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Note on monoidal localisation

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If a class Z of morphisms in a monoidal category A is closed under tensoring with the objects of A then the category obtained by inverting the morphisms in Z is monoidal. We note the immediate properties of this induced structure. The main application describes monoidal completions in terms of the ordinary category completions introduced by Applegate and Tierney. This application in turn suggests a "change-of-universe" procedure for category theory based on a given monoidal closed category. Several features of this procedure are discussed.

0. Introduction

The first step in this article is to apply a reflection theorem ([5], Theorem 2.1) for closed categories to the convolution structure of closed functor categories described in [3]. This combination is used to discuss *monoidal localisation* in the following sense. If a class Z of morphisms in a symmetric monoidal category B has the property that $s \in Z$ implies $1_B \otimes s \in Z$ for all objects $B \in B$ then the category B_Z of fractions of B with respect to Z (as constructed in [8], Chapter 1) is a monoidal category. Moreover, the projection functor $P : B \to B_Z$ then solves the corresponding universal problem in terms of *monoidal* functors; hence such a class Z is called monoidal.

To each class Z of morphisms in a monoidal category B there corresponds a monoidal interior Z^0, namely the largest monoidal class

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contained in Z . If B is monoidal closed and suitably complete then the monoidal projection $B \to B_{Z^0}$ has a right adjoint whenever the projection $B \to B_Z$ has one. Thus the interior operation on morphism classes in B provides a closure operation on the class of full reflective subcategories of B , embedding each such subcategory in a monoidal closed "completion".

In §2 the application to monoidal completion is described through work of Applegate and Tierney [2]. Results from [2] are used to obtain the right adjoints mentioned above, and then to deduce that any small monoidal category can be continuously embedded in a complete and cocomplete monoidal closed category in such a way as to preserve tensor products and also those colimits preserved originally by the tensor product.

At the same time it is seen that any dense and strongly cogenerated completion of a monoidal closed category is again monoidal closed. This provides a convenient method of enriching V-constructions that are illegitimately large with respect to a given monoidal closed category V .

The constructions given here are based on a category S of "small" sets and set maps. Under suitable hypotheses regarding V-localisation (as given by Wolff [9]), the category S can be replaced by a symmetric monoidal closed category V . In brief, we require that the functor category $[A, S]$ may be replaced by the V-functor category $[A, V]$ throughout the discussion. When A is a large V-category and the V-functor category $[A, V]$ is unavailable, the remarks concerning completions $\hat{V}$ of V may be used to introduce the $\hat{V}$-functor category $[A, \hat{V}]$. The need for the embedding $V \subset \hat{V}$ to preserve colimits is apparent at this stage from the dependence of V-localisation constructions on V-colimits (see [9] and [10]).

We describe the symmetric set-based version of the applications so it is supposed throughout the article that each monoidal structure has a symmetry. A non-symmetric (or "biclosed") version may be obtained by considering morphism classes Z for which $s \in Z$ implies $1 \otimes s$ and $s \otimes 1$ are in Z .

The terminology is essentially that of [3], [5] and [6], and familiarity with the representation theorem is assumed. The basic formulas from [3] and [5] are recalled in §1 below.

1. Reflection in closed functor categories

An adjoint pair of functors $\psi \dashv \phi : C \to B$ is called a *reflective embedding* if the right adjoint ϕ is full and faithful on morphisms. When B has a fixed monoidal closed structure $(B, \otimes, I, l, r, a, c, [-, -])$ the reflective embedding is called *normal* if there exists a monoidal closed structure on C and monoidal functor structures $(\psi, \tilde{\psi}, \psi^0)$ and $(\phi, \tilde{\phi}, \phi^0)$ on ψ and ϕ for which ϕ is a normal closed functor and the unit and counit of the adjunction are monoidal natural transformations. For convenience, the symbol ϕ is usually omitted and the adjunction unit is denoted simply by $\eta : 1 \Rightarrow \psi$.

