ and $\pi(Q) = \widehat{Q}$ because $Q - T \in \mathbb{K}_{\text{sa}}(\mathcal{H})$. In particular, for $T_0 = \rho_{\text{sa}}(\widehat{Q}_0)$ there is a symmetry $Q_0 = f(T_0)$ such that $\widehat{Q}_0 = \pi(Q_0)$. By continuity of the spectrum, there is a neighborhood $\mathcal{U}$ of $\widehat{Q}_0$ such that the function f can be chosen uniformly for all $\widehat{Q} \in \mathcal{U}$, and one obtains a continuous local map $\widehat{Q} \in \mathcal{U} \mapsto Q = f(\rho_{\text{sa}}(\widehat{Q})) \in \mathbb{U}_{\text{sa}}(\mathcal{H})$. As both Q and Q_0 are proper, there exists a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $Q = UQ_0U^*$ and the map $\widehat{Q} \in \mathcal{U} \mapsto U$ can be chosen continuously by Lemma 8.3.3. Then $\widehat{U} = \pi(U)$ is a unitary in the Calkin algebra and $\widehat{Q} = \widehat{U}\widehat{Q}_0\widehat{U}^*$. Thus $\widehat{\rho} : \mathcal{U} \to \mathbb{UQ}(\mathcal{H})$ defined by $\widehat{\rho}(\widehat{Q}) = \widehat{U}$ is a local section, namely $\widehat{\rho}$ is continuous and $\pi_0 \circ \widehat{\rho} = \text{id}$. This fact combined with the bundle structure theorem (see the paragraph after the proof of Theorem A.3.7) implies that $\pi_0 : \mathbb{UQ}(\mathcal{H}) \to \mathbb{UQ}(\mathcal{H})/G_0 \approx \mathbb{UQ}^*_{\text{sa}}(\mathcal{H})$ is a principal bundle. (Note that one can spell out a version of Lemma 8.3.3 directly for symmetries in the Calkin algebra and this shortens the proof a little.)

It remains to show that the base space is connected. Let $\widehat{Q}_0, \widehat{Q}_1 \in \mathbb{UQ}^*_{\text{sa}}(\mathcal{H})$ have symmetry lifts Q_0 and Q_1 (constructed as above). As both Q_0 and Q_1 are proper, there exists a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $UQ_1U^* = Q_0$. Deforming U to $\mathbf{1}$ (e. g., taking roots of U) one obtains a path of symmetries connecting Q_0 to Q_1, and consequently also a path connecting $\widehat{Q}_0$ to $\widehat{Q}_1$. Hence $\mathbb{UQ}^*_{\text{sa}}(\mathcal{H})$ is indeed pathwise connected.

Fact 3. The homotopy groups $\pi_k(\mathbb{FB}^*_{\text{sa}}(\mathcal{H}))$ are as stated.

The long exact sequence of homotopy theory for the principal bundle of Fact 2 combined with Fact 1 leads to

$$\cdots \to \pi_k(\mathbb{UQ}(\mathcal{H})) \to \pi_k(\mathbb{FB}^*_{\text{sa}}(\mathcal{H})) \to \pi_{k-1}(G_0) \to \pi_{k-1}(\mathbb{UQ}(\mathcal{H})) \to \cdots.$$

The set $\mathbb{UQ}(\mathcal{H})$ of unitaries in the Calkin algebra is a retract (using the polar decomposition) of the set $\mathbb{GQ}(\mathcal{H})$ of the invertibles in the Calkin algebra. The connected components of $\mathbb{GQ}(\mathcal{H})$ are $\mathbb{G}_n\mathbb{Q}(\mathcal{H}) = \{\widehat{T} \in \mathbb{GQ}(\mathcal{H}) : \rho(\widehat{T}) \in \mathbb{F}_n\mathbb{B}(\mathcal{H})\}$ for $n \in \mathbb{Z}$. These components are homeomorphic and therefore have the same homotopy groups. The homotopy groups of $\mathbb{G}_0\mathbb{Q}(\mathcal{H})$ were determined in the proof of Theorem 8.2.2, so that

$$\pi_k(\mathbb{U}_0\mathbb{Q}(\mathcal{H})) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k \geq 2 \text{ even}, \end{cases}$$

where $\mathbb{U}_0\mathbb{Q}(\mathcal{H}) = \mathbb{UQ}(\mathcal{H}) \cap \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ and therefore

$$\pi_k(\mathbb{UQ}(\mathcal{H})) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k \text{ even}. \end{cases} \tag{8.2}$$

Moreover, the stabilizer group G_0 consists of all $\widehat{U} \in \mathbb{UQ}(\mathcal{H})$ commuting with the projection $\widehat{P}_0 = \frac{1}{2}(\mathbf{1} - \widehat{Q}_0)$, hence all block diagonal unitaries with block form $\widehat{P}_0\mathbb{Q}(\mathcal{H})\widehat{P}_0$ and $(\mathbf{1} - \widehat{P}_0)\mathbb{Q}(\mathcal{H})(\mathbf{1} - \widehat{P}_0)$. The lift $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$ of P_0 with Q_0 as above is a projection, and one can identify $\widehat{P}_0\mathbb{Q}(\mathcal{H})\widehat{P}_0$ with $\mathbb{Q}(P_0\mathcal{H})$, and similarly with the other block. Hence

$$G_0 \cong \mathbb{UQ}(P_0\mathcal{H}) \oplus \mathbb{UQ}((\mathbf{1} - P_0)\mathcal{H}).$$

Consequently, the homotopy groups of G_0 can be read from (8.2) as

$$\pi_k(G_0) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z} \oplus \mathbb{Z}, & k \text{ even}. \end{cases}$$

Thus for k odd, the above exact sequence becomes

$$\cdots \to 0 \to \pi_k(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to \pi_{k-1}(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})) \to 0 \to \cdots,$$

where i_* is the induced map of the inclusion $i : G_0 \hookrightarrow \mathbb{UQ}(\mathcal{H})$. However, i_* is surjective (actually, it is just the addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ in the homotopy groups as one can check using the fact that $i : G_0 \cong \mathbb{UQ}(P_0\mathcal{H}) \oplus \mathbb{UQ}((\mathbf{1} - P_0)\mathcal{H}) \to \mathbb{UQ}(\mathcal{H})$ is the embedding as a block diagonal operator) and thus exactness implies that the homotopy groups of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ are as stated. □

Example 8.3.4. This example is a continuation of Example 5.7.4 in which $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $|n\rangle$, $n \in \mathbb{Z}$ and

$$Q_k = \sum_{n \geq k} |n\rangle\langle n| - \sum_{n<k} |n\rangle\langle n|.$$

Then $Q_k = U^* Q_0 U$ where U is the left-shift by k and $\mathrm{Sf}(Q_0, Q_k) = -k$. This path shall now be closed to a loop. For that purpose, let us first rotate the states in the subspace spanned by $|n\rangle$ and $|k - n\rangle$. For even k this is done by

$$V_s = \sum_{n>\frac{k}{2}} (|n-1\rangle |k-n\rangle) \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} \langle n-1| \\ \langle k-n| \end{pmatrix},$$

while for odd k

$$V_s = \sum_{n>\frac{k}{2}} (|n\rangle |k-n-1\rangle) \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} \langle n| \\ \langle k-n-1| \end{pmatrix} + |\tfrac{k-1}{2}\rangle\langle\tfrac{k-1}{2}|.$$

One readily checks that the rotation $s \in [0, 1] \mapsto V_s^* Q_k V_s$ connects Q_k to $V_1^* Q_k V_1 = -Q_0$. Let us concatenate with a second path, given in terms of

$$\tilde{V}_s = \sum_{n\geq 0} \big(|n\rangle|-1-n\rangle\big)\begin{pmatrix}\cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s)\\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s)\end{pmatrix}\begin{pmatrix}\langle n|\\ \langle -1-n|\end{pmatrix}.$$

Then $s\in[0,1]\mapsto \tilde{V}_s^*(-Q_0)\tilde{V}_s$ connects $-Q_0$ to $\tilde{V}_1^*(-Q_0)\tilde{V}_1 = Q_0$. These last two paths are isospectral, so there is no spectral flow. As a consequence, the spectral flow along the whole closed loop obtained after concatenation is equal to $-k$. ◇

In Example 8.3.4, a concrete path of self-adjoint Fredholm operators with nontrivial spectral flow was constructed from a pair of symmetries (Q, UQU^*) which satisfies $[U,Q]$ compact. Actually, there is a far more general statement that implies the existence of a such a path. For any fixed proper symmetry Q, let us set

$$\mathbb{U}_Q(\mathcal{H}) = \{U\in\mathbb{U}(\mathcal{H}) : [U,Q]\in\mathbb{K}(\mathcal{H})\}. \tag{8.3}$$

Note that $\mathbb{U}_Q(\mathcal{H})$ is a subgroup of $\mathbb{U}(\mathcal{H})$.

