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G-structures on orbifolds

Tese

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Apresenta

Sebastián Daza

Orientador

Ivan Struchiner (Dr. rer. nat.)

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Para Andrés.

Contents

Introduction

The objective of this thesis is to develop a language to study differential geometric properties in singular spaces. The singular spaces we are interested in are effective orbifolds. They appear in many fields of mathematics such as algebraic geometry, algebraic topology, groupoid theory, differential geometric problems in manifolds, among others ([ALR07], [MP97], [CR01], [LU04], [Thu79]). The language we are developing is the theory of G-structures. It relates the presence of a (linear) geometric structure on an orbifold with an action of a Lie subgroup $G < GL_n(\mathbb{R})$ on a subbundle of the frame orbibundle. The effectiveness hypothesis gives a manifold structure on the frame orbibundle. Hence, we will study differential geometric problems on singular spaces through non-singular ones.

This thesis has three chapters. In the first chapter, we establish the background needed to understand effective orbifolds: the fundamental definitions, maps between them, some examples, and properties that will help us develop the theory of G-structures. In particular, we are interested in orbifolds arising as quotients of manifolds by locally free and proper Lie group actions.

In the second chapter, we study the primary objects used to develop the theory of G-structures: orbibundles. We will show there is a 1-1 correspondence between cone and principal orbibundles. In particular, the tangent and the frame orbibundles are related. Besides, we will study connections on the tangent and frame orbibundles and show a 1-1 correspondence between them. Consequently, the geometric properties due to the presence of a connection have corresponding frameworks

on both the tangent and the frame orbibundles. Also, we will introduce two concepts of capital importance: the tautological form (characterizes G-structures) and reductions (the essence of what a G-structure is).
The third chapter introduces the theory of G-structures on effective orbifolds. We establish the basic concepts and relate them to the classical geometric structures. We will show that the tautological form characterizes when a principal G-bundle is a G-structure and when an isomorphism between principal G-bundles is an equivalence of G-structures. This gives a characterization of the category of G-structures on effective orbifolds. Later, we will use the theory constructed to talk about connections compatible with a geometric structure. We will compute the compatibility conditions for some classical geometries. Finally, we introduce a fundamental problem to be studied in the theory of G-structures: integrability. We will characterize when a G-structure is integrable only in terms of the frame bundle. Besides, we will talk about the torsion and intrinsic torsion, show that its vanishing is an obstruction for integrability, and compute these obstructions for some classical geometric structures.

Chapter 1

Orbifolds

Manifolds provide a proper framework which allows the use of tools from calculus in the study of geometric properties of a space. This is the scope of what is usually referred to as differential geometry. Many of the properties of a geometric structure on a manifold are captured by its group of symmetries. It is then natural to consider the quotient space of the manifold by the action of the symmetry group. Depending on the behavior of the group action, this quotient space might not be a smooth manifold. This leads us to consider more general spaces than manifolds. Orbifolds, first called V-manifolds, were introduced by Satake [Sat57] as a generalization of manifolds. While on manifolds there are local charts $\tilde{\phi} : \tilde{U} \subset \mathbb{R}^n \to U$ which are homeomorphisms, on orbifolds there is an extra information: a finite group Γ acting by diffeomorphisms on $\tilde{U}$. Then $\tilde{\phi} : \tilde{U} \to U$ is not a homeomorphism but is Γ-invariant and $\tilde{\phi} : \tilde{U}/\Gamma \to U$ is a homeomorphism. Consequently the compatibility conditions between the charts must take into account this extra information. Intuitively, orbifolds are spaces with good singularities which are codified by the points with non-trivial isotropy.

This thesis deals with geometric structures, known as G-structures, on orbifolds. In the first section we will give some basic definitions, examples and results about orbifolds (we refer to see [Sat57], [CJ19], [ALR07] for more details). In the second section we make a crucial remark for the development of the theory of G-structures on orbifolds: the frame

orbibundle of an effective orbifold is always a smooth manifold. This can be thought of as a desingularization process where we associate to a singular object, an orbifold, a non-singular one: a manifold. For this reason we will focus on effective orbifolds in this thesis. In the third section we will study maps between orbifolds. Be aware that there are different notions of maps between orbifolds. We will not cover this as deeply as we would like to, but instead we will focus on a particular notion that will be needed in the theory of G-structures. More informations about maps could be found in [BB13].
In this thesis we will use the more classical "charts" perspective on orbifolds. However, orbifolds can be treated in more ways: using pseudogroups, as in [CJ19]; and as proper, étale groupoids up to Morita equivalence, [MP97]. Information about maps between orbifolds, in the groupoid perspective, can be found in [Che06],[ALR07], [MP97].

1.1 Orbifold fundamentals

Let Γ be a group acting on a space X. The action is called effective if the only group element that acts as the identity is the identity. In other words

$$\gamma \cdot \tilde{x} = \tilde{x} \text{ , for all } \tilde{x} \Rightarrow \gamma = id.$$

Definition 1.1. *Let $\mathcal{O}$ be a topological Hausdorff space. A local model is a triple $(\tilde{U}, \Gamma, \tilde{\phi})$ where $0 \in \tilde{U} \subset \mathbb{R}^n$ is a connected open subset such that:*

1. *Γ a finite subgroup of the automorphism group of $\tilde{U}$, which fixes the origin and acts effectively on $\tilde{U}$.*

2. *$\tilde{\phi} : \tilde{U} \to U$ is a continuous map onto an open set $x \in U \subset \mathcal{O}$ such that $\tilde{\phi}(0) = x$.*

3. *$\tilde{\phi}$ is Γ-invariant and induces a homeomorphism $\phi : \tilde{U}/\Gamma \to U$.*

Remark: All open sets of $\mathbb{R}^n$ will be written with a tilde as $\tilde{U}$. Their corresponding images by $\tilde{\phi}$ will not have such a tilde. They will be denoted by U, and are open subsets of $\mathcal{O}$.

Example 1.2. Let $a \in \mathbb{R}^+$, define $\tilde{\phi}_a : (-a, a) \to [0, a)$ by

$$\tilde{\phi}_a(x) = \begin{cases} x & if \quad x \geq 0 \\ \\ -x & if \quad x < 0. \end{cases}$$

Define the action $\langle \gamma \rangle = \mathbb{Z}_2 \curvearrowright \mathbb{R}$ by $\gamma \cdot x = -x$ and

$$p_a : (-a, a) \to (-a, a) / \mathbb{Z}_2,$$

be the projection map. Because $\tilde{\phi}_a$ is constant on each fiber $p_a^{-1}(c)$, i.e., $\tilde{\phi}_a$ is invariant under the $\mathbb{Z}_2$ action, it induces the continuous function

$$\begin{tikzcd} (-a, a) \arrow[d, "p_a"] \arrow[dr, "\tilde{\phi}_a"] & \\ (-a, a) / \mathbb{Z}_2 \arrow[r, "\phi_a"] & {[0, a)}. \end{tikzcd}$$

The induced function $\phi_a : (-a, a) / \mathbb{Z}_2 \to [0, a)$ is a homeomorphism. It follows that $\left((-a, a), \mathbb{Z}_2, \tilde{\phi}_a\right)$ is a local model for $[0, a)$.

Example 1.3. Take $n \in \mathbb{N}$ and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by the angle $2\pi/n$ defined by

$$R(x, y) = \left(\cos\left(\frac{2\pi}{n}\right) x - \sin\left(\frac{2\pi}{n}\right) y, \sin\left(\frac{2\pi}{n}\right) x + \cos\left(\frac{2\pi}{n}\right) y\right).$$

For

$$W = \left\{(x, y) \in \mathbb{R}^2 \colon 0 \leq y \leq \tan\left(\frac{2\pi}{n}\right) x\right\},$$

define $W_k = \overbrace{(R \circ \ldots \circ R)}^{k}(W)$, where $k = 0, \ldots, n-1$. Then

$$\mathbb{R}^2 = \bigcup_{i=0}^{n-1} W_i.$$

Take the cone

$$C = \left\{(x, y, z) \in \mathbb{R}^3 \colon y^2 + z^2 = \left(\cot\left(\frac{\pi}{n}\right) x\right)^2\right\},$$

and define $\tilde{\phi}_0 : W - \{0\} \to C$ by

$$\tilde{\phi}_0(x,y) = \left(\sqrt{x^2+y^2},\ \cot\left(\frac{\pi}{n}\right)\sqrt{x^2+y^2}\cos\left(n\cdot\arctan\left(\frac{y}{x}\right)\right),\right.$$
$$\left.\cot\left(\frac{\pi}{n}\right)\sqrt{x^2+y^2}\sin\left(n\cdot\arctan\left(\frac{y}{x}\right)\right)\right).$$

We can extend $\tilde{\phi}_0(0,0) = (0,0,0)$ and obtain that $\tilde{\phi}_0\ |_W$ is a continuous map. Define $\tilde{\phi}_k : W_k \to C$ by $\tilde{\phi}_k = \tilde{\phi}_0 \circ R^{n-k}$. It is continuous being the composition of continuous functions. These functions coincide in the intersection and, by the pasting lemma, $\tilde{\phi} : \mathbb{R}^2 \to C$ defined by $\tilde{\phi}(x) = \tilde{\phi}_k(x)$, where $x \in W_k$, is a continuous function.
The map R induces an action $\langle R\rangle = \mathbb{Z}_n \curvearrowright \mathbb{R}^2$ with $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}_n$ the projection map. Because $\tilde{\phi}$ is constant on each fiber $p^{-1}(c)$, i.e., $\tilde{\phi}$ is invariant under the $\mathbb{Z}_n$ action, it induces a continuous function

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "p"] \arrow[dr, "\tilde{\phi}"] & \\
\mathbb{R}^2/\mathbb{Z}_n \arrow[r, "\phi"] & C.
\end{tikzcd}$$

The induced function $\phi : \mathbb{R}^2/\mathbb{Z}_n \to C$ is a homeomorphism. Therefore $(\mathbb{R}^2, \mathbb{Z}_n, \tilde{\phi})$ is a local model for the cone C.

Remark : Note that even though $\mathbb{R}^2/\mathbb{Z}_n$ and $\mathbb{R}^2/\mathbb{Z}_m$ are homeomorphic when $n \neq m$, they have different local models. We will see that they are not isomorphic as orbifolds.

Let $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ and $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$ be local models. To compare them we require that $\tilde{\phi}_\alpha(\tilde{U}_\alpha) = U_\alpha \subset U_\beta = \tilde{\phi}_\beta(\tilde{U}_\beta)$.

Definition 1.4. *A topological embedding $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ such that $\tilde{\phi}_\alpha = \tilde{\phi}_\beta \circ \tilde{\psi}_{\alpha\beta}$ is called an injection.*

An injection gives rise to the following diagram

$$\begin{tikzcd}
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"] \arrow[d, "\tilde{\phi}_\alpha"'] & \tilde{U}_\beta \arrow[d, "\tilde{\phi}_\beta"] \\
U_\alpha \arrow[r, "\subset"] & U_\beta.
\end{tikzcd}$$

Note that, by definition, every element $\gamma_\alpha \in \Gamma_\alpha$ induces an injection $\gamma_\alpha : (\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha) \to (\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$.

Lemma 1.5. *Let $\tilde{\psi}_1, \tilde{\psi}_2 : (\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha) \to (\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$ be two injections. Then there exist an unique $\gamma_\beta \in \Gamma_\beta$ such that $\tilde{\psi}_1 = \gamma_\beta \circ \tilde{\psi}_2$*

This lemma appears firstly in [Sat57] but its proof requires a dimensional hypothesis on the group. However, there is a proof of the statement without the dimensional hypothesis in [MP97].
Because the composition of injections is again an injection we have that $\tilde{\psi}_{\alpha\beta} \circ \gamma_\alpha$ and $\tilde{\psi}_{\alpha\beta}$ are both injections. It follows that there exists a unique $\gamma_\beta \in \Gamma_\beta$ such that $\tilde{\psi}_{\alpha\beta} \circ \gamma_\alpha = \gamma_\beta \circ \tilde{\psi}_{\alpha\beta}$. This can be restated as the existence of a monomorphism $\theta_{\alpha\beta} : \Gamma_\alpha \to \Gamma_\beta$, with $\theta_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$, such that

$$\begin{tikzcd}
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"] \arrow[d] & \tilde{U}_\beta \arrow[d] \\
\tilde{U}_\alpha/\Gamma_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}/\theta_{\alpha\beta}"] \arrow[d, "\phi_\alpha"] & \tilde{U}_\beta/\Gamma_\beta \arrow[d, "\phi_\beta"] \\
U_\alpha \arrow[r, "\subset"] & U_\beta
\end{tikzcd}$$

is a commutative diagram.

Example 1.6. Let $a, b \in \mathbb{R}^+$, with $a < b$. Take two local models $\left((-a, a), \mathbb{Z}_2, \tilde{\phi}_a\right)$ and $\left((-b, b), \mathbb{Z}_2, \tilde{\phi}_b\right)$ as in example 1.2. Then

$$\begin{tikzcd}
(-a, a) \arrow[r, "\tilde{\iota}"] \arrow[d, "p_a"] & (-b, b) \arrow[d, "p_b"] \\
(-a, a)/\mathbb{Z}_2 \arrow[r, "\tilde{\iota}/\mathbb{I}"] \arrow[d, "\tilde{\phi}_a"] & (-b, b)/\mathbb{Z}_2 \arrow[d, "\tilde{\phi}_b"] \\
{[0, a)} \arrow[r, "\iota"] & {[0, b)}
\end{tikzcd}$$

is an injection, where $\tilde{\iota}$ is the natural inclusion, $\mathbb{I} : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity morphism and ι is the induced inclusion on the quotient spaces.

Example 1.7. Let $\left(\mathbb{R}^2, \mathbb{Z}_n, \tilde{\phi}_1\right)$ be as in example 1.3.
Define $\tilde{\phi}_2 : \mathbb{R}^2 \to C$ by

$$\tilde{\phi}_2(x, y) = \left(\sqrt{x^2 + y^2},\ \cot\left(\frac{\pi}{n}\right) x, \cot\left(\frac{\pi}{n}\right) y\right).$$

It is a homeomorphism and then $\left(\mathbb{R}^2, \{e\}, \tilde{\phi}_2\right)$ is a local model for the cone C. However, there can not exists an injection between the two local models $\left(\mathbb{R}^2, \mathbb{Z}_n, \tilde{\phi}_1\right)$ and $\left(\mathbb{R}^2, \{e\}, \tilde{\phi}_2\right)$. If true, then there will exist a $\mathbb{Z}_n$-invariant map $\tilde{\psi} : \mathbb{R}^2 \to \mathbb{R}^2$, which means $\tilde{\psi}$ can not be injective.

The previous example shows we could have different local models for the same topological space that can not be linked by an injection.
As long as we are dealing with smooth structures, a *smooth local model* is a local model such that the group Γ acts by C^∞-diffeomorphisms. A *smooth injection* is an injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ between two local models $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$, $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$ such that $\tilde{\psi}_{\alpha\beta}$ is an embedding of manifolds.

Definition 1.8. *Let $\mathcal{O}$ be a topological Hausdorff paracompact space. An orbifold atlas associated with an open cover $\{U_\alpha\}_{\alpha\in I}$ of $\mathcal{O}$ is given by the following conditions:*

1. *The existence of a smooth local model $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ for every α, such that $\tilde{\phi}_\alpha(\tilde{U}_\alpha) = U_\alpha$.*

2. *For every two smooth local models $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ and $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$, with $U_\alpha \subset U_\beta$, there exists a smooth injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$.*

3. *For every $p \in U_\alpha \cap U_\beta$ there exists $\nu \in I$ such that $p \in U_\nu$ and $U_\nu \subset U_\alpha \cap U_\beta$.*

Orbifold atlases can be divided into equivalence classes. Two orbifold atlases belong to the same class if their union admits a refinement that is an orbifold atlas. These equivalence classes are represented maximal atlas.

Definition 1.9. *A n-dimensional orbifold is a topological space which is Hausdorff and paracompact, along with an n-dimensional maximal orbifold atlas on it.*

Example 1.10. $[0, \infty)$ has an orbifold structure induced by the local models $\left((-a, a), \mathbb{Z}_2, \tilde{\phi}_a\right)$ for all $a \in \mathbb{Z}^+$.

Example 1.11. The local models $(\mathbb{R}^2, \mathbb{Z}_n, \tilde{\phi}_1)$ and $(\mathbb{R}^2, \{e\}, \tilde{\phi}_2)$ define orbifold structures for the cone C. However, they are not the same orbifold structure because there does not exist a *smooth* injection between them.

Let $x \in \mathcal{O}$ and let $(\tilde{U}, \Gamma, \tilde{\phi})$ be an orbifold chart such that $\tilde{\phi}(\tilde{x}) = x$. The isotropy subgroup $\Gamma_{\tilde{x}} < \Gamma$ is given by

$$\Gamma_{\tilde{x}} = \{\gamma \in \Gamma \mid \gamma \cdot \tilde{x} = \tilde{x}\}.$$

For every $\gamma \in \Gamma$ the groups $\Gamma_{\tilde{x}} \cong \Gamma_{\gamma \cdot \tilde{x}}$, which implies $\Gamma_{\tilde{x}_1} \cong \Gamma_{\tilde{x}_2}$ for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{\phi}^{-1}(x)$.

Lemma 1.12. *For every $x \in \mathcal{O}$, there exists an orbifold chart $(\tilde{U}, \Gamma, \tilde{\phi})$ such that $\tilde{\phi}(\tilde{x}) = x$ and $\Gamma_{\tilde{x}} = \Gamma$.*

Proof. Since the group elements $\gamma_i \in \Gamma - \Gamma_{\tilde{x}}$ do not fix $\tilde{x}$ they will define a set $\{\gamma_1 \cdot \tilde{x}, \ldots, \gamma_r \cdot \tilde{x}\}$. Take a metric on $\tilde{U}$ such that Γ acts by isometries and an open neighborhood $B_\epsilon(\tilde{x})$ such that

$$B_\epsilon(\tilde{x}) \cap \gamma_i B_\epsilon(\tilde{x}) = \emptyset.$$

An injection between the orbifold chart $(B_\epsilon(\tilde{x}), \Gamma_{\tilde{x}}, \tilde{\phi})$ and $\tilde{U}$ is given by the inclusion $\tilde{\imath} : B_\epsilon(\tilde{x}) \hookrightarrow \tilde{U}$. It follows that $(B_\epsilon(\tilde{x}), \Gamma_{\tilde{x}}, \tilde{\phi})$ belongs to the same orbifold atlas. □

Hereafter we will work with these orbifold charts. The isotropy relative to the orbifold chart $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$, with $\tilde{\phi}_\beta(\tilde{y}) = x$, can be compared with the isotropy of the orbifold chart $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ passing through an orbifold chart $(\tilde{U}_\nu, \Gamma_\nu, \tilde{\phi}_\nu)$, with $U_\nu \subset U_\alpha \cap U_\beta$ and $\tilde{\phi}_\nu(0) = x$. Take smooth injections $\tilde{\psi}_{\nu\beta} : \tilde{U}_\nu \hookrightarrow \tilde{U}_\beta$ and $\tilde{\psi}_{\nu\alpha} : \tilde{U}_\nu \hookrightarrow \tilde{U}_\alpha$, with monomorphisms $\theta_{\nu\beta} : \Gamma_\nu \to \Gamma_\beta$ and $\theta_{\nu\alpha} : \Gamma_\nu \to \Gamma_\alpha$. As long as $\tilde{\psi}_{\nu\beta}(0) = \tilde{y}$ and $\tilde{\psi}_{\nu\alpha}(0) = \tilde{x}$, then, $\theta_{\nu\beta} : \Gamma_\nu \to (\Gamma_\beta)_{\tilde{y}}$ and $\theta_{\nu\alpha} : \Gamma_\nu \to (\Gamma_\alpha)_{\tilde{x}}$.
The proof for the following lemma can be found in [Sat57].

Lemma 1.13. *Let $\tilde{\psi}$ be an injection $\{\tilde{U}_1, \Gamma_1, \tilde{\phi}_1\} \to \{\tilde{U}_2, \Gamma_2, \tilde{\phi}_2\}$. If $\gamma_2(\tilde{\psi}(\tilde{U}_1)) \cap \tilde{\psi}(\tilde{U}_1) \neq \emptyset$ with $\gamma_2 \in \Gamma_2$, then $\gamma_2(\tilde{\psi}(\tilde{U}_1)) = \tilde{\psi}(\tilde{U}_1)$ and γ_2 belongs to the image of the monomorphism $\Gamma_1 \to \Gamma_2$.*

Consequently $\theta_{\nu\beta} : \Gamma_\nu \to (\Gamma_\beta)_{\tilde{y}}$ and $\theta_{\nu\alpha} : \Gamma_\nu \to (\Gamma_\alpha)_{\tilde{x}}$ are isomorphisms which implies $(\Gamma_\beta)_{\tilde{y}} \cong (\Gamma_\alpha)_{\tilde{x}}$. For this reason, the isotropy does not depend on the choice of orbifold charts, i.e., it is well-defined up to isomorphisms.

Definition 1.14. *Let $x \in \mathcal{O}$, the isotropy group Γ_x is by definition Γ_ν, where $(\tilde{U}_\nu, \Gamma_\nu, \tilde{\phi}_\nu)$ is an orbifold chart such that $\tilde{\phi}_\nu(0) = x$.*

Take the orbifold charts $(\tilde{\psi}_{\nu\beta}(\tilde{U}_\nu), (\Gamma_\beta)_{\tilde{y}}, \tilde{\phi}_\beta)$ and $(\tilde{\psi}_{\nu\alpha}(\tilde{U}_\nu), (\Gamma_\alpha)_{\tilde{x}}, \tilde{\phi}_\alpha)$. The diffeomorphism $\tilde{\psi}_{\nu\alpha} \circ \tilde{\psi}_{\nu\beta}^{-1} : \tilde{\psi}_{\nu\beta}(\tilde{U}_\nu) \to \tilde{\psi}_{\nu\alpha}(\tilde{U}_\nu)$, together with the isomorphism $\theta_{\beta\alpha} := \theta_{\nu\alpha} \circ \theta_{\nu\beta}^{-1} : (\Gamma_\beta)_{\tilde{y}} \to (\Gamma_\alpha)_{\tilde{x}}$, defines a $\theta_{\beta\alpha}$-equivariant smooth diffeomorphism.

Definition 1.15. *Two orbifold charts $(\tilde{U}_1, \Gamma_1, \tilde{\phi}_1)$ and $(\tilde{U}_2, \Gamma_2, \tilde{\phi}_2)$ are isomorphic if there exists a diffeomorphism $\tilde{\psi} : \tilde{U}_1 \to \tilde{U}_2$ and an isomorphism $\theta : \Gamma_1 \to \Gamma_2$ such that $\tilde{\psi}$ is θ-equivariant.*

For every triple $U_\nu \subset U_\alpha \cap U_\beta$, the orbifold charts

$$\left(\tilde{\psi}_{\nu\beta}(\tilde{U}_\nu), (\Gamma_\beta)_{\tilde{\psi}_{\nu\beta}(0)}, \tilde{\phi}_\beta\right) \cong \left(\tilde{U}_\nu, \Gamma_\nu, \tilde{\phi}_\nu\right) \cong \left(\tilde{\psi}_{\nu\alpha}(\tilde{U}_\nu), (\Gamma_\alpha)_{\tilde{\psi}_{\nu\alpha}(0)}, \tilde{\phi}_\alpha\right), \tag{1.1.1}$$

are isomorphic.

Example 1.16. Let $\left(\mathbb{R}^2, \mathbb{Z}_n, \tilde{\phi}_1\right)$ and $\left(\mathbb{R}^2 - \{0\}, \{e\}, \tilde{\phi}_2\right)$ be as in example 1.7. For every $x \in C$ its isotropy group is

$$\Gamma_x = \begin{cases} \mathbb{Z}_n & if \quad x = 0 \\ \\ e & if \quad x \neq 0. \end{cases}$$

Every orbifold chart around $0 \in C$ must have a group isomorphic to $\mathbb{Z}_n$. That is why the second orbifold chart can not be extended to all $\mathbb{R}^2$.

If $\Gamma_x = \{e\}$ for all $x \in \mathcal{O}$, then every orbifold chart have the form $(\tilde{U}, \{e\}, \tilde{\phi})$. The compatibility between the orbifold charts is given by embeddings, without any group monomorphism, and $\mathcal{O}$ is a manifold. The orbifold structures which are not manifolds appear when the isotropy groups are not trivial.

Example 1.17. A manifold M is an orbifold such that $\Gamma_x = \{e\}$, for all $x \in M$.

Since the theory of G-structures over an orbifold $\mathcal{O}$ relates the presence of a geometric structure with a group action on the frame orbibundle, it is fair to require that the local groups on the orbifold charts Γ_α acts by representation on some subgroup of $GL_n(\mathbb{R})$.

Lemma 1.18. *Every orbifold structure* $\left(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha\right)$ *admits a compatible orbifold structure* $\left(\tilde{U}^\triangleleft_\alpha, \Gamma^\triangleleft_\alpha, \tilde{\phi}^\triangleleft_\alpha\right)$ *such that* $\Gamma^\triangleleft_\alpha$ *acts by a representation of the orthogonal group.*

Proof. Let $\left(\tilde{U}, \Gamma, \tilde{\phi}\right)$ be an orbifold chart and $\langle \cdot, \cdot \rangle$ a riemannian metric on $\tilde{U}$. For $X, Y \in T\tilde{U}$ define the Riemannian metric

$$\langle X, Y \rangle^\triangleleft = \sum_{\gamma \in \Gamma} \langle d\gamma(X), d\gamma(Y) \rangle.$$

For all $\gamma \in \Gamma$ there is an induced action $\Gamma \curvearrowright T\tilde{U}$ given by $d\gamma : T\tilde{U} \to T\tilde{U}$, an isometric action with respect to $\langle \cdot, \cdot \rangle^\triangleleft$. Take the path $\eta(t) = exp_{\tilde{x}}(tX)$, with $X \in T_{\tilde{x}}\tilde{U}$ fixed. Because isometries carry geodesics to geodesics, $(\gamma \cdot \eta)(t)$ is also a geodesic passing through $\tilde{x}$ with velocity $d_{\tilde{x}}\gamma(X)$ so $(\gamma \cdot \eta)(t) = exp_{\tilde{x}}(td_{\tilde{x}}\gamma(X))$. The exponential map $exp_{\tilde{x}} : T_{\tilde{x}}\tilde{U} \to \tilde{U}$ becomes Γ-equivariant. Furthermore, there exist $\epsilon > 0$ such that $exp_{\tilde{x}}|_{B_\epsilon}$ is a diffeomorphism. Consequently, $\Gamma_{\tilde{x}} \curvearrowright T_{\tilde{x}}\tilde{U}$ acts by a representation of the orthogonal group and letting $\tilde{U}^\triangleleft = B_\epsilon(0)$, $\Gamma^\triangleleft = \Gamma$ and $\tilde{\phi}^\triangleleft = \tilde{\phi} \circ \exp_{\tilde{x}}$, we get an orbifold chart $(\tilde{U}^\triangleleft, \Gamma^\triangleleft, \tilde{\phi}^\triangleleft)$ on $U^\triangleleft \subset U$. The exponential map provides an injection $\exp_{\tilde{x}} : \tilde{U}^\triangleleft \hookrightarrow \tilde{U}$, which implies that this orbifold chart also belongs to the same orbifold structure. Having constructed these new charts we can glue them together using as injections $\tilde{\psi}^\triangleleft_{\alpha\beta} : \tilde{U}^\triangleleft_\alpha \hookrightarrow \tilde{U}^\triangleleft_\beta$ the maps $\tilde{\psi}^\triangleleft_{\alpha\beta} := d\tilde{\psi}_{\alpha\beta}$, with $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ the injections of the original orbifold atlas. $\square$

This charts are called *linear charts*. From now on our orbifold atlases will be given by linear orbifold charts.

1.2 Orbifolds as quotients

Let $\mu : P \times G \to G$ be an action of a Lie group.

- $P \curvearrowleft G$ denotes a (right) action of G on P.
- $\mu_p := \mu(p, \cdot) : G \to P$ denotes the map $g \mapsto \mu(p, g)$.
- $\mu^g := \mu(\cdot, g) : P \to P$ denotes the map $p \mapsto \mu(p, g)$.

The action is called *proper* when the map

$$\begin{aligned} \varphi : P \times G &\to P \times P \\ (g, p) &\mapsto (p, \mu(p, g)) \end{aligned}$$

is a proper map. Moreover, the action is *free* when the isotropy group

$$G_x = \{g \in G \mid \mu(x, g) = x\},$$

is trivial for every $p \in P$.