The significance of this situation is that normal enrichment is unique (up to monoidal isomorphism) and it exists if and only if one of the following equivalent conditions is satisfied for all objects $A \in A$, $B, B' \in B$, $C \in C$, and $D \in D$:

(1.1) $$\eta : [BC] \cong \psi[BC] ,$$

(1.2) $$\eta : [AD] \cong \psi[AD] ,$$

(1.3) $$[\eta, 1] : [\psi B, C] \cong [BC] ,$$

(1.4) $$\psi(1 \otimes \eta) : \psi(B' \otimes B) \cong \psi(B' \otimes \psi B) ,$$

(1.5) $$\psi(1 \otimes \eta) : \psi(A \otimes B) \cong \psi(A \otimes \psi B) ,$$

(1.6) $$\psi(\eta \otimes \eta) : \psi(B' \otimes B) \cong \psi(\psi B' \otimes \psi B) ,$$

where A is a strongly generating class of objects in B and D is strongly cogenerating in C . This is the reflection theorem established in [5]. The first condition requires that C be closed under exponentiation in B , while the final condition is equivalent to the enrichment $(\psi, \tilde{\psi}, \psi^0)$ preserving tensor products.

Let S denote the cartesian closed category of small sets and set maps, let $A = (A, \otimes, I, \ldots)$ be a small monoidal category, and let $[A, S]$ denote the category whose objects are the functors from A to S and whose morphisms are the natural transformations between them. This functor category admits a canonical monoidal closed structure whose tensor product and internal *hom* are given by the following formulas:

(1.7) $$F \otimes G = \int^{AA'} FA \times GA' \times \mathcal{A}(A \otimes A', -) \ ,$$

(1.8) $$[F,\ G] = \int_A S\big(FA,\ G(A \otimes -)\big) \ .$$

This structure is called the *convolution* closed structure on $[\mathcal{A}, S]$; when $\mathcal{A}$ itself is closed, the formula (1.7) takes the form $F \otimes G = \int^A FA \times G[A-]$. The convolution tensor product has the property that a $\otimes$-monoid in $[\mathcal{A}, S]$ corresponds to a monoidal functor from $\mathcal{A}$ to S . The details of these constructions are given in [3] and [4].

The preceding formulas are derived from the fact that each functor $F \in [\mathcal{A}, S]$ is canonically isomorphic to its coend expansion $\int^A FA \times \mathcal{A}(A, -)$ in terms of represented functors; this is one instance of the representation theorem. It also follows from this that the set of representable functors from $\mathcal{A}$ to S is dense, hence strongly generating, in $[\mathcal{A}, S]$.

On taking the category $\mathcal{B}$ in the reflection theorem to be the convolution $[\mathcal{A}, S]$, we obtain:

PROPOSITION 1.1. *A reflective embedding* $\psi \dashv \phi : \mathcal{C} \to [\mathcal{A}, S]$ *admits normal enrichment if and only if the functor* $F(A \otimes -)$ *has an isomorph in* $\mathcal{C}$ *whenever* $F \in \mathcal{C}$ *and* $A \in \mathcal{A}$.

Proof. This follows from condition (1.2), with $\mathcal{A}$ taken to be the set of all represented functors from $\mathcal{A}$ to S . By the representation theorem, we have

$$[\mathcal{A}(A,\ -),\ F] = \int_{A'} S\big(\mathcal{A}(A,\ A'),\ F(A' \otimes -)\big) \cong F(A \otimes -) \ ,$$

as required.

PROPOSITION 1.2. *Let* $P : \mathcal{A} \to \mathcal{A}_*$ *be a functor for which* $[P,\ 1] : [\mathcal{A}_*,\ S] \to [\mathcal{A},\ S]$ *is a full embedding. Then the following conditions are equivalent:*

(a) *the Kan adjunction* $\overline{P} \dashv [P,\ 1]$ *admits normal enrichment,*

(b) *the functor* $FP(A \otimes -) : \mathcal{A} \to S$ *factors through* P *for each*

$A \in A$ and $F \in [A_*, S]$,

(c) the functor P admits enrichment to a tensor product preserving monoidal functor.

Proof. *(a) ⇒ (c).* The left adjoint of $[P, 1]$ is given by the coend formula $\overline{P}(G) = \int^A A_*(PA, -) \times GA$ for $G \in [A, S]$. By the hypothesis *(a)*, the conditions of the reflection theorem are fulfilled and $\overline{P}$ becomes a tensor product preserving monoidal functor. Thus $\overline{P}^{op}$ has the same property. By the representation theorem, the functor $P : A \to A_*$ may be regarded as the restriction of $\overline{P}^{op}$ to represented functors, and the result *(c)* follows.