Proposition 8.3.5. *For any proper symmetry Q on $\mathcal{H}$ and any $k\in\mathbb{N}$, $\pi_k(\mathbb{U}_Q(\mathcal{H}))$ is isomorphic to $\pi_{k+1}(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$.*

Proof. First recall from the proof of Theorem 8.3.1 that $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is homotopy equivalent to the set $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ of proper symmetries in the Calkin algebra. Note that Q also provides a base point $\pi(Q)$ in $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$. Now let us define a map $\beta_Q : \mathbb{U}(\mathcal{H}) \to \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ via $\beta_Q(U) = \pi(UQU^*)$. By the arguments in the proof of Theorem 8.3.1, one can check that this is a principal bundle with structure group

$$\begin{aligned}(\beta_Q)^{-1}(\pi(Q)) &= \{U\in\mathbb{U} : \pi(UQU^*) = \pi(Q)\}\\ &= \{U\in\mathbb{U} : \pi\big([U,Q]U^*\big) = 0\}\\ &= \mathbb{U}_Q(\mathcal{H}).\end{aligned}$$

Hence one can use the long exact sequence of homotopy groups, which, due to the triviality of the homotopy groups of $\mathbb{U}(\mathcal{H})$, proves the proposition. (Let us note that there is an equivalent statement to Proposition 8.3.5 for $\mathbb{Z}_2$-valued index pairings given in [48] and Theorem 7.1 of [57]; the argument in these works was based on the claim that the connecting maps in the long exact sequence of homotopy groups of a fibre bundle are induced by a homotopy equivalence, which was not proved there; this claim is not used here.) □

8.4 An application: characterization of spectral flow

The aim of this section is to present an axiomatic characterization of the spectral flow that is due to Lesch [126], with modifications taken from [184]. Let us denote by $\Omega^*(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$ the set of norm-continuous paths in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$. Let us stress that the paths are not necessarily closed, which is why the notation Ω^* instead of Ω is used. Let us consider maps $\mu : \Omega^*(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ that are invariant under orientation-preserving reparametrizations of the paths. Such maps can have the following properties:

Homotopy invariance. If $(s,t) \in [0,1] \times [0,1] \mapsto H_{s,t} \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ is norm-continuous and $H_{s,0}$ as well as $H_{s,1}$ are invertible for all $s \in [0,1]$, then

$$\mu(t \in [0,1] \mapsto H_{0,t}) = \mu(t \in [0,1] \mapsto H_{1,t}).$$

Concatenation. If $t \in [0,2] \mapsto H_t$ is an element of $\Omega^*(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}))$, then

$$\mu(t \in [0,2] \mapsto H_t) = \mu(t \in [0,1] \mapsto H_t) + \mu(t \in [1,2] \mapsto H_t).$$

Integrality. If $t \in [0,1] \mapsto H_t$ is a path in $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ with invertible endpoints, then

$$\mu(t \in [0,1] \mapsto H_t) \in \mathbb{Z}.$$

Normalization. For every $H \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ and associated $H_t = H + t\mathbf{1}$, one has

$$\mu(t \in [-\delta_H, 0] \mapsto H_t) = \mu(t \in [0, \delta_H] \mapsto H_t) = \frac{1}{2}\dim(\mathrm{Ker}(H)),$$

where $\delta_H = \frac{1}{2}\min\{|\lambda| : 0 \neq \lambda \in \mathrm{spec}(H)\}$.

Note that the spectral flow is a map $\mathrm{Sf} : \Omega^*(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ that satisfies concatenation by Theorem 4.2.1 and homotopy invariance by Theorem 4.2.4. Moreover, integrality and normalization are immediate consequences of the definition of the spectral flow.

Theorem 8.4.1. *If $\mu : \Omega^*(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ satisfies homotopy invariance, concatenation, integrality, and normalization, then $\mu = \mathrm{Sf}$.*

Proof. Let $H_0 \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ have a one-dimensional kernel and set $\delta_0 = \delta_{H_0}$. Then let $t \in [-\delta_0, \delta_0] \mapsto H_t = H_0 + t\mathbf{1}$ be the corresponding path. By the concatenation, homotopy invariance, and integrality properties, it is clear that μ and Sf induce homomorphisms

$$\mu, \mathrm{Sf} : \pi_1(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}), H_{-\delta_0}) \to \mathbb{Z}, \tag{8.4}$$

where $\pi_1(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}), H_{-\delta_0})$ is the fundamental group. As the set of invertible operators in $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ is connected, there is a path $t \in [0,1] \mapsto H'_t$ of invertible operators connecting H_{δ_0} to $H_{-\delta_0}$ in $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$. Now the (time rescaled) concatenation $t \in [0,1] \mapsto (H * H')_t$ is an element in $\pi_1(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}), H_{-\delta_0})$ and thus

$$\begin{aligned} \mu(t \in [0,1] \mapsto (H * H')_t) &= \mu(t \in [0,1] \mapsto H_t) + \mu(t \in [0,1] \mapsto H'_t) \\ &= \mu(t \in [0,1] \mapsto H_t) \\ &= \dim(\mathrm{Ker}(H_0)) \\ &= \mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [0,1] \mapsto H'_t) \end{aligned}$$

$$= \mathrm{Sf}(t \in [0,1] \mapsto (H * H')_t),$$

where $\mu(t \in [0,1] \mapsto H'_t) = 0 = \mathrm{Sf}(t \in [0,1] \mapsto H'_t)$ was used which follows from the homotopy invariance and concatenation. As $\dim(\mathrm{Ker}(H_0)) = 1$, this firstly shows that $t \in [0,1] \mapsto (H * H')_t$ is a generator of the infinitely cyclic group $\pi_1(\mathbb{FB}^*_{\mathrm{sa}}, H_{-\delta_0})$ (see Theorem 8.3.1), and secondly that μ and Sf have the same value on it. Hence the maps in (8.4) coincide. Note that this also holds for any other invertible base point than $H_{-\delta_0}$, which follows by using once again that the set of invertible elements in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is connected.

Let now $t \in [0,1] \mapsto H_t$ be an arbitrary norm-continuous path in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$. Let us first consider the endpoints H_0 and H_1 and set

$$t \in [-\delta_0, 0] \mapsto H^0_t = H_0 + t\mathbf{1}, \quad t \in [0, \delta_1] \mapsto H^1_t = H_1 + t\mathbf{1},$$

where still $\delta_0 = \delta_{H_0}$ and $\delta_1 = \delta_{H_1}$. It follows from the normalization property that

$$\mathrm{Sf}(t \in [-\delta_0, 0] \mapsto H^0_t) = \mu(t \in [-\delta_0, 0] \mapsto H^0_t) = \frac{1}{2}\dim(\mathrm{Ker}(H_0))$$

and

$$\mathrm{Sf}(t \in [0, \delta_1] \mapsto H^1_t) = \mu(t \in [0, \delta_1] \mapsto H^1_t) = \frac{1}{2}\dim(\mathrm{Ker}(H_1)).$$

Let now $t \in [0,1] \mapsto \tilde{H}_t$ be a path of invertible operators in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ connecting $H^1_{\delta_1}$ to $H^0_{-\delta_0}$. It follows from the first part of the proof and the concatenation property that

$$\begin{aligned}
&\mu(t \in [0,1] \mapsto H_t)\\
&\quad = \mu(t \in [0,1] \mapsto (H^0 * H * H^1)_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1))\\
&\quad = \mu(t \in [0,1] \mapsto (H^0 * H * H^1 * \tilde{H})_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1))\\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto (H^0 * H * H^1 * \tilde{H})_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1))\\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto (H^0 * H * H^1)_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1))\\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto H_t),
\end{aligned}$$

and so the claim is shown. □

Let us note that Theorem 8.4.1 slightly differs from Lesch's work [126] as here it is not assumed that the endpoints of the paths in $\Omega^*(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$ are invertible. There are other axiomatic characterizations of the spectral flow. For example, Ciriza, Fitzpatrick, and Pejsachowicz showed in [60] that the spectral flow for paths in all three components of $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ is also uniquely determined by the homotopy invariance in Theorem 4.2.2, the two basic properties (i) and (v) in Theorem 4.2.1 and the fact that it is the difference of the Morse indices of the endpoints for paths in $\mathbb{FB}^+_{\rm sa}(\mathcal{H})$ (see Proposition 4.3.1). Finally, let us note that Georgescu proved a similar characterization for paths of unbounded self-adjoint Fredholm operators [93].

8.5 Homotopy groups of Fredholm pairs

Let us introduce a notation for the set of proper orthogonal projections which form a Fredholm pair with a fixed proper orthogonal projections $P_{\rm ref}$:

$$\mathbb{FP}(\mathcal{H}) = \{P : P \text{ proper orthogonal projection with } (P_{\rm ref}, P) \text{ Fredholm pair}\}.$$

This set will be equipped with norm topology $\mathcal{O}_N$ on $\mathbb{B}(\mathcal{H})$. Results on the homotopy groups of $\mathbb{FP}(\mathcal{H})$ and various modifications of it go back to Wojciechowski [207] and Abbondandolo and Majer [2].

Theorem 8.5.1. *The homotopy groups of* $\mathbb{FP}(\mathcal{H})$ *are given by*

$$\pi_k(\mathbb{FP}(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

To show this, let us introduce the set

$$\mathbb{FP}^{\rm C}(\mathcal{H}) = \{P \in \mathbb{FP}(\mathcal{H}) : P_{\rm ref} - P \in \mathbb{K}(\mathcal{H})\}$$

of orthogonal projections P such that $P - P_{\rm ref}$ is compact. The proof of Theorem 8.5.1 is then based on the following fact.

Proposition 8.5.2. *The space* $(\mathbb{FP}^{\rm C}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FP}(\mathcal{H}), \mathcal{O}_N)$.