Proposition 1.19. *Let $P \curvearrowleft G$ be a free and proper action of a Lie group on a manifold P. Then P/G has a manifold structure.*

For every $p \in P$, the properness hypothesis guarantees the existence of submanifolds $S_p \subset P$, called slices. The manifold structure in P/G is determined by the slices together with the embeddings $pr \circ \varphi : S_p \times G \to P$. More details about this proof can be found in section 3, theorem 3.34, [AB15].

Definition 1.20. *Take $p \in P$. A slice passing through p, denoted by S_p, is a G_p-invariant submanifold such that*

1. *$T_pP = d_e\mu_p(\mathfrak{g}) \oplus T_pS_p$ and $T_qP = d_e\mu_q(\mathfrak{g}) + T_qS_p$ for all $q \in S_p$.*
2. *If $q \in S_p$ and $g \in G$ are such that $\mu(q, g) \in S_p$ then $g \in G_p$.*

There are in general many slices through a point p. However, the manifold structure will be the same no matter which slice we have chosen. Slices will generate the topology on the quotient space P/G. They are related with opens in P by the tubular neighborhood associated with each slice.

Definition 1.21. *A tubular neighborhood for a slice is* $Tub(S) = \mu(S, G)$.

Define the action $S_p \times G \curvearrowleft G_p$ by

$$(s, g) \cdot g_p = (\mu(s, g_p), g_p^{-1} \cdot g).$$

If we denote $S_p \times_{G_p} G := (S_p \times G)/G_p$ then $S_p \times_{G_p} G \cong Tub(S_p)$ are diffeomorphic. As long as the action is free, we get

$$Tub(S_p)/G \cong S_p.$$

As we wil see bellow, if we allow locally free and proper actions, we will naturally obtain orbifolds instead of manifolds as quotients.

Take $\xi \in \mathfrak{g}$ and let $\Psi : P \times \mathfrak{g} \to TP$ denote the infinitesimal action associated to the G-action defined by

$$\Psi(p, \xi) = d_e\mu_p(\xi).$$

Definition 1.22. *A G-action on P is called locally free if Ψ is injective.*

Since for all $h \in G$ $ker\,(d_h\mu_p) = T_h\,(hG_p)$, a locally free action satisfies $\mathfrak{g}_p = ker\,(d_e\mu_p) = 0$. Thus, the isotropy group G_p is a 0-dimensional manifold, a discrete group. The properness hypothesis implies G_p is compact. Then G_p is finite whenever the action is locally free and proper.

Proposition 1.23. *Let $P \curvearrowleft G$ be a locally free and proper action of a Lie group on a manifold P. Then P/G has an orbifold structure.*

Proof. Let $p \in P$ and $\pi : P \to P/G$ be the projection map. Given that the topology on P/G is the quotient topology and

$$\pi^{-1}(\pi(U)) = \bigsqcup_{g \in G} gU,$$

π is a continuous, open map. Take a slice S_p at p and

$$\varphi_p = \mu|_{S_p} : S_p \times G \to Tub(S_p),$$

the restriction of the action map. Then

$$d_{(q,e)}\varphi_p(Y, \xi) = Y_q + \Psi(q, \xi).$$

Because S_p is a slice and the action is locally free, $d_{(q,e)}\varphi_p$ is an isomorphism. Furthermore, $\varphi_p(q, g) = \mu^g \circ \varphi_p(q, e)$, and then follows that $d_{(q,g)}\varphi_p$ is an isomorphism for every $q \in S_p$ and $g \in G$. By the inverse function theorem φ_p becomes a local diffeomorphism. Note that even though φ_p is not injective, it induces a diffeomorphism

$$\varphi_p : S_p \times_{G_p} G \overset{\cong}{\to} Tub(S_p).$$

Define $U_p = \pi(Tub(S_p))$. The diffeomorphism φ_p is G-invariant. Then $S_p/G_p \cong U_p$ are homeomorphic, which implies $(S_p, G_p, \tilde{\phi}_p)$ is a smooth local model for U_p, with $\tilde{\phi}_p := \pi|_{S_p}$.
The existence of smooth injections between the local models will guarantee an orbifold structure for P/G. For, let $(S_p, G_p, \tilde{\phi}_p)$ and $(S_q, G_q, \tilde{\phi}_q)$ be two local models with $U_p \subset U_q$. Since

$$\begin{array}{c} S_q \times G \curvearrowleft G_q \\ \downarrow \\ S_q \times_{G_q} G \end{array}$$

defines a principal bundle structure, there exist local sections

$$\delta_q : S_q \times_{G_q} G \to S_q \times G.$$

Consider the embeddings $\iota_p : S_p \hookrightarrow S_p \times_{G_p} G$ and $\psi_{pq} : S_p \hookrightarrow S_q$ defined by $\iota_p(x) = [x, e]$ and

$$\psi_{pq}(r) = \left(pr_1 \circ \delta_q \circ \varphi_q^{-1} \circ \iota \circ \varphi_p \circ \iota_p\right)(r).$$

In addition, let $\vartheta : S_p \to G$ be defined by

$$\vartheta(r) = \left(pr_2 \circ \delta_q \circ \varphi_q^{-1} \circ \iota \circ \varphi_p \circ \iota_p\right)(r).$$

Given that $\mu(p, \vartheta(p)^{-1}) = \psi_{pq}(p) = q$, if $g_p \in G_p$ then

$$\vartheta(p) \cdot g_p \cdot \vartheta(p)^{-1} \in G_q.$$

Fix g_p, the map $\hat{\theta} : S_p \to S_q \times G_q$ defined by

$$\hat{\theta}(r) = (\psi_{pq}(r), \vartheta(r) \cdot g_p \cdot \vartheta(r)^{-1}),$$

induces a monomorphism $\theta : G_p \to G_q$, as long as S_p is connected, $\hat{\theta}$ continue and G_q discrete. Furthermore, ψ_{pq} is a θ-equivariant map, i.e., $\psi_{pq} : S_p \hookrightarrow S_q$ is a smooth injection. $\square$

Example 1.24. Let $\mathbb{S}^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + |z_1|^2 + \ldots + |z_n|^2 = 1\}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{Z}^+$ be co-primes. Define the action $\mu : \mathbb{S}^{2n+1} \times \mathbb{S}^1 \to \mathbb{S}^{2n+1}$ by

$$\mu\left((z_0, z_1, \ldots, z_n), e^{i\theta}\right) = \left(e^{ia_0\theta} z_0, e^{ia_1\theta} z_1, \ldots, e^{ia_n\theta} z_n\right).$$

For $z = (z_0, z_1, \ldots, z_n)$ fixed we get an injective map

$$d_1\mu_z = \begin{pmatrix} ia_0 z_0 \\ ia_1 z_1 \\ \vdots \\ ia_n z_n \end{pmatrix},$$

for every $z \in \mathbb{S}^{2n+1}$. Thus, this action is locally free. Moreover, it is proper because $\mathbb{S}^1$ is compact. By the previous proposition, the quotient space $\mathbb{S}^{2n+1}/\mathbb{S}^1$ has an orbifold structure. Denote this orbifold structure by $\mathbb{WP}(a_0, \ldots, a_n)$. It is called the weighted projective space with weights $a_0, \ldots, a_n$.

Let $z \in \mathbb{S}^{2n+1}$ be a point with non-trivial isotropy. That means

$$\mu(z, e^{i\theta}) = z,$$

with $e^{i\theta} \neq 1$. Then

$$e^{ia_j\theta} z_j = z_j,$$

for all j. If $z_j \neq 0$ and $z_k \neq 0$ for $j \neq k$ then

$$a_k = \frac{a_j c_k}{c_j},$$

for two integers $c_j, c_k \in \mathbb{Z}$, with $0 < c_j < a_j$ and $0 < c_k < a_k$. Because the numbers a_j, a_k are co-primes, this equality can not hold. Then $z = (z_1, \ldots, z_n)$ have non-trivial isotropy when all but one z_j equals to zero.

Take $w = (0, 0, \ldots, 0, w_j, 0, \ldots, 0) \in \mathbb{S}^{2n+1}$. The isotropy group $\mathbb{S}^1_w$ is given by the elements $e^{i\theta} \in \mathbb{S}^1$ such that $e^{ia_j\theta} = 1$, the roots of the unity. Thus $\mathbb{S}^1_w \cong \left\langle e^{i2\pi/a_j} \right\rangle = \mathbb{Z}_{a_j}$.

Let $E_j = \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} \mid |z_j| = 1\}$, with $0 \leq j \leq n$. Its isotropies are

$$\Gamma_x = \begin{cases} \mathbb{Z}_{a_j} & if \quad x \in E_j \\ \\ e & if \quad x \notin E_j. \end{cases}$$

A slice passing through $w_j \in E_j$ is given by

$$S_{w_j} = \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} \mid |z_j| \neq 0, Arg(z_j) = Arg(w_j)\}.$$

The tubular neighborhoods associated to this slices are

$$Tub(S_{w_j}) = \{(z_0, \ldots, z_n) \in \mathbb{S}^{2n+1} \mid |z_j| \neq 0\}.$$

Take $\pi : \mathbb{S}^{2n+1} \to \mathbb{WP}(a_0, \ldots, a_n)$ the projection map and let

$$U_{w_j} = \pi(Tub(S_{w_j})).$$

Hence, $(S_{w_j}, \mathbb{Z}_{a_j}, \pi|_{S_{w_j}})$ is a local model for U_{w_j}. Choosing $w_j \in E_j$ for each equator, we get

$$\bigcup_{j=0}^{n} Tub(S_{w_j}) = \mathbb{S}^{2n+1}.$$

This means the local models $(S_{w_j}, \mathbb{Z}_{a_j}, \pi|_{S_{w_j}})$ generate the orbifold structure for the weighted projective space. In fact, $(Tub(S_{w_j}), \mathbb{Z}_{a_j}, \pi|_{S_{w_j}})$ generate the same orbifold structure, i.e., the orbifold charts look like the manifold charts for the projective space, plus isotropies due to the weights.

In particular, take $\mathbb{WP}(1, a) \cong \mathbb{S}^3/\mathbb{S}^1$. The points with non-trivial isotropy are the ones on the equator $E_1 = \{(w_0, w_1) \in \mathbb{S}^3 : |w_1| = 1\}$, with isotropy $\mathbb{Z}_a$. For all $w = (w_0, w_1) \in \mathbb{S}^3 - E_1$ there exist a unique $e^{i\theta}$ such that $\mu(w, e^{i\theta}) \in \mathbb{S}^2_+$, where $\mathbb{C} \times \mathbb{R} \supset \mathbb{S}^2_+ = \{(z, x) \in \mathbb{S}^2 : x \geq 0\}$. All the equator E_1 maps onto the equator of $\mathbb{S}^2_+$. Topologically, the quotient space will be $\mathbb{S}^2_+/\sim$, with $(z_1, x_1) \sim (z_2, x_2)$ if and only if $x_1 = x_2 = 0$. That means $\mathbb{S}^2_+/\sim \cong \mathbb{S}^2$ are homeomorphic. However, every chart of $\mathbb{S}^2$ that contains the south pole is of the form $(\tilde{U}, \mathbb{Z}_a, \phi_a)$, the local model for a cone. Therefore $\mathbb{WP}(1, a)$ is a 2-sphere with a cone point on the south pole or equivalently on the north pole. $\mathbb{WP}(1, a)$ looks like a teardrop.

1.3 Maps between orbifolds

Satake's works on orbifolds (V-manifolds!) suggested a close relationship with manifolds. Maps between orbifolds were defined locally. For any injection on the source orbifold, there is another on the target orbifold that commutes with the local lifts, [Sat57] section 2 (this is just the definition of manifold maps but on the orbifold context). However, maps are given up to an equivalence relation (this differs from the definition of maps on manifolds because many liftings exist over a chart, unlike the manifold case where only one exists). The way injections are related do not imply the existence of group homomorphisms between the orbifold charts. Furthermore, they do not allow us to define the pullback of an orbibundle uniquely.

A notion of orbifold map that considers these two situations is that of good orbifold map, see [CR01] definition 4.4.1. Remember orbifolds can be thought of as groupoids up to Morita invariance. Maps are smooth homomorphisms between the groupoids, up to Morita equivalence. In the groupoid context, this is the notion of a strong map, [MP97] section 5. It turns out that good maps and strong maps are equivalent, see [LU04] proposition 5.1.7. Because G-structure theory is based on the relation between the tangent orbibundle and frame orbibundle, we are interested in maps between principal orbibundles that come from an orbifold diffeomorphism between the basis orbifolds. To preserve the structures constructed, we will need our maps to be good maps.
There are more types of orbifold maps; one of the most natural is Sa-

take's definition but with extra information: group homomorphisms between the local charts. These different types of orbifold maps can be found on [BB13]. In particular, the notion of complete orbifold maps, where the equivalence between orbifold maps is given by the germs plus the equality of the group homomorphisms, coincides with the good orbifold maps. Reduced orbifold maps are equivalent to the notion of orbifold map given in [ALR07].

Let $\mathcal{O}_1$ be an n-dimensional orbifold and $\mathcal{O}_2$ be an m-dimensional orbifold with atlases $\{(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}^1_\alpha)\}_{\alpha\in J}$ and $\{(\tilde{V}_\upsilon, \Upsilon_\upsilon, \tilde{\phi}^2_\upsilon)\}_{\upsilon\in K}$, respectively.

Definition 1.25. *A smooth local lift for a continuous map $f : \mathcal{O}_1 \to \mathcal{O}_2$, around $x \in \mathcal{O}_1$, is given by:*

1. *Orbifold charts $\left(\tilde{U}, \Gamma, \tilde{\phi}^1\right)$ and $\left(\tilde{V}, \Upsilon, \tilde{\phi}^2\right)$ with $x \in U$, $f(x) \in V$ and $f(U) \subset V$.*
2. *A smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$ such that $f \circ \tilde{\phi}^1 = \tilde{\phi}^2 \circ \tilde{f}$.*
3. *For all $\gamma \in \Gamma$ there exists a $\hat{\gamma} \in \Upsilon$ with $\hat{\gamma}\tilde{f} = \tilde{f}\gamma$.*

The last condition does not imply the existence of a homomorphism-from Γ to Υ.

Example 1.26. Take the orbifolds $C = \mathbb{R}^2/\mathbb{Z}_k$ and $[0, \infty) = \mathbb{R}/\mathbb{Z}_2$. Define $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ as

$$\tilde{h}(x, y) = \sqrt{x^2 + y^2}.$$

Given that $\tilde{h}$ is $\mathbb{Z}_k$-invariant, it defines the lifts for a smooth orbifold map $h : C \to [0, \infty)$.

Take $a \in \mathbb{Z}^+$ and let $B_1(0) \subset \mathbb{R}^2$ be the open ball centered at the origin of radius 1. The rotation by any angle is a well-defined action on $B_1(0)$. Consequently $C_{1,a} = B_1(0)/\mathbb{Z}_a$ is an orbifold.

Example 1.27. Consider $C_{1,a}$ and $\mathbb{WP}(1, a)$. Recall that $(0, 1) \in \mathbb{WP}(1, a)$ have non-trivial isotropy equal to $\mathbb{Z}_a$. Also, let $S_{(0,1)}$ be a slice passing through $(0, 1)$ as in example 1.24. Let $\tilde{f} : B_1(0) \to S_{(0,1)}$ be

$$\tilde{f}(x, y) = (x + iy, \sqrt{1 - x^2 - y^2} + 0i).$$

The $\mathbb{Z}_a$-equivariant function $\tilde{f}$ induces a map $f : C_{1,a} \to \mathbb{WP}(1,a)$. This map stresses that the open sets containing the points with non-trivial isotropy looks like a cone.

Let $\tilde{f}_\beta : \tilde{U}_\beta \to \tilde{V}_\beta$ be a smooth local lift. If $U_\alpha \subset U_\beta$, there exists an orbifold chart $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ and an injection $\tilde{\psi}^1_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$. Define $\tilde{f}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{V}_\beta$ by $\tilde{f}_{\alpha\beta} := \tilde{f}_\beta \circ \tilde{\psi}^1_{\alpha\beta}$. Let $(\tilde{\phi}^2_\beta \circ \tilde{f}_{\alpha\beta})(\tilde{U}_\alpha) := V_\alpha$; it is a connected open subset $V_\alpha \subset V_\beta$. Thus, there exists an orbifold chart $(\tilde{V}_\alpha, \Upsilon_\alpha, \tilde{\phi}^2_\alpha)$ and an injection $\tilde{\psi}^2_{\alpha\beta} : \tilde{V}_\alpha \hookrightarrow \tilde{V}_\beta$. Take the lift $\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha$ defined by $\tilde{f}_\alpha := \left(\tilde{\psi}^2_{\alpha\beta}\right)^{-1} \circ \tilde{f}_{\alpha\beta}$. We obtain the following commutative diagram

$$\begin{tikzcd}
 & \tilde{U}_\beta \arrow[rr, "\tilde{f}_\beta"] \arrow[dd, dashed, "\tilde{\phi}^1_\beta" near start] & & \tilde{V}_\beta \arrow[dd, "\tilde{\phi}^2_\beta"] \\
\tilde{U}_\alpha \arrow[ur, "\tilde{\psi}^1_{\alpha\beta}"] \arrow[rr, "\tilde{f}_\alpha" near start] \arrow[dd, "\tilde{\phi}^1_\alpha"] & & \tilde{V}_\alpha \arrow[ur, "\tilde{\psi}^2_{\alpha\beta}"] \arrow[dd, "\tilde{\phi}^2_\alpha" near start] & \\
 & U_\beta \arrow[rr, dashed, "f" near end] & & V_\beta \\
U_\alpha \arrow[ur, dashed, "\subset"] \arrow[rr, "f"] & & V_\alpha \arrow[ur, "\subset"] &
\end{tikzcd}$$

We could have chosen a different orbifold chart over V_α. Its injection will be of the form $\tilde{\psi}^2_{\alpha\beta} \cdot \overline{\gamma}$, for a fixed $\overline{\gamma} \in \Upsilon_\alpha$, and $\overline{\gamma}^{-1} \cdot \tilde{f}_\alpha$ will be the smooth local lift induced by $\tilde{f}_\beta$.

Definition 1.28. *Take two smooth liftings of f, $\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha$ and $\tilde{f}_\beta : \tilde{U}_\beta \to \tilde{V}_\beta$. They are isomorphic if there are isomorphisms between the orbifold charts $\tilde{\psi}^1_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ and $\tilde{\psi}^2_{\alpha\beta} : \tilde{V}_\alpha \to \tilde{V}_\beta$ such that $\tilde{\psi}^2_{\alpha\beta} \circ \tilde{f}_\alpha = \tilde{f}_\beta \circ \tilde{\psi}^1_{\alpha\beta}$.*

Hence a smooth lift for $f_\beta : U_\beta \to V_\beta$ induces smooth lifts for every open $U_\alpha \subset U_\beta$. All the possible lifts $\tilde{f}_\alpha$ are isomorphic.

Definition 1.29. *Two lifts $\tilde{f}_\alpha$, $\tilde{f}_\beta$ are equivalent at $x \in U_\alpha \cap U_\beta$ as germs, denoted $\tilde{f}_\alpha \sim_x \tilde{f}_\beta$, if there exists an orbifold chart $x \in U_\sigma \subset$*

$U_\alpha \cap U_\beta$ such that the induced lifts $\tilde{f}_{\sigma\alpha} : \tilde{U}_\sigma \to \tilde{V}_\sigma$, from $\tilde{f}_\alpha$, and $\tilde{f}_{\sigma\beta} : \tilde{U}_\sigma \to \tilde{V}_\sigma$, from $\tilde{f}_\beta$, are isomorphic lifts.

Having defined local liftings for orbifold maps we can define a lift for an orbifold map.

Definition 1.30. *Let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be a continuous function. A smooth lift $\tilde{f}$ is given by a system of smooth local liftings $\tilde{f}_x : \tilde{U}_x \to \tilde{V}_x$ for every $x \in \mathcal{O}_1$ such that $\tilde{f}_x \sim_z \tilde{f}_y$, for every $z \in U_x \cap U_y$.*

Definition 1.31. *An orbifold map is a continuous map $f : \mathcal{O}_1 \to \mathcal{O}_2$ together with a germ of liftings $\tilde{f}$. If the liftings are of class C^k then the orbifold map is said to be of class C^k.*

Because all the injections between orbifold charts and group actions are taken to be C^∞, then a lift $\tilde{f}_\alpha$ is of class C^k if and only if any other lift $\tilde{f}_\beta$ is of class C^k.

Example 1.32. Define $\tilde{\eta}_1, \tilde{\eta}_2 : (-1,1) \to \mathbb{R}^2$ by $\tilde{\eta}_1(t) = (t, |t|)$ and $\tilde{\eta}_2(t) = (t,t)$. Both $\tilde{\eta}_1, \tilde{\eta}_2$ project onto the same path on $\mathbb{R}^2/\mathbb{Z}_4$. They are lifts of the same map $\eta : (-1,1) \to \mathbb{R}^2/\mathbb{Z}_4$ but are not on the same equivalence class because they are not equal as germs of functions around $0 \in \mathbb{R}^2$. The regularity of η, as a map between orbifolds, depends on the choice of the lift.

On orbifold category, a map will have certain property if this property holds for every lifting. For example, if one lift of an orbifold map has constant rank, then all lifts will have the same property. It follows that having constant rank is a well-defined property of an orbifold map. In particular, that the lifts are immersions is a well-defined property of an orbifold map. Take a lift $\tilde{f} : \tilde{U} \to \tilde{V}$, and the smooth map $d\tilde{f} : T\tilde{U} \to T\tilde{V}$. If the local lifts of the orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$ are immersions, then for $\gamma_1, \gamma_2 \in \Gamma$ and $\theta : \Gamma \to \Upsilon$ the equation

$$\theta(\gamma_1) \cdot d\tilde{f}(\tilde{X}) = \theta(\gamma_2) \cdot d\tilde{f}(\tilde{X}),$$

implies

$$d\tilde{f}(\gamma_1 \cdot \tilde{X}) = d\tilde{f}(\gamma_2 \cdot \tilde{X}),$$

where the Γ action means $\gamma \cdot \tilde{X} := d\gamma(\tilde{X})$. Given that the differential is injective $\gamma_1\gamma_2^{-1} \cdot \tilde{X} = \tilde{X}$. If we have chosen a point with trivial isotropy then $\gamma_1 = \gamma_2$. If not, this will be true for almost every point, which implies (see the proof in lemma 1 [Sat57]) $\gamma_1 = \gamma_2$; that means θ is injective. However, if the lifts of f are submersions, it is not true that the homomorphism θ is surjective. For example, take $\mathbb{R}^2/\mathbb{Z}_4$ and $\mathbb{R}^2/\mathbb{Z}_8$ with the identity map $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ as injection and the homomorphism $\theta : \mathbb{Z}_4 \hookrightarrow \mathbb{Z}_8$ defined by $1 \bmod 4 \mapsto 2 \bmod 8$. Clearly ι is a submersion, but θ is not surjective.

Definition 1.33. *A smooth orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a submersion (immersion) if the local lifts are submersions (immersions) and the local homomorphisms are surjective (injective). It is a diffeomorphism if the local lifts are diffeomorphisms and the local homomorphisms are isomorphisms.*

The conditions imposed for submersions (immersions) on the homomorphisms guarantees that the maps satisfies classical theorems satisfied on manifold category: the local form submersion (immersion) theorem ([CJ19] section 3.2), the regular value theorem ([BB12] section 4) among other classical results. This is due to the close relationship between manifolds and analysis on $\mathbb{R}^n$, and orbifolds and manifolds. The techniques are similar, i.e., using analysis locally and proving that the property does not depend on the choise of a local chart. Furthermore, locally, some geometric constructions can induce globally defined geometric structures by gluing together the local informations. The gluing process requires the existence of partitions of unity. Take an orbifold atlas $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ and a partition of the unity $\rho_\alpha : \tilde{U}_\alpha \to \mathbb{R}$ subordinate to this atlas (whose existence is guaranteed by the Hausdorff and paracompactness hypothesis). Now define $\lambda_\alpha : \tilde{U}_\alpha \to \mathbb{R}$ by

$$\lambda_\alpha(\tilde{x}) = \frac{1}{|\Gamma_\alpha|} \sum_{\gamma \in \Gamma_\alpha} \rho_\alpha(\gamma \cdot \tilde{x}).$$

They are continuous functions that define a Γ_α-invariant partition of the unity.

Because the orbifold structures that appears on G-structure theory are quotients of manifolds by locally free and proper actions P/G, the orbifold maps we are interested enjoys particular properties. For, let $\mathcal{O}_1 = P_1/G$, $\mathcal{O}_2 = P_2/G$ be two orbifolds and $\theta : G \to G$ a group isomorphism. If $\tilde{f} : P_1 \to P_2$ is a θ-equivariant map, then it send slices in slices and for an arbitrary $g \in G_p$

$$\theta(g) \cdot \tilde{f}(p) = \tilde{f}(g \cdot p) = \tilde{f}(p).$$

It follows that for every $p \in P$ the restriction $\theta_p : G_p \to G_{\tilde{f}(p)}$ is an isomorphism. Thus, $\tilde{f}$ induces an orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$. The lifts of $f : \mathcal{O}_1 \to \mathcal{O}_2$ are pairs $\{\tilde{f}, \theta_p\}$, with $\tilde{f}$ a θ_p-equivariant map. This last condition is a consequence of a more general fact. Take two injections $\psi_{pq} : S_p \hookrightarrow S_q$ and $\psi_{qr} : S_q \hookrightarrow S_r$ between slices of P_1. If we write $\theta(g) = \overline{g}$, the homomorphism condition becomes

$$\overline{g_1 \cdot g_2} = \overline{g_1} \cdot \overline{g_2}. \tag{1.3.1}$$

Let $\tilde{f}(S_p) = S_{\tilde{f}(p)}$ (the same for the subindices q and r). An injection $\psi_{pq} : S_p \hookrightarrow S_q$ induces an injection

$$\overline{\psi_{pq}} : S_{\tilde{f}(p)} \hookrightarrow S_{\tilde{f}(q)}.$$

The induced injections satisfy

$$\overline{\psi_{qr} \circ \psi_{pq}} = \overline{\psi_{qr}} \circ \overline{\psi_{pq}}. \tag{1.3.2}$$

Every isotropy element $g \in G_p$ can be though of as an injection. Then condition (1.3.1) is a consequence of condition (1.3.2). That means that for every lift $\tilde{f} : S_p \to S_{\tilde{f}(p)}$, the existence of a group homomorphism $\theta_p : G_p \to H_{\tilde{f}(p)}$ such that $\tilde{f}$ is θ_p-equivariant is a consequence of the compatibility condition (1.3.2) between the injections on the orbifolds.

Returning to the general setting of orbifolds, without thinking them as quotients, let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be an orbifold map. Locally

$$\begin{tikzcd}
& \tilde{U}_\beta \arrow[rr, "\tilde{f}_\beta"] \arrow[dd, dashed, "\tilde{\phi}^1_\beta" near end] & & \tilde{V}_\beta \arrow[dd, "\tilde{\phi}^2_\beta"] \\
\tilde{U}_\alpha \arrow[ur, "\tilde{\psi}^1_{\alpha\beta}"] \arrow[rr, "\tilde{f}_\alpha" near start] \arrow[dd, "\tilde{\phi}^1_\alpha"'] & & \tilde{V}_\alpha \arrow[ur, "\tilde{\psi}^2_{\alpha\beta}"] \arrow[dd, "\tilde{\phi}^2_\alpha" near start] & \\
& U_\beta \arrow[rr, dashed, "f" near end] & & V_\beta \\
U_\alpha \arrow[ur, dashed, "\subset"] \arrow[rr, "f"] & & V_\alpha \arrow[ur, "\subset"] &
\end{tikzcd}$$

The injection $\tilde{\psi}^1_{\alpha\beta}$ induces the injection $\tilde{\psi}^2_{\alpha\beta}$ whenever the lifts $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ are chosen. We can though of as condition (1.3.2) as associating the injections induced by a system of lifts $(\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha)_{\alpha \in I}$ by the rule

$$\tilde{\psi}^1_{\beta\sigma} \circ \tilde{\psi}^1_{\alpha\beta} \mapsto \tilde{\psi}^2_{\beta\sigma} \circ \tilde{\psi}^2_{\alpha\beta}. \tag{1.3.3}$$

Definition 1.34. *A system of lifts $(\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha)_{\alpha \in I}$ for an orbifold map $f : \mathcal{O}_1 \to \mathcal{O}_2$ is called a compatible system if the induced injections satisfy* (1.3.3)*.*

Then, for $\gamma_1, \gamma_2 \in \Gamma_\alpha$ one gets

$$\gamma_1 \mapsto \hat{\gamma}_1 \quad , \gamma_2 \mapsto \hat{\gamma}_2 \quad , \gamma_1 \cdot \gamma_2 \mapsto \hat{\gamma}_1 \cdot \hat{\gamma}_2,$$

which implies that $\theta_\alpha : \Gamma_\alpha \to \Upsilon_\alpha$, defined $\theta_\alpha(\gamma_i) = \hat{\gamma}_i$, is a homomorphism of groups. Moreover, $\tilde{f}_\alpha$ is a θ_α-equivariant map. Hence, a compatible system associates for every orbifold chart $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ a pair $\{\tilde{f}_\alpha, \theta_\alpha\} : \tilde{U}_\alpha \to \tilde{V}_\alpha$, with $\tilde{f}_\alpha$ a θ_α-equivariant map.