(c) ⇒ (b). Let $\tilde{P} : PA \otimes PA' \cong P(A \otimes A')$ be the tensor component of the monoidal structure on P . Then $FP(A \otimes -) \cong F(PA \otimes P-) : A \to S$ for each $A \in A$ and $F \in [A_*, S]$, as required.

(b) ⇒ (a). By Proposition 1.1.

We now consider categories of fractions and ask when projection onto a category of fractions is compatible with a given monoidal structure on the domain category. Let A be a (symmetric) monoidal category and let Z be a class of morphisms in A .

DEFINITION 1.3. The *monoidal interior* of Z is the class $Z^0 = \{s \in Z ; 1_A \otimes s \in Z$ for all $A \in A\}$. The class Z is called *monoidal* if $Z = Z^0$.

COROLLARY 1.4. *If Z is monoidal in A then the projection functor $P = P_Z : A \to A_Z$ admits enrichment to a tensor product preserving monoidal functor.*

Proof. The functor $[P_Z, 1]$ is characteristically a full embedding of $[A_Z, S]$ onto the class of those functors in $[A, S]$ which invert the members of Z (Gabriel and Zisman [8]). Thus the result follows on taking $P = P_Z : A \to A_Z$ in Proposition 1.2.

This fact, combined with condition (1.6), gives the following:

COROLLARY 1.5. *If Z is a monoidal class of morphisms in a monoidal closed category $\mathcal{B}$ and if the projection $P : \mathcal{B} \to \mathcal{B}_Z$ has a right adjoint R then this adjunction admits normal enrichment.*

We shall suppose that the category S of sets may be enlarged so that the preceding results are established when $\mathcal{A}$ has a class of objects and Z is a monoidal class of morphisms in $\mathcal{A}$.

It follows readily from Corollary 1.4, and the axioms for a monoidal functor, that any monoidal functor on $\mathcal{A}$ which factors (uniquely) through a monoidal projection $\mathcal{A} \to \mathcal{A}_Z$ receives a unique monoidal functor structure. Thus the construction of $\mathcal{A}_Z$ with Z monoidal is universal in the category of monoidal categories and functors.

A reflective embedding $\psi \dashv \phi : \mathcal{C} \to \mathcal{B}$ establishes a category equivalence between $\mathcal{C}$ and $\mathcal{B}_Z$, where Z is the class of unit components of the adjunction, thus condition (1.4) and Corollary 1.5 together mean that normal reflective embeddings in $\mathcal{B}$ correspond to those monoidal projections $\mathcal{B} \to \mathcal{B}_Z$ which have a right adjoint.

2. Monoidal completion

Throughout this section we consider a monoidal closed category $\mathcal{B} = (\mathcal{B}, \otimes, I, l, r, a, c, [-, -])$ which is assumed to be complete and to contain a small strongly generating set $\mathcal{A}$. The tensor-hom adjunction of $\mathcal{B}$ then associates to each adjoint pair $S \dashv T : \mathcal{C} \to \mathcal{B}$ over $\mathcal{B}$, a new adjoint pair $S_* \dashv T_* : \mathcal{C}^{|\mathcal{A}|} \to \mathcal{B}$ defined by $S_*(B)_A = S(A \otimes B)$ and $T_*(C_A) = \prod_A [A, TC_A]$ for each $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $(C_A)_{A \in \mathcal{A}} \in \mathcal{C}^{|\mathcal{A}|}$. In this section an adjunction $S \dashv T$ over $\mathcal{B}$ is fixed, and Z is the class of morphisms inverted by S .

LEMMA 2.1. *The class Z_* of morphisms inverted by S_* is the monoidal interior of Z .*

Proof. Clearly $Z^0 \subset Z_*$. To show $Z_* \subset Z^0$, take $s : B_1 \to B_2$ in Z_* and consider the following commuting diagram:

$$\begin{array}{ccc}
C(S(A \otimes B_2),\, C) & \xrightarrow{C(S(1 \otimes s),\, 1)} & C(S(A \otimes B_1),\, C) \\
\wr\| & & \wr\| \\
B(A \otimes B_2,\, TC) & & B(A \otimes B_1,\, TC) \\
\wr\| & & \wr\| \\
B(A,\, [B_2,\, TC]) & \xrightarrow[B(1,\, [s,\, 1])]{} & B(A,\, [B_1,\, TC])\ .
\end{array}$$