Proof. (Based on the proof of Proposition 5.3.19.) For $P \in \mathbb{FP}(\mathcal{H})$, let $Q = \mathbf{1} - 2P$ be the associated symmetry and let $Q_{\rm ref} = \mathbf{1} - 2P_{\rm ref}$ be the symmetry associated to $P_{\rm ref}$ and then set

$$R = QQ_{\rm ref} + Q_{\rm ref}Q = 2\mathbf{1} - 4(P - P_{\rm ref})^2.$$

Then one has $[R, Q] = 0 = [R, Q_{\rm ref}]$. Let us set $a = \sup \operatorname{spec}_{\rm ess}((P - P_{\rm ref})^2)$. Because $(P_{\rm ref}, P)$ is a Fredholm pair, $1 \notin \operatorname{spec}_{\rm ess}((P - P_{\rm ref})^2)$ and therefore $a \in [0, 1)$. Then also $b = \min\{\frac{a+1}{2}, 2a\} \in [0, 1)$ and the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \chi_{[0,a]}(x) + (x - b)(a - b)^{-1}\chi_{(a,b)}(x) \tag{8.5}$$

is continuous for $a > 0$. For

$$H_t = \mathbf{1} + R\cos\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big)\sin\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big), \quad t \in [0,1],$$

clearly, $[H_t, (P-P_{\mathrm{ref}})^2] = 0$, and next it is shown that H_t is invertible for all t. Setting $\mathcal{H}' = \mathrm{Ran}(\chi((P-P_{\mathrm{ref}})^2 \geq b))$ the restriction of H_t to this space is $\mathbf{1}_{\mathcal{H}'}$. On the orthogonal complement $(\mathcal{H}')^\perp$ one has $(P-P_{\mathrm{ref}})^2 \leq b$. Therefore $R\mathbf{1}_{(\mathcal{H}')^\perp} > (2-4b)\mathbf{1}_{(\mathcal{H}')^\perp}$ and, because $\|\cos(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2))\sin(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2))\| \leq \frac{1}{2}$, one gets $H_t\mathbf{1}_{(\mathcal{H}')^\perp} > (1+\frac{1}{2}(2-4b))\mathbf{1}_{(\mathcal{H}')^\perp} = (2-2b)\mathbf{1}_{(\mathcal{H}')^\perp} > 0$. Combined with the fact that $[H_t, (P-P_{\mathrm{ref}})^2] = 0$ and therefore H_t is diagonal with respect to the grading $\mathcal{H} = \mathcal{H}' \oplus (\mathcal{H}')^\perp$, this implies that H_t is invertible for all t. Therefore one can set

$$Q_t = (H_t)^{-\frac{1}{2}}\Big(Q\cos\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}}\sin\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big)\Big), \quad t \in [0,1].$$

Clearly, $Q_t^* = Q_t$ and computing the square shows $Q_t^2 = \mathbf{1}$, so this is a path of symmetries. Moreover, $Q_0 = Q$. To show that (P_{ref}, P_t) is a Fredholm pair for all $t \in [0,1]$ where $P_t = \frac{1}{2}(\mathbf{1} - Q_t)$, let us compute

$$(P_t - P_{\mathrm{ref}})^2 = \frac{1}{2}\mathbf{1} - \frac{1}{4}(H_t)^{-\frac{1}{2}}\Big(R\cos\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big) + 2\sin\Big(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\Big)\mathbf{1}\Big).$$

Suppose that $a > 0$. Then the right-hand side is a continuous function of the self-adjoint operator $(P-P_{\mathrm{ref}})^2$. Namely,

$$(P_t - P_{\mathrm{ref}})^2 = g_t((P-P_{\mathrm{ref}})^2)$$

for the continuous function $g_t : [0,1] \to [0,1]$ defined by

$$\begin{aligned} g_t(x) = \frac{1}{2} - \frac{1}{4}\Big(1 + (2-4x)\cos\Big(\frac{\pi}{2}tf(x)\Big)\sin\Big(\frac{\pi}{2}tf(x)\Big)\Big)^{-\frac{1}{2}} \\ \cdot \Big((2-4x)\cos\Big(\frac{\pi}{2}tf(x)\Big) + 2\sin\Big(\frac{\pi}{2}tf(x)\Big)\Big). \end{aligned} \tag{8.6}$$

By the spectral mapping theorem in the Calkin algebra, one gets $\mathrm{spec}_{\mathrm{ess}}((P_t - P_{\mathrm{ref}})^2) = g_t(\mathrm{spec}_{\mathrm{ess}}((P-P_{\mathrm{ref}})^2))$ and therefore

$$\sup \mathrm{spec}_{\mathrm{ess}}((P_t - P_{\mathrm{ref}})^2) \leq \sup_{t\in[0,1],x\in[0,a]} g_t(x) \leq \sup_{t\in[0,1],x\in[0,a]} h_t(x)$$

for

$$h_t(x) = \frac{1}{2} - \frac{1}{4}\Big(1 + (2-4x)\cos\Big(\frac{\pi}{2}t\Big)\sin\Big(\frac{\pi}{2}t\Big)\Big)^{-\frac{1}{2}}\Big((2-4x)\cos\Big(\frac{\pi}{2}t\Big) + 2\sin\Big(\frac{\pi}{2}t\Big)\Big). \tag{8.7}$$

The supremum is given by

$$\sup_{t\in[0,1],x\in[0,a]} h_t(x) = \sup_{x\in[0,a]} h_0(x) = \sup_{x\in[0,a]} \frac{1}{2} - \frac{1}{4}(2-4x) = a < 1.$$

Then $\sup \operatorname{spec}_{\mathrm{ess}}((P_t - P_{\mathrm{ref}})^2) < 1$ by Corollary 5.3.13 implies that (P_{ref}, P_t) is a Fredholm pair for all $t \in [0,1]$. Moreover, $\operatorname{spec}_{\mathrm{ess}}((P_1 - P_{\mathrm{ref}})^2) = g_1(\operatorname{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2)) = \{0\}$ and therefore $P_1 - P_{\mathrm{ref}}$ is compact.

Suppose that $P - P_{\mathrm{ref}}$ is compact, or equivalently that $a = 0$. Then it follows that $f((P_t - P_{\mathrm{ref}})^2) = \chi((P_t - P_{\mathrm{ref}})^2 = 0)$ is the projection onto $\operatorname{Ker}((P_t - P_{\mathrm{ref}})^2) = \operatorname{Ker}(P_t - P_{\mathrm{ref}})$. We show that in this case $t \in [0,1] \mapsto Q_t$ is constant. Clearly, Q_t commutes with $(P_t - P_{\mathrm{ref}})^2$ and thus Q_t is diagonal with respect to the grading $\mathcal{H} = \operatorname{Ker}(P_t - P_{\mathrm{ref}}) \oplus \operatorname{Ker}(P_t - P_{\mathrm{ref}})^{\perp}$. On $\operatorname{Ker}(P_t - P_{\mathrm{ref}})^{\perp}$ one has $f((P_t - P_{\mathrm{ref}})^2) = 0$ and thus $Q_t = Q$. On $\operatorname{Ker}(P_t - P_{\mathrm{ref}})$ one has $Q = Q_{\mathrm{ref}}$ and therefore

$$Q_t = \left(\mathbf{1} + 2\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}} \left(Q\cos\left(\frac{\pi}{2}t\right) + Q\sin\left(\frac{\pi}{2}t\right)\right) = Q.$$

Thus $Q_t = Q$ for all $t \in [0,1]$ on all of $\mathcal{H}$.

Next let us consider the homotopy

$$h : \mathbb{FP}(\mathcal{H}) \times [0,1] \to \mathbb{FP}(\mathcal{H}), \quad h(P,t) = \frac{1}{2}(\mathbf{1} - Q_t)$$

It is shown that h is continuous at any point $(P,t) \in \mathbb{FP}(\mathcal{H}) \times [0,1]$. This is verified by a rather lengthy argument in the remainder of the proof which an experienced reader may want to skip.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{FP}(\mathcal{H})$ converging to P. Associated to it is the sequence $(Q_n)_{n\in\mathbb{N}}$ of symmetries, where $Q_n = \mathbf{1} - 2P_n$. Moreover, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ converging to t. Let us first assume that $a = \sup \operatorname{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2) > 0$. Then for n sufficiently large, $a_n = \sup \operatorname{spec}_{\mathrm{ess}}((P_n - P_{\mathrm{ref}})^2) > 0$ and $b_n = \min\{\frac{a_n+1}{2}, 2a_n\} \in (a_n, 1)$, and the function $f_n : [0,1] \to [0,1]$ defined by

$$f_n(x) = \chi_{[0,a_n]}(x) + (x - b_n)(a_n - b_n)^{-1}\chi_{(a_n,b_n)}(x)$$

is continuous. Moreover,

$$\sup_{x\in[0,1]} \left|f(x) - f_n(x)\right| \to 0 \tag{8.8}$$

for $f : [0,1] \to [0,1]$ as in (8.5). Furthermore, let us set

$$R_n = Q_n Q_{\mathrm{ref}} + Q_{\mathrm{ref}} Q_n = 2\mathbf{1} - 4(P_n - P_{\mathrm{ref}})^2$$

and

$$H_{n,t} = \mathbf{1} + R_n \cos\left(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\right) \sin\left(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\right), \quad t \in [0,1],$$

which by the same argument as above is invertible with inverse bounded by

$$\|H_{n,t}^{-1}\| \le \min\{1, 2-2b_n\}^{-1}. \tag{8.9}$$

Clearly,

$$\big\|h(P_n,t_n)-h(P,t)\big\| \le \big\|h(P_n,t_n)-h(P_n,t)\big\| + \big\|h(P_n,t)-h(P,t)\big\|.$$