Definition 1.35. *A map $f : \mathcal{O}_1 \to \mathcal{O}_2$ is called a good map if it admits a compatible system of liftings that belongs to the same germ.*

Remark: Not all the continuous functions between orbifolds are good maps. For a counterexample, see [CR01] example 4.4.2a.

We will see that by construction, every orbibundle will be defined in terms of good orbifold maps. In addition, constructions associated with orbibundles, such as tensor products or pullbacks, are given in terms of good orbifold maps.

Proposition 1.36. *Let $P_1 \circlearrowleft G$, $P_2 \circlearrowleft G$ be two manifolds with a locally free and proper action of a Lie group. Take the Lie group isomorphism $\theta : G \to G$. There is a 1-1 correspondence between θ-equivariant maps $\tilde{f} : P_1 \to P_2$ and good orbifold maps $f : P_1/G \to P_2/G$ with respect to θ.*

Proof. ($\Rightarrow$) Done.
($\Leftarrow$) Take a lift $\tilde{f}_p : S_p \to S_{\tilde{f}(p)}$ that belongs to the compatible system. Recall that $Tub(S_p) \cong S_p \times_{G_p} G$. Then we can extend $\tilde{f}$ to all P_1 using the tubular neighborhoods and leting $\tilde{f} : S_p \times_{G_p} G \to S_{\tilde{f}(p)} \times_{G_{\tilde{f}(p)}} G$ be defined by

$$\tilde{f}([s, g]) = \left[\tilde{f}_p(s), \theta(g)\right].$$

$\square$

Chapter 2

Orbibundles

The theory of G-structures happens on orbibundles. It relates the presence of (some) geometric structures on $\mathcal{O}$ with a principal subbundle $Q \subset Fr(\mathcal{O})$ of the frame orbibundle. Due to the effectiveness hypothesis on the orbifold charts, the frame orbibundle is a manifold ([ALR07] theorem 1.23). In addition to geometric structures on $\mathcal{O}$, also compatible connections on $T\mathcal{O}$ are related to connections on the frame bundle $Fr(\mathcal{O})$. Consequently, many objects we can address to our geometric structures, such as torsion or curvature, will also have their corresponding object on $Fr(\mathcal{O})$.

In the first section we study cone orbibundles, morphisms, and present some cone orbibundles constructions whose sections are of particular interest (they allow us to define geometric structures). The second section describes cone connections, gives an equivalent approach by connection matrices and studies parallel translation in the orbibundle setting. On the other hand, in the third section, we define principal orbibundles and morphisms. Since we are interested in subbundles of the frame orbibundle (a manifold) we will develop the theory of principal orbibundles in a specific context: a manifold with a proper and locally free action of a Lie group. Later we show how to construct from a principal orbibundle a fiber orbibundle, through associated bundles. This construction gives a 1-1 correspondence between cone orbibundles and principal orbibundles. One important object that appears is the tautological form, which comes from the identity morphisms $Id : T\mathcal{O} \to T\mathcal{O}$, represented by a section of $T^*\mathcal{O} \otimes T\mathcal{O}$. Finally, we will discuss reductions of the structural group, a critical concept that connects the existence of geometric structures with a subbundle $Q \curvearrowleft G \to \mathcal{O}$ of the frame orbibundle $Fr(\mathcal{O})$. The fourth section defines principal orbibundle connections from the perspective of horizontal distribution and principal connection. Then we show the 1-1 correspondence between cone and principal orbibundle connections.

This chapter's organization is inspired by the lecture notes [Cra15]. The material presented in this chapter develops the theory of orbibundles as close as possible to the theory of fibre bundles over manifolds. The orbibundle definition given by Satake were not enough to construct a well-defined orbibundle theory since morphisms are not well-behaved. Nevertheless, it turns out that the category of orbibundles with morphisms provided by good orbifold maps is well-defined. We encourage the reader to see the appendix of [CR01]. Besides, when orbifolds are treated as groupoids, there is a natural notion of orbibundles and morphisms; for example, see chapter 2 of [ALR07].

2.1 Cone orbibundles

Let $\mathcal{O}$ be an n-dimensional orbifold with atlas $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)_{\alpha \in I}$. The tangent orbibundle will be our guiding example to talk about the theory of cone orbibundles.

2.1.1 The tangent orbibundle

The Γ_α-action on $\tilde{U}_\alpha$ lifts to $T\tilde{U}_\alpha$ by $\gamma \cdot \tilde{X} = d_{\tilde{x}}\gamma(\tilde{X})$. The natural projection $\tilde{\pi}_\alpha : T\tilde{U}_\alpha \to \tilde{U}_\alpha$ becomes Γ_α-equivariant, inducing a map on the quotients such that

$$\begin{tikzcd}
T\tilde{U}_\alpha \arrow[r, "\tilde{\pi}_\alpha"] \arrow[d] & \tilde{U}_\alpha \arrow[d] \\
T\tilde{U}_\alpha/\Gamma_\alpha \arrow[r, "\pi_\alpha"'] & \tilde{U}_\alpha/\Gamma_\alpha
\end{tikzcd}$$

is a commutative diagram. Take an orbifold chart $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$ such that $U_\alpha \subset U_\beta$. Thus, there exists a smooth injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ which induces the smooth map $d\tilde{\psi}_{\alpha\beta} : T\tilde{U}_\alpha \to T\tilde{U}_\beta$. Because every injection has an associated monomorphism $\theta_{\alpha\beta} : \Gamma_\alpha \to \Gamma_\beta$, for every $\gamma \in \Gamma_\alpha$, we get

$$\begin{aligned} d\tilde{\psi}_{\alpha\beta}(\gamma \cdot \tilde{X}) &= d\tilde{\psi}_{\alpha\beta}(d\gamma(\tilde{X})) \\ &= d(\tilde{\psi}_{\alpha\beta} \circ \gamma)(\tilde{X}) \\ &= d(\theta_{\alpha\beta}(\gamma) \circ \tilde{\psi}_{\alpha\beta})(\tilde{X}) \\ &= \theta_{\alpha\beta}(\gamma) \cdot d\tilde{\psi}_{\alpha\beta}(\tilde{X}). \end{aligned}$$

Therefore $(d\tilde{\psi}_{\alpha\beta}, \theta_{\alpha\beta})$ defines a smooth injection. For every α, the quotient $T\tilde{U}_\alpha/\Gamma_\alpha$ inherits the quotient topology. Define the topological space

$$Y := \bigsqcup_{\alpha \in I} \alpha \times \left(T\tilde{U}_\alpha/\Gamma_\alpha\right),$$

where every $\alpha \times T\tilde{U}_\alpha/\Gamma_\alpha$ is an open. The relations between the opens of Y allow the construction of a new topological space as follows. Let $(\alpha, [\tilde{X}_\alpha])$, $(\beta, [\tilde{X}_\beta]) \in Y$ and define the equivalence relation generated by $(\alpha, [\tilde{X}_\alpha]) \sim (\beta, [\tilde{X}_\beta])$ if there exists an injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ such that $d\tilde{\psi}_{\alpha\beta}(\tilde{X}_\alpha) = \tilde{X}_\beta$. Take the quotient

$$T\mathcal{O} := Y/\sim,$$

and the projection map $p : Y \to T\mathcal{O}$. Every open on Y is given by arbitrary unions of elements of the form $\alpha \times W \subset Y$, where $W \subset T\tilde{U}_\alpha$ is open. Moreover

$$p^{-1}(p(\alpha \times W)) = \bigsqcup_{\beta \in I} \beta \times \left(d\tilde{\psi}_{\alpha\beta}(W)/\Gamma_\beta\right),$$

and therefore $p : Y \to T\mathcal{O}$ is an open continuous map. Take the restriction $p_\alpha := p|_{\alpha \times (T\tilde{U}_\alpha/\Gamma_\alpha)}$, if $p_\alpha([\tilde{X}_1]) = p_\alpha([\tilde{X}_2])$ then there exists $\gamma \in \Gamma_\alpha$ such that $\gamma \cdot \tilde{X}_1 = \tilde{X}_2$ which means $[\tilde{X}_1] = [\tilde{X}_2]$. Hence, we get a homeomorphism $p : \alpha \times (T\tilde{U}_\alpha/\Gamma_\alpha) \to T\mathcal{O}$ onto its image. Given that the opens $p_\alpha\Big(\alpha \times (T\tilde{U}_\alpha/\Gamma_\alpha)\Big) := TU_\alpha$ forms a basis for the quotient topology on $T\mathcal{O}$, we get a $2n$-dimensional orbifold structure on $T\mathcal{O}$. The Γ_α-equivariant projections $\tilde{\pi}_\alpha : T\tilde{U}_\alpha \to \tilde{U}_\alpha$ induces an orbifold map $\pi : T\mathcal{O} \to \mathcal{O}$ because, locally, we have the following commutative diagram

$$\begin{tikzcd}
T\tilde{U}_\alpha \arrow[r, "\tilde{\pi}_\alpha"] \arrow[d] & \tilde{U}_\alpha \arrow[d] \\
T\tilde{U}_\alpha/\Gamma_\alpha \arrow[r, "\pi_\alpha"'] \arrow[d] & \tilde{U}_\alpha/\Gamma_\alpha \arrow[d] \\
TU_\alpha \arrow[r, "\pi"'] & U_\alpha.
\end{tikzcd}$$

If $\tilde{x} \in \tilde{U}_\alpha$ projects onto $x \in \mathcal{O}$, each tangent space $T_{\gamma\cdot\tilde{x}}\tilde{U}_\alpha$ projects onto x too. Consequently, each fiber is given by the quotient

$$\pi^{-1}(x) = T_{\tilde{x}}\tilde{U}_\alpha/(\Gamma_\alpha)_{\tilde{x}}.$$

Remark: If $(\Gamma_\alpha)_{\tilde{x}} \neq \{e\}$ then this is not a vector space but what is called a *cone.*

2.1.2 Cone orbibundles and morphisms

The key concepts that allow the construction of the orbifold structure on the tangent orbibundle are, firstly, the bundle structure

$$\begin{tikzcd}
\Gamma_\alpha \curvearrowright T\tilde{U}_\alpha \arrow[d, "\tilde{\pi}_\alpha"] \\
\Gamma_\alpha \curvearrowright \tilde{U}_\alpha
\end{tikzcd}$$

on an orbifold atlas such that the projection is Γ_α-equivariant. Secondly, for each injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ there exists an associated injection $d\tilde{\psi}_{\alpha\beta} : T\tilde{U}_\alpha \hookrightarrow T\tilde{U}_\beta$ such that

$$\begin{tikzcd}
T\tilde{U}_\alpha \arrow[r, "d\tilde{\psi}_{\alpha\beta}"] \arrow[d, "\pi_\alpha"'] & T\tilde{U}_\beta \arrow[d, "\pi_\beta"] \\
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"'] & \tilde{U}_\beta
\end{tikzcd}$$

commutes. The action by an element $\gamma \in \Gamma_\alpha$ is an injection which means that the induced action on $T\tilde{U}_\alpha$ is a consequence of this compatibility between injections.

Definition 2.1. *An orbifold $\mathcal{E}$ defines an orbibundle over $\mathcal{O}$ if*

1. *For each orbifold chart $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ there exists an orbifold chart $(\tilde{\mathcal{E}}_\alpha, \Gamma_\alpha, \tilde{\varphi}_\alpha)$ on $\mathcal{E}$ and a map $\tilde{\pi}_\alpha : \tilde{\mathcal{E}}_\alpha \to \tilde{U}_\alpha$ defining a bundle structure.*
2. *Each injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ defines an injection $\tilde{\psi}^{\mathcal{E}}_{\alpha\beta} : \tilde{\mathcal{E}}_\alpha \hookrightarrow \tilde{\mathcal{E}}_\beta$ such that*

$$\begin{tikzcd} \tilde{\mathcal{E}}_\alpha \arrow[r, "\tilde{\psi}^{\mathcal{E}}_{\alpha\beta}"] \arrow[d, "\tilde{\pi}_\alpha"'] & \tilde{\mathcal{E}}_\beta \arrow[d, "\tilde{\pi}_\beta"] \\ \tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"'] & \tilde{U}_\beta \end{tikzcd}$$

   commutes.

A *cone orbibundle* of rank k is an orbibundle $\mathcal{E}$ such that, locally, $\tilde{\mathcal{E}}_\alpha \cong \tilde{U}_\alpha \times \mathbb{R}^k$ are diffeomorphic and each induced injection $\tilde{\psi}^{\mathcal{E}}_{\alpha\beta}$ is linear over the fibers. A trivialization, called a local frame, for a cone orbibundle is a system of local sections $\tilde{s}^\alpha_i : \tilde{U}_\alpha \to \tilde{\mathcal{E}}_\alpha$ such that $(\tilde{s}^\alpha_i(\tilde{x}))^k_{i=1}$ is a basis for $(\tilde{\mathcal{E}}_\alpha)_{\tilde{x}}$. Explicitly the diffeomorphism $\xi_\alpha : \tilde{U}_\alpha \times \mathbb{R}^k \to \tilde{\mathcal{E}}_\alpha$ is given by

$$\xi_\alpha(\tilde{x}, v^1, \ldots, v^k) = \sum_{i=1}^k v^i \tilde{s}^\alpha_i(\tilde{x}).$$

Take two local frames $(\xi_\alpha(x, e_i))^k_{i=1}$ and $(\xi_\beta(x, e_i))^k_{i=1}$ for $\tilde{U}_\alpha$ and $\tilde{U}_\beta$. An injection $\tilde{\psi}^{\mathcal{E}}_{\alpha\beta} : \mathcal{E}|_{\tilde{U}_\alpha} \hookrightarrow \mathcal{E}|_{\tilde{U}_\beta}$ induces the injection

$$\tilde{\psi}^{\times}_{\alpha\beta} : \tilde{U}_\alpha \times \mathbb{R}^k \hookrightarrow \tilde{U}_\beta \times \mathbb{R}^k,$$

defined by

$$\tilde{\psi}^{\times}_{\alpha\beta}(\tilde{x}, v) = \left(\xi^{-1}_\beta \circ \tilde{\psi}^{\mathcal{E}}_{\alpha\beta} \circ \xi_\alpha\right)(\tilde{x}, v).$$

There is an explicit description for $\tilde{U}_\alpha \times \mathbb{R}^k \overset{\tilde{\psi}^\times_{\alpha\beta}}{\to} \tilde{U}_\beta \times \mathbb{R}^k$ in terms of a smooth map $g_{\alpha\beta} : \tilde{U}_\alpha \to GL_k(\mathbb{R})$, because of the linearity of the injections, given by

$$\tilde{\psi}^\times_{\alpha\beta}(\tilde{x}, v) = (\tilde{\psi}_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x})v).$$

Hence every $\gamma \in \Gamma_\alpha$ induces an action on $\tilde{U}_\alpha \times \mathbb{R}^k$, encoded on the smooth map $g_\gamma : \tilde{U}_\alpha \to GL_k(\mathbb{R})$. The maps $g_{\alpha\beta}$ are called *transition functions* associated with the orbibundle structure $\mathcal{E} \to \mathcal{O}$.
Take injections $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$, $\tilde{\psi}_{\beta\eta} : \tilde{U}_\beta \to \tilde{U}_\eta$ and $\tilde{\psi}_{\alpha\eta} : \tilde{U}_\alpha \to \tilde{U}_\eta$. Given that the action is effective there exists a unique $\gamma \in \Gamma_\alpha$ such that $\theta_{\alpha\eta}(\gamma) \circ \tilde{\psi}_{\alpha\eta} = \tilde{\psi}_{\beta\eta} \circ \tilde{\psi}_{\alpha\beta}$, which implies

$$g_{\theta_{\alpha\eta}(\gamma)}(\tilde{\psi}_{\alpha\eta}(\tilde{x})) \cdot g_{\alpha\eta}(\tilde{x}) = g_{\beta\eta}(\tilde{\psi}_{\alpha\beta}(\tilde{x})) \cdot g_{\alpha\beta}(\tilde{x}). \tag{2.1.1}$$

This equation is the orbifold version of the classical cocycle condition used to describe vector bundles over manifolds.

Definition 2.2. *A cone orbibundle $\mathcal{E}$ of rank k over an orbifold $\mathcal{O}$ consists of*

1. *A system of vector bundles $\pi_\alpha : \tilde{\mathcal{E}}_\alpha \to \tilde{U}_\alpha$ of rank k for an orbifold atlas on $\mathcal{O}$.*

2. *The existence of a smooth map $g_{\alpha\beta} : \tilde{U}_\alpha \to GL_k(\mathbb{R})$, for each injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$, such that (2.1.1) is satisfied.*

Example 2.3. The cotangent orbibundle.
Each orbifold chart defines a local trivialization for the tangent orbibundle structure $T\tilde{U}_\alpha \cong \tilde{U}_\alpha \times \mathbb{R}^n$ denoted $\xi_\alpha(\tilde{x}, e_i) := \left.\dfrac{\partial}{\partial x^i}\right|_{\tilde{x}} \in T_{\tilde{x}}\tilde{U}_\alpha$ and defined

$$\left.\frac{\partial}{\partial x^i}\right|_{\tilde{x}} f = \left.\frac{d}{dt}\right|_{t=0} \tilde{f}_\alpha(\tilde{x} + te_i),$$

where $f : \mathcal{O} \to \mathbb{R}$ is an orbifold map. The tangent orbibundle structure is codified on the transition maps $g_{\alpha\beta} : \tilde{U}_\alpha \to GL_n(\mathbb{R})$

$$g_{\alpha\beta}(\tilde{x}) = d_{\tilde{x}}\tilde{\psi}_{\alpha\beta},$$

where the matrix is the one that represents the linear transformation $d_{\tilde{x}}\tilde{\psi}_{\alpha\beta}$ on the basis $\left.\dfrac{\partial}{\partial x^i}\right|_{\tilde{x}}$. The duals $dx_i \in T^*\tilde{U}_\alpha$, characterized by

$$dx_i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j,$$

define a frame on $T^*\tilde{U}_\alpha$. Define $g^*_{\alpha\beta} : \tilde{U}_\alpha \to GL_n(\mathbb{R})$ by

$$g^*_{\alpha\beta} = (g^{-1}_{\alpha\beta})^T.$$

They satisfy the cocycle condition (2.1.1) and define an injection $\tilde{\psi}^*_{\alpha\beta}$ such that

$$\begin{tikzcd}
T^*\tilde{U}_\alpha \arrow[r, "\tilde{\psi}^*_{\alpha\beta}"] \arrow[d, "\tilde{\pi}_\alpha"'] & T^*\tilde{U}_\beta \arrow[d, "\tilde{\pi}_\beta"] \\
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"] & \tilde{U}_\beta
\end{tikzcd}$$

commutes. This gives a cone orbibundle structure $T^*\mathcal{O} \xrightarrow{\pi} \mathcal{O}$.

A natural way to compare two vector bundles is by collecting linear maps between the fibers that vary smoothly on the base. Because the cone orbibundles could have some fibers that are not vector spaces, but a cone, linear maps between the fibers have to consider this information. Furthermore, the smooth variation on the base adapts onto the orbifold case.

Definition 2.4. *Take $\pi_1 : \mathcal{E}_1 \to \mathcal{O}_1$ and $\pi_2 : \mathcal{E}_2 \to \mathcal{O}_2$ two cone orbibundles of rank k_1 and k_2. A morphism is given by orbifold maps $f : \mathcal{O}_1 \to \mathcal{O}_2$ and $h : \mathcal{E}_1 \to \mathcal{E}_2$ satisfying*

1. *For all $x \in \mathcal{O}_1$ and $f(x) \in \mathcal{O}_2$ there exist orbifold charts $(\tilde{U}, \Gamma, \tilde{\phi})$ and $(\tilde{V}, \Upsilon, \tilde{\varphi})$ such that*

$$\begin{tikzcd}
\tilde{U} \times \mathbb{R}^{k_1} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{\pi}_1"'] & \tilde{V} \times \mathbb{R}^{k_2} \arrow[d, "\tilde{\pi}_2"] \\
\tilde{U} \arrow[r, "\tilde{f}"'] & \tilde{V}
\end{tikzcd}$$

commutes and $\tilde{h}_{\tilde{x}} : \{\tilde{x}\} \times \mathbb{R}^{k_1} \to \{\tilde{f}(\tilde{x})\} \times \mathbb{R}^{k_2}$ is linear for all $\tilde{x} \in \tilde{U}$.

2. *Every injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ induces an injection $\tilde{\tau}_{\alpha\beta} : \tilde{V}_\alpha \to \tilde{V}_\beta$ such that $\tilde{h}_\beta \circ \tilde{\psi}^\times_{\alpha\beta} = \tilde{\tau}^\times_{\alpha\beta} \circ \tilde{h}_\alpha$. This means that*

$$\begin{tikzcd}
 & \tilde{U}_\beta \times \mathbb{R}^{k_1} \arrow[rr, "\tilde{h}_\beta"] \arrow[dd, dashed, "\tilde{\pi}_1"] & & \tilde{V}_\beta \times \mathbb{R}^{k_2} \arrow[dd, "\tilde{\pi}_2"] \\
\tilde{U}_\alpha \times \mathbb{R}^{k_1} \arrow[ur, "\tilde{\psi}^\times_{\alpha\beta}"] \arrow[rr, "\tilde{h}_\alpha"] \arrow[dd, "\tilde{\pi}_1"] & & \tilde{V}_\alpha \times \mathbb{R}^{k_2} \arrow[ur, "\tilde{\tau}^\times_{\alpha\beta}"] \arrow[dd, "\tilde{\pi}_2"] & \\
 & \tilde{U}_\beta \arrow[rr, dashed, "\tilde{f}_\beta"] & & \tilde{V}_\beta \\
\tilde{U}_\alpha \arrow[ur, dashed, "\tilde{\psi}_{\alpha\beta}"] \arrow[rr, "\tilde{f}_\alpha"] & & \tilde{V}_\alpha \arrow[ur, "\tilde{\tau}_{\alpha\beta}"] &
\end{tikzcd}$$

commutes.

Because the maps $\tilde{h}_\alpha$ are linear for each fiber then

$$\tilde{h}_\alpha(\tilde{x}, v) = (\tilde{f}_\alpha(\tilde{x}), k_\alpha(\tilde{x})v), \tag{2.1.2}$$

where $k_\alpha : \tilde{U}_\alpha :\to M_{k_1,k_2}(\mathbb{R})$ is a smooth map representing the linear transformations $(\tilde{h}_\alpha)_{\tilde{x}}$. Take the transition maps on $\mathcal{E}_1$ and $\mathcal{E}_2$ given by $g^1_{\alpha\beta} : \tilde{U}_\alpha \to GL_{k_1}(\mathbb{R})$ and $g^2_{\alpha\beta} : \tilde{V}_\alpha \to GL_{k_2}(\mathbb{R})$. The definition of a morphism implies that

$$k_\beta \left(\tilde{\psi}_{\alpha\beta}(\tilde{x})\right) = g^2_{\alpha\beta}(\tilde{f}_\alpha(\tilde{x})) \cdot k_\alpha(\tilde{x}) \cdot \left(g^1_{\alpha\beta}(\tilde{x})\right)^{-1}. \tag{2.1.3}$$

Proposition 2.5. *Take $\pi_1 : \mathcal{E}_1 \to \mathcal{O}_1$ and $\pi_2 : \mathcal{E}_2 \to \mathcal{O}_2$ two cone orbibundles of rank k_1 and k_2. An orbifold map $h : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism if there exists a system of smooth maps $k_\alpha : \tilde{U}_\alpha \to M_{k_1,k_2}(\mathbb{R})$ over an orbifold atlas such that $\tilde{h}_\alpha$ satisfies (2.1.2) and (2.1.3).*

Example 2.6. Let $\pi_1 : T\mathcal{O} \to \mathcal{O}$ and $\pi_2 : T^*\mathcal{O} \to O$ be the tangent and cotangent orbibundles with the structures already given. If the charts on $\mathcal{O}$ are such that $g_{\alpha\beta}(\tilde{x}) \in O(n)$, then

$$g^*_{\alpha\beta} = \left(g^{-1}_{\alpha\beta}\right)^T = g_{\alpha\beta}.$$

Choose $k_\alpha = Id$ for all charts. Then, because equation (2.1.3) is satisfied, we obtain an orbibundle morphism $h : T\mathcal{O} \to T^*\mathcal{O}$ covering the identity map $Id : \mathcal{O} \to \mathcal{O}$.

We will see that reducing the Lie group $GL_n(\mathbb{R})$ to the Lie group $O(n)$ on the frame orbibundle is the same as requiring the condition supposed in the previous example.

2.1.3 Operations with cone orbibundles

Let $\pi : \mathcal{E} \to \mathcal{O}$ be a cone orbibundle with transition functions $g_{\alpha\beta}$ associated with the orbifold structure $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)_{\alpha\in J}$ and the local frames $(\tilde{s}_i^\alpha)_{i=1}^k$. We want to construct new cone orbibundles out of $\mathcal{E}$.

Example 2.7. Tensor product.
Take the (r, s)-type tensor product of $\mathcal{E}$, that is

$$\mathcal{E}^{r,s} := \overbrace{\mathcal{E}^* \otimes \cdots \otimes \mathcal{E}^*}^{r} \otimes \underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{s}.$$

To make sense of this space, we want to construct an orbifold structure that induces the orbibundle structure $\pi_{r,s} : \mathcal{E}^{r,s} \to \mathcal{O}$. Let $\tilde{s}^i_\alpha \in \tilde{\mathcal{E}}^*_\alpha$ be characterized by

$$\tilde{s}^i_\alpha(\tilde{s}^\alpha_j) = \delta^i_j.$$

Define

$$\tilde{s}^{i_1,\ldots,i_{r+s}}_\alpha := \tilde{s}^{i_1}_\alpha \otimes \cdots \otimes \tilde{s}^{i_r}_\alpha \otimes \tilde{s}^\alpha_{i_{r+1}} \otimes \cdots \otimes \tilde{s}^\alpha_{i_{r+s}},$$

for $1 \leq i_1 \leq i_2 \leq \ldots \leq i_{r+s} \leq k$. All the possible values it takes defines a local frame for $\tilde{\mathcal{E}}^{r,s}_\alpha$, inducing the diffeomorphism $\tilde{U}_\alpha \times \mathbb{R}^{k(r+s)} \cong \tilde{\mathcal{E}}^{r,s}_\alpha$. For every injection $\tilde{\psi}_{\alpha\beta}$ let $g^{r,s}_{\alpha\beta} : \tilde{U}_\alpha \to GL_{k(r+s)}(\mathbb{R})$ be

$$g^{r,s}_{\alpha\beta}(\tilde{x}) = \overbrace{\left(g^{-1}_{\alpha\beta}(\tilde{x})\right)^T \otimes \cdots \otimes \left(g^{-1}_{\alpha\beta}(\tilde{x})\right)^T}^{r} \otimes \underbrace{g_{\alpha\beta}(\tilde{x}) \otimes \cdots \otimes g_{\alpha\beta}(\tilde{x})}_{s},$$

where $\otimes$ stands for the Kronecker product of matrices. Since $g_{\alpha\beta}$ satisfies (2.1.1), it follows that $g^{r,s}_{\alpha\beta}$ also satisfies (2.1.1). This gives a cone orbibundle structure $\mathcal{E}^{r,s} \to \mathcal{O}$ of rank $k(r + s)$.

