The Structure of Fusion Categories via Topological Quantum Field Theories

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Joint with Christopher Douglas and Noah Snyder
Duality: Adjoint Functors

Definition

An adjunction is a pair of functors

\[ F : C \leftrightarrow D : G \]

and a natural bijection

\[ \text{Hom}_D(Fx, y) \cong \text{Hom}_C(x, Gy). \]

\( F \) is left adjoint to \( G \).
Equivalent Formulation

\[ \text{Hom}_D(Fx, Fx) \cong \text{Hom}_C(x, GFx) \quad \text{Hom}_C(Gy, Gy) \cong \text{Hom}_D(FGy, y) \]

\[ id_{Fx} \mapsto (\eta_x : x \to GFx) \quad \text{id}_{Gy} \mapsto (\varepsilon_y : FGy \to y) \]
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\[ \text{id}_{Fx} \mapsto (\eta_x : x \rightarrow GFx) \quad \text{id}_{Gy} \mapsto (\varepsilon_y : FGy \rightarrow y) \]

Natural transformations...

**unit** \( \eta : \text{id}_C \rightarrow GF \)  
**counit** \( \varepsilon : FG \rightarrow \text{id}_D \)

Satisfying equations...

\[
\begin{align*}
F \xrightarrow{1 \ast \eta} & \quad FGF \xrightarrow{\varepsilon \ast 1} F = F \xrightarrow{id} F \\
G \xrightarrow{\eta \ast 1} & \quad GFG \xrightarrow{1 \ast \varepsilon} G = G \xrightarrow{id} G
\end{align*}
\]
Definition
An adjunction is a pair of 1-morphisms

\[ F : C \leftrightarrow D : G \]

and 2-morphisms

\[ \eta : id_C \to GF \quad \varepsilon : FG \to id_D \]

satisfying ‘Zig-Zag’ equations:

\[ F \xrightarrow{1 \ast \eta} FGF \xrightarrow{\varepsilon \ast 1} F = F \xrightarrow{id} F \]

\[ G \xrightarrow{\eta \ast 1} GFG \xrightarrow{1 \ast \varepsilon} G = G \xrightarrow{id} G \]
Higher Category Theory

Use the theory of $(\infty, n)$-categories.

Generalizes both topological spaces and categories.

Hueristically:
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Hueristically:

- Objects: $a, b, c, \ldots$

- 1-morphisms $f, g, h, \ldots$

- 2-morphisms, 3-morphisms, etc. (invertible above $n$)

- Compositions...
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\[ \begin{array}{ccc}
\downarrow & \quad f \\
\ast & \quad \ast \\
\downarrow & \quad g \\
\ast & \quad \ast
\end{array} \]

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Higher Category Theory

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$\Rightarrow$

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Examples of $(\infty, n)$-categories

Example

$\text{Cat}$ the 2-category of small categories.
More generally, any bicategory.

Example ($\text{Spaces} = (\infty, 0)$-categories)

$X$ a space

- objects = points of $X$
- 1-morphisms = paths in $X$
- 2-morphisms = paths between paths
- etc.
Duality in any bicategory

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Monoidal Category \((M, \otimes)\) \(\rightarrow\) one object bicategory \(BM\).

- 1-morphisms = objects of \(M\)  
  composition given by \(\otimes\)
- 2-morphism = morphisms of \(M\)
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**Dual objects** in \(M \leftrightarrow\) dual 1-morphisms in \(BM\): \(x, x^*, \) and...

\[
\eta : 1 \to x \otimes x^* \quad \text{evaluation} \quad \epsilon : x^* \otimes x \to 1
\]

satisfying ‘Zig-Zag’ equations.
Example

Monoidal Category $(M, \otimes) \leadsto$ one object bicategory $BM$.

- 1-morphisms $= \text{objects of } M$
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Dual objects in $M \leftrightarrow$ dual 1-morphisms in $BM$: $x, x^*$, and...

  coevaluation $\eta : 1 \rightarrow x \otimes x^*$  evaluation $\varepsilon : x^* \otimes x \rightarrow 1$

satisfying ‘Zig-Zag’ equations.