For the first summand, one has

$$\begin{aligned}
&\big\|h(P_n,t_n)-h(P_n,t)\big\| \\
&\quad= \frac{1}{2}\bigg\|(H_{n,t_n})^{-\frac{1}{2}}\bigg(Q_n\cos\Big(\frac{\pi}{2}t_nf_n((P_n-P_{\mathrm{ref}})^2)\Big)+Q_{\mathrm{ref}}\sin\Big(\frac{\pi}{2}t_nf_n((P_n-P_{\mathrm{ref}})^2)\Big)\bigg) \\
&\qquad-(H_{n,t})^{-\frac{1}{2}}\bigg(Q_n\cos\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)+Q_{\mathrm{ref}}\sin\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)\bigg)\bigg\| \\
&\quad\le \frac{1}{2}\bigg\|(H_{n,t})^{-\frac{1}{2}}\bigg[Q_n\bigg(\cos\Big(\frac{\pi}{2}t_nf_n((P_n-P_{\mathrm{ref}})^2)\Big)-\cos\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)\bigg) \\
&\qquad+Q_{\mathrm{ref}}\bigg(\sin\Big(\frac{\pi}{2}t_nf_n((P_n-P_{\mathrm{ref}})^2)\Big)-\sin\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)\bigg)\bigg]\bigg\| \\
&\qquad+\frac{1}{2}\bigg\|\Big((H_{n,t_n})^{-\frac{1}{2}}-(H_{n,t})^{-\frac{1}{2}}\Big) \\
&\qquad\cdot\bigg(Q_n\cos\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)+Q_{\mathrm{ref}}\sin\Big(\frac{\pi}{2}tf_n((P_n-P_{\mathrm{ref}})^2)\Big)\bigg)\bigg\| \\
&\quad\le \frac{1}{2}\sup_{n,t}\|(H_{n,t})^{-\frac{1}{2}}\|\bigg(\sup_{x\in[0,1]}\Big|\cos\Big(\frac{\pi}{2}t_nx\Big)-\cos\Big(\frac{\pi}{2}tx\Big)\Big|+\sup_{x\in[0,1]}\Big|\sin\Big(\frac{\pi}{2}t_nx\Big)-\sin\Big(\frac{\pi}{2}tx\Big)\Big|\bigg) \\
&\qquad+\big\|(H_{n,t_n})^{-\frac{1}{2}}-(H_{n,t})^{-\frac{1}{2}}\big\|.
\end{aligned}$$

Clearly, the first summand converges to 0 for $t_n \to t$ (uniformly in P_n). To bound the second summand, let us first note that $\lim_{n\to\infty} b_n = b$ and therefore by (8.9) $\|H_{n,t}^{-1}\|$ is uniformly bounded in t and P_n, namely the is a constant $C \in \mathbb{R}_{>0}$ such that

$$\sup_{t\in[0,1],n\in\mathbb{N}} \|H_{n,t}^{-1}\| < C.$$

Then

$$\begin{aligned}
\big\|(H_{n,t_n})^{-\frac{1}{2}}-(H_{n,t})^{-\frac{1}{2}}\big\| &= \big\|(H_{n,t_n})^{-\frac{1}{2}}\big((H_{n,t})^{\frac{1}{2}}-(H_{n,t_n})^{\frac{1}{2}}\big)(H_{n,t})^{-\frac{1}{2}}\big\| \\
&\le \big\|(H_{n,t_n})^{-\frac{1}{2}}\big\|\big\|(H_{n,t})^{\frac{1}{2}}-(H_{n,t_n})^{\frac{1}{2}}\big\|\big\|(H_{n,t})^{-\frac{1}{2}}\big\| \\
&\le C\|H_{n,t}-H_{n,t_n}\|^{\frac{1}{2}},
\end{aligned} \tag{8.10}$$

where the last step follows from Proposition A.2.2. Because

$$\begin{aligned}\|H_{n,t} - H_{n,t_n}\| &= \Big\| R_n \Big[\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) \\ &\qquad - \cos\Big(\frac{\pi}{2} t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big)\sin\Big(\frac{\pi}{2} t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le 2 \sup_{x\in[0,1]} \Big|\cos\Big(\frac{\pi}{2} t x\Big)\sin\Big(\frac{\pi}{2} t x\Big) - \cos\Big(\frac{\pi}{2} t_n x\Big)\sin\Big(\frac{\pi}{2} t_n x\Big)\Big|,\end{aligned}$$

one has

$$\lim_{m\to 0} \|H_{n,t_m} - H_{n,t}\| = 0$$

uniformly in n. Thus $\lim_{t_n\to t} \|h(P_n, t_n) - h(P_n, t)\| = 0$ uniformly in P_n. It remains to show $\lim_{n\to\infty} \|h(P_n, t) - h(P, t)\| = 0$. One has

$$\begin{aligned}&\big\|h(P_n, t) - h(P, t)\big\| \\ &= \Big\|(H_{n,t})^{-\frac{1}{2}}\Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big] \\ &\quad - (H_t)^{-\frac{1}{2}}\Big[Q \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le \Big\|(H_t)^{-\frac{1}{2}}\Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - Q \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big) \\ &\quad + Q_{\mathrm{ref}}\Big(\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big)\Big]\Big\| \\ &\quad + \Big\|((H_{n,t})^{-\frac{1}{2}} - (H_t)^{-\frac{1}{2}}) \\ &\quad \cdot \Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le \|(H_t)^{-\frac{1}{2}}\|\Big(\Big\|(Q_n - Q)\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \Big\|Q\Big(\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big)\Big\| \\ &\quad + \Big\|\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big\|\Big) \\ &\quad + 2\big\|(H_{n,t})^{-\frac{1}{2}} - (H_t)^{-\frac{1}{2}}\big\| \\ &\le \|(H_t)^{-\frac{1}{2}}\| \|Q_n - Q\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\| \Big\|\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\| \Big\|\cos\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\| \Big\|\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big)\Big\|\end{aligned}$$

$$+\|(H_t)^{-\frac{1}{2}}\|\left\|\sin\left(\frac{\pi}{2}tf((P_n-P_{\mathrm{ref}})^2)\right)-\sin\left(\frac{\pi}{2}tf((P-P_{\mathrm{ref}})^2)\right)\right\|$$
$$+C\|H_{n,t}-H_t\|^{\frac{1}{2}},$$

where the last step follows from a similar argument as that leading to (8.10). Clearly, the first and last summands converge to 0 for $n \to \infty$. Because f is a continuous function on $[0,1] \supset \bigcup_{n\in\mathbb{N}} \operatorname{spec}((P_n-P_{\mathrm{ref}})^2) \cup \operatorname{spec}((P-P_{\mathrm{ref}})^2))$ the third and fifth summands converge to 0. Finally the second and fourth summands converge to 0 by the same argument using (8.8). This shows the claim for $a > 0$. Finally, let us consider the case $a = 0$. Then, by the above, $P_t = P$ for all $t \in [0,1]$. One has to show

$$\|h(P_n,t_n)-P\| \to 0,$$

or equivalently

$$\|\tilde{Q}_n - Q\| \to 0,$$

for $\tilde{Q}_n = \mathbf{1} - 2h(P_n,t_n)$. Note that by the spectral radius theorem in the Calkin algebra $a_n = \sup \operatorname{spec}_{\mathrm{ess}}((P_n-P_{\mathrm{ref}})^2) \to 0$. For $\frac{1}{2} > \epsilon > 0$, there is $\lambda_\epsilon \in (0,\epsilon) \setminus \operatorname{spec}(P_{\mathrm{ref}}-P)^2$. Note that

$$\|\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \to 0.$$

Then for n sufficiently large, $a_n < \frac{1}{2}\lambda_\epsilon$ and therefore

$$\tilde{Q}_n(\mathbf{1}-\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)) = Q_n(\mathbf{1}-\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)).$$

One then has

$$\|\tilde{Q}_n - Q\| \le \|(\tilde{Q}_n-Q)\chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|(\tilde{Q}_n-Q)(\mathbf{1}-\chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\|$$

and the second term is bounded by

$$\begin{aligned}
&\|(\tilde{Q}_n-Q)(\mathbf{1}-\chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad \le \|\tilde{Q}_n(\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\qquad + \|(Q_n-Q)(\mathbf{1}-\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\qquad + \|Q(\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad \le 2\|\chi((P_n-P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|Q_n - Q\| \\
&\quad \to 0.
\end{aligned}$$

For the first summand, one has

$$\|(\tilde{Q}_n-Q)\chi((P-P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\|$$

$$\begin{aligned}
&= \|(Q_{\mathrm{ref}} - \tilde{Q}_n)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - (Q_{\mathrm{ref}} - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \\
&\le \|(Q_{\mathrm{ref}} - \tilde{Q}_n)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|(Q_{\mathrm{ref}} - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \\
&\le \|(Q_{\mathrm{ref}} - \tilde{Q}_n)(\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad + \|(Q_{\mathrm{ref}} - \tilde{Q}_n)\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|(Q_{\mathrm{ref}} - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\|.
\end{aligned}$$

The first summand is bounded by

$$\begin{aligned}
&\|(Q_{\mathrm{ref}} - \tilde{Q}_n)(\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad \le 2\|\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \\
&\quad \to 0.
\end{aligned}$$

For the third summand, one has

$$\begin{aligned}
\|(Q_{\mathrm{ref}} - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| &= \|(2P_{\mathrm{ref}} - 2P)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \\
&= 2\|(P_{\mathrm{ref}} - P)^2\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\|^{\frac{1}{2}} \\
&\le 2\lambda_\epsilon^{\frac{1}{2}} \\
&\le 2\epsilon^{\frac{1}{2}}.
\end{aligned}$$

And finally, the second summand is bounded by

$$\begin{aligned}
\|(Q_{\mathrm{ref}} - \tilde{Q}_n)\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| &= \|(2P_{\mathrm{ref}} - 2h(P_n, t_n))\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \\
&= 2\|(P_{\mathrm{ref}} - h(P_n, t_n))^2\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\|^{\frac{1}{2}} \\
&\le 2\Big(\sup_{t\in[0,1], x\in[0,\lambda_\epsilon]} h_t(x)\Big)^{\frac{1}{2}} \\
&= 2\lambda_\epsilon^{\frac{1}{2}} \\
&\le 2\epsilon^{\frac{1}{2}},
\end{aligned}$$

where $h_t(x)$ is defined in (8.7). Because $\epsilon > 0$ was arbitrary $\|\tilde{Q}_n - Q\| \to 0$ follows and therefore the considered homotopy is continuous. Thus, one can conclude that $(\mathbb{FP}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FP}(\mathcal{H}), \mathcal{O}_N)$. □

For $n \in \mathbb{Z}$ let us introduce the sets

$$\mathbb{F}_n\mathbb{P}(\mathcal{H}) = \{P \in \mathbb{FP}(\mathcal{H}) : \mathrm{Ind}(P_{\mathrm{ref}}, P) = n\}$$

and

$$\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H}) = \{P \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H}) : \mathrm{Ind}(P_{\mathrm{ref}}, P) = n\}.$$

The next result shows that these are the connected components of $\mathbb{FP}(\mathcal{H})$ and $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$, respectively.