In this diagram the vertical isomorphisms are those of the $S \dashv T$ and tensor-hom adjunctions over B. The top arrow is an isomorphism for all $A \in A$ and $C \in C$ because $s \in Z_*$. Thus $[s, 1] : [B_2, TC] \to [B_1, TC]$ is an isomorphism in B for all $C \in C$, because A is strongly generating in B. Now reverse the direction of the argument, replacing $A \in A$ by an arbitrary object $B \in B$. By the representation theorem applied to $C \in C$ we then have that $S(1_B \otimes s) : S(B \otimes B_1) \to S(B \otimes B_2)$ is an isomorphism in C for all $B \in B$. Thus $s \in Z^0$, as required.

PROPOSITION 2.2. *The monoidal projection functor* $P : B \to B_{Z^0}$ *has a right adjoint.*

Proof. First, the category B is assumed to be complete and to contain a small strongly generating set A. The latter assumption means that the functor $B \to [A^{op}, S]$ which sends $B \in B$ to $B(-B) : A^{op} \to S$, is an isomorphism-reflecting embedding. This implies that B is well-powered. Secondly, by Lemma 2.1, the class Z^0 is the class of morphisms inverted by a functor, namely S_*, which already has a right adjoint. Thus the hypotheses of [2] Theorem 3.1 are satisfied, whence $B \to B_{Z^0}$ has a right adjoint.

The results of Corollary 1.5 and Proposition 2.2 could be summarised by forming the partially ordered class RB of reflective embeddings into B and observing that the monoidal interior operation on Z's determines a closure operation on RB which embeds each B_Z (Z saturated) in its *monoidal closure* B_{Z^0}, and thus determines the subclass NRB of normally enriched reflections in B.

Returning to the situation $S \dashv T : C \to B$, we have:

PROPOSITION 2.3. *Let D be a strongly cogenerating class of objects in C . Then Z is monoidal if the functor $C(S(A \otimes -), D) : B^{op} \to S$ factors through $S^{op} : B^{op} \to C^{op}$ for each $A \in A$ and $D \in D$.*

Proof. By hypothesis, if $s \in Z$ then $C(S(1_A \otimes s), D)$ is an isomorphism in S for each $A \in A$ and $D \in D$. Thus $S(1_A \otimes s)$ is an isomorphism in C for all $A \in A$, because D is strongly cogenerating in C . Thus $s \in Z^0$ by Lemma 2.1.

For the remainder of this section let A be a small monoidal category and let $B = [A^{op}, S]$ be the convolution of the monoidal dual of A with S . By the representation theorem, the Yoneda embedding $Y : A \to [A^{op}, S]$ is dense and preserves monoidal structure so that the convolution is an extension of the monoidal structure on A .

Suppose that the category C in the adjunction $S \dashv T : C \to B$ is cocomplete. Then the functors S and T take the form $SF \cong \int^A FA \cdot MA$ and $TC \cong C(M-, C)$ respectively, where M denotes the composite functor $SY : A \to C$. Conversely, any functor $M : A \to C$ yields an adjunction $S \dashv T$ by means of these formulas. This is the "models" situation studied by Applegate and Tierney [2] for which we now discuss monoidal structure. Again, Z is the class of morphisms in $[A^{op}, S]$ inverted by the cocontinuous functor S .

PROPOSITION 2.4. *If the given monoidal category A is closed and the functor $M : A \to C$ is a full embedding which strongly cogenerates C then Z is monoidal.*

Proof. This is a computation using the definition (1.7) of the convolution tensor product. For each $A \in A$ and $F \in [A^{op}, S]$ we have that

$$\mathrm{A}(-, A) \otimes F = \int^{A'A''} FA' \times \mathrm{A}(A'', A) \times \mathrm{A}(-, A' \otimes A'')$$
$$\cong \int^{A'} FA' \times \mathrm{A}(-, A' \otimes A)$$

by the representation theorem. But

$$S\big(\mathrm{A}(-, A) \otimes F\big) = \int^{A''} \left(\int^{A'} FA' \times \mathrm{A}(A'', A' \otimes A)\right) \cdot MA''$$
$$\cong \int^{A'} FA' \cdot M(A' \otimes A)$$