Example

$M = Vect$, a vector space $x$ is dualizable $\iff x$ is finite dimensional
Fusion Categories

Definition

A Fusion Category is a monoidal semi-simple $k$-linear category, with
- finitely many isom. classes of simples,
- $\text{End}(1) \cong k$,
- left and right duals for all objects.

For simplicity, $k = \mathbb{C}$.

Sources of Fusion Categories:
- Quantum Groups
- Operator Algebras
- Conformal Field Theory
- Representations of Loop Groups
Theorem (Etingof-Nikshych-Ostrik)

In any Fusion category, the functor

\[ X \mapsto X^{****} \]

is canonically monoidally equivalent to id.
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In any Fusion category, the functor

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is canonically monoidally equivalent to \( \text{id} \).

Why?
Definition
A fusion category is **pivotal** if it admits a **pivotal structure**, i.e. a natural monoidal isomorphism $X \cong X^{**}$.

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A fusion category is **spherical** if it admits a pivotal structure compatible with canonical $X \cong X^{****}$. 
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Manifold Invariants

Locality of manifold invariants: Reconstruct $Z_W$ from $Z_{W_1}$ and $Z_{W_2}$?

$Z_W = \langle Z_{W_1}, Z_{W_2} \rangle$
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$$Z_W = \langle Z_{W_1}, Z_{W_2} \rangle$$
Objects are closed compact $(d - 1)$-manifolds $Y$ with germ of $d$-manifold.
The Cobordism Category

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- Morphisms are compact \(d\)-manifolds \(W\), with \(\partial W = Y_1 \sqcup Y_2\) up to equivalence.
The Cobordism Category

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variants:
- extra structures: orientations, spin structures, etc
- higher categories of cobordisms
Topological Quantum Field Theories

Definition
A TQFT is a symmetric monoidal functor:

\[ \text{Bord} \rightarrow \text{C} \]

\[ \text{Cobordism Category} \rightarrow \text{Target Category} \]

M. Atiyah

G. Segal
Definition
A TQFT is a symmetric monoidal functor:

\[ \text{Bord} \to \text{C} \]

Cobordism Category \hspace{1cm} Target Category

\[ \emptyset \in \text{Bord} \mapsto 1 \in \text{C} \]

\[ M \text{ closed} \mapsto (1 \xrightarrow{Z_M} 1) \]

M. Atiyah

G. Segal
Distinguishing Manifolds?

- 0D, 1D, and 2D TFTs distinguish manifolds.
- 4D (unitary) TFTs cannot detect smooth structures. [Freedman-Kitaev-Nayak-Slingerland-Walker-Wang]
- 5D (unitary) TFTs can detect, if $\pi_1 = 0$. [Kreck-Teichner]
- $\geq 6$D (unitary) TFTs cannot detect homotopy type. [Kreck-Teichner]

Open Problem: Can 3D TFTs distinguish 3-manifolds?  
Evidence suggest “yes?”. [Calegari-Freedman-Walker]
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Input:

$C$ a Spherical Category

- triangulate your 3-manifold
- Label using data from $C$
- Weighted average over all labelings gives invariant.
Turaev-Viro-Barrett-Westbury Construction: a 3D TQFT

Input:

$C$ a Spherical Category

- triangulate your 3-manifold
- Label using data from $C$
- Weighted average over all labelings gives invariant.

In 2010...

Theorem (Turaev-Virelizier, Balsam-Kirillov)

This gives a tqft which is local down to 1-manifolds.
Theorem (Douglas-SP-Snyder)

*Fusion, Pivotal, and Spherical Categories all give rise to fully local extended 3D TQFTs.*

Moreover the *structure* of the TQFTs reflects the structure of fusion categories.
A manifold $M$ has a tangent bundle $\tau$ classified by a map

$$M \xrightarrow{\tau} BO(n)$$
Tangential Structures on Manifolds

A manifold $M$ has a tangent bundle $\tau$ classified by a map $G \to O(n)$

$BO(n)$

- $G = SO(n) \leadsto$ Orientation
- $G = Spin(n)$ (universal cover of $SO(n)$) $\leadsto$ Spin structure
- $G = 1 \leadsto$ framing
- etc
different sorts of fusion categories give different tqfts.

