Modular tensor categories over number fields

Orit Davidovich
Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

May 20, 2009

DRAFT VERSION ONLY

1 Modular tensor categories over number fields

This talk is meant to offer one perspective on classification of modular tensor categories.

Why should we be interested in this classification?

Theorem 1. *(Turaev) MTC* $\leftrightarrow 3-2-1$ *TQFT e* $\leftrightarrow Z(S^1)$

We use this theorem to discuss key features of modular tensor categories:

The first obvious features are:

- $\mathbb{C}$-linear
- $\otimes$-cat
- semi-simple
- finitely many $\cong$ classes of simples
- braiding
- rigidity (we have duals)
- non-degeneracy
- plus a few more...

$$Z(\text{pants})\colon Z(S^1) \times Z(S^1) \to Z(S^1)$$

Figures of elbows and zig-zag plus zorro's lemma.

Consider a 2-bordism M and a diffeomorphism of M, $f \in Diff(M rel \partial M)$.

NICE FIGURE OF $M \times I$ and identify boundaries by Id and f. Then our TQFT assigns to this a natural isomorphism

$$Z(M \times^f I)\colon Z(M) \xrightarrow{\sim} Z(M)$$

Consider a particular example where M is a pair of pants. Consider a diffeomorphism f that interchanges the two incoming circles. Here we get

$$Z(f) maps \otimes \xrightarrow{\sim} \otimes^{op}$$

I.e., for any pair of objects in $\mathrm{Ob}(\mathcal{C})$,

$$Z(f)_{X,Y} \colon X \otimes Y \xrightarrow{\sim} T \otimes X$$

the braiding.

Given a homotopy between two homeomorphisms $f_1 \to f_2$, we get a diffeomorphism of 3-bordisms and since we are not in the infinity world and quotient by diffeomorphisms, we have $Z(f_1) \cong Z(f_2)$.

1.1 Ocneanu Rigidity

Fixing the Grothendieck ring of a modular tensor category, the number of equivalence classes of modular tensor categories withthis Grothendieck ring is finite.

Goal: We are going to construct new modular tensor categories from old ones fixing the Grothendieck ring.

A theorem that allows us to do that is the following:

Theorem 2. *(Z. Wang, T. Hagge, D-) Given a modular tensor category $\mathcal{C}$ over $\mathbb{C}$, there exists a number field $\mathfrak{k}$ and the modular tensor category $\mathcal{C}_{alg}$ defined over $\mathfrak{k}$ such that*

$$\mathcal{C} \cong \mathcal{C}_{alg} \otimes_{\mathfrak{k}} \mathbb{C}$$

Remark 3.

- *Toen gave is $\otimes$ of cats*
- *End(simple)$\cong \mathfrak{k}$, $\not\vDash$ simple*
- *$\cong$ means isomorphism of modular tensor categories - must respect the monoidal and braiding structure*

Let $\mathcal{C}$ be a category defined over $\mathbb{C}$, and let $\mathcal{C}_{alg}$,$\mathfrak{k}$ as in theorem. Then for $\sigma \in Fal(\mathfrak{k}/\mathbb{C})$

$$\mathcal{C}^{\sigma} := \mathcal{C}_{alg} \otimes^{\sigma}_{\mathfrak{k}} \mathbb{C}.$$

Sketch of proof:

- Use Moore-Seiberg data to construct modular parametrizing modular tensor category with fixed Grothendieck ring
- this is an affine variety defined over $\mathbb{Q}$ equipped with G-action
- G has finitely many orbits in $V^{modular}$ (Ocneanu's rigidity)
- $\mathcal{C}$ locate in a G-orbit and find an algebraic point in that orbit, $\mathcal{C}_{alg}$
- $\mathfrak{k} = \mathbb{Q}(\mathcal{C}_{alg})$

- By construction, $\mathcal{C} \cong \mathcal{C}_{alg} \otimes_{\mathfrak{k}} \mathbb{C}$

Questions:

- How much of the moduli space of modular tensor categories with fixed Grothendieck ring does the Galois action "see"? (Is this action transitive?)
- Can we find interesting examples where we have fixed points? If we can then $Gal(\mathfrak{k}/\mathbb{Q}) \to End(Id_{\mathcal{C}})$.
- $V^{modular} \to V^{fusion}$ What can we say about this cover?
- What can we say about $\mathfrak{k}$? In all known examples $Gal(\mathfrak{k}/\mathbb{Q})$ is abelian. Is this generally true?

Construction of $V^{modular}$:

Example: $\mathbb{C}$linear, $\{mathbb1, \epsilon\}$ two $\cong$ classes of simples

The Grothedieck ring: $\epsilon \otimes \epsilon$
Our categoryis strict monoidal

$$x_1 \otimes (x_2 \otimes x_3) = (x_1 \otimes x_2) \otimes x_3$$

Our monoidal structure is captured in tensoring of morphisms.

Remark 4. $\hom(a_1 \otimes \ldots \otimes a_n, b_1 \otimes \ldots \otimes b_m)$, *$(a_i, b_i$ simple) as (n, m) morphism space*

$(1, 1)$-morphisms

$$\hom(a, b) = [0, a \neq b \text{ and } \mathbb{C}, a \cong b$$

$(2, 1)$-morphisms

$$\hom(a \otimes b, c) \text{dimensions of homs are fixed...}$$

$\hom(\epsilon \otimes \epsilon \otimes \epsilon, \epsilon)$ with $g, g' \in \hom(\epsilon \otimes \epsilon, \epsilon)$. Then compose $g' \circ (g \otimes id_{\epsilon}$.
FIGURE OF TREE DIAGRAM
$g \in \hom(\epsilon \otimes \epsilon, \nVdash)$
$g' \in \hom(\nVdash \otimes \epsilon, \epsilon)$
And some more... plus trees
We get two isomorphisms from trees:

$$\oplus_a \hom(\epsilon \otimes \epsilon, a) \otimes \hom(a \otimes \epsilon, \epsilon) \xrightarrow{\sim} \hom(\epsilon \otimes \epsilon \otimes \epsilon, \nVdash)$$

and

$$\oplus_a \hom(\epsilon \otimes a, \epsilon) \otimes \hom(\epsilon \otimes \epsilon, a) \xrightarrow{\sim} \hom(\epsilon \otimes \epsilon \otimes \epsilon, \epsilon)$$

Idea:
These two isomorphisms capture (strict) monoidal structure up to coherence on higher levels.

FIGURE OF PENTAGON OF TREES with F arrows

Choice of bases of $(2,1)-$hom spaces

Then F is a matrix of basis changes in $\hom(\epsilon\otimes\epsilon\otimes\epsilon,\epsilon)$ (and any other $(3,1)$-hom space)/

We get an algebraic relation from the pentagon of tress which is defined over $\mathbb{Z}$.

In this example, $F_1^{\epsilon\epsilon\epsilon}=(1)$

SOME MATRIX FOR $F_\epsilon^{\epsilon\epsilon\epsilon}$ and some equation on entries.

G change of bases in $(2,1)homs$

$$G = GL(1)\times GL(1)$$

$$(\lambda,\lambda'\colon z\mapsto \frac{\lambda^2}{\lambda'}z$$