Remark: The action of an element $\gamma \in \Gamma$ on $\tilde{s} \in \tilde{\mathcal{E}}^*$ is characterized by

$$(\gamma \cdot \tilde{s})(\tilde{X}) = \tilde{s}(\gamma^{-1} \cdot \tilde{X}),$$

where $\tilde{X} \in \tilde{\mathcal{E}}$.

Example 2.8. Symmetric and alternating tensor products. Let S^r be the permutations on r-letters. Define the symmetric tensor product by

$$\Sigma^r(\mathcal{E}^*) = \{v^1 \otimes \cdots \otimes v^r \in \mathcal{E}^{r,0} \mid v^1 \otimes \cdots \otimes v^r = v^{\sigma(1)} \otimes \cdots \otimes v^{\sigma(r)},\ \sigma \in S^r\},$$

and the alternating tensor product by

$$\begin{aligned}\Lambda^r(\mathcal{E}^*) = \{v^1 \otimes \cdots \otimes v^r \in \mathcal{E}^{r,0} \mid \\ v^1 \otimes \cdots \otimes v^r = (-1)^{sgn(\sigma)} v^{\sigma(1)} \otimes \cdots \otimes v^{\sigma(r)}, \sigma \in S^r\}.\end{aligned}$$

Locally, any $s \in \Sigma^r(\mathcal{E}^*)$ is given by

$$\tilde{s} = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \tilde{s}_{i_1,\ldots,i_r} \tilde{s}^{i_1} \otimes \cdots \otimes \tilde{s}^{i_r}.$$

$\tilde{s}$ can be though of as a multilinear map $\tilde{s} : \tilde{\mathcal{E}} \times \cdots \times \tilde{\mathcal{E}} \to \mathbb{R}$ defined by

$$\tilde{s}(\tilde{X}_1, \ldots, \tilde{X}_r) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \tilde{s}_{i_1,\ldots,i_r} \tilde{s}^{i_1}(\tilde{X}_1) \cdots \tilde{s}^{i_r}(\tilde{X}_r).$$

If $\gamma \cdot \tilde{X}_i = \tilde{X}_i$ for all i and $\gamma \in \Gamma$ then

$$(\gamma \cdot \tilde{s})(\tilde{X}_1, \ldots, \tilde{X}_r) = \tilde{s}(\tilde{X}_1, \ldots, \tilde{X}_r).$$

It follows that $s(X_1, \ldots, X_r)$ is well-defined. Furthermore, because s is invariant under permutations the induced map $s : \mathcal{E}^{inv} \times \cdots \times \mathcal{E}^{inv} \to \mathbb{R}$ is a multilinear symmetric map. Similarly, every $\omega \in \Lambda^r(\mathcal{E}^*)$ can be tough of as a multilinear antisymmetric map $\omega : \mathcal{E}^{inv} \times \cdots \times \mathcal{E}^{inv} \to \mathbb{R}$.

Let $\mathcal{F} \to \mathcal{O}$ be a cone orbibundle with orbifold atlas $\left(\tilde{\mathcal{F}}_\alpha, \Gamma_\alpha, \tilde{\varphi}_\alpha\right)$.

Example 2.9. Direct sum.
The direct sum, denoted by $\mathcal{E} \oplus \mathcal{F}$, has $\tilde{\mathcal{E}}_\alpha \oplus \tilde{\mathcal{F}}_\alpha$ as orbifold charts , and $g^{\oplus}_{\alpha\beta}$ as transition maps, defined by

$$g^{\oplus}_{\alpha\beta}(\tilde{x}) := \begin{pmatrix} g^{\mathcal{E}}_{\alpha\beta}(\tilde{x}) & 0 \\ 0 & g^{\mathcal{F}}_{\alpha\beta}(\tilde{x}) \end{pmatrix}$$

In particular, the Γ-action is given by

$$\gamma \cdot v \oplus w = (\gamma \cdot v) \oplus (\gamma \cdot w).$$

Example 2.10. Symmetric and alternating tensor products with coefficients on a cone orbibundle.
Locally an element $s \otimes f \in \Sigma^r(\mathcal{E}^*) \otimes \mathcal{F}$ is given by

$$\tilde{s} \otimes \tilde{f} = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \tilde{s}_{i_1,\ldots,i_r} \tilde{s}^{i_1} \otimes \cdots \otimes \tilde{s}^{i_r} \otimes \tilde{f}_{i_1,\ldots,i_r}.$$

It defines the symmetric multilinear map $\tilde{s} : \tilde{\mathcal{E}} \times \cdots \times \tilde{\mathcal{E}} \to \tilde{\mathcal{F}}$ by

$$\tilde{s}(\tilde{X}_1, \ldots, \tilde{X}_r) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \tilde{s}_{i_1,\ldots,i_r} \tilde{s}^{i_1}(\tilde{X}_1) \cdots \tilde{s}^{i_r}(\tilde{X}_r) \tilde{f}_{i_1,\ldots,i_r}.$$

Then $s \otimes f : \mathcal{E}^{inv} \times \cdots \mathcal{E}^{inv} \to \mathcal{F}^{inv}$ is a multilinear symmetric map. Similarly, $\Lambda^r(\mathcal{E}^*) \otimes \mathcal{F} \ni \omega \otimes f : \mathcal{E}^{inv} \times \cdots \mathcal{E}^{inv} \to \mathcal{F}^{inv}$ is a multilinear antisymmetric map.

Remark: The orbibundle $Hom(\mathcal{E}, \mathcal{F})$ is $\mathcal{E}^* \otimes \mathcal{F}$.

Example 2.11. Pullbacks.
Let $\mathcal{O}_2$ be an orbifold with orbifold atlas $(\tilde{U}^2_\alpha, \Gamma^2_\alpha, \tilde{\phi}^2_\alpha)$ and $f : \mathcal{O}_2 \to \mathcal{O}$ be a good orbifold map with a compatible system $\{\tilde{f}_\alpha, \theta_{f,\alpha}\}$. Every injection $\tilde{\psi}^2_{\alpha\beta} : \tilde{U}^2_\alpha \hookrightarrow \tilde{U}^2_\beta$ induces a unique injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ because of the definition of a good map. Take the pullback vector bundle over $\tilde{U}^2_\alpha$

$$\begin{tikzcd} \tilde{f}^*_\alpha\left(\tilde{\mathcal{E}}_\alpha\right) \arrow[r] \arrow[d] & \tilde{\mathcal{E}}_\alpha \arrow[d, "pr_1"] \\ \tilde{U}^2_\alpha \arrow[r, "\tilde{f}_\alpha"'] & \tilde{U}_\alpha \end{tikzcd}$$

Define $g^2_{\alpha\beta} : \tilde{U}^2_\alpha \to GL_k(\mathbb{R})$ by

$$g^2_{\alpha\beta}(\tilde{y}) = g_{\alpha\beta}(\tilde{f}_\alpha(\tilde{y})).$$

This system of maps satisfies condition (2.1.1), as long as the orbifold map is a good one, and then defines a cone orbibundle structure $f^*(\mathcal{E})$ over $\mathcal{O}_2$.

2.1.4 Orbisections.

An *orbisection* for the orbibundle $\pi : \mathcal{E} \to \mathcal{O}$ is a (good) orbifold map $s : \mathcal{O} \to \mathcal{E}$ such that $\pi \circ s = id$. The space of all orbisections is denoted by $Sec(\mathcal{E})$.

Example 2.12. Cone fields.
Cone fields are orbisections of a cone orbibundle. Take $(\tilde{U}, \Gamma, \tilde{\phi})$ an orbifold chart, $\tilde{\pi} : \tilde{\mathcal{E}} \to \tilde{U}$, $\tilde{s} : \tilde{U} \to \tilde{\mathcal{E}}$ local lifts with $\theta_\pi : \Gamma \to \Gamma$, $\theta_s : \Gamma \to \Gamma$ their associated isomorphisms. The local lifts for the identity are given by $\tilde{x} \mapsto \gamma_0 \cdot \tilde{x}$, with $\gamma_0 \in \Gamma$ fixed, which implies $\tilde{s}(\tilde{x}) \in \tilde{\mathcal{E}}_{\gamma_0 \cdot \tilde{x}}$. Because of $\theta_\pi(\gamma) = \gamma$ we get $\theta_s(\gamma) = \gamma_0 \gamma \gamma_0^{-1}$. It follows that $\tilde{s}(\tilde{x}) = [\tilde{s}(\tilde{x})]$ is a well-defined element on the cone space $\tilde{\mathcal{E}}_{\gamma_0 \cdot \tilde{x}} / \Gamma_{\gamma_0 \cdot \tilde{x}}$, i.e., $s(x) \in \mathcal{E}^{inv}_x$. The space of cone fields will be denoted by $Sec(\mathcal{E})$; it is a $C^\infty(\mathcal{O})$-module.

Remark: It is always possible to take as lifts for the identity map $\tilde{x} \mapsto \tilde{x}$ because we can compose the lift $\tilde{x} \mapsto \gamma_0 \cdot \tilde{x}$ with the injection $\gamma_0^{-1} : \tilde{U} \to \tilde{U}$. With this lift of the identity map, the monomorphism θ_s becomes the identity too. In addition, if one takes the orbibundle $T\mathcal{O} \to \mathcal{O}$ then $Sec(T\mathcal{O}) := \mathfrak{X}(\mathcal{O})$.

Example 2.13. Differential forms.
A differential form of degree k is an orbisection of $\Lambda^k(\mathcal{E}^*)$. Locally we have a section $\tilde{\omega} : \tilde{U} \to \Lambda^k(\tilde{\mathcal{E}}^*)$, or equivalently, an alternating multilinear smooth function $\tilde{\omega}(\tilde{x}) : \tilde{\mathcal{E}}_{\tilde{x}} \times \overset{k}{\cdots} \times \tilde{\mathcal{E}}_{\tilde{x}} \to \mathbb{R}$. The Γ-action is given by

$$\tilde{\omega}^\gamma(\tilde{x})\left(X_1, \ldots, X_k\right) = \tilde{\omega}(\tilde{x})\left(\gamma^{-1} \cdot X_1, \ldots, \gamma^{-1} \cdot X_k\right).$$

Then, if $\gamma \in \Gamma_{\tilde{x}}$

$$\tilde{\omega}(\tilde{x})\left(\gamma \cdot X_1, \ldots, \gamma \cdot X_k\right) = \tilde{\omega}(\tilde{x})\left(X_1, \ldots, X_k\right).$$

Hence, $\omega(X_1, \ldots, X_k) \in C^\infty(\mathcal{O})$, with $X_1, \ldots, X_k \in Sec(\mathcal{E})$.

Example 2.14. Riemannian metrics.
An orbisection $s \in \Sigma^2(T^*\mathcal{O})$ is a symmetric bilinear form. Locally is given by $\tilde{s} : T\tilde{U} \oplus T\tilde{U} \to \mathbb{R}$ and for every $\gamma \in (\Gamma)_{\tilde{x}}$

$$\tilde{s}(\tilde{x})\,(\gamma \cdot X_1, \gamma \cdot X_2) = \tilde{s}(\tilde{x})\,(X_1, X_2)\,.$$

If $\tilde{s}(\tilde{x})$ is positive definite, then we have a Riemannian metric on the bundle $T\tilde{U} \to \tilde{U}$. Denotes $\tilde{s}(\cdot, \cdot) := \langle \cdot, \cdot \rangle$ and take $X, Y \in \mathfrak{X}(\mathcal{O})$. Its inner product is given by $\langle X, Y \rangle \in C^\infty(\mathcal{O})$. Then a Riemannian metric can be though as a smooth orbibundle map $\langle \cdot, \cdot \rangle : T\mathcal{O} \oplus T\mathcal{O} \to \mathbb{R}$ satisfying the same properties Riemannian tensor satisfies.

Example 2.15. Differential forms with coefficients.
An orbisection of $\Lambda^k\,(T^*\mathcal{O}) \otimes \mathcal{E}$ is a differential form with coefficients on the cone orbibundle $\mathcal{E}$. Locally $\tilde{\omega} : \tilde{U} \to \Lambda^k(T^*\tilde{U}) \otimes \tilde{\mathcal{E}}$, which is the same as a multilinear alternating map $\tilde{\omega}(\tilde{x}) : T_{\tilde{x}}\tilde{U} \times \cdots \times T_{\tilde{x}}\tilde{U} \to \tilde{\mathcal{E}}_{\tilde{x}}$. The Γ-action is given by

$$\tilde{\omega}^\gamma(\tilde{x})\,(X_1, \ldots, X_k) = \gamma \cdot \tilde{\omega}(\tilde{x})\left(\gamma^{-1} \cdot X_1, \ldots, \gamma^{-1} \cdot X_k\right).$$

Hence, for every $\gamma \in (\Gamma)_{\tilde{x}}$

$$\tilde{\omega}(\tilde{x})\,(\gamma \cdot X_1, \ldots, \gamma \cdot X_k) = \gamma \cdot \tilde{\omega}(\tilde{x})\,(X_1, \ldots, X_k)\,,$$

where the action on the right is the one on $\tilde{\mathcal{E}}$. If $X_1, \ldots, X_k \in \mathfrak{X}(\mathcal{O})$ then $\omega(X_1, \ldots, X_k) \in Sec(\mathcal{E})$ is a section for the orbibundle structure $\mathcal{E} \to \mathcal{O}$.

Proposition 2.16. *There is a* 1-1 *correspondence between*

$$Sec(\Sigma^r(\mathcal{E}^*) \otimes \mathcal{F}) \overset{1-1}{\longleftrightarrow} Hom_{C^\infty(\mathcal{O})}\left(Sec(\Sigma^r(\mathcal{E}^*)), Sec(\mathcal{F})\right)$$

$$Sec(\Lambda^r(\mathcal{E}^*) \otimes \mathcal{F}) \overset{1-1}{\longleftrightarrow} Hom_{C^\infty(\mathcal{O})}\left(\Omega^r(\mathcal{E}), Sec(\mathcal{F})\right)$$

Proof. We will do the symmetric part. The skew-symmetric one is analogue.
($\Rightarrow$) We already saw that an element $s \otimes f \in \Sigma^r(\mathcal{E}^*) \otimes \mathcal{F}$ can be though

of as a multilinear symmetric map $s \otimes f : \mathcal{E}^{inv} \times \cdots \mathcal{E}^{inv} \to \mathcal{F}^{inv}$. Furthermore, if $X_i \in Sec(\mathcal{E})$, for $i \in \{1, \ldots, r\}$, then $X_i \in \mathcal{E}^{inv}$. It follows that $s \otimes f(X_1, \ldots, X_r) \in \mathcal{F}^{inv}$, i.e., it belongs to $Sec(\mathcal{F})$. Given that the multilinear map was given in terms of tensor products, the homomorphisms is $C^\infty(\mathcal{O})$-multilinear.

($\Leftarrow$) Let $h \in Hom_{C^\infty(\mathcal{O})}(Sec(\Sigma^r(\mathcal{E}^*)), Sec(\mathcal{F}))$. It induces the multilinear map $h : \mathcal{E}^{inv} \times \cdots \times \mathcal{E}^{inv} \to \mathcal{F}^{inv}$. It is symmetric by definition. Because it was $C^\infty(\mathcal{O})$-multilinear, it defines a tensor. Then $h \in Sec(\Sigma^r(\mathcal{E}^*) \otimes \mathcal{F})$. □

Example 2.17. The identity 1-form.
Take the local section $\tilde{\omega} : \tilde{U} \to T^*\tilde{U} \otimes T\tilde{U}$, defined by

$$\tilde{\omega}(\tilde{x})(X) = X.$$

It defines a local orbisection; gluing together these orbisections, we get a 1-form with values on $T\mathcal{O}$, i.e., $\omega : \mathcal{O} \to T^*\mathcal{O} \otimes T\mathcal{O}$, the identity 1-form.

Remark: If, for a fixed $\gamma_1 \in \Gamma$, the local section is defined

$$\tilde{\omega}(\tilde{x})(X) = \gamma_1 \cdot X,$$

then it is an orbisection only if $\gamma_1 \Gamma = \Gamma \gamma_1$.

Example 2.18. Orbiframes.
A local trivialization for the local orbibundle structure $\tilde{\mathcal{E}}|_{\tilde{U}} \cong \tilde{U} \times \mathbb{R}^k$ is the same as a system of sections that form a basis for each fiber. Notice that we have not made any assumption about the compatibility between the action on the base and the bundle for each section. Some sections are not Γ-equivariant. For example, let $\mathbb{R}^2/\mathbb{Z}_n$ be the cone; it's tangent orbibundle structure is isomorphic to the cone itself. However, at the origins, no vector is $\mathbb{Z}_n$-equivariant except from the zero vector itself. Consequently, $\mathbb{R}^2/\mathbb{Z}_n$ admits no local orbiframe around the zero. Thus, the existence of a local orbiframe is not the same as having a trivialization for the cone orbibundle structure. However, a trivialization in the usual (manifold) sense induces a Γ-action on the trivialized chart $\tilde{U} \times \mathbb{R}^k$. A local orbiframe is a collection of orbisections $\tilde{s}^i : \tilde{U} \to \tilde{\mathcal{E}}|_{\tilde{U}}$

such that for every $\tilde{x} \in \tilde{U}$ the set $\{\tilde{s}^1(\tilde{x}), \ldots, \tilde{s}^k(\tilde{x})\}$ is a basis for $\tilde{\mathcal{E}}_{\tilde{x}}$. Locally, every section $s \in Sec(\mathcal{E}|_U)$ can be written as

$$s(x) = \sum_{i=1}^{k} a_i(x) s^i(x),$$

for $a_i \in C^\infty(\mathcal{O})$, as long as the local orbiframe exists.

2.2 Cone connections

Take the trivial cone orbibundle $pr_1 : \mathcal{O} \times \mathbb{R}^k \to \mathcal{O}$ and an orbisection $s : \mathcal{O} \to \mathcal{O} \times \mathbb{R}^k$. It is characterized by the orbifold map $f_s : \mathcal{O} \to \mathbb{R}^k$. There is a natural way of differentiating this orbisection along a cone field. Consider $df_s : T\mathcal{O} \to \mathbb{R}^k$ and take a cone field $X \in \mathfrak{X}(\mathcal{O})$. We get the orbisection

$$\nabla_X s = df_s(X).$$

Given that the space of orbisections of a cone orbibundle have a $C^\infty(\mathcal{O})$-module structure, for $h \in C^\infty(\mathcal{O})$, ∇ satisfies

$$\nabla(hs) = h\nabla s + dh \otimes s,$$

and

$$\nabla_{hX} s = h\nabla_X s.$$

However, there is no canonical way of differentiating a section of a non-trivial cone orbibundle . A cone connection allow us to differentiate sections of cone orbibundles along cone fields.

Let $\pi : \mathcal{E} \to \mathcal{O}$ be a cone orbibundle structure.

Definition 2.19. *Take a system of local connections $(\nabla_\alpha)_{\alpha \in I}$ for every trivializing orbifold chart. If $\nabla_\alpha : \mathfrak{X}(\tilde{U}_\alpha) \times Sec(\tilde{\mathcal{E}}_\alpha) \to Sec(\tilde{\mathcal{E}}_\alpha)$ are such that for every injection $\tilde{\psi}_{\alpha\beta}$*

$$\nabla_\alpha = \tilde{\psi}^*_{\alpha\beta} \nabla_\beta,$$

then they define a lift for the connection $\nabla : \mathfrak{X}(\mathcal{O}) \times Sec(\mathcal{E}) \to Sec(\mathcal{E})$.

Condition $\gamma^*\nabla = \nabla$ means

$$\nabla_{\gamma\cdot\tilde{X}}\gamma \cdot \tilde{s} = \gamma \cdot \left(\nabla_{\tilde{X}}\tilde{s}\right).$$

Just as orbifold maps, connections are defined up to an equivalence relation.

Definition 2.20. *Let $x \in \mathcal{O}$ and take two lifts ∇^1, ∇^2 of a cone connection. They are equivalent at x as germs, denoted by $\nabla^1 \sim_x \nabla^2$, if there exists an orbifold chart $x \in U \subset U_\alpha \cap U_\beta$ such that $\nabla^1_\alpha = \lambda^*\nabla^2_\beta$, where $\lambda : \tilde{\psi}_\alpha(\tilde{U}) \to \tilde{\psi}_\beta(\tilde{U})$ is the diffeomorphism induced by the injections $\tilde{\psi}_\alpha : \tilde{U} \hookrightarrow \tilde{U}_\alpha$ and $\tilde{\psi}_\beta : \tilde{U} \hookrightarrow \tilde{U}_\beta$.*

Two connections are equivalent, denoted by $\nabla^1 \sim \nabla^2$, if $\nabla^1 \sim_x \nabla^2$ for every $x \in \mathcal{O}$.

Definition 2.21. *A cone connection is the class $[\nabla]$ of an orbifold lift ∇_α for a cone connection ∇.*

We will omit the equivalence relation notation $[\nabla]$ by lifting a connection ∇ representing the class.

2.2.1 Connection matrices

Take a connection ∇ on a cone orbibundle structure $\pi : \mathcal{E} \to \mathcal{O}$. The trivialization $(\tilde{s}_i^\alpha)_{i=1}^n$ of $\tilde{\mathcal{E}}_\alpha$ induces smooth functions $\eta_\alpha^i \in C^\infty(\tilde{U}_\alpha)$ such that $\tilde{\sigma}_\alpha \in \Gamma(\tilde{\mathcal{E}}_\alpha)$ is given by

$$\tilde{\sigma}_\alpha(\tilde{x}) = \sum_i \tilde{a}_\alpha^i(\tilde{x})\tilde{s}_i^\alpha(\tilde{x}).$$

Take a local vector field $\tilde{X} \in \mathfrak{X}(\tilde{U}_\alpha)$. By the Leibniz rule

$$\nabla_{\tilde{X}}\tilde{\sigma}_\alpha = \sum_{j=1}^n \left[d\tilde{a}_\alpha^j(\tilde{X}) + \sum_{i=1}^k \tilde{a}_\alpha^j \omega_{ij}^\alpha(\tilde{X})\right]\tilde{s}_j, \tag{2.2.1}$$

with $\omega_{ij}^\alpha \in \Omega^1(\tilde{U}_\alpha)$. It follows that the 1-form matrix $\omega_\alpha = (\omega_{ij}^\alpha)_j^i$ determines a connection over the frame $(\tilde{s}_i^\alpha(\tilde{x}))_{i=1}^n$. The compatibility condition

$$\nabla_{\gamma\cdot\tilde{X}}\gamma \cdot \tilde{\sigma}_\alpha = \gamma \cdot \nabla_{\tilde{X}}\tilde{\sigma}_\alpha,$$

is given in terms of the connection matrices by

$$\omega_\alpha(\gamma \cdot X) = \gamma^{-1}\omega_\alpha(X)\gamma - \gamma^{-1}d\gamma(X).$$

The smooth map $g_\gamma : \tilde{U}_\alpha \to GL_n(\mathbb{R})$ gives the Γ_α-action, induced from the trivialization $(\tilde{s}_i^\alpha)_{i=1}^n$.

Take two local frames $(\tilde{s}_i^\alpha)_{i=1}^n$ and $(\tilde{s}_i^\beta)_{i=1}^n$ over two orbifold charts $\tilde{U}_\alpha$ and $\tilde{U}_\beta$ with non-trivial intersection $\tilde{U} = \tilde{U}_\alpha \cap \tilde{U}_\beta$. There exists a smooth function $A : \tilde{U} \to GL_n(\mathbb{R})$ such that

$$\tilde{s}_i^\beta(\tilde{x}) = \sum_{k=1}^n \tilde{s}_k^\alpha(\tilde{x}) A_i^k(\tilde{x}).$$

The connection matrices ω_α and ω_β associated to the frames s^α and s^β are related by

$$\omega_\beta(X) = A^{-1}\omega_\alpha(X)A + A^{-1}dA(X). \tag{2.2.2}$$

Proposition 2.22. *There is a* 1-1 *correspondence between cone connections ∇ on $\mathcal{E}$ and a system of connection matrices ω_α over an orbifold atlas $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ such that (2.2.2) is true for every non-trivial intersection.*

Proof. Equation (2.2.1) and the Leibniz hypothesis gives a 1-1 correspondence between connection matrices ω_α and connections ∇_α over an orbifold chart $\tilde{U}_\alpha$. Condition (2.2.2) guarantees that the system of local connections ∇_α belongs to the same germ at every point $x \in \mathcal{O}$. $\square$

2.2.2 Parallel transport

Take an orbifold path $\eta : I \subset \mathbb{R} \to \mathcal{O}$, with lifts $\eta_\alpha : I_\alpha \to \tilde{U}_\alpha$. The isomorphism $\gamma : \tilde{U}_\alpha \to \tilde{U}_\alpha$ induces the commutative diagram

$$\begin{tikzcd}
I_\alpha \arrow[r, "id"] \arrow[d, "\eta_\alpha"'] & I_\alpha \arrow[d, "\gamma\cdot\eta_\alpha"] \\
\tilde{U}_\alpha \arrow[r, "\gamma"'] & \tilde{U}_\alpha
\end{tikzcd}$$

Consequently, $\gamma \cdot \eta_\alpha$ belongs to the same germ as η_α for all $\gamma \in \Gamma_\alpha$ (in fact, they are all the possible lifts over $\tilde{U}_\alpha$). Let $\mathcal{E} \to \mathcal{O}$ be a cone orbibundle with the connection ∇. The vector bundle structure $\eta^*\mathcal{E} \to I$ is given locally by

$$\begin{tikzcd}
\eta_\alpha^* \tilde{\mathcal{E}}_\alpha \arrow[r] \arrow[d, "p_\alpha"'] & \tilde{\mathcal{E}}_\alpha \arrow[d, "\pi_\alpha"] \\
I_\alpha \arrow[r, "\eta_\alpha"] & \tilde{U}_\alpha
\end{tikzcd}$$

Be aware that the pullback vector bundle $\eta_\alpha^* \tilde{\mathcal{E}}_\alpha$ is taken without any group action Γ_α since $I_\alpha$is a manifold. The pullback connections $\eta_\alpha^* \nabla_\alpha$ on $\eta_\alpha^* \tilde{\mathcal{E}}_\alpha \to I_\alpha$ defines a connection $\eta^*\nabla$ on the vector bundle $\eta^*\mathcal{E} \to I$. Take a section $\eta^* s \in \Gamma(\eta^*\mathcal{E})$. It defines the section

$$\frac{\nabla}{dt}\eta^* s := (\eta^*\nabla)_{d/dt}\,(\eta^* s),$$

locally characterized by

$$(\nabla_\alpha)_{\eta'_\alpha(t)}\, s_\alpha(\eta_\alpha(t)).$$

Notice that taking another local lift $\gamma \cdot \eta_\alpha$ yields the local expression

$$(\nabla_\alpha)_{\gamma\cdot\eta'_\alpha(t)}\, s_\alpha(\gamma \cdot \eta_\alpha(t)) = \gamma \cdot \Big((\nabla_\alpha)_{\eta'_\alpha(t)}\, s_\alpha(\eta_\alpha(t)) \Big).$$

Definition 2.23. *A section $\eta^* s \in \Gamma(\eta^*\mathcal{E})$ is called parallel along the path $\eta : I \to \mathcal{O}$ if*

$$\frac{\nabla}{dt}\eta^* s = 0.$$

for all $t \in I$.