**Theorem (Douglas-SP-Snyder)**

<table>
<thead>
<tr>
<th>$G$</th>
<th>name of structure</th>
<th>kind of category</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(3)^{†}$</td>
<td>Orientation</td>
<td>Spherical</td>
</tr>
<tr>
<td>$SO(2)$</td>
<td>Combing</td>
<td>Pivotal</td>
</tr>
<tr>
<td>$1 = SO(1)$</td>
<td>Framing</td>
<td>Fusion</td>
</tr>
</tbody>
</table>

$^{†}$ This group might change slightly.
Theorem (Folklore)

The category of (non-local) oriented 2D tqfts in $C$ is equivalent to category of commutative Frobenius algebras in $C$.

[R. Dijkgraaf, L. Abrams, S. Sawin, B. Dubrovin, Moore-Segal, ...]
**1D TQFTs**

**Theorem (1D Cobordism Hypothesis)**

The category of 1D oriented tqfts in $C$ is equivalent to the groupoid of dualizable objects of $C$, denoted $k(C^{fd})$

**Zig-Zag equations:**

\[
\begin{align*}
\text{coevaluation} & \quad \subseteq_+ \quad \text{evaluation} \quad +\subseteq \\
\end{align*}
\]

\[
\begin{align*}
F \xrightarrow{1 \ast \eta} FGF \xrightarrow{\varepsilon \ast 1} F = F \xrightarrow{id} F \\
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\end{align*}
\]
Like 1D tqfts, but with 2D bordisms too.

- Objects (0-manifolds) have duals
2D Local TQFTs

Like 1D tqfts, but with 2D bordisms too.

- Objects (0-manifolds) have duals
- 1-morphisms (1-manifolds) also have duals

Zig-Zag Equation:
Layers of dualizability

3-Cat
Layers of dualizability

3-Cat \rightarrow \text{dual 2-mor}
Layers of dualizability

3-Cat

<table>
<thead>
<tr>
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Layers of dualizability

monoidal 3-Cat

---

dual 2-mor

dual 1-mor

dual objects
Fully-dualizable is dualizable on all levels:

**Definition**

If $C$ is a symmetric monoidal $n$-category, there is a filtration

$$C^{fd} = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{n-1} \subseteq C$$

where $C_i = $ the maximal sub-$n$-category where $j$-morphisms have both duals if $i \leq j \leq n - 1$. 
“Bord$_n$ is the free symmetric monoidal $n$-category with duality”

Theorem (Hopkins-Lurie)

$\text{Fun}(\text{Bord}_n^{fr}, C) \simeq k(C^{fd})$
Theorem (Douglas-SP-Snyder)

Fusion categories are fully-dualizable objects in the symmetric monoidal 3-category $TC$. (Tensor Categories)
Theorem (Douglas-SP-Snyder)

*Fusion categories are fully-dualizable objects in the symmetric monoidal 3-category $\mathcal{T}C$. (Tensor Categories)*

Corollary

*Fusion categories give rise to fully-local extended 3D tqfts.*

**What is $\mathcal{T}C$?**
The 3-category of Tensor Categories

Example

Algebras, Bimodules, Bimodule maps = a (monoidal) 2-category
The 3-category of Tensor Categories

Example

Algebras, Bimodules, Bimodule maps = a (monoidal) 2-category

Definition

$TC =$

- objects: Tensor Categories (monoidal $k$-linear)
- 1-morphisms: Bimodule Categories
- 2-morphisms and 3-morphisms: Bimodule Functors and Bimodule Natural Transformations

Monoidal for Deligne tensor product.