Proposition 8.5.3. *The sets* $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ *and* $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ *are connected with respect to the operator norm. Moreover, the space* $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ *is homeomorphic to* $\mathbb{F}_0\mathbb{P}(\mathcal{H})$ *and* $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ *is homeomorphic to* $\mathbb{F}_0\mathbb{P}^{\mathrm{C}}(\mathcal{H})$.

Proof. The argument leading to Proposition 5.3.23 shows that both $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ and $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ are connected.

To show that $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ is homeomorphic to $\mathbb{F}_0\mathbb{P}(\mathcal{H})$, let $P_{\mathrm{ref},n} \in \mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ be a fixed projection. Then by Corollary 5.3.13, for any projection $P \in \mathbb{P}(\mathcal{H})$, (P_{ref}, P) is a Fredholm pair if and only if the pair $(P_{\mathrm{ref},n}, P)$ is Fredholm. By Proposition 5.3.15,

$$\mathbb{F}_n\mathbb{P}(\mathcal{H}) = \{P \in \mathbb{P}(\mathcal{H}) : (P_{\mathrm{ref},n}, P)\ \text{Fredholm}, \mathrm{Ind}(P_{\mathrm{ref},n}, P) = 0\}.$$

Moreover, by Proposition 5.1.7, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $U^* P_{\mathrm{ref},n} U = P_{\mathrm{ref}}$. Then, by the above, one has $P \in \mathbb{F}_n\mathbb{P}(\mathcal{H})$ if and only if $U^*PU \in \mathbb{F}_0\mathbb{P}(\mathcal{H})$. Therefore

$$f : \mathbb{F}_n\mathbb{P}(\mathcal{H}) \to \mathbb{F}_0\mathbb{P}(\mathcal{H}), \quad P \mapsto U^*PU$$

is a homeomorphism. Thus the claim on $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ is shown. Restricting f to $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ implies the last claim. □

Proof of Theorem 8.5.1. By Proposition 8.5.2, the homotopy groups of $\mathbb{FP}(\mathcal{H})$ and $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ coincide, namely $\pi_k(\mathbb{FP}(\mathcal{H})) = \pi_k(\mathbb{FP}^{\mathrm{C}}(\mathcal{H}))$ for all $k \in \mathbb{N}_0$.

Recall from (8.3) that $\mathbb{U}_{Q_{\mathrm{ref}}}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : [U, Q_{\mathrm{ref}}] \in \mathbb{K}(\mathcal{H})\}$ denotes the set of all unitaries that have a compact commutator with $Q_{\mathrm{ref}} = \mathbf{1} - 2P_{\mathrm{ref}}$. Then the map $\pi_0 : \mathbb{U}_{Q_{\mathrm{ref}}}(\mathcal{H}) \to \mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ defined by

$$\pi_0(U) = UP_{\mathrm{ref}}U^*$$

is the base projection of a fiber bundle with fiber given by

$$\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : [U, Q_{\mathrm{ref}}] = 0\}. \tag{8.11}$$

For the justification, let us first note that π_0 is surjective. For $P_0 \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\mathrm{ref}}U^* = P_0\}.$$

For the neighborhood

$$\mathcal{U} = \{P \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H}) : \|P - P_0\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\mathrm{ref}}U^* \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \pi_0^{-1}(\mathcal{U})$, $P \mapsto V_P$ such that $P_{\mathrm{ref}} = V_P^* P V_P$. For $U \in \pi_0^{-1}(P)$, one gets $UP_{\mathrm{ref}}U^* = P = V_P P_{\mathrm{ref}} V_P^*$ and therefore $[P_{\mathrm{ref}}, U^* V_P] = 0$. Thus

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$$

defined by

$$\phi(U) = \big(UP_{\mathrm{ref}}U^*, U^* V_{UP_{\mathrm{ref}}U^*}\big)$$

is a homeomorphism.

Even though the base space $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ is not connected, Proposition 8.5.3 implies that the homotopy groups of all connected components are the same. Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains isomorphisms $\pi_k(\mathbb{FP}^{\mathrm{C}}(\mathcal{H})) \cong \pi_k(\mathbb{U}_{Q_{\mathrm{ref}}}(\mathcal{H}))$ because $\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$ is contractible by Kuiper's theorem and hence has vanishing homotopy groups. Then Proposition 8.3.5 combined with Theorem 8.3.1 shows the claim. □

Recall from (5.19) that $\mathbb{FPP}(\mathcal{H})$ denotes the set of Fredholm pairs of proper orthogonal projections. This set is then equipped with the topology $\mathcal{O}_N \times \mathcal{O}_N$ where $\mathcal{O}_N$ denotes the norm topology on $\mathbb{B}(\mathcal{H})$. The following result goes back to Abbondandolo and Majer [2], but the proof below is different.

Theorem 8.5.4. *The homotopy groups of* $\mathbb{FPP}(\mathcal{H})$ *are given by*

$$\pi_k\big(\mathbb{FPP}(\mathcal{H})\big) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

The proof is based on the following fact on the homotopy groups of the set:

$$\mathbb{P}(\mathcal{H}) = \{P = P^* = P^2 \in \mathbb{B}(H) : \dim(\mathrm{Ran}(P)) = \dim(\mathrm{Ker}(P))\}$$

of all proper orthogonal projections on $\mathcal{H}$, which goes back to [15].

Proposition 8.5.5. *The space* $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ *is contractible.*

Proof. First note that, by Proposition 5.1.7, the space $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is connected. Let, as above, $P_{\mathrm{ref}} \in \mathbb{P}(\mathcal{H})$ be one fixed proper orthogonal projection. Then $\pi_0 : \mathbb{U}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ defined by

$$\pi_0(U) = UP_{\mathrm{ref}}U^*$$

is the base projection of a fiber bundle with connected base space and fiber given by $\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$. For the justification, let us first note that π_0 is surjective by Proposition 5.1.7. For $P_0 \in \mathbb{P}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\mathrm{ref}}U^* = P_0\}.$$

For the neighborhood

$$\mathcal{U} = \{P \in \mathbb{P}(\mathcal{H}) : \|P - P_0\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\mathrm{ref}}U^* \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \mathbb{U}(\mathcal{H})$, $P \mapsto V_P$, such that $P_0 = V_P^* P V_P$. Moreover, there is a unitary $U_0 \in \mathbb{U}(\mathcal{H})$ such that $U_0^* P_0 U_0 = P_{\mathrm{ref}}$. Then for $P \in \mathcal{U}$ and $U \in \pi_0^{-1}(P)$, one has $UP_{\mathrm{ref}}U^* = P = V_P P_0 V_P^*$ and therefore $V_P^* UP_{\mathrm{ref}}U^* V_P = P_0$, or equivalently $U_0^* V_P^* UP_{\mathrm{ref}}U^* V_P U_0 = P_{\mathrm{ref}}$. Thus $U^* V_P U_0 \in \mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$ where as in (8.11) $\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$ denotes the set of unitaries that commute with P_{ref} and

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$$

defined by

$$\phi(U) = \big(UP_{\mathrm{ref}}U^*, U^* V_{UP_{\mathrm{ref}}U^*} U_0\big)$$

is a homeomorphism.

Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains that $\pi_k(\mathbb{P}(\mathcal{H})) = 0$ for all $k \in \mathbb{N}_0$ because $\mathbb{U}(\mathcal{H})$ and $\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H})$ have vanishing homotopy groups by Kuiper's theorem. As $\mathbb{P}(\mathcal{H})$ is a metrizable Banach manifold (see [2]), $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is contractible by Theorem A.3.5 combined with Theorem A.3.6. □

Proof of Theorem 8.5.4. The map $\pi_0 : \mathbb{FPP}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ defined by

$$\pi_0((P_0, P_1)) = P_0$$

is the base projection of a fiber bundle with connected base space and fiber given by $\mathbb{FP}(\mathcal{H})$. For the justification, let us first note that π_0 is surjective. For $P_0' \in \mathbb{P}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0') = \{(P_0, P_1) \in \mathbb{FPP}(\mathcal{H}) : P_0 = P_0'\}.$$

For the neighborhood

$$\mathcal{U} = \{P_0 \in \mathbb{P}(\mathcal{H}) : \|P_0 - P_0'\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{(P_0, P_1) \in \mathbb{FPP}(\mathcal{H}) : P_0 \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \mathbb{U}(\mathcal{H}), P_0 \mapsto V_{P_0}$, such that $P_0' = V_{P_0}^* P V_{P_0}$. Moreover, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $U^* P_0' U = P_{\text{ref}}$. Then for $P_0 \in \mathcal{U}$ and $P_1 \in \mathbb{P}(\mathcal{H})$, the pair (P_0, P_1) is Fredholm if and only if $(P_0', V_{P_0}^* P_1 V_{P_0})$ is Fredholm, which is equivalent to the Fredholm property of $(P_{\text{ref}}, U^* V_{P_0}^* P_1 V_{P_0} U)$. Thus

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{FP}(\mathcal{H})$$

defined by

$$\phi((P_0, P_1)) = (P_0, U^* V_{P_0}^* P_1 V_{P_0} U)$$

is a homeomorphism.

Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains isomorphisms $\pi_k(\mathbb{FPP}(\mathcal{H})) \cong \pi_k(\mathbb{FP}(\mathcal{H}))$ because $\mathbb{P}(\mathcal{H})$ has vanishing homotopy groups by Proposition 8.5.5. Then Theorem 8.5.1 allows finishing the proof. □

8.6 Homotopy groups of unbounded self-adjoint Fredholm operators

On the set of unbounded self-adjoint Fredholm operators $\mathbb{F}_{\text{sa}}(\mathcal{H})$, there are two natural topologies, the Riesz and gap topologies, see Section 6.3. As to the Riesz topology, Proposition 6.3.3 already shows that $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ is homotopy equivalent to the set of bounded self-adjoint Fredholm operators $(\mathbb{FB}_{\text{sa}}(\mathcal{H}), \mathcal{O}_N)$. Hence their homotopy groups coincide, and one immediately deduces the next result.

Theorem 8.6.1. *The homotopy groups of* $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ *are the same as the homotopy groups of* $\mathbb{FB}_{\text{sa}}(\mathcal{H})$*, namely* $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ *has three connected components and the homotopy groups of the nontrivial component are as given by Theorem* 8.3.1.

Note that the proof of Theorem 8.6.1 merely implements self-adjointness in the proof of Theorem 8.2.4, because the same can be said already about Propositions 6.3.3 and 6.2.18.

The homotopy groups of the space $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$ are much more difficult to access. It was already proved in Theorem 6.3.16 that $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$ is connected, which is a striking difference to $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$. Of course, this reflects the fact that the Riesz topology is strictly finer than the gap topology. Moreover, item (ii) of Theorem 7.1.7 and Theorem 7.1.8 directly imply that the spectral flow on closed loops establishes a homomorphism $\text{Sf} : \pi_1(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G) \to \mathbb{Z}$. Whether this captures the whole fundamental group was an open question, as pointed out in [31, 126]. An affirmative answer was given in a paper by Joachim [108] which actually computed all the homotopy groups of $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$. This paper is placed in the more general framework of Hilbert modules and, unfortunately, this made parts of the paper difficult to understand for many (including ourselves). Very recently, Prokhorova provided a new and independent proof in a Hilbert space framework [154]. The arguments of [154] are in spirit close to the approach used in this book and are therefore followed closely in the remainder of this section. The outcome is the following:

Theorem 8.6.2. *Spaces* $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ *and* $(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ *are homotopy equivalent. In particular, their homotopy groups coincide and are as given by Theorem* 8.3.1.

Theorem 8.6.2 follows from the next result on the set-theoretic preimage

$$\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}) = \mathcal{F}^{-1}\big(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})\big)$$

of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ under the bounded transform.

Theorem 8.6.3. *The embedding* $I : (\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ *is a homotopy equivalence.*

Proof of Theorem 8.6.2. By Theorem 8.6.3, $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ is homotopy equivalent to $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$. Because $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ is homotopy equivalent to $(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ by Proposition 6.3.3 this implies the claim. □

It now remains to prove Theorem 8.6.3, and this will make up the remainder of this section. Let us state at the very beginning that the main novel ingredient of [154] is to use a technique of tom Dieck to prove homotopy equivalence, stated in Theorem A.3.3 in Appendix A.3. This motivates many of the constructions that follow. Let us begin by analyzing the space $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ where for $a \geq 0$,

$$\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}) = \{H \in \mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap [-a, a] = \emptyset\}$$

denotes the set of operators in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ with spectral gap around 0 of size a.

Proposition 8.6.4. *The space* $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ *is contractible for every* $a \geq 0$.

Proof. Consider the map $f : (\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ defined by $f(H) = \chi(H > 0)$. Then f is well defined and continuous because for $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ one actually has $\chi(H > 0) = \chi(\mathcal{F}(H) > 0) \in \mathbb{P}(\mathcal{H})$ as $\mathcal{F}(H) \in \mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ has positive and negative essential spectrum. This map is a homotopy equivalence with homotopy inverse given by the map $g : (\mathbb{P}(\mathcal{H}), \mathcal{O}_N) \to (\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ defined by $g(P) = (2a + 1)(2P - \mathbf{1})$. Clearly, $f \circ g$ is the identity on $\mathbb{P}(\mathcal{H})$. And $g \circ f$ is homotopic to the identity on $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ via the homotopy $h : \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}) \times [0, 1] \to \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ defined by

$$h(H, t) = \mathcal{F}^{-1}\big(t\mathcal{F}(H) + (1 - t)\mathcal{F}\big((2a + 1)(2\chi(H > 0) - \mathbf{1})\big)\big),$$

where the argument in $\mathcal{F}^{-1}$ has no eigenvalues at ±1 by spectral calculus. By the spectral mapping theorem,

$$\mathrm{spec}(\mathcal{F}(H)) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset$$

and

$$\operatorname{spec}(\mathcal{F}((2a+1)(2\chi(H>0)-\mathbf{1}))) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset.$$

Therefore and as $\chi(H>0) = \chi(\mathcal{F}(H)>0)$,

$$\operatorname{spec}\Big(t\mathcal{F}(H) + (1-t)\mathcal{F}((2a+1)(2\chi(H>0)-\mathbf{1}))\Big) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset$$

for all $t \in [0,1]$. Then, again by the spectral mapping theorem, $h(H,t)$ is indeed in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ for all $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ and $t \in [0,1]$. Thus h is well defined. By Proposition 6.2.17, h is continuous with respect to $\mathcal{O}_R$. Because $h(H,0) = (2a+1)(2\chi(H>0)-\mathbf{1}) = (g \circ f)(H)$ and $h(H,1) = H$ for all $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$, one can conclude that f is a homotopy equivalence with homotopy inverse g. By Proposition 8.5.5, $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is contractible and combined with the above this concludes the proof. □

Next it is shown that the space $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ is contractible where for $a \geq 0$,

$$\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H}) : \operatorname{spec}(H) \cap [-a,a] = \emptyset\}$$

denotes the set of operators in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ with spectral gap around 0 of size a. The argument is based on the following fact.

Proposition 8.6.5. *The map* $f : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}(\mathcal{H}), \mathcal{O}_N)$ *given by* $f(H) = H^{-1}$ *is continuous. It provides a homeomorphism*

$$f_0 : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N),$$

where $\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) = \{H \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H}) : H \text{ injective}\}$ *denotes the set of bounded self-adjoint injective operators. For* $a > 0$*, the restriction of* f *gives another homeomorphism*

$$f_a : (\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N),$$

where $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}) = \{H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) : \|H\| < a^{-1}\}$.

Proof. For any $H \in \mathbb{F}_{\mathrm{sa},0}(\mathcal{H})$, the bounded linear map $g : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ defined by $g(\phi, \psi) = (\psi, \phi)$ maps the graph of H onto the graph of its inverse H^{-1}. This implies that the map $\tilde{f} : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}(\mathcal{H}), \mathcal{O}_G)$ defined by $\tilde{f}(H) = H^{-1}$ is continuous. However, $\mathcal{O}_G$ and $\mathcal{O}_N$ coincide on $\mathbb{B}(\mathcal{H})$ by Theorem 6.1.10 and therefore f is continuous. Because $H \in \mathbb{F}_{\mathrm{sa},0}(\mathcal{H})$ has a spectral gap containing 0, one has $H^{-1} \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. For $H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$ its inverse H^{-1} is, moreover, densely defined, closed, and symmetric. As $\operatorname{Ran}(H^{-1}) = \mathcal{H}$, this implies that H^{-1} is self-adjoint. Thus f_0 is a homeomorphism. By the spectral radius theorem, $H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$ has norm less than a^{-1} if and only if $\operatorname{spec}(H) \subset (-a^{-1}, a^{-1})$. By the spectral mapping theorem, this is equivalent to the property $\operatorname{spec}(H^{-1}) \cap [-a,a] = \emptyset$. Thus also f_a is a homeomorphism. □

Proposition 8.6.6. *The space* $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ *is contractible for all* $a \geq 0$.