A section $\eta^* s$ is parallel if and only if

$$(\nabla_\alpha)_{\eta'_\alpha(t)}\, s_\alpha(\eta_\alpha(t)) = 0,$$

for all $\alpha \in I$. In terms of a local frame $(e_i^\alpha)_{i=1}^n$ over $\tilde{U}_\alpha$ we can write

$$s_\alpha(\eta_\alpha(t)) = \sum_{i=1}^n \tilde{a}_\alpha^i(t) e_i^\alpha(\eta_\alpha(t)),$$

with $\tilde{a}^k_\alpha : I_\alpha \to \mathbb{R}$ smooth functions. Then the parallel condition equation yields the ODE system

$$\frac{d\tilde{a}^i_\alpha}{dt} = -\sum_{k=1}^{n} \omega^\alpha_{ik}\left(\eta'_\alpha(t)\right)\tilde{a}^k_\alpha(t)\ , \quad i = 1, \dots n,$$

where ω_α is the 1-form matrix connection associated to the frame e^α_i. Take $\tilde{v} \in \mathcal{E}_{\eta_\alpha(t_0)}$, it can be decomposed in terms of the local frame

$$\tilde{v} = \sum_{k=1}^{n} \tilde{v}^k e^\alpha_k(\eta_\alpha(t_0)).$$

This give us the initial conditions $\tilde{a}^k_\alpha(t_0) = \tilde{v}^k$ to the ODE system whose local solution give rise to the local section

$$u_{\tilde{v}}(t) = \sum_{k=1}^{n} \tilde{a}^k_\alpha(t) e^\alpha_k(\eta_\alpha(t)).$$

Define $\tilde{T}^{t_0,t_1}_\eta : (\tilde{\mathcal{E}}_\alpha)_{\eta_\alpha(t_0)} \to (\tilde{\mathcal{E}}_\alpha)_{\eta_\alpha(t_1)}$ by

$$\tilde{T}^{t_0,t_1}_\eta(\tilde{v}) = u_{\tilde{v}}(t_1).$$

Since the fibers have equal dimension and by the uniqueness of the solutions for the ODE, $\tilde{T}^{t_0,t_1}_\eta$ is an isomorphism. However, for every $\gamma \in \Gamma_\alpha$

$$\tilde{T}^{t_0,t_1}_\eta(\gamma \cdot \tilde{v}) = \gamma \cdot \tilde{T}^{t_0,t_1}_{\gamma^{-1}\cdot\eta}(\tilde{v}), \tag{2.2.3}$$

which implies $\tilde{T}^{t_0,t_1}_\eta$ is not Γ_α-equivariant unless $\Gamma_\alpha \cdot \eta_\alpha = \eta_\alpha$.

Proposition 2.24. *Let $\eta : I \to \mathcal{O}$ be an orbifold path that defines a cone field over $\mathcal{O}$. Then $T^{t_0,t_1}_\eta : \mathcal{E}_{t_0} \to \mathcal{E}_{t_1}$ is a homeomorphism.*

Proof. Since η defines a cone field, its lifts are Γ_α-invariant. It follows that

$$\gamma \cdot \eta = \eta,$$

for all $\gamma \in \Gamma_\alpha$, $\alpha \in I$. Equation (2.2.3) implies that the isomorphism $\tilde{T}^{t_0,t_1}_\eta : (\tilde{\mathcal{E}}_\alpha)_{\eta_\alpha(t_0)} \to (\tilde{\mathcal{E}}_\alpha)_{\eta_\alpha(t_1)}$ is Γ_α-equivariant. Hence, it induces the homeomorphism $T^{t_0,t_1}_\eta : \mathcal{E}_{t_0} \to \mathcal{E}_{t_1}$ between the fibers. $\square$

2.3 Principal orbibundles

Among the fiber orbibundles, we have already studied some properties of cone orbibundles. The word cone means that locally the fibers are vector spaces. The word principal means that locally the fibers are a Lie group G. The G-structure theory is based on the relation between the tangent orbibundle, a cone orbibundle, and the frame orbibundle, a principal orbibundle. Therefore, our guiding example will be the frame orbibundle structure. However, because our orbifolds are effective, the frame orbibundle enjoys an exceptional property that not all principal orbibundles enjoy: being a manifold. From now on, the frame orbibundle will be called frame bundle because of its non-singular structure. Its orbifold behavior is encoded on the isotropies of the Lie group action G together with its transversal geometry, which means the existence and relations between the slices (see chapter 2, section 2.2).

2.3.1 The frame orbibundle

Let $(\tilde{U}, \Gamma, \tilde{\phi})$ be an orbifold chart. Sections $\tilde{s}_i : \tilde{U} \to T\tilde{U}$ such that $(\tilde{s}_i(\tilde{x}))_{i=1}^n$ is a basis for $T_{\tilde{x}}\tilde{U}$ gives a trivialization for $T\tilde{U}$. This trivialization, called a frame, induces an action $\Gamma \curvearrowright \tilde{U} \times \mathbb{R}^n$ given by the smooth maps $g_\gamma : \tilde{U} \to GL_n(\mathbb{R})$. Each frame could be thought of as a system of linear isomorphisms $p_{\tilde{x}} : \mathbb{R}^n \to T\tilde{U}$ defined by

$$p_{\tilde{s}}(v^1, \dots, v^n) = \sum_i v^i \tilde{s}_i.$$

Take another frame $(\tilde{e}_i)_{i=1}^n$ for $T\tilde{U}$. They are related by the smooth coefficients $a_{ij} : \tilde{U} \to \mathbb{R}$

$$\tilde{e}_i(\tilde{x}) = \sum_j a_{ij}(\tilde{x}) s_j(\tilde{x}).$$

Let $A : \tilde{U} \to GL_n(\mathbb{R})$ be $A(\tilde{x}) = (a_{ij}(\tilde{x}))_i^j$. If $p_{\tilde{e}}$ is the frame induced by $(\tilde{e}_i)_{i=1}^n$, then

$$p_{\tilde{e}} \circ A = p_{\tilde{s}}.$$

Define the frame bundle over $\tilde{x}$ as

$$Fr_{\tilde{x}}(\tilde{U}) = \{p : \mathbb{R}^n \to T_{\tilde{x}}\tilde{U} \mid p \textit{ is an isomorphism.}\}.$$

Every matrix $A \in GL_n(\mathbb{R})$ defines an element in $Fr_{\tilde{x}}(\tilde{U}_\alpha)$ by $p_{\tilde{e}} \circ A$. They are all the possible isomorphisms between $\mathbb{R}^n$ and $T_{\tilde{x}}\tilde{U}$. Then the $GL_n(\mathbb{R})$-action on $Fr_{\tilde{x}}(\tilde{U})$ is transitive. Let

$$Fr(\tilde{U}) := \bigsqcup_{\tilde{x} \in \tilde{U}} Fr_{\tilde{x}}(\tilde{U}).$$

The diffeomorphism $\xi : \tilde{U} \times GL_n(\mathbb{R}) \to Fr(\tilde{U})$ gives a smooth structure on $Fr(\tilde{U})$ by

$$\xi(\tilde{x}, A) = (p_{\tilde{e}} \circ A)(\tilde{x}).$$

The Γ-action $\Gamma \curvearrowright Fr(\tilde{U})$ is

$$\gamma \cdot p_{\tilde{e}} = d\gamma \circ p_{\tilde{e}} \in Fr(\tilde{U}).$$

Lemma 2.25. *The action $\Gamma \curvearrowright Fr(\tilde{U})$ is free.*

Proof. Take $e \neq \gamma \in \Gamma$, $p \in Fr(\tilde{U})$ such that $\gamma \cdot p = p$ and the Riemannian metric taken in Lemma 1.18. The exponential map defines the Γ-equivariant diffeomorphism $\exp_{\tilde{x}} : B_\epsilon(0) \subset T_{\tilde{x}}\tilde{U} \to \tilde{U}$. For all $\tilde{y} \in \exp_{\tilde{x}}(B_\epsilon(0))$ there exists a $X_{\tilde{x}} \in B_\epsilon(0)$ such that $\exp_{\tilde{x}}(X(\tilde{x})) = \tilde{y}$. Furthermore, there exists $v \in \mathbb{R}^n$ such that $p_{\tilde{s}}(v)(\tilde{x}) = X_{\tilde{x}}$. Consequently

$$\gamma \cdot \tilde{y} = \gamma \cdot \exp_{\tilde{x}}\left(p_{\tilde{s}}(v)(\tilde{x})\right) = \exp\left(d_{\tilde{x}}\gamma \circ p_{\tilde{s}}(v)(\tilde{x})\right) = \exp(p_{\tilde{s}}(v)(\tilde{x})) = \tilde{y}.$$

It follows that there is an open subset $W \subset \tilde{U}$ such that $\gamma|_W = e$. As long as the action is effective $\gamma = e$. $\square$

Then $Fr(\tilde{U})/\Gamma$ is a manifold! An injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ between two orbifold charts induces an injection $\tilde{\psi}^*_{\alpha\beta} : Fr(\tilde{U}_\alpha) \hookrightarrow Fr(\tilde{U}_\beta)$ given by

$$\tilde{\psi}^*_{\alpha\beta}(p_\alpha) = d\tilde{\psi}_{\alpha\beta} \circ p_\alpha.$$

Take the same constructions as for the tangent orbibundle. Then

$$Fr(\mathcal{O}) = \bigsqcup_{\alpha \in I} \alpha \times (Fr(\tilde{U}_\alpha)/\Gamma_\alpha)/\sim$$

has the structure of manifold. It is called the frame bundle of $\mathcal{O}$.

The right action $Fr(\tilde{U}_\alpha) \curvearrowleft GL_n(\mathbb{R})$ and the left action $\Gamma_\alpha \curvearrowright Fr(\tilde{U}_\alpha)$ commute. This implies the existence of a well-defined right action $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$.

Proposition 2.26. *The (right) action $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$ is locally free and proper.*

Proof. Let $p \in Fr(\mathcal{O})$, $A \in GL_n(\mathbb{R})_p$, i.e., $p \circ A = p$. Given that $p \in Fr(\tilde{U})/\Gamma$, there exist $\gamma \in \Gamma$ such that $\gamma \cdot p = p \circ A$. It follows that

$$A_\gamma = \begin{pmatrix} | & | & & | \\ d\gamma(p(e_1)) & d\gamma(p(e_2)) & \cdots & d\gamma(p(e_n)) \\ | & | & & | \end{pmatrix}.$$

Consequently, $p \circ A = p$ if and only if $A \in \{A_{\gamma_1}, \ldots, A_{\gamma_r}\} \cong \Gamma$. As long as the isotropy group $GL_n(\mathbb{R})_p$ is discrete, its Lie algebra will be zero dimensional. The infinitesimal action $\Psi : Fr(\mathcal{O}) \times \mathfrak{gl}_\mathfrak{n}(\mathbb{R}) \to TFr(\mathcal{O})$ satisfies

$$\dim(\ker \Psi_p) = \dim(\mathfrak{gl}_\mathfrak{n}(\mathbb{R})_p) = 0.$$

Then $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$ is locally free.

Take sequences $p_k \in Fr(\mathcal{O})$, $g_k \in GL_n(\mathbb{R})$ such that $p_k \to p$ and $p_k \cdot g_k \to q$. The action is transitive on the fibers, so there exists $A \in GL_n(\mathbb{R})$ such that $p \circ A = q$. Given that the action is smooth, there exists a subsequence of g_{k_i} such that $g_{k_i} \to A$. Then $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$ is proper. $\square$

By proposition 1.23, $Fr(\mathcal{O})/GL_n(\mathbb{R})$ has an orbifold structure. As long as the $GL_n(\mathbb{R})$ and Γ actions commute, we get the homeomorphism

$$\left(Fr(\tilde{U})/\Gamma\right)/GL_n(\mathbb{R}) \cong \left(Fr(\tilde{U})/GL_n(\mathbb{R})\right)/\Gamma.$$

But $Fr(\tilde{U})/GL_n(\mathbb{R}) = \tilde{U}$. It follows that $Fr(\mathcal{O})/GL_n(\mathbb{R}) = \mathcal{O}$.

Let $\pi : Fr(\mathcal{O}) \xrightarrow{/GL_n(\mathbb{R})} \mathcal{O}$ be the quotient map. Locally, it is given by the following commutative diagram

$$\begin{tikzcd}
Fr(\tilde{U}) \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\varphi}"'] & \tilde{U} \arrow[d, "\tilde{\phi}"] \\
Fr(\tilde{U})/\Gamma \arrow[r] \arrow[d] & \tilde{U}/\Gamma \arrow[d] \\
Fr(U) \arrow[r, "\pi"'] & U
\end{tikzcd}$$

Theorem 2.27. *Every effective orbifold is the quotient of a manifold by a smooth, locally free and proper action of a Lie group.*

2.3.2 Principal orbibundles and morphisms.

We will define principal orbibundles in the same spirit as cone orbibundles. However, one crucial fact about the theory of G-structures on effective orbifolds is that its frame bundle is a manifold. We are especially interested in this structure, so we will focus on this particular setting as we advance through the theory.

Definition 2.28. *A principal orbibundle $\mathcal{P}$ over $\mathcal{O}$, with structure group G, is given by*

1. *For each orbifold chart $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ a G-principal bundle structure $\pi_\alpha : \tilde{\mathcal{P}}_\alpha \to \tilde{U}_\alpha$.*

2. *Every injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ induces a G-equivariant injection $\tilde{\psi}^{\times}_{\alpha\beta} : \tilde{\mathcal{P}}_\alpha \hookrightarrow \tilde{\mathcal{P}}_\beta$ such that*

$$\begin{tikzcd}
\tilde{\mathcal{P}}|_\alpha \arrow[r, "\tilde{\psi}^{\times}_{\alpha\beta}"] \arrow[d, "\pi_\alpha"'] & \tilde{\mathcal{P}}|_\beta \arrow[d, "\pi_\beta"] \\
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"'] & \tilde{U}_\beta
\end{tikzcd}$$

   commutes.

A local section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$ induces a trivialization. Explicitly we have the diffeomorphisms $\tilde{U} \times G \cong \tilde{\mathcal{P}}$ defined by

$$(\tilde{x}, a) \mapsto \tilde{s}(\tilde{x}) \cdot a.$$

This diffeomorphism induces two actions on $\tilde{U} \times G$. The G action, on the right, given by

$$(\tilde{x}, a) \cdot b = (\tilde{x}, a \cdot b),$$

and the left Γ-action defined by

$$\gamma \cdot (\tilde{x}, a) = (\gamma \cdot \tilde{x}, g_\gamma(\tilde{x}, a)),$$

with $g_\gamma : \tilde{U} \times G \to G$. Given that the actions commute, for every $a, b \in G$ it is true that

$$g_\gamma(\tilde{x}, a \cdot b) = g_\gamma(\tilde{x}, a) \cdot b.$$

Then, if $g_\gamma(\tilde{x}) := g_\gamma(\tilde{x}, e) : \tilde{U} \to G$ the Γ-action is given by

$$\gamma \cdot (\tilde{x}, a) = (\gamma \cdot \tilde{x}, g_\gamma(\tilde{x}) \cdot a).$$

Proposition 2.29. *A principal orbibundle $\mathcal{P}$, with structure group G, over an orbifold $\mathcal{O}$ is given by the following two conditions:*

1. *For every injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ there exists a smooth map $g_{\alpha\beta} : \tilde{U}_\alpha \cap \tilde{U}_\beta \to G$.*
2. *These maps satisfy*

$$g_{\theta_{\alpha\eta}(\gamma)} g_{\alpha\eta} = g_{\beta\eta} g_{\alpha\beta},$$

with a unique $\gamma \in \Gamma_\alpha$, determined by the two injections $\tilde{\psi}_{\alpha\beta}$ and $\tilde{\psi}_{\beta\eta}$.

Proof. Take a principal orbibundle structure $(\mathcal{P}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$. An injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ induces a map $g_{\alpha\beta} : \tilde{U}_\alpha \cap \tilde{U}_\beta \to G$ such that

$$\tilde{\psi}^{\times}_{\alpha\beta}(\tilde{x}, g) = (\tilde{\psi}_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x}) \cdot g). \tag{2.3.1}$$

Given that
$$\theta_{\alpha\eta}(\gamma)\psi_{\alpha\eta} = \psi_{\beta\eta} g_{\alpha\beta},$$
with $\gamma \in \Gamma_\alpha$ uniquely determined, then
$$g_{\theta_{\alpha\eta}(\gamma)} g_{\alpha\eta} = g_{\beta\eta} g_{\alpha\beta}.$$
Equation (2.3.1) gives the equivalence between principal orbibundle definition and the proposition conditions. $\square$

Example 2.30. Take the local frames $(\tilde{s}^i_\alpha)_{i=1}^n$ and $(\tilde{s}^i_\beta)_{i=1}^n$ trivializing the local cone orbibundle structures $T\tilde{U}_\alpha \to \tilde{U}_\alpha$ and $T\tilde{U}_\beta \to \tilde{U}_\beta$. The injections $\psi^*_{\alpha\beta} : T\tilde{U}_\alpha \to T\tilde{U}_\beta$ induce the transition maps $g_{\alpha\beta}$ defined by
$$\begin{aligned}\tilde{U}_\alpha \times \mathbb{R}^n &\to \tilde{U}_\beta \times \mathbb{R}^n \\ (\tilde{x}, v) &\mapsto (\psi_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x})(v).\end{aligned}$$
They satisfy condition 2 of the proposition. Let $\tilde{s}^\alpha : \tilde{U}_\alpha \to Fr(\tilde{U}_\alpha)$ be defined by
$$\tilde{s}^\alpha(\tilde{x})(v_1, \ldots, v_n) = \sum_{i=1}^n v_i \tilde{s}^i_\alpha(\tilde{x}).$$
It defines the trivialization $Fr(\tilde{U}_\alpha) \cong \tilde{U}_\alpha \times GL_n(\mathbb{R})$. Let
$$\begin{aligned}\psi^\times_{\alpha\beta} : \tilde{U}_\alpha \times GL_n(\mathbb{R}) &\to \tilde{U}_\beta \times GL_n(\mathbb{R}) \\ (\tilde{x}, A) &\mapsto (\psi_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x}) \cdot A).\end{aligned}$$
They define injections $\psi^\times_{\alpha\beta} : Fr(\tilde{U}_\alpha) \hookrightarrow Fr(\tilde{U}_\beta)$ such that the conditions of the proposition are satisfied. It follows that the $T\mathcal{O}$ structure induces the $Fr(T\mathcal{O})$ structure. We will denote $Fr(\mathcal{O}) := Fr(T\mathcal{O})$.

Example 2.31. The orbifold structure of the cotangent bundle $T^*\mathcal{O}$ is related to the tangent bundle by $g^*_{\alpha\beta} = (g^T_{\alpha\beta})^{-1}$. Because the maps $g^*_{\alpha\beta}$ also satisfy the conditions of the previous proposition, they define a principal orbibundle structure on $Fr(T^*\mathcal{O})$.

Proposition 2.32. *Every cone orbibundle structure $\mathcal{E} \to \mathcal{O}$ induces a principal orbibundle structure $Fr(\mathcal{E}) \curvearrowleft GL_n(\mathbb{R}) \to \mathcal{O}$.*

Proof. Locally, the cone orbibundle structure is given by systems of vector bundles $\tilde{\mathcal{E}} \to \tilde{U}$ together with the transition maps $g_{\alpha\beta}$. Take the principal bundle structure $Fr(\tilde{\mathcal{E}}) \circlearrowleft GL_n(\mathbb{R}) \to \tilde{U}$ with the same transition maps $g_{\alpha\beta}$. They satisfy the same relationships we need to construct a principal orbibundle structure. This defines the principal orbibundle structure we were looking for. $\square$

Because the Γ-action and G-action commute, we have an orbifold map $\mu : \mathcal{P} \times G \to \mathcal{P}$ satisfying the axioms of an action. An element $b \in G$ fixes a point $p \in \mathcal{P}$ if $\tilde{p} \cdot b = \gamma \cdot \tilde{p}$ over an orbifold chart. In a trivialization

$$(\tilde{x}, a) \cdot b = (\tilde{x}, a \cdot b) = (\gamma \cdot \tilde{x}, g_\gamma(\tilde{x}) \cdot b).$$

This means that $\gamma \in \Gamma_{\tilde{x}}$ and $b = a^{-1} \cdot g_\gamma(\tilde{x}) \cdot a$. If $\tilde{\mathcal{P}}$ was trivialized by the section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$ then $b \in G_p$ if and only if

$$b = \delta\Big(\tilde{s}\left(\tilde{\pi}(\tilde{p})\right), \tilde{p}\Big)^{-1} \cdot g_\gamma\Big(\tilde{\pi}(\tilde{p})\Big) \cdot \delta\Big(\tilde{s}\left(\tilde{\pi}(\tilde{p})\right), \tilde{p}\Big),$$

where $\delta : \tilde{\mathcal{P}} \times_{\tilde{\pi}} \tilde{\mathcal{P}} \to G$ is the smooth map $\delta := pr_2 \circ (id \times \mu)^{-1}$ and characterized by

$$\tilde{p} \cdot \delta(\tilde{p}, \tilde{q}) = \tilde{q}. \tag{2.3.2}$$

It follows that $G_p \cong \Gamma_x$, so the G-action on $\mathcal{P}$ is locally free. Locally, the quotient $\mathcal{P}/G$ is given by $(\tilde{\mathcal{P}}/\Gamma)/G$. Because the actions commute, this quotient is homeomorphic to $(\tilde{\mathcal{P}}/G)/\Gamma$. Nevertheless, $\tilde{\mathcal{P}}/G \cong \tilde{U}$ are diffeomorphic. Then the orbifold structure we get onto the quotient is locally given by $(\tilde{U}, \Gamma, \tilde{\phi})$. Gluing together these charts, one has that $\mathcal{P}/G \cong O$ are diffeomorphic as orbifolds.

Proposition 2.33. *Let $P \times_\pi P := \{(p, q) \in P \times P \mid \pi(p) = \pi(q)\}$. The map $id \times \mu : P \times G \to P \times_\pi P$, defined by $(id \times \mu)(p, g) = (p, p \cdot g)$ is a local diffeomorphism.*

Proof. Take $(p, g) \in P \times G$, $X \in T_pP$ and $\xi \in \mathfrak{g}$. Then

$$d_{(p,g)}(id \times \mu)(X, d_eR_g(\xi)) = \begin{pmatrix} Id & 0 \\ d_p\mu_g & d_g\mu^p \end{pmatrix} \begin{pmatrix} X \\ d_eR_g(\xi) \end{pmatrix}$$

where $\mu_g(p) := (id \times \mu)(p, g)$ and $\mu^p(g) = (id \times \mu)(p, g)$. Furthermore,

$$d_g\mu^p(d_eR_g(\xi)) = d_p\mu_g \circ \Psi(p, \xi),$$

where $\Psi(p, \xi)$ is the infinitesimal action of $\mathfrak{g}$ on P. Because the action is locally free, it follows that $d_{(p,g)}(id \times \mu)$ is an isomorphism. Then $id \times \mu$ is a local diffeomorphism. $\square$

Let us come back to the setup of the frame bundle structure of an effective orbifold $\mathcal{O}$. It was given by a manifold $Fr(O)$, a (right) locally free and proper action $GL_n(\mathbb{R})$ such that $Fr(\mathcal{O})/GL_n(\mathbb{R}) \cong \mathcal{O}$ are diffeomorphic as orbifolds.

Definition 2.34. *A principal bundle P over $\mathcal{O}$, with structural group G, is a manifold P with a locally free, proper action $P \curvearrowleft G$ such that $P/G \cong \mathcal{O}$ are diffeomorphic as orbifolds.*

The base orbifold structure could be omitted because it is codified on the quotient P/G. In addition, the map $\pi : P \to \mathcal{O}$ is the quotient map.

Definition 2.35. *A morphism between two principal bundles $P \curvearrowleft G$, $Q \curvearrowleft H$ is given by a homomorphism $\theta : G \to H$ and a smooth map $F : P \to Q$ that is θ-equivariant.*

In proposition 1.36 we proved that such a map induces the following commutative diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "/G"'] & Q \arrow[d, "/H"] \\ P/G \arrow[r, "f"] & Q/H \end{tikzcd}$$

In G-structure theory, we deal with different bundle structures over the same orbifold on the base. Furthermore, we want to compare all possible orbifold structures compatible with a specific geometric structure, which means the structural group is the same on each principal bundle. Then, in the G-structure framework, $F : P \to Q$ is a G-equivariant map such that

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "/G"'] & Q \arrow[d, "/G"] \\ \mathcal{O} \arrow[r, "f"] & \mathcal{O} \end{tikzcd}$$

commutes.

2.3.3 Associated bundles.

By proposition 2.32, from a cone orbibundle $\mathcal{E} \to \mathcal{O}$ of rank k we can construct a principal bundle $Fr(\mathcal{E}) \curvearrowleft GL_k(\mathbb{R}) \to \mathcal{O}$. The associated bundle allows us to construct from a principal bundle and a manifold with an action $G \curvearrowright F$, an orbibundle with fibers F/G_x over $\mathcal{O}$. For $F = \mathbb{R}^n$, this gives us a 1-1 correspondence between cone orbibundles and principal orbibundles. In particular, $T\mathcal{O}$ and $Fr(\mathcal{O})$ are in 1-1 correspondence.

Let $\pi : P \curvearrowleft G \to \mathcal{O}$ be a principal orbibundle. Define the (right) action $P \times F \curvearrowleft G$ by

$$(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f).$$

Since $(p, f) \cdot g = (p, f)$ if and only if $p \cdot g = p$ and $g^{-1} \cdot f = f$, then the action is locally free. Moreover, take sequences $(p_k, f_k) \to (p, f)$ in $P \times F$ and g_k in G such that $(p_k \cdot g_k, g_k^{-1} \cdot f_k) \to (\hat{p}, \hat{f})$. Then $p_k \cdot g_k \to \hat{p}$. Provided that the action $P \curvearrowleft G$ is proper, there exists a subsequence $g_{k_l} \to g$ that converges. Consequently, the action $P \times F \curvearrowleft G$ is locally free and proper so the quotient is an orbifold.

Definition 2.36. *Let $P \curvearrowleft G$ be a principal bundle and $G \curvearrowright F$ a manifold with a G-action. The associated bundle of the principal bundle with fiber F, denoted by $E(P, F, G)$, is the orbifold given by the quotient $(P \times F)/G$.*

When the structure group is understood from the context, we will denote the associated bundle by $E(P, F) := E(P, F, G)$.

The smooth map $pr : P \times F \to P$ is G-equivariant inducing an orbifold map

$$\begin{tikzcd} E(P,F,G) \arrow[d] \\ P/G \end{tikzcd}$$

Example 2.37. Let $\mathcal{O}$ be an orbifold with atlas $(\tilde{U}, \Gamma, \tilde{\phi})$ and take a cone orbibundle of rank k with orbifold charts $(\tilde{\mathcal{E}}, \Gamma, \tilde{\phi})$. Its frame bundle has an orbifold structure codified on the orbifold atlas $(Fr(\tilde{\mathcal{E}}), \Gamma, \tilde{\phi})$. Take the smooth map $\tilde{\varphi} : Fr(\tilde{\mathcal{E}}) \times \mathbb{R}^k \to \tilde{\mathcal{E}}$ given by

$$\tilde{\varphi}(\tilde{p}, v) = \tilde{p}(v).$$

It induces the Γ-equivariant diffeomorphism

$$\tilde{\varphi} : Fr(\tilde{\mathcal{E}}) \times_{GL_k(\mathbb{R})} \mathbb{R}^k \overset{\cong}{\to} \tilde{\mathcal{E}}.$$

Then the orbifold structure of the associated bundle is generated by the orbifold charts $(\tilde{\mathcal{E}}, \Gamma, \tilde{\phi})$, which means, $E(Fr(\mathcal{E}), \mathbb{R}^k) \cong \mathcal{E}$ are isomorphic. In particular $E(Fr(\mathcal{O}), \mathbb{R}^n) \cong T\mathcal{O}$.

Proposition 2.38. *There is a* 1-1 *correspondence between isomorphism classes of cone bundles of rank k and isomorphism classes of principal $GL_k(\mathbb{R})$-orbibundles over a fixed orbifold $\mathcal{O}$.*

Proof. In proposition 2.32, we showed how to obtain a principal orbibundle $Fr(\mathcal{E}) \curvearrowleft GL_k(\mathbb{R})$ from a cone orbibundle $\mathcal{E}$ of rank k. A principal orbibundle $\mathcal{P} \curvearrowleft GL_k(\mathbb{R})$, generates the cone orbibundle of rank k defined by the associated bundle $E(\mathcal{P}, \mathbb{R}^k, GL_k(\mathbb{R}))$. In the previous example we prove that

$$E(Fr(\mathcal{E}), \mathbb{R}^k) \cong \mathcal{E}.$$

On the other hand, take $\tilde{\mathcal{P}}_\alpha \to \tilde{U}_\alpha$ the local principal orbibundle structure of $\mathcal{P}$, with transition matrices $g_{\alpha\beta}$. The associated bundle has the local structure $\tilde{U}_\alpha \times \mathbb{R}^k$, with $\psi^*_{\alpha\beta} : \tilde{U}_\alpha \times \mathbb{R}^k \hookrightarrow \tilde{U}_\beta \times \mathbb{R}^k$ given by

$$\psi^*_{\alpha\beta}(\tilde{x}, v) = (\psi_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x})(v)).$$

If we construct its frame orbibundle $Fr(E(\mathcal{P}, \mathbb{R}^k))$, we get the local structure $\tilde{U}_\alpha \times GL_k(\mathbb{R})$, with $\psi^\times_{\alpha\beta} : \tilde{U}_\alpha \times GL_k(\mathbb{R}) \hookrightarrow \tilde{U}_\beta \times GL_k(\mathbb{R})$ given by

$$\psi^\times_{\alpha\beta}(\tilde{x}, A) = (\psi_{\alpha\beta}(\tilde{x}), g_{\alpha\beta}(\tilde{x}) \cdot A).$$

Then $Fr(E(\mathcal{P}, \mathbb{R}^k)) \cong \mathcal{P}$. That gives us the 1-1 correspondence between cone orbibundles and principal orbibundles. $\square$

2.3.4 Sections and forms on the associated bundle.

Many of the main objects in G-structure theory are given in terms of differential forms on $\mathcal{O}$ with values in some orbibundle. For example, we will see that the torsion of a connection on $T\mathcal{O}$ is an element of $\Omega^2(\mathcal{O}, T\mathcal{O})$ and its curvature of $\Omega^2(\mathcal{O}, Hom(T\mathcal{O}, T\mathcal{O}))$. The idea is to study these objects on the frame bundle, given that it is a manifold. Moreover, their properties show up in different perspectives that allow the use of linear algebra and calculus to find obstructions and invariants of certain geometric structures. Some differential forms on $\mathcal{O}$ will lift to differential forms on $Fr(\mathcal{O})$ because

$$\Omega^\bullet(\mathcal{O}, E(Fr(\mathcal{O}), V)) \overset{\cong}{\longleftrightarrow} \Omega^\bullet_{bas}(Fr(\mathcal{O}), V)$$

are isomorphic, where $\Omega^\bullet_{bas}(Fr(\mathcal{O}), V) \subset \Omega^\bullet(Fr(\mathcal{O}), V)$ is a special type of differential form and V a vector space.