by the representation theorem. Thus

$$C\big(S\big(\mathrm{A}(-A) \otimes F\big), MA''\big) \cong C\left(\int^{A'} FA' \cdot M(A' \otimes A), MA''\right)$$
$$\cong \int_{A'} S\big(FA', C\big(M(A' \otimes A), MA''\big)\big)$$
$$\cong \int_{A'} S\big(FA', \mathrm{A}(A' \otimes A, A'')\big)$$

because M is a full embedding,

$$\cong \int_{A'} S\big(FA', \mathrm{A}(A', [AA''])\big)$$ because A is closed,

$$\cong \int_{A'} S\big(FA', C(MA', M[AA''])\big)$$

because M is a full embedding,

$$\cong C\left(\int^{A'} FA' \cdot MA', M[AA'']\right)$$
$$= C(SF, M[AA''])$$

for all $A, A'' \in \mathrm{A}$ and $F \in [\mathrm{A}^{\mathrm{op}}, S]$. Thus the result follows from Proposition 2.3 since the image of M is strongly cogenerating in C .

The hypothesis of Proposition 2.3 is satisfied whenever $M : \mathrm{A} \to C$ is a tensor product preserving monoidal functor for which the canonical map:

$$\int^{A} FA \cdot (MA \otimes MA') \to \left(\int^{A} FA \cdot MA\right) \otimes MA'$$

is an isomorphism. For example, this is true of the inclusion functor from A = {compact Hausdorff spaces and continuous maps} into

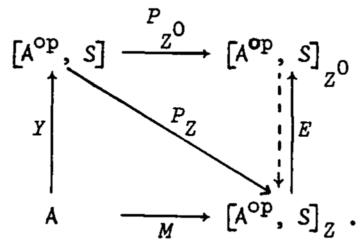
C = {all topological spaces and continuous maps} . When these categories are taken with the cartesian monoidal structure we have that $- \times A : C \to C$ preserves topological colimits for each compact Hausdorff space A . The resulting category $[A^{op}, S]_Z$ is (equivalent to) the usual cartesian closed category of compactly generated topological spaces (see [1] and [5]). Many other closed categories may be constructed in this way. Another cartesian example appears in [7], Theorem 8.9.

An important example is the (dense) cotriple-tower completion (see Applegate and Tierney [2]) of a small monoidal closed category A with respect to the codense Yoneda embedding $A \to [A, S]^{op}$. Such a completion $\hat{A}$, when expressed as a reflective subcategory $[A^{op}, S]_Z$ of $[A^{op}, S]$, is again monoidal closed by Proposition 2.4. Moreover, Corollary 1.5 implies that $\hat{A}$ is a normal reflective subcategory of the convolution, hence the conditions (1.1) and (1.6) mean that $\hat{A}$ is closed under exponentiation in $[A^{op}, S]$ and the reflection preserves tensor products. Thus, because the Yoneda embedding $A \to [A^{op}, S]$ preserves both the tensor product and internal hom of A , so does the embedding of A into $\hat{A}$.

In the case where A is known only to be monoidal, the process of taking the monoidal closure of a completion gives the following:

PROPOSITION 2.5. *A small monoidal category A can be continuously embedded in a complete and cocomplete monoidal closed category such that the embedding preserves the monoidal structure of A and any colimits in A that are preserved by the tensor product of A .*

Proof. Let $M : A \to [A^{op}, S]_Z$ be a (dense) Lambek completion of A , with Z saturated. The exterior of the following diagram commutes to within a natural isomorphism:



In this diagram, E is the right adjoint to the factorisation of P_Z through P_{Z^0} . Then the composite functor EM is an embedding of the required type. First, it preserves all limits because each factor does. Secondly, it is naturally isomorphic to $P_{Z^0} \cdot Y$ which preserves tensor products. Finally, if $K : K \to A$ is a functor and $\lambda_k : Kk \to \operatorname{colim}_k Kk$ is a colimit cone in A then the canonical transformation

$$s : \operatorname{colim}_k A(-, Kk) \to A\left(-, \operatorname{colim}_k Kk\right)$$

is in Z because $[A^{op}, S]_Z$ is a completion, which implies that P_Z inverts s . If this cone is mapped to a colimit cone in A by each functor $- \otimes A : A \to A$ then the map s is in Z^0 . Hence P_{Z^0} inverts s so that EM will preserve the colimit of K .