Proof. Let us first focus on the case $a = 0$. By Proposition 8.6.5, it is sufficient to show that $(\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N)$ is contractible. Let $K \in \mathbb{K}(\mathcal{H})$ be a positive semidefinite injective compact operator with $\|K\| < 1$, as already used in the proof of Proposition 6.4.7. Then

$$h_1 : \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) \times [0,1] \to \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \quad h_1(H,t) = \big((1-t)\mathbf{1} + tK\big)H\big((1-t)\mathbf{1} + tK\big),$$

is well defined because $(1-t)\mathbf{1} + tK$ is a convex combination of positive semidefinite injective operators and therefore injective for all $t \in [0,1]$. Thus $h_1(H,t) \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. Clearly, h_1 is a norm-continuous homotopy such that $h_1(H,0) = H$ and $h_1(H,1) = KHK$ is an injective compact operator lying in $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) = \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) \cap \mathbb{K}(\mathcal{H}) \subset \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. It is thus sufficient to show that $(\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N)$ is contractible. To do so, let us identify $\mathcal{H}$ with $L^2([0,1])$, but suppress the unitary in the following. For $t \in (0,1]$, let us consider (inspired by [73] and as in the proof of Proposition 6.4.16) the partial isometry

$$V_t(\phi)(x) = \begin{cases} t^{-\frac{1}{2}}\phi(\frac{x}{t}), & \text{for } x \le t, \\ 0, & \text{for } x > t. \end{cases}$$

Similarly, for $t \in [0,1)$,

$$W_t(\phi)(x) = \begin{cases} 0, & \text{for } x \le t, \\ (1-t)^{-\frac{1}{2}}\phi(\frac{x-t}{1-t}), & \text{for } x > t, \end{cases}$$

is also a partial isometry that is complementary to V_t. Clearly, V_t, V_t^*, W_t, and W_t^* continuously depend on t in the strong operator topology. Moreover, the projections $P_t = V_t V_t^*$ and $Q_t = W_t W_t^*$ fulfill

$$s\text{-}\lim_{t\to 0} P_t = 0, \quad s\text{-}\lim_{t\to 1} Q_t = 0,$$

as well as

$$P_1 = V_1 = \mathbf{1} = W_0 = Q_0, \quad P_t + Q_t = \mathbf{1}. \tag{8.12}$$

For $H_0 \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$, we define the homotopy $h_2 : \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) \times [0,1] \to \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ by

$$h_2(H,t) = \begin{cases} H_0, & \text{for } t = 0, \\ tV_tHV_t^* + (1-t)W_tH_0W_t^*, & \text{for } t \in (0,1), \\ H, & \text{for } t = 1. \end{cases}$$

Clearly, $h_2(H,t)$ is self-adjoint and compact. Moreover, $V_tHV_t^*$ is an injective operator on $\mathrm{Ran}(P_t)$ and $W_tH_0W_t^*$ is an injective operator on $\mathrm{Ran}(Q_t)$ by (8.12) and therefore $h_2(H,t) \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ so that h_2 is well defined. Since H and H_0 are compact and the maps $t \in (0,1) \mapsto V_t$ and $t \in (0,1) \mapsto W_t$ are strongly continuous, h_2 is norm-continuous on $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) \times (0,1)$. Moreover, for every $H, \tilde{H} \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ and $t \in (0,1]$,

$$\begin{aligned}\|h_2(H,1)-h_2(\tilde{H},t)\| &\le \|H-h_2(H,t)\|+\|h_2(H,t)-h_2(\tilde{H},t)\|\\ &\le \|H-tV_tHV_t^*\|+(1-t)\|W_tH_0W_t^*\|+\|H-\tilde{H}\|.\end{aligned}$$

As $\lim_{t\to 1}\|H-tV_tHV_t^*\| = 0$, this implies that h_2 is continuous at all points $(H,1)$ for $H \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$. Similarly, one shows that h_2 is continuous at all points $(H,0)$ for $H \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$. Thus h_2 is continuous on the whole domain $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})\times[0,1]$ and therefore $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ is contractible. Thus $(\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}),\mathcal{O}_N)$ is contractible and the claim on $(\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}),\mathcal{O}_G)$ follows from Proposition 8.6.5.

For $a>0$, the homotopy h_1 defined as above maps $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H})\times[0,1]$ to the set $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H})$. Furthermore $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H})\times\{1\}$ is mapped to $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})=\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H})\cap\mathbb{K}(\mathcal{H})$. For $H_0\in\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})$ the homotopy h_2 maps $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})\times[0,1]$ to $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})$ because $tV_tHV_t^*\in\mathbb{K}_{a,\mathrm{sa,inj}}(\mathrm{Ran}(P_t))$ and $(1-t)W_tH_0W_t^*\in\mathbb{K}_{a,\mathrm{sa,inj}}(\mathrm{Ran}(Q_t))$. Therefore the same argument as for the case $a=0$ shows that $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}),\mathcal{O}_G)$ is contractible for all $a\ge 0$. ☐

Now all is prepared to complete the

Proof of Theorem 8.6.3. The proof is based on Theorem A.3.3. Note that both spaces $(\mathbb{F}_{\mathrm{sa}}^*(\mathcal{H}),\mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_G)$ are metric and therefore paracompact. Thus every open covering of these spaces is numerable. Let $\mathcal{T}$ denote the set of all finite symmetric (with respect to 0) nonempty subsets of $\mathbb{R}$, such as $\tau=\{-a,a\}$ and $\tau=\{-a,-b,b,a\}$ for $a>b>0$. For $\tau\in\mathcal{T}$, let $\overline{\tau}$ denote the convex hull of τ which is a closed symmetric interval in $\mathbb{R}$. Then

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})=\{H\in\mathbb{F}_{\mathrm{sa}}(\mathcal{H}):\mathrm{spec}(H)\cap\tau=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\overline{\tau}\}$$

is open in the gap topology $\mathcal{O}_G$ because

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})=\mathcal{F}^{-1}(\{H\in\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H}):\mathrm{spec}(H)\cap\mathcal{F}(\tau)=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\mathcal{F}(\overline{\tau})\}),$$

$\mathcal{F}:(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_G)\to(\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H}),\mathcal{O}_E)$ is continuous by Corollary 6.3.4 and, moreover, $\{H\in\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H}):\mathrm{spec}(H)\cap\mathcal{F}(\tau)=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\mathcal{F}(\overline{\tau})\}$ is an open subset of $\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H})$ with respect to the extended gap topology by the spectral mapping theorem and because τ is symmetric. Thus $(\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}))_{\tau\in\mathcal{T}}$ is an open and, by the above, numerable covering of $(\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}),\mathcal{O}_G)$. On the other hand,

$$\mathbb{F}_{\mathrm{sa},\tau}^*(\mathcal{H})=\{H\in\mathbb{F}_{\mathrm{sa}}^*(\mathcal{H}):\mathrm{spec}(H)\cap\tau=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\overline{\tau}\}$$

is open in the Riesz topology $\mathcal{O}_R$ because

$$\mathbb{F}_{\mathrm{sa},\tau}^*(\mathcal{H})=\mathcal{F}^{-1}(\{H\in\mathbb{FB}_{1,\mathrm{sa}}^{*,0}(\mathcal{H}):\mathrm{spec}(H)\cap\mathcal{F}(\tau)=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\mathcal{F}(\overline{\tau})\}),$$

the bounded transform $\mathcal{F}:(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_R)\to(\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H}),\mathcal{O}_N)$ is continuous by Corollary 6.3.2 and $\{H\in\mathbb{FB}_{1,\mathrm{sa}}^{*,0}(\mathcal{H}):\mathrm{spec}(H)\cap\mathcal{F}(\tau)=\emptyset=\mathrm{spec}_{\mathrm{ess}}(H)\cap\mathcal{F}(\overline{\tau})\}$ is an open subset of $\mathbb{FB}_{1,\mathrm{sa}}^0(\mathcal{H})$ with respect to the norm topology. Thus $(\mathbb{F}_{\mathrm{sa},\tau}^*(\mathcal{H}))_{\tau\in\mathcal{T}}$ is an open and, by the above, numerable covering of $(\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R)$. For $\tau, \tau' \in \mathcal{T}$ one clearly has $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \cap \mathbb{F}^*_{\mathrm{sa},\tau'}(\mathcal{H}) = \mathbb{F}^*_{\mathrm{sa},\tau\cup\tau'}(\mathcal{H})$ and $\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) \cap \mathbb{F}_{\mathrm{sa},\tau'}(\mathcal{H}) = \mathbb{F}_{\mathrm{sa},\tau\cup\tau'}(\mathcal{H})$ with $\tau \cup \tau' \in \mathcal{T}$. Moreover, $I(\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H})) \subset \mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})$. Thus, by Theorem A.3.3, it is sufficient to show that the embedding $I_\tau : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G)$ is a homotopy equivalence for every fixed $\tau \in \mathcal{T}$.

Let $\mathbb{P}_{\mathrm{fin}}(\mathcal{H}) = \{P = P^* = P^2 \in \mathbb{B}(\mathcal{H}) : \dim(\mathrm{Ran}(P)) < \infty\}$ denote the set of finite-dimensional orthogonal projections on $\mathcal{H}$. Then $\pi_R : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{P}_{\mathrm{fin}}(\mathcal{H}), \mathcal{O}_N)$ defined by

$$\pi_R(H) = \chi_{\overline{\tau}}(H)$$

is the base projection of a fiber bundle with fiber over $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ given by

$$\mathbb{F}^*_{\mathrm{sa},\tau,P}(\mathcal{H}) = \pi_R^{-1}(P) = \{H \in \mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) : \chi_{\overline{\tau}}(H) = P\}.$$

Similarly, $\pi_G : (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{P}_{\mathrm{fin}}(\mathcal{H}), \mathcal{O}_N)$ defined by

$$\pi_G(H) = \chi_{\overline{\tau}}(H)$$

is the base projection of a fiber bundle with fiber over $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ given by

$$\mathbb{F}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \pi_G^{-1}(P) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) : \chi_{\overline{\tau}}(H) = P\}.$$