Proposition (Douglas-SP-Snyder)

$TC$ is a symmetric monoidal $(\infty, 3)$-category.
A Basic Principle

If $G$ acts on $B$, then $G$ acts on $\text{Map}(B, C)$. 
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$O(3)$ acts on $\text{Bord}_{3}^{fr}$ by change of framing.

\[ O(3) \rightarrow \text{Aut}(k(C^{fd})) \]
A Basic Principle and a Theorem

If $G$ acts on $B$, then $G$ acts on $\text{Map}(B, C)$.

$O(3)$ acts on $\text{Bord}_3^{fr}$ by change of framing.

$$O(3) \to \text{Aut}(k(C^{fd}))$$

**Theorem (Hopkins-Lurie)**

$$\text{Fun}(\text{Bord}_n^G, C) \simeq [k(C^{fd})]^{hG}.$$
So $O(3)$ acts on the “space” of fusion categories. 
What is the action?
So $O(3)$ acts on the “space” of fusion categories. What is the action?

- points in $O(3) \rightsquigarrow$ self-equivalences $k(C^{fd}) \to k(C^{fd})$
So $O(3)$ acts on the “space” of fusion categories.  

**What is the action?**

- Points in $O(3) \rightsquigarrow$ self-equivalences $k(C^{fd}) \rightarrow k(C^{fd})$
- Paths in $O(3) \rightsquigarrow$ natural isomorphisms
- Paths between paths in $O(3) \rightsquigarrow$ natural 2-isomorphism
- Etc.
In more detail...

- $\pi_0 O(3) = \mathbb{Z}/2$, non-trivial element: $(F, \otimes) \mapsto (F, \otimes^{op})$.
- $\pi_1 O(3)$ gives the *Serre automorphism* (natural automorphism of identity functor).
In more detail...

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  in components

  $$S_F : F \to F$$

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  in components

  $$S_F : F \rightarrow F$$

  is an invertible $F$-$F$-bimodule category.

- $\pi_2 O(3) = 0$
- $\pi_3 O(3) = \mathbb{Z}$ gives the anomaly. $\sim a_F \in \mathbb{C}^\times$

No other data since $TC$ is just a 3-category.
Theorem (Douglas-SP-Snyder)

The Serre Automorphism of a fusion category $F$ is the bimodulification of

$$(F, \otimes) \rightarrow (F, \otimes)$$

$x \mapsto x^{**}$

$
\pi_1 O(3) \cong \mathbb{Z}/2 \Rightarrow \text{square of the Serre is trivial!}
$

Corollary

The bimodulification of $x \mapsto x^{****}$ is trivial.
Some 3D structure groups

1

Spin(3) \rightarrow Spin^c(3) \rightarrow SO(3) \rightarrow O(3)

SO(2) \rightarrow Orpo

\begin{align*}
&\text{kill } w_2 \\
&\text{kill } \beta w_2 \\
&\text{kill } p_1 \\
&\text{kill } w_1
\end{align*}
Some 3D structure groups

\[ \text{Spin}(3) \quad \text{Spin}^c(3) \quad \text{SO}(3) \quad \text{SO}(2) \quad \text{Orpo} \quad \text{O}(3) \]

1

[framing]

[combing]

[Atiyah 2-framing]

if our TQFT agrees with Turaev-Viro Conjecture: anomaly vanishes

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\downarrow & \\
\text{O}(3) & \\
\end{array}
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- fusion
- pivotal
- spherical

if our TQFT agrees with Turaev-Viro Conjecture: anomaly vanishes

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Some 3D structure groups

\[ \text{1 fusion} \]

\[ \text{Spin}(3) \quad \text{SO}(2) \quad \text{Orpo} \]

\[ \text{Spin}^c(3) \quad \text{spherical} \]

\[ \text{SO}(3) \quad \text{pivotal} \]

\[ \text{O}(3) \]

Conjecture: anomaly vanishes

if our TQFT agrees with Turaev-Viro
A new version of ENO conjecture

Conjecture

All framed extended 3D tqfts in TC can be extended to oriented tqfts.

Evidence one dimension lower...

Theorem

All framed extended 2D tqfts in Alg can be extended to oriented tqfts.
The End