3. A chain rule for localisations

Suppose that $H \dashv K : B \to B'$ is an S-adjunction between categories B and B' , with counit $\alpha : HK \Rightarrow 1$ and unit $\beta : 1 \Rightarrow KH$, and let Z be a class of morphisms in B with projection $P : B \to B_Z$. Let Z' be the class of morphisms in B' inverted by PH .

PROPOSITION 3.1. *If $P\alpha$ is an isomorphism then B_Z is category equivalent to $B'_{Z'}$.*

Proof. Consider the following diagram where $P' : B' \to B'_{Z'}$ is the canonical projection:

$$\begin{array}{ccc} \mathsf{B}' & \underset{K}{\overset{H}{\rightleftarrows}} & \mathsf{B} \\ {\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle P} \\ \mathsf{B}'_{Z'} & \underset{K_Z}{\overset{H_Z}{\rightleftarrows}} & \mathsf{B}_Z \end{array} .$$

The functor H_Z is defined because PH inverts Z'. The hypothesis is that $P\alpha : PHK \cong P$ so $s \in Z$ implies $Ks \in Z'$. This defines the functor K_Z. Moreover, $P'\beta : P' \Rightarrow P'KH$ is an isomorphism because the triangle identity $\alpha H \cdot H\beta = 1$ implies that $P\alpha H \cdot PH\beta = 1$; hence $PH\beta$ is an isomorphism so, $\beta \in Z'$. Hence there are natural isomorphisms $H_Z K_Z \cong 1$ and $K_Z H_Z \cong 1$.

To interpret this result in the "models" situation, suppose that $N : \mathsf{A}' \to \mathsf{A}$ and $M : \mathsf{A} \to \mathsf{C}$ are functors with C a suitably cocomplete category (having the coends required below). As in §2, the Kan extensions of M and MN respectively provide cocontinuous functors $S : \mathsf{B} = [\mathsf{A}^{op}, S] \to \mathsf{C}$ and $S' : \mathsf{B}' = [\mathsf{A}'^{op}, S] \to \mathsf{C}$.

$$\begin{array}{ccccc} [\mathsf{A}'^{op}, S] & \underset{[N,\,1]}{\overset{\overline{N}}{\rightleftarrows}} & [\mathsf{A}^{op}, S] & & \\ {\scriptstyle Y}\uparrow & & {\scriptstyle Y}\uparrow & \searrow{\scriptstyle S} & \\ \mathsf{A}' & \xrightarrow[N]{} & \mathsf{A} & \xrightarrow[M]{} & \mathsf{C} \end{array} .$$

If H is taken to be the left Kan adjoint $\overline{N}$ of $[N, 1]$ then $S' \cong SH$. Again, let Z and Z' be the classes of morphisms inverted by S and S' respectively.

COROLLARY 3.2. *The categories* $[\mathsf{A}^{op}, S]_Z$ *and* $[\mathsf{A}'^{op}, S]_{Z'}$ *are equivalent if the canonical map*

$$(3.1) \qquad \int^{A'} \mathsf{A}(NA', A) \cdot MNA' \xrightarrow{\cong} MA$$

is an isomorphism for all $A \in \mathsf{A}$.

Proof. The counit of the adjunction $\overline{N} \dashv [N, 1]$ is the canonical transformation whose component at $F \in [A^{op}, S]$ is

$$\alpha_{F,A} : \int^{A'} A(A, NA') \times FNA' \to FA .$$

Thus $S\alpha$ is the canonical transformation

$$\int^{A} \left(\int^{A'} A(A, NA') \times FNA'\right) \cdot MA \to \int^{A} FA \cdot MA .$$

By the representation theorem applied to $A \in A$, $S\alpha$ is an isomorphism if and only if (3.1) is an isomorphism. The result follows by Proposition 3.1.

For example, let $M : A \to C$ be a full embedding whose image strongly cogenerates C . Then (3.1) is an isomorphism when the induced map

$$C(MA, MB) \to C\left(\int^{A'} A(NA', A) \cdot MNA', MB\right)$$

is an isomorphism; that is, when the transformation

$$A(A, B) \to \int_{A'} S\big(A(NA', A), A(NA', B)\big)$$

is an isomorphism for all $A, B \in A$. Thus the chain rule implies that any dense subcategory of A is again dense in a (dense) completion of A .