Both fiber bundles are locally trivial in the sense that, by Lemma 8.3.3, for $P_0 \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ there is a continuous map $\{P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) : \|P - P_0\| < 1\} \to \mathbb{U}(\mathcal{H})$, $P \mapsto V_P$, such that $P_0 = V_P^* P V_P$. Let

$$\mathcal{H} \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) = \mathcal{H}' \oplus \mathcal{H}''$$

be the canonical decomposition of the trivial Hilbert bundle over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ into the direct sum of two vector bundles, whose fibers are $\mathcal{H}'_P = \mathrm{Ran}(P)$ and $\mathcal{H}''_P = \mathrm{Ker}(P)$. Let $\mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ be the fiber bundles over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ associated with $\mathcal{H}'$, respectively $\mathcal{H}''$, with fibers given by

$$\mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathrm{Ran}(P)) : \mathrm{spec}(H) \subset \overline{\tau} \setminus \tau\}$$

and

$$\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}^*_{\mathrm{sa},\tau}(\mathrm{Ker}(P)) : \mathrm{spec}(H) \cap \overline{\tau} = \emptyset\},$$

where both fibers are equipped with the Riesz topology. By Lemma 8.3.3, these fiber bundles are again locally trivial. Then taking fiberwise the direct sums, $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H})$ can be seen as fiber product bundle over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ of the form

$$\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) = \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \times_{\mathbb{P}_{\mathrm{fin}}(\mathcal{H})} \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$$

with bundle maps $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ given by the restriction $H \mapsto H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))}$ and $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ given by $H \mapsto H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))}$. In exactly the same way, one can view $\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})$ as a fiber product

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) = \mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H}) \times_{\mathbb{P}_{\mathrm{fin}}(\mathcal{H})} \mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H}),$$

where $\mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H})$ are the fiber bundles over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ with fibers given by $\mathbb{F}'_{\mathrm{sa},\tau,P}(\mathcal{H}) = \mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H})$ and

$$\mathbb{F}''_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathrm{Ker}(P)) : \mathrm{spec}(H) \cap \overline{\tau} = \emptyset\},$$

where both fibers are equipped with the gap topology. Now the embedding $I_\tau : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G)$ is a product of the maps

$$I'_\tau : \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H}), \quad H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))} \mapsto H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))}$$

and

$$I''_\tau : \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H}), \quad H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))} \mapsto H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))}.$$

On $\mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \mathbb{F}'_{\mathrm{sa},\tau,P}(\mathcal{H})$, the Riesz topology and the gap topology coincide with the norm topology by Theorem 6.1.10, therefore I'_τ is a homeomorphism.

To show that I''_τ is a homotopy equivalence note that the Hilbert bundle $\mathcal{H}''$ over the (metric and thus) paracompact space $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ has infinite-dimensional separable fibers. Thus by Theorem A.3.12, it is a trivial Hilbert bundle. The trivialization map can be chosen to be unitary, namely there is a norm-continuous map $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \mapsto W_P \in \mathbb{B}(\mathcal{H})$ where W_P is a partial isometry with $\mathrm{Ker}(W_P) = \mathrm{Ran}(P)$ and $\mathrm{Ran}(W_P) = \mathcal{H}$. Then the map $(\phi, P) \in \mathcal{H}'' \mapsto (W_p\phi, P) \in \mathcal{H} \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ is a trivialization of $\mathcal{H}''$. Therefore the fiber bundles $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H})$ over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ are also trivial, namely isomorphic to the trivial bundles $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$, respectively $\mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$, via the trivalization maps $A \in \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) \mapsto (W_P A W_P^*, P) \in \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ and $A \in \mathbb{F}''_{\mathrm{sa},\tau,P}(\mathcal{H}) \mapsto (W_P A W_P^*, P) \in \mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$. After this, isomorphism I''_τ transposes to $I''_{\tau,0} : \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ simply given by $(H, P) \mapsto (H, P)$. As $\overline{\tau} = [-a, a]$ for some $a \geq 0$, one gets $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) = \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) = \mathbb{F}_{\mathrm{sa},a}(\mathcal{H})$. By Propositions 8.6.4 and 8.6.6, the spaces $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ are contractible. This implies that I''_τ is a homotopy equivalence and therefore I_τ is a homotopy equivalence as it is the product of two homotopy equivalences. This concludes the argument. □

Remark 8.6.7. Let us point out that the proof of Theorem 8.4.1 merely uses that the fundamental group of $(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ is infinitely cyclic. Thus, by the results of this section, Theorem 8.4.1 also holds for $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$. ◇

8.7 Resumé: homotopy equivalences of operator classes

For the convenience of the reader, this section summarizes various of the homotopy equivalences of sets of Fredholm operators proved in this and earlier chapters. Let us begin with a diagram for self-adjoint Fredholm operators. It is quite extended, even though not all results proved in this book are included. For sake of compactness of the presentation, we drop the specification of the Hilbert space $\mathcal{H}$.

$$\begin{tikzcd}
(\mathbb{F}\mathbb{B}_{\mathrm{sa}}, \mathcal{O}_N) \arrow[r, leftrightarrow, "\text{retr.}", "3.6.2"'] & (\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}, \mathcal{O}_N) \arrow[r, leftrightarrow, "\mathcal{F}", "6.3.2"'] & (\mathbb{F}_{\mathrm{sa}}, \mathcal{O}_R) \arrow[r, leftrightarrow, "i", "6.3.3"'] & (\mathbb{F}\mathbb{B}_{\mathrm{sa}}, \mathcal{O}_N) \\
(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}, \mathcal{O}_N) \arrow[u, hook, "\text{incl.}"'] \arrow[d, leftrightarrow, "6.3.3"', "i"] & & & \\
(\mathbb{F}^*_{\mathrm{sa}}, \mathcal{O}_R) \arrow[r, leftrightarrow, "\mathcal{F}", "6.3.2"'] \arrow[d, leftrightarrow, "I"', "8.6.3"] & (\mathbb{F}\mathbb{B}^{*,0}_{1,\mathrm{sa}}, \mathcal{O}_N) & & \\
(\mathbb{F}_{\mathrm{sa}}, \mathcal{O}_G) \arrow[r, leftrightarrow, "\mathcal{F}", "6.3.4"'] \arrow[d, leftrightarrow, "i"', "6.4.2"] & (\mathbb{F}\mathbb{B}^{0}_{1,\mathrm{sa}}, \mathcal{O}_E) \arrow[r, leftrightarrow, "\mathrm{id}", "6.4.6"'] & (\mathbb{F}\mathbb{B}^{0}_{1,\mathrm{sa}}, \mathcal{O}_{SE}) \arrow[r, leftrightarrow, "\mathcal{G}", "4.6.12"'] & (\mathbb{F}\mathbb{U}^0, \mathcal{O}_N) \\
(\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}, \mathcal{O}_G) \arrow[r, leftrightarrow, "\mathcal{F}", "6.4.4"'] & (\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\mathrm{sa}}, \mathcal{O}_E) \arrow[r, leftrightarrow, "\mathrm{id}", "6.4.6"'] & (\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\mathrm{sa}}, \mathcal{O}_{SE}) \arrow[r, leftrightarrow, "\mathcal{G}", "4.6.15"'] \arrow[d, leftrightarrow, "\mathrm{id}"', "6.4.7"] & (\mathbb{U}^{\mathrm{C},0}, \mathcal{O}_N) \\
 & (\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}, \mathcal{O}_K) \arrow[r, leftrightarrow, "\mathrm{id}", "6.4.15"'] & (\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}, \mathcal{O}_{SE}) \arrow[r, leftrightarrow, "\text{retr.}", "6.4.12"'] \arrow[d, leftrightarrow, "\text{retr.}"', "6.4.16"] & (\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}, \mathcal{O}_{SE}) \\
 & & (\mathbb{F}\mathbb{B}^{\mathrm{C},\infty}_{1,\mathrm{sa}}, \mathcal{O}_{SE}) &
\end{tikzcd}$$

The diagram splits in the top row and the rest. However, they are tightly connected as it was shown in Proposition 3.6.1 that $\mathbb{F}\mathbb{B}_{\mathrm{sa}} = \mathbb{F}\mathbb{B}^*_{\mathrm{sa}} \cup \mathbb{F}\mathbb{B}^+_{\mathrm{sa}} \cup \mathbb{F}\mathbb{B}^-_{\mathrm{sa}}$ is a disjoint union in which the two components $\mathbb{F}\mathbb{B}^{\pm}_{\mathrm{sa}}$ are contractible. Hence the nontrivial part of the higher homotopy groups (of degree greater or equal to $k = 1$) of the upper row stems from the component $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}$, namely the lower part of the diagram. These homotopy groups have been computed in Section 8.3. Let us also note that many of the homotopy equivalences in the diagram also hold for the operator sets without Fredholm properties. The corresponding statements can always be found near by those on the Fredholm operators.

Next let us come to the set of (not necessarily self-adjoint) Fredholm operators. The results are summarize as follows:

$$\begin{tikzcd}
(\mathbb{FB}(\mathcal{H}), \mathcal{O}_N) \arrow[r, leftrightarrow, "i", "6.2.18"'] & (\mathbb{F}(\mathcal{H}), \mathcal{O}_R) \arrow[r, leftrightarrow, "\mathcal{F}", "6.2.17"'] \arrow[d, hook, "\mathrm{id}"'] & (\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_N) \arrow[r, leftrightarrow, "f", "6.4.7"'] & (\mathbb{FB}_1(\mathcal{H}), \mathcal{O}_N) \\
 & (\mathbb{F}(\mathcal{H}), \mathcal{O}_G) \arrow[r, leftrightarrow, "\mathcal{F}", "6.2.17"'] & (\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_E). &
\end{tikzcd}$$

Here the two-sided errors designate homotopy equivalences by the maps on top of them and the corresponding statement below them, while the hook error is a homotopy equivalence by [154]. Hence the homotopy groups of all spaces are given in Section 8.2.