Take a principal bundle $\pi : P \curvearrowleft G \to \mathcal{O}$ and its associated orbibundle $\pi_E : E(P, V) \to \mathcal{O}$.

Proposition 2.39. *The orbifold structures $E(P, V)_{\pi(p)} \cong V/G_p$ are diffeomorphic.*

Proof. Take an orbifold chart $(\tilde{P}, G_p, \tilde{\phi})$, an element $\tilde{P} \ni g_p \cdot \tilde{p} \mapsto p \in P$ and define $\tilde{\varphi}_{g_p \cdot \tilde{p}} : E(\tilde{P}, V)_{g_p \cdot \tilde{\pi}(\tilde{p})} \to V$ by

$$\tilde{\varphi}_{g_p \cdot \tilde{p}}([\tilde{q}, v]) = \delta_{\tilde{q}}(g_p \cdot \tilde{p})^{-1} \cdot v = g_p \cdot \delta_{\tilde{q}}(\tilde{p})^{-1} \cdot v,$$

with δ defined by equation (2.3.2). Then $\tilde{\varphi}_{\tilde{p}}$ is a G_p-equivariant diffeomorphism. All the group elements $g_p \in G_p$ induce isomorphic lifts and then the orbifold diffeomorphism $\varphi_p : E(P, V)_{\pi(p)} \to V/G_p$ induced by $\tilde{\varphi}_{\tilde{p}}$ is well-defined. $\square$

In addition, let $\varphi_p : \pi^*(E(P,V))_{\pi(p)} \to \{p\} \times V/G_p$ be defined by

$$\varphi_p(p,[q,v]) = (p, \varphi_p([q,v])).$$

Locally, for all $g_p \in G_p$, the diffeomorphism

$$\tilde{\varphi}_{\tilde{p}} : \tilde{\pi}^*(E(\tilde{P},V)) \to \tilde{P} \times V$$

satisfies

$$\tilde{\varphi}_{\tilde{p}}(g_p \cdot (\tilde{p},[\tilde{q},v])) = (g_p \cdot \tilde{p}, \delta_{g_p \cdot \tilde{q}}(g_p \cdot \tilde{p})^{-1} \cdot v) = (g_p \cdot \tilde{p}, \delta_{\tilde{q}}(\tilde{p})^{-1} \cdot v).$$

That means the induced action on $\tilde{P} \times V$ is given by

$$g_p \cdot (\tilde{p}, v) = (g_p \cdot \tilde{p}, v).$$

Hence, $\varphi : \pi^*(E(P,V)) \to P \times V$ is a vector bundle diffeomorphism (of manifolds!). Consequently, we have the following commutative diagram

$$\begin{tikzcd}
P \times V \arrow[r, "/G"] \arrow[d, "pr"'] & E(P,V) \arrow[d, "\pi_E"] \\
P \arrow[r, "/G", "\pi"'] & \mathcal{O}
\end{tikzcd}$$

Take a k-form $\omega \in \Omega^k(\mathcal{O}, E(P,V))$. Its pullback belongs to

$$\pi^*\omega \in \Omega^k\left(P, \pi^*(E(P,V))\right).$$

But $\pi^*(E(P,V)) \overset{\varphi}{\cong} P \times V$. Denote $\pi^\star\omega := \varphi(\pi^*\omega)$ (note there are two different symbols here: $\star$ and $*$). Then

$$\pi^\star\omega \in \Omega^k(P,V).$$

The k-form $\pi^\star\omega$ satisfy two properties that define it: being horizontal and G-equivariant. Horizontal means it vanishes on the vertical bundle. The vertical bundle is defined by $\Psi(P \times \mathfrak{g}) = T^V P$, where Ψ is the infinitesimal action field. In fact, since $\pi(p \cdot g) = \pi(p)$ for every $g \in G$ it follows that

$$d_p\pi(\Psi(p,\xi)) = 0.$$

Then $T^V P \subset \ker(d\pi)$ is a subbundle. In addition,

$$\dim(\ker d_p\pi) = \dim(P) - \dim(\mathcal{O}) = \dim(G).$$

Because the action is locally free we have that $T^V P = \ker(d\pi)$ so it is a trivial vector bundle.

Definition 2.40. *A k-form $\omega \in \Omega^k(P, V)$ is horizontal if*

$$\iota_{\Psi(*,\xi)}\omega = \omega(\Psi(*,\xi), *, \ldots, *) = 0,$$

for all $\xi \in \mathfrak{g}$.

Because for all $p \in P$ and $\xi \in \mathfrak{g}$ it is true that $d_p\pi(\Psi(p,\xi)) = 0$, then the k-form $\pi^\star\omega$ is horizontal. Furthermore, for all $g \in G$

$$\begin{aligned} R_g^*(\pi^\star\omega)_p(X_1, \ldots, X_k) &= (\pi^\star\omega)_{p\cdot g}\Big(R_g^*(X_1), \ldots, R_g^*(X_k)\Big) \\ &= \varphi_{p\cdot g}\Big(\omega_{\pi(p)}\big(d_p\pi(X_1), \ldots, d_p\pi(X_k)\big)\Big). \end{aligned}$$

Locally

$$\tilde{\varphi}_{\tilde{p}\cdot g}([\tilde{q}, f]) = \delta_{\tilde{q}}(\tilde{p}\cdot g)^{-1} \cdot f = g^{-1} \cdot \tilde{\varphi}_{\tilde{p}}([\tilde{q}, f]),$$

which implies $\varphi_{p\cdot g} = g^{-1} \cdot \varphi_p$. Then

$$R_g^*(\pi^\star\omega)_p(X_1, \ldots, X_k) = g^{-1} \cdot (\pi^\star\omega)_p(X_1, \ldots, X_k).$$

Definition 2.41. *A k-form $\eta \in \Omega^k(P, F)$ is:*

1. *G-equivariant if $R_g^*\omega = g^{-1} \cdot \omega$.*

2. *Basic if it is horizontal and G-equivariant.*

The space of basic k-forms is denoted by $\Omega^k_{bas}(P, F)$.

Proposition 2.42. *$\pi^\star : \Omega^k(\mathcal{O}, E(P, F)) \to \Omega^k_{bas}(P, F)$ is an isomorphism.*

Proof. The pullback $\pi^* : \Omega^k(\mathcal{O}, E(P, F)) \to \Omega^k(P, \pi^*(E(P, F)))$ and $\varphi : \pi^*(E(P, F)) \to P \times F$ are bundle morphisms. Then $\pi^\star := \varphi \circ \pi^*$ is a morphism.

We already showed that the image of $\pi^\star$ is contained in the space of basic k-forms. For the injectivity take $\overline{\omega_1}, \overline{\omega_2} \in \Omega^k(\mathcal{O}, E(P,F))$. If $\pi^\star(\overline{\omega_1}) = \pi^\star(\overline{\omega_2})$, then for every $q \in P$, $X_i \in T_qP$ we have that

$$\begin{aligned}
\varphi_q^{-1}\Big((\overline{\omega_1})_{\pi(q)}\big(d_q\pi(X_1), \dots, d_q\pi(X_k)\big)\Big)& \\
= \varphi_q^{-1}\Big((\overline{\omega_2})_{\pi(q)}\big(d_q\pi(X_1), \dots, d_q\pi(X_k)\big)\Big)&.
\end{aligned}$$

Thus $(\overline{\omega_1})_{\pi(q)}(X_1, \dots, X_k) = (\overline{\omega_2})_{\pi(q)}(X_1, \dots, X_k)$. That means $\pi^\star$ is injective.

For the surjectivity, take $\omega \in \Omega^k_{bas}(P,F)$. Let $X_i \in \mathfrak{X}(\mathcal{O})$ and $x \in \mathcal{O}$. Because $\pi : P \to \mathcal{O}$ is a submersion there exists $q \in P$ and $Y_i \in \mathfrak{X}(\mathcal{P})$ such that $\pi(q) = x$ and $d\pi(Y_i) = X_i$. Define $\overline{\omega} \in \Omega^k(\mathcal{O}, E(P,F))$ by

$$\overline{\omega}_x(X_1, \dots, X_k) = [q, \omega_q(Y_1, \dots, Y_k)]\,.$$

Let us prove this definition is independent on the coices of q and Y_i. Take other lifts $\hat{Y}_i \in \mathfrak{X}(P)$ of X_i. Given that

$$d\pi(Y_i) = X_i = d\pi(\hat{Y}_i),$$

then $Y_i - \hat{Y}_i \in T^VP$. Because ω is basic

$$\iota_{Y_i - \hat{Y}_i}\omega = 0.$$

We can write

$$\begin{aligned}
\omega(Y_1, \dots, Y_k) - \omega(\hat{Y}_1, \dots, \hat{Y}_k) = \omega(Y_1 - \hat{Y}_1, Y_2, \dots, Y_k)& \\
+\, \omega(\hat{Y}_1, Y_2 - \hat{Y}_2, Y_3, \dots, Y_k) + \cdots + \omega(\hat{Y}_1, \dots, \hat{Y}_{k-2}, Y_{k-1} - \hat{Y}_{k-1}, Y_k)& \\
+\, \omega(\hat{Y}_1, \dots, \hat{Y}_{k-1}, Y_k - \hat{Y}_k) = 0&.
\end{aligned}$$

Hence, $\omega(Y_1, \dots, Y_k) = \omega(\hat{Y}_1, \dots, \hat{Y}_k)$. On the other hand, the elements q that projects onto $x \in \mathcal{O}$ are of the form $q \cdot g$. If we take $q \cdot g$ instead of q, and reminding that ω is G-equivariant, we obtain

$$\begin{aligned}
\overline{\omega}_x(X_1, \dots, X_k) &= \big[q \cdot g, \omega_{q \cdot g}(R_g^*(Y_1), \dots, R_g^*(Y_k))\big] \\
&= \big[q \cdot g, R_g^*\omega_q(Y_1, \dots, Y_k)\big] \\
&= \big[q \cdot g, g^{-1} \cdot \omega_q(Y_1, \dots, Y_k)\big] \\
&= [q, \omega_q(Y_1, \dots, Y_k)]\,.
\end{aligned}$$

Then $\overline{\omega}$ is well-defined and belongs to $\overline{\omega} \in \Omega^k(\mathcal{O}, E(P,F))$. By construction, $\pi^\star\overline{\omega} = \omega$. Consequently, $\pi^\star$ is a bijection and then an isomorphism. $\square$

2.3.5 The tautological form.

Take the identity 1-form $\overline{\theta} \in \Omega^1(\mathcal{O}, T\mathcal{O})$ and a local frame $(\tilde{s}_i)_{i=1}^n$ for the local orbibundle structure $T\tilde{U} \to \tilde{U}$. It induces the section $\tilde{p}_{\tilde{s}} : \tilde{U} \to Fr(\tilde{U})$, defined by

$$\tilde{p}_{\tilde{s}}(\cdot)(v_1, \dots, v_n) = \sum_i \tilde{s}_i(\cdot)v_i.$$

The cone orbibundle diffeomorphism $T\mathcal{O} \cong E(Fr(\mathcal{O}), \mathbb{R}^n)$ is locally given by

$$(\tilde{x}, \tilde{X}) \mapsto [\tilde{p}_{\tilde{s}}(\tilde{x}), \tilde{p}_{\tilde{s}}(\tilde{x})^{-1}(\tilde{X})].$$

It allows us to consider $\overline{\theta}$ as an element of $\Omega^1(\mathcal{O}, E(Fr(\mathcal{O}), \mathbb{R}^n))$. Given that

$$\Omega^k(\mathcal{O}, E(Fr(\mathcal{O}), \mathbb{R}^n)) \overset{\pi^\star, \cong}{\longleftrightarrow} \Omega^k_{bas}(Fr(\mathcal{O}), \mathbb{R}^n),$$

then $\theta := \pi^\star(\overline{\theta})$ is a well defined basic 1-form. Locally, θ is defined by

$$\tilde{\theta}(\tilde{p}, \tilde{Y}) = \tilde{p}^{-1}\left(d_{\tilde{p}}\tilde{\pi}(\tilde{Y})\right).$$

Definition 2.43. *Let $p \in Fr(\mathcal{O})$ and $Y \in T_pFr(\mathcal{O})$. The tautological form is the 1-form $\theta \in \Omega^1_{bas}(Fr(\mathcal{O}), \mathbb{R}^n)$ defined by*

$$\theta_p(Y) = p^{-1}(d_p\pi(Y)).$$

The tautological form is *strongly horizontal*, which means, it only vanishes on the vertical vectors

$$\theta(X) = 0 \Longleftrightarrow X \in T^V P.$$

A differential form is called *tensorial* if it is strongly horizontal and G-equivariant. Then $\theta \in \Omega^1_{ten}(P, \mathbb{R}^n)$. Let $P \curvearrowleft G$ be a manifold with a locally free and proper action such that $n = \dim P/G$ and $G < GL_n(\mathbb{R})$. We will show that it is a G-structure if and only if it has a tensorial form $\theta \in \Omega^1_{ten}(P, \mathbb{R}^n)$. Furthermore, morphisms between G-structures are diffeomorphisms that pullback one tautological form to the other.

2.3.6 Reductions

Take a closed Lie subgroup $H < G$, $\iota : H \to G$ the inclusion and an H-principal bundle $\pi_Q : Q \curvearrowleft H \to \mathcal{O}$. The associated bundle $\iota_*(Q) := E(Q, G, H)$ has a manifold structure. Moreover, it has an action $\iota_*(Q) \curvearrowleft G$ defined by

$$[q, g] \cdot \tilde{g} = [q, g\tilde{g}] \, .$$

It follows that

$$\begin{tikzcd} \iota_*(Q) \curvearrowleft G \arrow[d, "\pi_Q"] \\ \mathcal{O} \end{tikzcd}$$

defines a principal bundle structure.

Definition 2.44. *A reduction of the principal bundle $\pi_P : P \curvearrowleft G \to \mathcal{O}$ to a closed subgroup $H < G$ is an H-principal bundle $\pi_Q : Q \curvearrowleft H \to \mathcal{O}$ such that $\iota_*(Q) \cong P$.*

Remark: The map $\phi : Q \to \iota_*(Q)$ defined by $\phi(q) = [q, e]$ is an embedding. Then Q can be thought of as a submanifold of $Q \subset P$. It follows that a reduction $H < G$ of $P \curvearrowleft G$ is equivalent to have an H-invariant subbundle $Q \subset P$.

Proposition 2.45. *There is a* 1-1 *relation between reductions $Q \curvearrowleft H$ of $P \curvearrowleft G$ and orbisections $s : \mathcal{O} \to P/H$.*

Proof. Take a reduction $Q \curvearrowleft H$ of $P \curvearrowleft G$ and $\varphi : \iota_*(Q) \to P$ an isomorphism. The local sections $\tilde{s}_\alpha : \tilde{U}_\alpha \to \tilde{Q}_\alpha$ of $\pi_Q : Q \to \mathcal{O}$ induces local maps $\tilde{s}_\alpha : \tilde{U}_\alpha \to \iota_*(\tilde{Q}_\alpha)$ defined by

$$\tilde{s}_\alpha(\tilde{x}) = [\tilde{s}_\alpha(\tilde{x}), e].$$

However, over a non-trivial intersection of two charts $\tilde{U}_\alpha \cap \tilde{U}_\beta$, it could happen that

$$\tilde{s}_\alpha(\tilde{x}) = [\tilde{s}_\alpha(\tilde{x}), e] = [\tilde{s}_\beta(\tilde{x}) \cdot h, e] = [\tilde{s}_\beta(\tilde{x}), h] \neq \tilde{s}_\beta(\tilde{x}).$$

$\tilde{s}_\alpha$ and $\tilde{s}_\beta$ belong to the same fiber and the element that takes $\tilde{s}_\alpha$ to $\tilde{s}_\beta$

$$\delta(\tilde{s}_\alpha(\tilde{x}), \tilde{s}_\beta(\tilde{x})) \in H.$$

Consequently, the local maps $\tilde{s}_\alpha : \tilde{U}_\alpha \to \iota_*(\tilde{Q}_\alpha)$ defines a global orbisection

$$s : \mathcal{O} \to \iota_*(Q)/H.$$

Nevertheless, φ is a G-equivariant isomorphism. Then $\iota_*(Q)/H \cong P/H$ and s induces the orbisection $s : \mathcal{O} \to P/H$.
Conversely, take the orbisection $s : \mathcal{O} \to P/H$ and define $Q \subset P$ by

$$Q := \{p \in P \mid s(\pi_P(p)) \in [p]_H\}.$$

It is a non-empty H-invariant set. Take a local section $\tilde{\sigma} : \tilde{P}_H \to \tilde{P}$ of the principal bundle $P \to P/H$ and a lift $\tilde{s}_H : \tilde{U} \to \tilde{\mathcal{P}}_H$ of the orbisection $s : \mathcal{O} \to P/H$. The smooth map $\tilde{s} := \tilde{\sigma} \circ \tilde{s}_H$ is a local section $\tilde{s} : \tilde{U} \to \tilde{P}$ of the principal bundle structure $P \curvearrowleft G \to \mathcal{O}$. Moreover, the induced map $s : U \to P$ satisfies $s(U) \subset Q$. Then Q has a manifold structure such that $Q \curvearrowleft H \to \mathcal{O}$ is a principal bundle. $\square$

Example 2.46. An $\{e\}$-reduction for $\pi : P \curvearrowleft G \to \mathcal{O}$ is an orbisection $s : \mathcal{O} \to P$. A local lift $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$ of s is a *local orbisection.* Then the diffeomorphism $\varphi : \tilde{U} \times G \to \tilde{\mathcal{P}}$ induces the Γ-action $\Gamma \curvearrowright \tilde{U} \times G$ given by

$$\gamma \cdot (\tilde{x}, g) = (\gamma \cdot \tilde{x}, g).$$

Then $(\tilde{U}/\Gamma) \times G \cong \tilde{\mathcal{P}}/\Gamma$. However, $\tilde{\mathcal{P}}/\Gamma$ is a manifold which implies that $\mathcal{O}$ has a manifold structure.

Therefore, it is not always possible to take reductions. We will prove that reductions are in 1-1 correspondence with geometric structures.

2.4 Principal connections.

The G-action on P induces the trivial vector bundle $T^V\mathcal{P} \subset TP$. A connection is a choice of a G-invariant distribution $\mathcal{H} \subset TP$ such that

$\mathcal{H} \oplus T^V P$ is isomorphic to TP. Given that $T^V P \cong P \times \mathfrak{g}$, the presence of a horizontal distribution induces the projection

$$TP \longrightarrow T^V P \longrightarrow P \times \mathfrak{g} \longrightarrow \mathfrak{g}.$$

It defines a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ called principal connection. We will show that principal connections $\omega \in \Omega(Fr(\mathcal{E}), \mathfrak{gl}_k(\mathbb{R}))$ are in 1-1 correspondence with cone connections ∇ on $\mathcal{E} \to \mathcal{O}$.

2.4.1 The Atiyah sequence.

Take the short exact sequence of vector bundles over P given by

$$0 \to P \times \mathfrak{g} \xrightarrow{\Psi} TP \xrightarrow{d\pi} \pi^*(T\mathcal{O}) \to 0,$$

with Ψ the infinitesimal action and $\pi : P \to \mathcal{O}$. The G-action on P induces the locally free and proper G-actions $P \times \mathfrak{g} \curvearrowleft G$, $TP \curvearrowleft G$ and $\pi^*(T\mathcal{O}) \curvearrowleft G$ given by

$$\begin{aligned} (p, \xi) \cdot g &= (p \cdot g, Ad(g^{-1})\xi), \\ X \cdot g &= R_g^* X, \\ (q, Y) \cdot g &= (q \cdot g, Y). \end{aligned}$$

Given that $\Psi : \mathcal{P} \times \mathfrak{g} \to T^V \mathcal{P}$ and $d\pi : T\mathcal{P} \to \pi^*(T\mathcal{O})$ are G-equivariant, then we can take the quotient by G and obtain

$$0 \to (P \times \mathfrak{g})/G \xrightarrow{\Psi} TP/G \xrightarrow{d\pi} \pi^*(T\mathcal{O})/G \to 0.$$

Given that the action is locally free and proper, they are not vector bundles but orbibundles over $\mathcal{O}$. Denote by $Ad(P) := (P \times \mathfrak{g})/G$ and notice that $\pi^*(T\mathcal{O})/G \cong T\mathcal{O}$.

Definition 2.47. *A short exact sequence of cone orbibundles $A \to \mathcal{O}$, $B \to \mathcal{O}$ and $C \to \mathcal{O}$ is given by two morphisms $j : A \to B$ and $k : B \to C$ covering the identity $id : \mathcal{O} \to \mathcal{O}$, such that locally*

$$0 \to \tilde{A} \xrightarrow{\tilde{j}} \tilde{B} \xrightarrow{\tilde{k}} \tilde{C} \to 0$$

is a short exact sequence of vector bundles.

Given that the G-action and the Γ-action commutes over $\tilde{P}$ then

$$0 \to Ad(P) \overset{\Psi}{\to} TP/G \overset{d\pi}{\to} T\mathcal{O} \to 0 \tag{2.4.1}$$

is locally given by

$$0 \to \tilde{P} \times_G \mathfrak{g} \to T\tilde{P}/G \to T\tilde{U} \to 0,$$

and called the Atiyah sequence.

2.4.2 Splittings and connections.

The short exact sequence of vector bundles

$$0 \longrightarrow P \times \mathfrak{g} \xrightarrow{\rho} TP \underset{h}{\overset{d\pi}{\rightleftarrows}} \pi^*(T\mathcal{O}) \longrightarrow 0$$

splits if there exists a vector bundle morphism $h : \pi^*(T\mathcal{O}) \to TP$, covering the identity $id : P \to P$, such that $d\pi \circ h = id$. There is a 1-1 correspondence between splittings and horizontal distributions of $\mathcal{H} \subset TP$, with $\mathcal{H} = h(\pi^*(T\mathcal{O}))$. Every short exact sequence of vector bundles splits. Then there always exists a horizontal distribution of TP. However, the G-invariant hypothesis requires that not only this sequence but the Atiyah sequence 2.4.1 splits.

Definition 2.48. *A short exact sequence of cone orbibundles splits if for every $x \in \mathcal{O}$ there are orbifold charts such that*

$$0 \to \tilde{A} \overset{\tilde{j}}{\to} \tilde{B} \overset{\tilde{k}}{\to} \tilde{C} \to 0$$

splits.

As long as every short exact sequence of vector bundle splits, every short exact sequence of cone orbibundles splits.

Definition 2.49. *A connection on a principal bundle $\pi : \mathcal{P} \to \mathcal{O}$ is a choice of a horizontal distribution $\mathcal{H} \subset T\mathcal{P}$ ($\mathcal{H} \oplus T^V\mathcal{P} \cong T\mathcal{P}$) such that*

$$d_pR_g(\mathcal{H}_p) = \mathcal{H}_{p\cdot g},$$

for all $g \in G$.

Remark: Take $p \in P$. Its isotropy $G_p \subset G$ defines an action $G_p \curvearrowright \mathcal{H}_p$ such that the continuous function $d_p\pi : \mathcal{H}_p \to T_{\pi(p)}\mathcal{O}$ becomes the homeomorphism $d_p\pi : \mathcal{H}_p/G_p \overset{\cong}{\to} T_{\pi(p)}\mathcal{O}$.

Proposition 2.50. *There is a* 1-1 *correspondence between splittings*

$$0 \longrightarrow Ad(P) \xrightarrow{\Psi} TP/G \underset{h}{\overset{d\pi}{\rightleftarrows}} T\mathcal{O} \longrightarrow 0$$

of the Atiyah sequence and connections $\mathcal{H} \subset TP$ on the principal bundle $\pi : P \to \mathcal{O}$.

Proof. Locally, take the splitting $\tilde{h}$ of the short exact sequence

$$0 \longrightarrow Ad(\tilde{P}) \xrightarrow{\Psi} T\tilde{P}/G \underset{\tilde{h}}{\overset{d\tilde{\pi}}{\rightleftarrows}} T\tilde{U} \longrightarrow 0.$$

Let

$$\tilde{\mathcal{H}}_{\tilde{p}} := \{\tilde{X} \in T_{\tilde{p}}\tilde{P} \mid [\tilde{X}]_G = \tilde{h}(d_{\tilde{p}}\tilde{\pi}(\tilde{X}))\}. \tag{2.4.2}$$

Given that $T\tilde{P} \cong Ad(\tilde{P}) \oplus T\tilde{U}$, the vector bundle structures on $Ad(\tilde{P})$ and $T\tilde{U}$ induces a vector bundle structure on $T\tilde{P}/G$ such that $\tilde{h}$ is a monomorphism. Hence, $\tilde{\mathcal{H}}_p$ has a vector space structure of rank equal to the rank of $T\tilde{U}$. If $\tilde{X} \in \tilde{\mathcal{H}}_p$ is such that $d_p\tilde{\pi}(\tilde{X}) = 0$, then

$$[\tilde{X}]_G = \tilde{h}(d_{\tilde{p}}\tilde{\pi}(\tilde{X})) = \tilde{h}_{\tilde{p}}(0) = 0.$$

Thus, $\tilde{\mathcal{H}}_p$ is a horizontal distribution. Furthermore, for every $g \in G$

$$\tilde{h}(d\tilde{\pi}(R_g^*\tilde{X})) = \tilde{h}(d\tilde{\pi}(\tilde{X})) = \left[\tilde{X}\right]_G = \left[R_g^*\tilde{X}\right]_G.$$

It follows that $R_g^*(\tilde{\mathcal{H}}_p) \subset \tilde{\mathcal{H}}_{\tilde{p}\cdot g}$. In addition, the inverse $R_{g^{-1}}^*$ satisfies $R_{g^{-1}}^*(\tilde{\mathcal{H}}_p) \subset \tilde{\mathcal{H}}_{\tilde{p}\cdot g^{-1}}$ and then $d_pR_g(\mathcal{H}_p) = \mathcal{H}_{p\cdot g}$. Define

$$\tilde{\mathcal{H}} := \bigsqcup_{\tilde{p}\in\tilde{P}} \tilde{\mathcal{H}}_{\tilde{p}}.$$

It has a smooth structure given by the bijections $d_{\tilde{p}}\pi : \tilde{\mathcal{H}}_{\tilde{p}} \to T_{\tilde{\pi}(\tilde{p})}\tilde{U}$ and the smooth structure on $T\tilde{U}$. The natural projection $\tilde{\pi} : \tilde{\mathcal{H}} \to \tilde{P}$ is a well defined Γ-equivariant map. That gives us the local bundle structure of the horizontal G-equivariant distribution $\mathcal{H} \subset T\mathcal{P} \to \mathcal{P}$.
On the other hand, if $\mathcal{H} \subset TP$ is a connection then $d\pi|_{\mathcal{H}} : \mathcal{H} \to \pi^*(T\mathcal{O})$ is an isomorphism of vector bundles. Its inverse provides a G-equivariant splitting $h = d\pi^{-1}|_{\mathcal{H}}$. The definition (2.4.2) gives the 1-1 correspondence between these two constructions. □

Corollary 2.51. *Every principal bundle $\pi : \mathcal{P} \to \mathcal{O}$ admits a connection.*

2.4.3 Connection form.

Take a connection $\mathcal{H} \oplus T^V P \cong TP$. It defines the vertical projection $\upsilon : TP \to T^V P$. A principal connection associates to a vector $Y \in TP$ the Lie algebra element $\xi \in \mathfrak{g}$ such that

$$\Psi(\cdot, \xi) = \upsilon(Y(\cdot)),$$

with Ψ the infinitesimal action. The existence of a connection is in 1-1 correspondence with the splitting $h : T\mathcal{O} \to TP/G$ of the Atiyah sequence

$$\begin{tikzcd} 0 \arrow[r] & Ad(P) \arrow[r, "\Psi"] & TP/G \arrow[r, shift left, "d\pi"] & T\mathcal{O} \arrow[l, shift left, dashed, "h"] \arrow[r] & 0. \end{tikzcd}$$

Given that

$$d\pi\left([Y]_G - (h \circ d\pi)([Y]_G)\right) = 0,$$

then $\Psi^{-1}([Y]_G - h(d\pi([Y]_G)))$ is well-defined. Let $\omega : TP/G \to Ad(P)$ be

$$\omega([Y]_G) = \Psi^{-1}\left([Y]_G - (h \circ d\pi)([Y]_G)\right). \qquad (2.4.3)$$

It satisfies $\omega \circ \Psi = id|_{Ad(P)}$.