In a similar manner, if A has a monoidal structure, then the monoidal closure $[A^{op}, S]_{Z^0}$ of $[A^{op}, S]_Z$ may also be presented as a full reflective subcategory of $[A'^{op}, S]$ if the transformation

$$(3.2) \qquad \int^{A'} A(NA', A) \cdot M(NA' \otimes -) \xrightarrow{\cong} M(A \otimes -)$$

is an isomorphism for all $A \in A$. This reduces to (3.1) on evaluation at the identity object of A . The condition (3.2) ensures that $\alpha : HK \Rightarrow 1$ is in the monoidal interior of Z so that Proposition 3.1 applies to Z^0 .

The information in Corollaries 2.5 and 3.2 gives a procedure for enriching "V-constructions" that are too large for a given symmetric monoidal closed category V (based on S). This is done by replacing V with its completion $\hat{V}$ with respect to the Yoneda embedding

$V \to [V, \hat{S}]^{op}$, where $\hat{S}$ is a suitable enlargement of the given category S of sets. Then Corollary 3.2 ensures that $\hat{V}$ is category equivalent to $\hat{S}$ when V is taken to be S ; this follows on taking the functor N to be the dense inclusion of a one-point category into S . The utility of such a "change of V-universe" was suggested to the author by G.M. Kelly.

4. Completion of functor categories

More generally, suppose that A is a category whose class of objects is small with respect to $\hat{S}$, and let Z be a class of natural transformations in $[A, S]$. As in [7], §8, a functor $F : A \to S$ is called *Z-continuous* if for each $s : G \Rightarrow H$ in Z and natural $\alpha : G \Rightarrow F$ there exists a unique natural $\beta : H \Rightarrow F$ such that $\alpha = \beta \cdot s$.

Let C be the full subcategory of $[A, S]$ determined by the Z-continuous functors and suppose that the inclusion $C \subset [A, S]$ is isomorphic to $C \mapsto C(N-, C)$ for some (necessarily dense) functor $N : A^{op} \to C$. Then, by Corollary 3.2, the completion $\hat{C}$ of C can be described as a full reflective subcategory of $[A, \hat{S}]$. As such, $\hat{C}$ is equivalent to the category of $\hat{Z}$-continuous functors in $[A, \hat{S}]$ where $\hat{Z}$ is the class of natural transformations inverted by the functor $S : [A, \hat{S}] \to [C, \hat{S}]^{op}$, $S(F) = \int_A \hat{S}(FA, C(NA, -))$. But S inverts all the members of Z , thus we obtain the following:

PROPOSITION 4.1. *There is a canonical full embedding* $E : \hat{C} \to B$ *of the completion of* C *into the full subcategory* B *of* Z*-continuous functors from* A *to* $\hat{S}$. *This embedding is an equivalence if and only if* B *is* $\hat{S}$*-cocomplete and is strongly cogenerated by* C .

Proof. If E is an equivalence then C strongly cogenerates B because C strongly cogenerates $\hat{C}$ which is also cocomplete. Conversely, if C strongly cogenerates B then each Z-continuous functor $F : A \to \hat{S}$ admits an embedding $F \to \prod_{x \in X} G_x$ with $G_x \in C$ and $X \in \hat{S}$. Because B is cocomplete, this embedding has a cokernel pair in B, whose codomain is embedded in a product $\prod_{y \in Y} H_y$ with $H_y \in C$ and $Y \in \hat{S}$. Thus F is the equaliser in $[A, \hat{S}]$ of a pair of morphisms from $\prod_{x \in X} G_x$ to $\prod_{y \in Y} H_y$. This implies that F is $\hat{Z}$-continuous, whence E is an equivalence.

In particular, the completion of a functor category $[A, S]$ is equivalent to $[A, \hat{S}]$ because the class of functors $\{S(A(-A), X); A \in A, X \in S\}$ strongly cogenerates the larger functor category. If A is a (finitary) algebraic theory and Z is the class of natural transformations defined by (finite) products in A then the embedding of the completion of the category C of A-algebras in S into the category B of A-algebras in $\hat{S}$ may or may not be an equivalence. For example, if A is the theory of groups then C is not strongly cogenerating in B because each functor $B(-, C)$ with $C \in C$ sends the constant map of a simple group in $\hat{S}$, not in S , to an isomorphism. On the other hand, if A is the theory of R-modules for a ring R in S then the category of R-module structures in $\hat{S}$ is equivalent to the completion of the category of R-module structures in S .

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