Definition 2.52. *Let*

$$0 \longrightarrow A \xrightarrow{j} B \xrightarrow{k} C \longrightarrow 0$$

be a short exact sequence of cone orbibundles over $\mathcal{O}$. A morphism $\omega : B \to A$ is a splitting of j if $\omega \circ j = id_A$.

Lemma 2.53. *Take a short exact sequence of cone orbibundles*

$$0 \longrightarrow A \underset{\omega}{\overset{j}{\rightleftarrows}} B \underset{h}{\overset{k}{\rightleftarrows}} C \longrightarrow 0.$$

There exists a 1-1 *correspondence between splittings* $\omega : B \to A$ *and* $h : C \to B$.

Proof. Equation (2.4.3) gives the 1-1 correspondence. □

Hence, a connection $\mathcal{H} \subset TP$ is in 1-1 correspondence with an orbibundle morphism $\omega : TP/G \to Ad(P)$. Take the diffeomorphism

$$Ad(P)_{\pi(p)} \overset{\varphi_p}{\cong} \mathfrak{g}/G_p,$$

between the orbifold structures as in proposition 2.39. Identify $\omega_p(Y)$ with the composition

$$T_pP \overset{/G}{\to} T_pP/G_p \overset{\omega_p}{\to} Ad(P)_{\pi(p)} \overset{\varphi_p}{\to} \mathfrak{g}/G_p.$$

It is locally given by

$$\tilde{\omega}_{\tilde{p}}(\tilde{Y}) = (\tilde{\varphi}_{\tilde{p}} \circ \omega_{\tilde{p}})([\tilde{Y}]_G).$$

It follows that

$$\begin{aligned}
\tilde{\omega}_{\gamma\cdot\tilde{p}}(\gamma \cdot \tilde{Y}) &= (\tilde{\varphi}_{\gamma\cdot\tilde{p}} \circ \omega_{\gamma\cdot\tilde{p}})(\gamma \cdot [\tilde{Y}]_G) \\
&= \gamma^{-1} \cdot \tilde{\varphi}_{\tilde{p}}(\gamma \cdot \omega_{\tilde{p}}([\tilde{Y}]_G)) \\
&= \gamma^{-1} \cdot \gamma \cdot \tilde{\varphi}_{\tilde{p}}(\omega_{\tilde{p}}([\tilde{Y}]_G)) \\
&= \tilde{\omega}_{\tilde{p}}(\tilde{Y}).
\end{aligned}$$

Then $\tilde{\omega} : T\tilde{P} \to \mathfrak{g}$ is Γ-invariant, which implies $\omega \in \Omega^1(P, \mathfrak{g})$. It satisfies two properties that characterize the fact that comes from a connection:

1. $R_g^*\omega = Ad_{g^{-1}}\omega$ for all $g \in G$.

$$\begin{aligned}
R_g^*\omega(X) &= \varphi_{p\cdot g}\left(\omega_{p\cdot g}(\left[R_g^*X\right]_G)\right) \\
&= g^{-1} \cdot \varphi_p\left(\omega_p([X]_G)\right) \\
&= Ad_{g^{-1}}(\omega(X)).
\end{aligned}$$

2. $\omega(\Psi(p,\xi)) = \xi$ for all $\xi \in \mathfrak{g}$.

$$\begin{aligned}\omega(\Psi(p,\xi)) &= \varphi_p\left(\omega\left([\Psi(p,\xi)]_G\right)\right) \\ &= \varphi_p([p,\xi]) \\ &= \xi.\end{aligned}$$

Definition 2.54. *A 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ such that :*

1. *$R_g^*\omega = Ad_{g^{-1}}\omega$ for all $g \in G$,*
2. *$\omega(\Psi(p,\xi)) = \xi$ for all $\xi \in \mathfrak{g}$,*

is known as connection form.

Proposition 2.55. *There is a 1-1 correspondence between connections $\mathcal{H} \subset TP$ on a principal bundle $P \curvearrowleft G \to \mathcal{O}$ and connection forms $\omega \in \Omega^1(P,\mathfrak{g})$.*

Proof. ($\Rightarrow$) Already done.
($\Leftarrow$) Take a connection form $\omega \in \Omega^1(P,\mathfrak{g})$. Let $X \in T_pP$ and define $\breve{\omega} : TP/G \to Ad(P)$ by

$$\breve{\omega}([X]_G) = [p, \omega(X)]$$

It is well-defined because

$$\begin{aligned}\breve{\omega}([d_pR_gX]_G) &= [p\cdot g, \omega(d_pR_g(X))] \\ &= \left[p\cdot g, Ad_{g^{-1}}\omega(X)\right] \\ &= [p, \omega(X)] \\ &= \breve{\omega}([X]_G).\end{aligned}$$

In addition

$$\begin{aligned}(\breve{\omega}\circ\Psi)([p,\xi]) &= \breve{\omega}(\Psi(p,\xi)) \\ &= [p, \omega(\Psi(p,\xi))] \\ &= [p,\xi]\,,\end{aligned}$$

which means that $\breve{\omega} : TP/G \to Ad(P)$ is a splitting for the Atiyah sequence. Hence, it induces a connection on P. $\square$

2.4.4 $Fr(\mathcal{E})$ connections $\to$ $\mathcal{E}$ cone connections

A principal connection $\omega \in \Omega^1(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$ on the frame bundle $Fr(\mathcal{E})$ induces a connection ∇ on the cone orbibundle $\mathcal{E}$. That happens because, firstly, a connection on a cone orbibundle could be given in terms of connection matrices (see proposition 2.22). Secondly, a principal connection $\omega \in \Omega^1(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$ assigns to each vector $TFr(\mathcal{E})$ a matrix! Then, going from $T\mathcal{O}$ to $TFr(\mathcal{E})$ will give us a way to construct the connection matrices.
Take a local frame $(\tilde{s}_i^\alpha)_{i=1}^n$ over an orbifold chart $\tilde{U}_\alpha$. Define the smooth map $\tilde{s}_\alpha : \tilde{U}_\alpha \to Fr(\tilde{\mathcal{E}}_\alpha)$ by

$$\tilde{s}_\alpha(\tilde{x})\left(v^1, \ldots, v^n\right) = \sum_{i=1}^n v^i \tilde{s}_i^\alpha(\tilde{x}).$$

Its differential induces a map $d\tilde{s}_\alpha : T\tilde{U}_\alpha \to TFr(\tilde{\mathcal{E}}_\alpha)$. Define the connection matrix $\tilde{\omega}_\alpha^{\tilde{s}} \in \Omega^1(T\tilde{U}_\alpha, \mathfrak{gl}_n(\mathbb{R}))$ associated to the local frame $\tilde{s}_\alpha$ by

$$\tilde{\omega}_{\tilde{s}}^\alpha(\tilde{Y}) = \tilde{\omega}\left(\left[d\tilde{s}_\alpha(\tilde{Y})\right]_{\Gamma_\alpha}\right)^T.$$

Let $(\tilde{s}_i^\beta)_{i=1}^n$ be a local frame over $\tilde{U}_\beta$, with $\tilde{U} = \tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$. There exists a smooth function $g : \tilde{U} \to GL_n(\mathbb{R})$ such that $s_\alpha \cdot g = s_\beta$. Consequently

$$\begin{aligned}
\omega_{\tilde{s}}^\beta(\tilde{Y}) &= \omega\left(\left[d\tilde{s}_\beta(\tilde{Y})\right]_{\Gamma_\beta}\right) \\
&= \omega\left(R_g^*\left[(d\tilde{s}_\alpha(\tilde{Y}))\right]_{\Gamma_\alpha} + \left[\Psi(s_\beta, g^{-1}dg(\tilde{Y}))\right]_{\Gamma_\alpha}\right) \\
&= Ad_{g^{-1}}\omega_{\tilde{s}}^\alpha(\tilde{Y}) + g^{-1}dg(\tilde{Y}) \\
&= g^{-1}\omega_{\tilde{s}}^\alpha(\tilde{Y})g + g^{-1}dg(\tilde{Y}).
\end{aligned}$$

Given that the system of local matrices $(\omega_{\tilde{s}}^\alpha)_{\alpha\in J}$ satisfies proposition 2.22, then we get a connection ∇ on $\mathcal{E}$.

Proposition 2.56. *A principal connection $\omega \in \Omega^1(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$ induces a cone connection ∇ on $\mathcal{E}$.*

Corollary 2.57. *Every cone orbibundle $\mathcal{E}$ admits a connection.*

2.4.5 $\mathcal{E}$ cone connections $\leftrightarrows$ $Fr(\mathcal{E})$ connections

Despite the manifold case, a connection on a cone orbibundle $\mathcal{E}$ does not lift a path $\eta : I \to \mathcal{O}$ to a unique path $u : I \to Fr(\mathcal{E})$. Nevertheless, it lifts cone fields $X \in \mathfrak{X}(\mathcal{O})$ to vector fields $Y \in \mathfrak{X}(Fr(\mathcal{E}))$. In addition, the cone vectors not represented by cone fields do not lift to a unique vector but a finite collection of vectors. They define a connection $\mathcal{H} \subset TFr(\mathcal{E})$. This gives a 1-1 correspondence between cone connections ∇ on $\mathcal{E} \to \mathcal{O}$ and principal bundle connections $\mathcal{H} \subset TFr(\mathcal{E})$.

Take a local cone orbibundle connection ∇ on $\tilde{\mathcal{E}} \to \tilde{U}$, a lift $\tilde{\eta} : I \to \tilde{U}$ for the path $\eta : I \to \mathcal{O}$, $\tilde{\eta}(0) = \tilde{x}$ and a trivialization $(\tilde{s}_i)_{i=1}^k$ for $\tilde{\mathcal{E}}$. The trivialization induces the frame $\tilde{s} : \tilde{U} \to Fr(\tilde{\mathcal{E}})$. The parallel transport along $\tilde{\eta}$, with initial conditions $\tilde{s}(\tilde{x})(e_i)$, gives paths $\tilde{p}_i : I \to \tilde{\mathcal{E}}$ defined by

$$\tilde{p}_i(t) := \tilde{T}^{0,t}_{\tilde{\eta}}(\tilde{s}(\tilde{x})(e_i)).$$

They induce the frame $\tilde{p}_{\tilde{s}} : I \to Fr(\tilde{\mathcal{E}})$ given by

$$\tilde{p}_{\tilde{s}}(t)(v^1, v^2, \ldots, v^n) = \sum_{i=1}^{r} v^i \tilde{p}_i(t),$$

and called parallel frame along $\tilde{\eta}$ starting at $\tilde{s}(\tilde{x})$. We have chosen $\tilde{\eta}$ instead of $\gamma \cdot \tilde{\eta}$ and $\tilde{p}_{\tilde{s}}$ depends on this choice. Let $\tilde{p}^i_\gamma(t) = \tilde{T}^{0,t}_{\gamma\cdot\eta}(\tilde{s}_i(\tilde{x}))$ and $\tilde{p}_{\gamma\tilde{s}} : I \to Fr(\tilde{\mathcal{E}})$ be defined by

$$\tilde{p}_{\gamma\tilde{s}}(t)(v^1, v^2, \ldots, v^n) := \sum_{i=1}^{n} v^i \tilde{p}^i_\gamma(t).$$

Given that $\tilde{T}^{0,t}_{\gamma\cdot\tilde{\eta}} = \gamma \cdot \tilde{T}^{0,t}_{\tilde{\eta}} \cdot \gamma^{-1}$ then

$$\tilde{p}_{\gamma\tilde{s}} = \gamma \cdot \tilde{p}_{\tilde{s}} \cdot g_{\gamma^{-1}}(\tilde{x}).$$

Let $p_{\gamma s}(t) = [\tilde{p}_{\gamma\tilde{s}}(t)]_\Gamma$ be the lifts for $\gamma \cdot \eta$ on $Fr(\mathcal{E})$. They are related by

$$p_{\gamma s}(t) = p_s(t) \cdot g_{\gamma^{-1}}(\tilde{x}).$$

Hence, a connection on the cone orbibundle $\mathcal{E} \to \mathcal{O}$ allows lifting a path $I \to \mathcal{O}$ to a finite set of paths $I \to Fr(\mathcal{E})$

$$\eta \mapsto \begin{cases} p_s \\ p_s \cdot g_{\gamma_1^{-1}}(\tilde{x}) \\ \vdots \\ p_s \cdot g_{\gamma_l^{-1}}(\tilde{x}) \end{cases}$$

Define $\tilde{h} : \tilde{\pi}^*(T\tilde{U}) \to TFr(\tilde{\mathcal{E}})$ by

$$\tilde{h}\left(\tilde{s}, \tilde{\eta}'(0)\right) = \tilde{h}_{\tilde{s}}(\tilde{\eta}'(0)) = \tilde{p}_{\tilde{s}}'(0).$$

Because the connection is linear and the uniqueness of solutions to ODE implies that

$$\tilde{h}_{\tilde{s}}\left(\tilde{\eta}_1'(0) + \tilde{\eta}_2'(0)\right) = \tilde{h}_{\tilde{s}}\left(\tilde{\eta}_1'(0)\right) + \tilde{h}_{\tilde{s}}\left(\tilde{\eta}_2'(0)\right).$$

Given that $\nabla_{\tilde{\eta}'(t)}\tilde{p}^i_{\tilde{s}}(t) = 0$ and $\gamma \cdot 0 = 0$ then

$$\nabla_{\gamma\cdot\tilde{\eta}'(t)}\gamma \cdot \tilde{p}^i_{\tilde{s}}(t) = 0,$$

which means

$$\tilde{h}(\gamma \cdot \tilde{s}, \gamma \cdot \tilde{\eta}'(0)) = \gamma \cdot \tilde{h}(\tilde{s}, \tilde{\eta}'(0)).$$

In addition, it is true that

$$d_{\tilde{s}(\tilde{x})}\tilde{\pi}\left(\tilde{h}_{\tilde{s}}\left(\tilde{\eta}'(0)\right)\right) = \tilde{\eta}'(0).$$

Then the short exact sequence

$$0 \longrightarrow T^V Fr(\tilde{\mathcal{E}}) \xrightarrow{\ \iota\ } TFr(\tilde{\mathcal{E}}) \underset{\tilde{h}}{\overset{d\tilde{\pi}}{\rightleftarrows}} \tilde{\pi}^*(T\tilde{U}) \longrightarrow 0$$

splits. Define the horizontal vector subspace by $\tilde{\mathcal{H}}_{\tilde{s}(\tilde{x})} := \tilde{h}_{\tilde{s}}(T_{\tilde{x}}\tilde{U})$. Take $g \in GL_n(\mathbb{R})$, the frame $\tilde{p}_{\tilde{s}} \cdot g : I \to Fr(\tilde{\mathcal{E}})$ is given by

$$(\tilde{p}_{\tilde{s}} \cdot g)(t)(v^1, \ldots, v^n) = \sum_{i=1}^{n} v^i \tilde{p}^i_{\tilde{s}\cdot g}(t),$$

with $\tilde{p}^i_{\tilde{s}\cdot g}(t) = \sum_{j=1}^n g_{ij}\tilde{p}^i_{\tilde{s}}(t)$. It is a frame along $\tilde{\eta}$ which allow us to calculate

$$\frac{\nabla}{dt}\tilde{p}^i_{\tilde{s}\cdot g}(t) = \sum_{j=1}^n g_{ij}\frac{\nabla}{dt}\tilde{p}^i_{\tilde{s}}(t) = 0.$$

Then $\tilde{p}_{\tilde{s}\cdot g}$ is a parallel frame along $\tilde{\eta}$ with $\tilde{s}(\tilde{x})\cdot g$ as the initial point, which implies

$$\tilde{h}(\tilde{s}\cdot g, \tilde{X}) = R^*_g\tilde{h}(\tilde{s}, \tilde{X}).$$

If we prove $\tilde{h}$ is smooth, by proposition 2.50, we get a principal connection $\mathcal{H} \subset TFr(\mathcal{E})$. For, firstly notice that the composition $\tilde{s}\circ\tilde{\eta}$ is a frame along $\tilde{\eta}$. Given that the parallel frame $\tilde{p}_{\tilde{s}}$ is also a frame along $\tilde{\eta}$, there exists a smooth map $A : I \to GL_n(\mathbb{R})$ such that

$$(\tilde{s}\circ\tilde{\eta})(t) = \tilde{p}_{\tilde{s}}(t)\cdot A(t).$$

By construction $A(0) = Id$; furthermore

$$\tilde{p}'_{\tilde{s}}(0) = d_{\tilde{x}}\tilde{s}(\tilde{\eta}'(0)) - \Psi(\tilde{s}(\tilde{x}), A'(0)).$$

Even though this equation tells us the smooth behavior of the horizontal vectors, it uses the infinitesimal action associated to $A'(0)$, which depends on the horizontal lift $\tilde{p}_{\tilde{s}}$. To avoid cyclic arguments, notice that

$$\nabla_{\tilde{\eta}'(t)}\tilde{s}_i(\tilde{\eta}(t)) = \sum_{j=1}^n (\omega_{\tilde{s}})_{ij}(\tilde{\eta}'(t))\tilde{s}_j(\tilde{\eta}(t)),$$

with $\omega_{\tilde{s}}$ the connection matrix associated to the frame $\tilde{s}$ and

$$\begin{aligned}
\nabla_{\tilde{\eta}'(t)}\tilde{s}_i(\tilde{\eta}(t)) &= \sum_{j=1}^n \nabla_{\tilde{\eta}'(t)}\tilde{p}^j_{\tilde{s}}(t)A(t)_{ji} \\
&= \sum_{j=1}^n \left(A'_{ji}(t)\tilde{p}^j_{\tilde{s}}(t) + A(t)_{ji}\nabla_{\tilde{\eta}'(t)}\tilde{p}^j_{\tilde{s}}(t)\right) \\
&= \sum_{j=1}^n A'_{ji}(t)\tilde{p}^j_{\tilde{s}}(t).
\end{aligned}$$

Evaluating at zero and comparing both equations we obtain

$$A'_{ji}(0) = (\omega_{\tilde{s}})_{ij}\,(\tilde{\eta}'(0)),$$

or equivalently, $A'(0) = \omega_{\tilde{s}}(\tilde{\eta}'(0))^T$. Hence

$$\tilde{p}'_{\tilde{s}}(0) = d_{\tilde{x}}\tilde{s}_\alpha(\tilde{\eta}'(0)) - \Psi(\tilde{s}(\tilde{x}), \omega_{\tilde{s}}(\tilde{\eta}'(0))^T).$$

We conclude that for $\tilde{X} \in \mathfrak{X}(\tilde{U})$, its horizontal lift is given by

$$\tilde{h}(\tilde{s}, \tilde{X}) = \tilde{s}_*(\tilde{X}) - \Psi(\tilde{s}, \omega_{\tilde{s}}(\tilde{X})^T). \tag{2.4.4}$$

The previous equations proves $\tilde{h} : \tilde{\pi}^*(T\tilde{U}) \to TFr(\tilde{\mathcal{E}})$ is smooth.

Proposition 2.58. *A cone orbibundle connection ∇ on $\mathcal{E} \to \mathcal{O}$ induces a principal connection $\mathcal{H} \subset TFr(\mathcal{E})$ on $Fr(\mathcal{E}) \to \mathcal{O}$.*

Proof. Take orbifold charts $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$, $(\tilde{U}_\beta, \Gamma_\beta, \tilde{\phi}_\beta)$ such that $U_\alpha \subset U_\beta$ and an injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$. Let $\tilde{\mathcal{E}}_\alpha \to \tilde{U}_\alpha$ be a cone orbibundle structure trivialized by $(\tilde{s}_i^\alpha)_{i=1}^n$. It induces the frame $\tilde{s}^\alpha : \tilde{U}_\alpha \to Fr(\tilde{\mathcal{E}}_\alpha)$. Take the path $\tilde{\eta}_\alpha : I_\alpha \to \tilde{U}_\alpha$ and $\tilde{h}_\alpha : \tilde{\pi}_\alpha^*(T\tilde{U}_\alpha) \to TFr(\tilde{\mathcal{E}}_\alpha)$ the horizontal lift. The injection $\tilde{\psi}_{\alpha\beta}^\times : \tilde{\mathcal{E}}_\alpha \to \tilde{\mathcal{E}}_\beta$ induces a trivialization $\tilde{s}_i^\beta = \tilde{\psi}_{\alpha\beta}^\times(\tilde{s}_i^\alpha)$. The frame it defines $\tilde{s}^\beta : \tilde{\psi}_{\alpha\beta}(\tilde{U}_\alpha) \to Fr(\tilde{\mathcal{E}}_\beta)$ is given by $\tilde{s}^\beta = d\tilde{\psi}_{\alpha\beta} \circ \tilde{s}^\alpha$. The path $\tilde{\eta}_\alpha$ goes to $\tilde{\eta}_\beta(t) = \tilde{\psi}_{\alpha\beta}(\tilde{\eta}_\alpha(t))$ and the injection $\tilde{\psi}_{\alpha\beta}^* : \tilde{\pi}_\alpha^*(T\tilde{U}_\alpha) \to \tilde{\pi}_\beta^*(T\tilde{U}_\beta)$ is characterized by

$$\tilde{\psi}_{\alpha\beta}^*\left(\tilde{s}_\alpha(\tilde{x}), \tilde{\eta}'_\alpha(0)\right) = \left(d_{\tilde{x}}\tilde{\psi}_{\alpha\beta} \circ \tilde{s}_\alpha(\tilde{x}), d_{\tilde{x}}\tilde{\psi}_{\alpha\beta}(\tilde{\eta}'_\alpha(0))\right).$$

Take $\tilde{p}_{i,\alpha}(t) := \tilde{T}_{\tilde{\eta}}^{0,t}(\tilde{s}_\alpha(\tilde{x})(e_i))$ and $\tilde{p}_{\tilde{s},\alpha}$ the parallel frame along $\tilde{\eta}_\alpha$ with initial conditions $\tilde{s}_\alpha(\tilde{x})$. As long as $\nabla_\beta = \left(\tilde{\psi}_{\alpha\beta}^{-1}\right)^* \nabla_\alpha$ we get

$$(\nabla_\beta)_{\tilde{\eta}'_\beta(t)}\,\tilde{\psi}_{\alpha\beta}^\times(\tilde{p}_{i,\alpha}(t)) = (\nabla_\alpha)_{\tilde{\eta}'_\alpha(t)}\,\tilde{p}_{i,\alpha}(t) = 0.$$

Let $\tilde{p}_{\tilde{s},\beta}(t) := \tilde{\psi}_{\alpha\beta}^\times(\tilde{p}_{\tilde{s},\alpha}(t))$. If $\tilde{\psi}_{\alpha\beta}^\times : Fr(\tilde{\mathcal{E}}_\alpha) \to Fr(\tilde{\mathcal{E}}_\beta)$ is the induced injection on the principal orbibundle structure, then

$$\tilde{h}_\beta\left(d_{\tilde{x}}\tilde{\psi}_{\alpha\beta} \circ \tilde{s}_\alpha(\tilde{x}), d_{\tilde{x}}\tilde{\psi}_{\alpha\beta}(\tilde{\eta}'_\alpha(0))\right) = \tilde{p}'_{\tilde{s},\beta}(0).$$

Consequently

$$\begin{tikzcd}
TFr(\tilde{\mathcal{E}}_\alpha) \arrow[r, "d\tilde{\psi}^\times_{\alpha\beta}"] & TFr(\tilde{\mathcal{E}}_\beta) \\
\tilde{\pi}^*_\alpha(T\tilde{U}_\alpha) \arrow[u, "\tilde{h}_\alpha"] \arrow[r, "\tilde{\psi}^*_{\alpha\beta}"] & \tilde{\pi}^*_\beta(T\tilde{U}_\beta) \arrow[u, "\tilde{h}_\beta"']
\end{tikzcd}$$

is a commutative diagram. It follows that the horizontal distributions $\tilde{\mathcal{H}}_\alpha$ and $\tilde{\mathcal{H}}_\beta$ belong to the same diffeomorphism class. The Γ_α-action on $TFr(\tilde{\mathcal{E}}_\alpha)$ is free and then

$$\mathcal{H} = \bigsqcup_\alpha \tilde{\mathcal{H}}_\alpha / \Gamma_\alpha$$

is a well-defined principal connection. $\square$

Applying the principal connection $\omega \in \Omega^1(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$ to both sides of (2.4.4) yields

$$\omega(\tilde{s}_*(\tilde{X})) = \omega_{\tilde{s}}(\tilde{X})^T,$$

with $\omega_{\tilde{s}}$ the connection matrix associated to the frame $\tilde{s} : \tilde{U} \to Fr(\tilde{\mathcal{E}})$. Then

$$\tilde{s}^*\omega = \omega_{\tilde{s}}^T. \tag{2.4.5}$$

Proposition 2.59. *There is a* 1-1 *correspondence between cone connections* ∇ *on* $\mathcal{E} \to \mathcal{O}$ *and principal connections* $\omega \in \Omega^1(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$ *on* $Fr(\mathcal{E}) \to \mathcal{O}$.

Proof. ($\Rightarrow$) Take a cone connection ∇. Let ω_∇ be the principal connection that comes from ∇ and $\overline{\nabla}$ be the connection induced by the principal connection ω_∇. Define by $\overline{\omega}_{\tilde{s}}$ the connection matrices associated to $\overline{\nabla}$. By construction

$$\overline{\omega}_{\tilde{s}} = (\tilde{s}^*\omega_\nabla)^T.$$

Moreover, because of (2.4.5), if $\omega_{\tilde{s}}$ is the connection matrix associated to ∇ then

$$\omega_{\tilde{s}} = (\tilde{s}^*\omega_\nabla)^T.$$

It follows that the connections matrices of ∇ and $\overline{\nabla}$ are equal and then $\nabla = \overline{\nabla}$.
($\Leftarrow$) Take a principal connection $\omega \in \Omega(Fr(\mathcal{E}), \mathfrak{gl}_n(\mathbb{R}))$. Let ∇_ω be the induced connection by ω. Its connection matrix $\omega_{\tilde{s}}$ over the frame $\tilde{s} : \tilde{U} \to Fr(\tilde{\mathcal{E}})$ is

$$\omega_{\tilde{s}}(\tilde{X})^T = \omega(\tilde{s}_*(\tilde{X})).$$

If $\overline{\omega}$ denotes the principal connection obtained from the connection ∇_ω, by (2.4.5), it follows

$$\tilde{s}^*\overline{\omega} = \omega_{\tilde{s}}^T.$$

Then $\tilde{s}^*\overline{\omega} = \tilde{s}^*\omega$. For every $\tilde{Y} \in Fr(\tilde{\mathcal{E}})$

$$d\tilde{\pi}\left(\tilde{Y} - (\tilde{s} \circ \tilde{\pi})_*(\tilde{Y})\right) = 0,$$

which guarantees the existence of $\xi \in \mathfrak{gl}_n(\mathbb{R})$ such that

$$\tilde{Y} = (\tilde{s} \circ \tilde{\pi})_*(\tilde{Y}) + \Psi(\tilde{s}, \xi).$$

Consequently

$$\overline{\omega}(\tilde{Y}) = \overline{\omega}\left((\tilde{s}_* \circ \tilde{\pi}_*)(\tilde{Y})\right) + \xi = \omega\left((\tilde{s}_* \circ \tilde{\pi}_*)(\tilde{Y})\right) + \omega(\Psi(\tilde{s}, \xi)) = \omega(\tilde{Y}),$$

which implies $\overline{\omega} = \omega$. $\square$

Chapter 3

G-structures

The 1-1 relation
$$(Fr(\mathcal{O}), \omega) \leftrightarrow (T\mathcal{O}, \nabla_\omega)$$
between the frame bundle $Fr(\mathcal{O})$ with connection $\omega \in \Omega(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ and the tangent orbibundle $T\mathcal{O}$ with connection ∇_ω transforms differential geometric problems in $T\mathcal{O}$ into differential geometric problems in $Fr(\mathcal{O})$. One of the main advantages is that, on the one hand, the differential geometric problems on $T\mathcal{O}$ involves orbifold theory, but, on the other hand, $Fr(\mathcal{O})$ is a manifold! Some geometric structures on $T\mathcal{O}$ are in 1-1 correspondence with reductions of the structural group on $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$. For example, Riemannian structures over $\mathcal{O}$ are the same as $O(n)$-reductions $P \curvearrowleft O(n) \to \mathcal{O}$. If we vary the structural group, we obtain other geometric structures. For example, distributions $(GL_{k,n-k}(\mathbb{R}))$, orientations $(GL_n^+(\mathbb{R}))$, volume forms $(SL_n(\mathbb{R}))$, almost symplectic structures $(Sp_k(\mathbb{R}))$, almost complex structures $(GL_k(\mathbb{C}))$, hermitian and almost Kähler structures $(U(k))$, frames and coframes $(\{Id\})$.

In the first section, we will define G-structures, give some examples involving different structural groups and characterize when a principal bundle is a G-structure. In the second, we will define equivalences of G-structures, give some examples and characterize when an isomorphism of principal bundles is an equivalence of G-structures. That allows us to characterize the category of G-structures over a fixed orbifold. In the third section, we will define connections compatible with a G-structure and find an explicit description of it means that they are compatible with the geometric structure. Finally, we will introduce a central problem on G-structure theory: integrability. We are going to characterize this condition only in terms of the manifold defining the G-structure. Besides, we will discuss the first obstruction for integrability: the intrinsic torsion; we will calculate the intrinsic torsion to find their first-order obstructions for integrability.

A good reference for G-structures theory on manifolds is the lecture notes [Cra15], our principal guide. For further reading, classical texts about G-structure theory are [Ste99] and [Kob12].

3.1 G-structures

Let G be a closed Lie subgroup of the general linear Lie group $GL_n(\mathbb{R})$.

Definition 3.1. *A G-structure is a reduction of $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$ to the group G.*

A reduction to the structural group G is in 1-1 correspondence with an orbisection $s : \mathcal{O} \to Fr(\mathcal{O})/G$ (see 2.45). By definition a G-structure is a principal G-subbundle $P \subset Fr(\mathcal{O})$.
The idea is to take a linear geometric structure on $\mathcal{O}$. It comes from a canonical geometric structure on $\mathbb{R}^n$. There are canonically defined adapted frames on $\mathbb{R}^n$, and they differ by elements $A \in G$ (the symmetries of the geometric structure). A system of local sections $\tilde{s}_\bullet : \tilde{U}_\bullet \to Fr(\tilde{U})_\bullet$ induces adapted frames on $T\mathcal{O}$. Over non-trivial intersections, the local sections will differ by elements $A \in G$ too. Consequently, we have a principal subbundle $P \curvearrowleft G$ (the space of adapted frames) of the frame bundle $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$ over $\mathcal{O}$.

We will use the following arguments constantly. Let $P \curvearrowleft G$ be a G-structure. Given that $P/G \cong \mathcal{O}$, then the local Γ-action over an arbitrary orbifold chart $(\tilde{U}, \Gamma, \tilde{\phi})$ is given by a representation on the group G. Besides, the G-structure induces a system of local sections

$\tilde{s}^\alpha : \tilde{U}_\alpha \to Fr(\tilde{U}_\alpha)$ over an orbifold atlas $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)_{\alpha \in J}$ such that $\delta(\tilde{s}^\alpha, \tilde{s}^\beta) \in G$ over every non-trivial intersection $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$, where

$$\tilde{s}^\alpha \cdot \delta(\tilde{s}^\alpha, \tilde{s}^\beta) = \tilde{s}^\beta.$$

The section $\tilde{s}^\alpha : \tilde{U}_\alpha \to Fr(\tilde{U})_\alpha$ induces the trivialization $(\tilde{s}^i_\alpha)_{i=1}^n$ of $T\tilde{U}_\alpha$ defined by

$$\tilde{s}^i_\alpha(\tilde{x}) = \tilde{s}^\alpha(\tilde{x})(e_i).$$

It satisfies

$$\tilde{s}^\alpha(\gamma \cdot \tilde{x}) = \gamma \cdot \tilde{s}^\alpha(\tilde{x}) \cdot g_\gamma^{-1}(\tilde{x}).$$

An injection $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \hookrightarrow \tilde{U}_\beta$ induces the commutative diagram

$$\begin{tikzcd}
Fr(\tilde{U}_\alpha) \arrow[r, "\tilde{\psi}^\times_{\alpha\beta}"] \arrow[d, "\tilde{\pi}_\alpha"'] & Fr(\tilde{U}_\beta) \arrow[d, "\tilde{\pi}_\beta"] \\
\tilde{U}_\alpha \arrow[r, "\tilde{\psi}_{\alpha\beta}"] & \tilde{U}_\beta
\end{tikzcd}$$

Take the transition matrices $g^\times_{\alpha\beta}$ defining the injections $\tilde{\psi}^\times_{\alpha\beta}$. If $g^*_{\alpha\beta}$ are the transition matrices associated to the injections $\tilde{\psi}^*_{\alpha\beta} : T\tilde{U}_\alpha \to T\tilde{U}_\beta$, then

$$g^*_{\alpha\beta} = g^\times_{\alpha\beta}.$$

3.1.1 $\{e\}$-structures

As we already mentioned in section 2.3.6 reductions subsection, an $\{e\}$-structure induces a manifold structure on $\mathcal{O}$. In fact, in this case

$$\mathcal{O} \cong P/\{e\} \cong P.$$

Definition 3.2. *A trivialization of the principal bundle $P \curvearrowleft G \to \mathcal{O}$ is a manifold structure on $\mathcal{O}$ together with an isomorphism $\mathcal{O} \times G \cong P$.*

Then, an e-reduction induces a trivialization of $Fr(\mathcal{O}) \curvearrowleft GL_n(\mathbb{R})$. Conversely, the trivialization $\varphi : \mathcal{O} \times GL_n(\mathbb{R}) \to Fr(\mathcal{O})$ induces the section $s : \mathcal{O} \to Fr(\mathcal{O})$ defined by

$$s(x) = \varphi(x, e).$$

Proposition 3.3. *There is a* 1-1 *correspondence between trivializations of $Fr(\mathcal{O})$ and $\{e\}$-structures over $\mathcal{O}$.*

3.1.2 $GL_n^+(\mathbb{R})$-structures

Take a $GL_n^+(\mathbb{R})$-structure $P \curvearrowleft GL_n^+(\mathbb{R})$. Given that Γ acts by a representation of $GL_n^+(\mathbb{R})$, then it preserves the orientation on $\tilde{U}$. We have a system of trivializations $(\tilde{s}_i^\alpha)_{\alpha\in J}$ of $T\mathcal{O} \to \mathcal{O}$ such that the transition maps $g_{\alpha\beta} \in GL_n^+(\mathbb{R})$ over every non-trivial intersection.

Definition 3.4. *An orientation for $\mathcal{O}$ is a choice of trivializations $(\tilde{s}_i^\alpha)_{\alpha\in J}$ for the orbibundle structure $T\mathcal{O} \to \mathcal{O}$ such that the transition maps $g_{\alpha\beta}$ associated to the injections $\tilde{\psi}_{\alpha\beta}$ has positive determinant.*

Then the $GL_n^+(\mathbb{R})$-structure $P \curvearrowleft GL_n^+(\mathbb{R})$ induces an orientation on $\mathcal{O}$. Conversely, take an orientation on $\mathcal{O}$. The local sections

$$\tilde{s}^\alpha : \tilde{U}_\alpha \to Fr(\tilde{U})_\alpha,$$

induced by the trivializations $(\tilde{s}_\alpha^i)_{i=1}^n$, induces the principal bundle structures

$$\begin{array}{c} Fr(\tilde{U}_\alpha) \\ \Big\downarrow \\ \tilde{U}_\alpha \end{array}$$

The transition matrices $g_{\alpha\beta}$ associated with the injections $\tilde{\psi}_{\alpha\beta}^\times$ belong to $GL_n^+(\mathbb{R})$. Hence, we have a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft GL_n^+(\mathbb{R}) \\ \Big\downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.5. *There is a* 1-1 *correspondence between orientations on $\mathcal{O}$ and $GL_n^+(\mathbb{R})$-structures over $\mathcal{O}$.*

3.1.3 $SL_n(\mathbb{R})$-structures

Let $P \curvearrowleft SL_n(\mathbb{R})$ be a $SL_n(\mathbb{R})$-structure. The canonical volume element on $\mathbb{R}^n$ is defined by

$$\mu_{can} = dx_1 \wedge \ldots \wedge dx_n.$$

Take a section $\tilde{s} : \tilde{U} \to Fr(\tilde{U})$ and define $\mu \in \Omega^n(\tilde{U})$ by

$$\mu(\tilde{X}_1, \ldots, \tilde{X}_n)(\tilde{x}) = \mu_{can}\left(\tilde{s}(\tilde{x})^{-1}(\tilde{X}_1), \ldots, \tilde{s}(\tilde{x})^{-1}(\tilde{X}_n)\right).$$

Let $\gamma \in \Gamma$, we have that

$$\begin{aligned}
\mu(\gamma \cdot \tilde{X}_1, \ldots, &\gamma \cdot \tilde{X}_n)(\gamma \cdot \tilde{x}) \\
&= \mu_{can}\left(\tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}_1), \ldots, \tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}_n)\right) \\
&= \mu_{can}\left(\tilde{s}(\tilde{x})^{-1}(\tilde{X}_1), \ldots, \tilde{s}(\tilde{x})^{-1}(\tilde{X}_n)\right) \\
&= \mu(\tilde{X}_1, \ldots, \tilde{X}_n)(\tilde{x}).
\end{aligned}$$

Besides, over a non-trivial intersection $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$

$$\begin{aligned}
\mu_\beta(\tilde{X}_1, \ldots, \tilde{X}_n) &= \mu_{can}\left(\tilde{s}^\beta(\tilde{x})^{-1}(\tilde{X}_1), \ldots, \tilde{s}^\beta(\tilde{x})^{-1}(\tilde{X}_n)\right) \\
&= \mu_{can}\left(g_{\alpha\beta}^{-1} \cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}_1), \ldots, g_{\alpha\beta}^{-1} \cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}_n)\right) \\
&= \det(g_{\alpha\beta}^{-1}) \cdot \mu_{can}\left(\tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}_1), \ldots, \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}_n)\right) \\
&= \mu_\alpha(\tilde{X}_1, \ldots, \tilde{X}_n).
\end{aligned}$$

It follows that the system of n-forms $(\mu_\alpha)_{\alpha \in J}$ gives a well-defined global n-form $\mu \in \Omega^n(\mathcal{O})$.

Definition 3.6. *A volume form over an orbifold $\mathcal{O}$ is a no where vanishing top degree differential form $\mu \in \Omega^n(\mathcal{O})$.*

Then, an $SL_n(\mathbb{R})$-structure $P \curvearrowleft SL_n(\mathbb{R}) \to \mathcal{O}$ induces a volume form $\mu \in \Omega^n(\mathcal{O})$. Conversely, take a volume form $\mu \in \Omega^n(\mathcal{O})$ and a trivialization $(\tilde{s}^i)_{i=1}^n$ of $T\tilde{U}$. Every lift μ of μ satisfies

$$\mu(\tilde{s}^1, \ldots, \tilde{s}^n) \neq 0.$$

Without loss of generality, let us assume $\tilde{c}(\tilde{x}) := \mu(\tilde{s}^1, \ldots, \tilde{s}^n)(\tilde{x}) > 0$ (if not interchange two elements of the basis). Define $\tilde{\sigma}^i : \tilde{U} \to T\tilde{U}$ by

$$\tilde{\sigma}^i(\tilde{x}) = \sqrt[n]{\tilde{c}(\tilde{x})}\tilde{s}^i(\tilde{x}).$$

The trivialization $(\tilde{\sigma}^i)_{i=1}^n$ is such that

$$\mu(\tilde{\sigma}^1, \dots, \tilde{\sigma}^n) = 1.$$

Over a non-trivial intersection $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$ we have that

$$\begin{aligned}
\mu_\beta(\tilde{\sigma}^1_\beta, \dots, \tilde{\sigma}^n_\beta) &= \mu_\alpha(\tilde{\sigma}^1_\alpha, \dots, \tilde{\sigma}^n_\alpha) \\
&= \mu_\alpha(g_{\alpha\beta} \cdot \tilde{\sigma}^1_\beta, \dots, g_{\alpha\beta} \cdot \tilde{\sigma}^n_\beta) \\
&= \det(g_{\alpha\beta}) \cdot \mu_\alpha(\tilde{\sigma}^1_\beta, \dots, \tilde{\sigma}^n_\beta) \\
&= \det(g_{\alpha\beta}).
\end{aligned}$$

Then the transition matrices $g_{\alpha\beta}$ associated to the principal bundle injections $\tilde{\psi}^\times_{\alpha\beta} : Fr(\tilde{U}_\alpha) \hookrightarrow Fr(\tilde{U}_\beta)$ belong to $SL_n(\mathbb{R})$. Hence, we have a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft SL_n(\mathbb{R}) \\ \big\downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.7. *There is a* 1-1 *correspondence between volume forms on* $\mathcal{O}$ *and* $SL_n(\mathbb{R})$*-structures over* $\mathcal{O}$.

3.1.4 $GL_{k,n-k}(\mathbb{R})$-structures

Let $P \curvearrowleft GL_{k,n-k}(\mathbb{R}) \to \mathcal{O}$ be a $GL_{k,n-k}(\mathbb{R})$-structure. The canonical k-distribution of $\mathbb{R}^n$ is $\mathbb{R}^k \times \{0\}$. Take a section $\tilde{s}^\bullet : \tilde{U} \to Fr(\tilde{U})$ and define the distribution $\tilde{\mathcal{D}} \subset T\tilde{U}$ by

$$\tilde{\mathcal{D}}_{\tilde{x}} = \tilde{s}(\tilde{x}) \left(\mathbb{R}^k \times \{0\}\right).$$

A matrix $A \in GL_{k,n-k}(\mathbb{R})$ has the form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

with $A_1 \in GL_k(\mathbb{R})$, $A_2 \in M_{k,n}(\mathbb{R})$ and $A_3 \in GL_{n-k}(\mathbb{R})$. Let i vary over $\{1, \dots, k\}$. On a non-trivial intersection we have that

$$\tilde{s}^\beta(\tilde{x})(e_i) = \tilde{s}^\alpha(\tilde{x})(A \cdot e_i) = \tilde{s}^\alpha(\tilde{x}) \left(\sum_{j=1}^k A_{ji} e_j\right) = \sum_{j=1}^k A_{ji} \tilde{s}^\alpha(\tilde{x})(e_j).$$

It follows that

$$\tilde{\mathcal{D}}_{\alpha,\tilde{x}} = \tilde{s}^{\alpha}(\tilde{x})(\mathbb{R}^k \times \{0\}) = \tilde{s}^{\beta}(\tilde{x})(\mathbb{R}^k \times \{0\}) = \tilde{\mathcal{D}}_{\beta,\tilde{x}}.$$

Furthermore, given that Γ acts by representations on $GL_{k,n-k}(\mathbb{R})$, then the Γ-action on $\tilde{U} \times \mathbb{R}^n$ induces an action $\Gamma \curvearrowright \tilde{U} \times \mathbb{R}^k$. Thus, Γ acts on $\tilde{\mathcal{D}}$. The topological space

$$D = \bigsqcup_{\alpha} \tilde{\mathcal{D}}_{\alpha}/\Gamma_{\alpha},$$

has an orbifold structure given by the trivializations $(\tilde{s}^i_{\alpha})_{i=1}^k$.

Definition 3.8. *A distribution of rank k over $\mathcal{O}$ is a cone orbibundle $D \subset T\mathcal{O}$ of rank k such that the inclusion map $\iota : D \to T\mathcal{O}$ is an embedding.*

Then, the $GL_{k,n-k}(\mathbb{R})$-structure $P \curvearrowleft GL_{k,n-k}(\mathbb{R}) \to \mathcal{O}$ induces a distribution $D \subset T\mathcal{O}$ on $\mathcal{O}$. Conversely, take a distribution $D \subset T\mathcal{O}$ of rank k with trivializations $(\tilde{s}^i)_{i=1}^k$ over $\tilde{\mathcal{D}} \to \tilde{U}$. We can embed $\tilde{\mathcal{D}} \hookrightarrow T\tilde{U}$ so we get k linearly independent sections $\tilde{s}^i : \tilde{U} \to T\tilde{U}$. Complete them and form a trivialization $(\tilde{s}^i)_{i=1}^n$ over $T\tilde{U}$. Let i vary over $\{1, \dots, k\}$. D is a distribution, and then

$$\tilde{s}^i_{\beta}(\tilde{x}) = \sum_{j=1}^{k} A_{ij}(\tilde{x})\tilde{s}^j_{\alpha}(\tilde{x}). \tag{3.1.1}$$

Complete the change of basis matrix $A = (A_{ij})^i_j$, which means

$$\tilde{s}^{\beta} = \tilde{s}^{\alpha} \cdot A.$$

Because of (3.1.1), it follows that $A \in GL_{k,n-k}(\mathbb{R})$. But $A = g_{\alpha\beta}$ is the transition matrix associated with the orbibundle structure $T\mathcal{O} \to \mathcal{O}$. We get a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft GL_{k,n-k}(\mathbb{R}) \\ \downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.9. *There is a* 1-1 *correspondence between distributions on $\mathcal{O}$ and $GL_{k,n-k}(\mathbb{R})$-structures over $\mathcal{O}$.*

3.1.5 $O(n)$-structures

Let $P \curvearrowleft O(n) \to \mathcal{O}$ be an $O(n)$-structure. The canonical $O(n)$-structure on $\mathbb{R}^n$ is given by

$$\langle (v_1, \dots, v_n), (w_1, \dots, w_n) \rangle_{can} = \sum_{i=1}^{n} v_i w_i.$$

Define $\langle \cdot, \cdot \rangle \in \Sigma^2(T\tilde{U}^*)$ by

$$\langle \tilde{X}, \tilde{Y} \rangle(\tilde{x}) = \langle \tilde{s}(\tilde{x})^{-1}(\tilde{X}), \tilde{s}(\tilde{x})^{-1}(\tilde{Y}) \rangle_{can}.$$

Take $\gamma \in \Gamma$; it follows that

$$\begin{aligned}
\langle \gamma \cdot \tilde{X}, \gamma \cdot \tilde{Y} \rangle(\gamma \cdot \tilde{x}) &= \langle \tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}), \tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{Y}) \rangle_{can} \\
&= \langle g_\gamma \cdot \tilde{s}(\tilde{x})^{-1}(\tilde{X}), g_\gamma \cdot \tilde{s}(\tilde{x})^{-1}(\tilde{Y}) \rangle_{can} \\
&= \langle \tilde{X}, \tilde{Y} \rangle(\tilde{x}).
\end{aligned}$$

Hence, $\langle \cdot, \cdot \rangle$ defines a Riemannian metric on $T\tilde{U} \to \tilde{U}$. Furthermore

$$\begin{aligned}
\langle \tilde{X}, \tilde{Y} \rangle_\beta(\tilde{x}) &= \langle \tilde{s}^\beta(\tilde{x})^{-1}(\tilde{X}), \tilde{s}^\beta(\tilde{x})^{-1}(\tilde{Y}) \rangle_{can} \\
&= \langle g_{\alpha\beta}^{-1} \cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}), g_{\alpha\beta}^{-1} \cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{Y}) \rangle_{can} \\
&= \langle \tilde{X}, \tilde{Y} \rangle_\alpha(\tilde{x}).
\end{aligned}$$

It follows that the system of positive definite 2-symmetric tensors

$$(\langle \cdot, \cdot \rangle_\alpha)_{\alpha \in J},$$

induces a well-defined global positive definite 2-symmetric tensor

$$\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\mathcal{O}).$$

Definition 3.10. *A Riemannian structure on $\mathcal{O}$ is a positive definite 2-symmetric tensor $\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\mathcal{O})$.*

Then, a $O(n)$-structure $P \curvearrowleft O(n) \to \mathcal{O}$ induces a Riemannian metric $\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\mathcal{O})$. Conversely, let $\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\mathcal{O})$ be a Riemannian metric over $\mathcal{O}$ and take a local lift $\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\tilde{U})$. By the Gram-Schmidt process there exist a trivialization $(\tilde{\sigma}^i)_{i=1}^n$ such that

$$\langle \tilde{\sigma}^i, \tilde{\sigma}^j \rangle = \delta^i_j.$$

It follows that

$$\begin{aligned} \langle \tilde{\sigma}^\alpha(e_i), \tilde{\sigma}^\alpha(e_j) \rangle_\alpha &= \langle \tilde{\sigma}^\beta(e_i), \tilde{\sigma}^\beta(e_j) \rangle_\beta \\ &= \langle g_{\alpha\beta} \cdot \tilde{\sigma}^\alpha(e_i), g_{\alpha\beta} \cdot \tilde{\sigma}^\alpha(e_j) \rangle_\alpha, \end{aligned}$$

and then $g_{\alpha\beta} \in O(n)$. Hence, we have a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft O(n) \\ \downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.11. *There is a* 1-1 *correspondence between Riemannian metrics on $\mathcal{O}$ and $O(n)$-structures over $\mathcal{O}$.*

3.1.6 $Sp_{2k}(\mathbb{R})$-structures

Let $P \curvearrowleft Sp_{2k}(\mathbb{R}) \to \mathcal{O}$ be an $Sp_{2k}(\mathbb{R})$-structure. Take the canonical basis $(x_1, y_1, \dots, x_k, y_k)$ of $\mathbb{R}^{2k}$ and dx^i, dy^i its duals. The canonical $Sp_{2k}(\mathbb{R})$-structure on $\mathbb{R}^{2k}$ is given by

$$\omega_{can} = \sum_{i=1}^k dx^i \wedge dy^i.$$

Define $\omega \in \Omega^2(T\tilde{U}^*)$ by

$$\omega(\tilde{X}, \tilde{Y})(\tilde{x}) = \omega_{can}(\tilde{s}(\tilde{x})^{-1}(\tilde{X}), \tilde{s}(\tilde{x})^{-1}(\tilde{Y})).$$

Take $\gamma \in \Gamma$; it follows that

$$\begin{aligned} \omega(\gamma \cdot \tilde{X}, \gamma \cdot \tilde{Y})(\gamma \cdot \tilde{x}) &= \omega_{can}(\tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}), \tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{Y})) \\ &= \omega_{can}(g_\gamma^{-1} \cdot \tilde{s}(\tilde{x})^{-1}(\tilde{X}), g_\gamma^{-1} \cdot \tilde{s}(\tilde{x})^{-1}(\tilde{Y})) \\ &= \omega(\tilde{X}, \tilde{Y})(\tilde{x}). \end{aligned}$$

Hence, ω defines an almost symplectic form on $T\tilde{U} \to \tilde{U}$. Furthermore

$$\begin{aligned}\omega_\beta(\tilde{X},\tilde{Y})(\tilde{x}) &= \omega_{can}(\tilde{s}^\beta(\tilde{x})^{-1}(\tilde{X}), \tilde{s}^\beta(\tilde{x})^{-1}(\tilde{Y})) \\ &= \omega_{can}(g_{\alpha\beta}^{-1}\cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}), g_{\alpha\beta}^{-1}\cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{Y})) \\ &= \omega_\alpha(\tilde{X},\tilde{Y})(\tilde{x}).\end{aligned}$$

It follows that the system of non-degenerate 2-forms

$$(\omega_\alpha)_{\alpha\in J},$$

induces a well-defined global non-degenerate 2-form

$$\omega \in \Omega^2(T^*\mathcal{O}).$$

Definition 3.12. *An almost symplectic structure on $\mathcal{O}$ is a non-degenerate 2-form $\omega \in \Omega^2(T^*\mathcal{O})$.*

Then, an $Sp_{2k}(\mathbb{R})$-structure $P \curvearrowleft Sp_{2k}(\mathbb{R}) \to \mathcal{O}$ induces an almost symplectic structure $\omega \in \Omega^2(T^*\mathcal{O})$. Conversely, let $\omega \in \Omega^2(T^*\mathcal{O})$ be an almost symplectic structure over $\mathcal{O}$ and take a local lift $\omega \in \Omega^2(T^*\tilde{U})$. We can always find a symplectic basis, which means a trivialization $(\tilde{\sigma}^i, \tilde{\rho}^i)_{i=1}^k$ such that

$$\begin{aligned}\omega(\tilde{\sigma}^i,\tilde{\sigma}^j) &= 0 \\ \omega(\tilde{\rho}^i,\tilde{\rho}^j) &= 0 \\ \omega(\tilde{\sigma}^i,\tilde{\rho}^j) &= \delta^i_j.\end{aligned}$$

It follows that

$$\begin{aligned}\omega_{can}(x_i,y_j) &= \omega_\beta(\tilde{\sigma}^i_\beta,\tilde{\rho}^j_\beta)(\tilde{x}) \\ &= \omega_\alpha(\tilde{\sigma}^i_\alpha,\tilde{\rho}^j_\alpha)(\tilde{x}) \\ &= \omega_{can}(g_{\alpha\beta}^{-1}\cdot x_i, g_{\alpha\beta}^{-1}\cdot y_j).\end{aligned}$$

Similarly

$$\begin{aligned}\omega_{can}(x_i,x_j) &= \omega_{can}(g_{\alpha\beta}^{-1}\cdot x_i, g_{\alpha\beta}^{-1}\cdot x_j), \\ \omega_{can}(y_i,y_j) &= \omega_{can}(g_{\alpha\beta}^{-1}\cdot y_i, g_{\alpha\beta}^{-1}\cdot y_j),\end{aligned}$$

and then $g_{\alpha\beta} \in Sp_{2k}(\mathbb{R})$. Hence, we have a principal bundle structure

$$
\begin{tikzcd}
Fr(\mathcal{O}) \subset P \curvearrowleft Sp_{2k}(\mathbb{R}) \arrow[d] \\
\mathcal{O}
\end{tikzcd}
$$

Proposition 3.13. *There is a* 1-1 *correspondence between almost symplectic structures on* $\mathcal{O}$ *and* $Sp_{2k}(\mathbb{R})$*-structures over* $\mathcal{O}$.

3.1.7 $GL_k(\mathbb{C})$-structures

Let $P \curvearrowleft GL_k(\mathbb{C}) \to \mathcal{O}$ be a $GL_k(\mathbb{C})$-structure. Every $z \in \mathbb{C}$ can be thought of as a real 2×2 matrix

$$z \mapsto M_z = \begin{pmatrix} \Re(z) & -\Im(z) \\ \Im(z) & \Re(z) \end{pmatrix},$$

where $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z. This assignment carries complex multiplication into matrix multiplication. In particular

$$i \mapsto J_{can} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In general, $GL_k(\mathbb{C}) \cong GL_{2k}(\mathbb{R})$ are isomorphic, the isomorphism given by

$$Z \mapsto Z_{\mathbb{R}}$$
$$\begin{pmatrix} z_{11} & \dots & z_{1k} \\ z_{21} & \dots & z_{2k} \\ \vdots & \ddots & \vdots \\ z_{k1} & \dots & z_{kk} \end{pmatrix} \mapsto \begin{pmatrix} M_{z_{11}} & \dots & M_{z_{1k}} \\ M_{z_{21}} & \dots & M_{z_{2k}} \\ \vdots & \ddots & \vdots \\ M_{z_{k1}} & \dots & M_{z_{kk}} \end{pmatrix}.$$

Identify

$$J_{can} := \begin{pmatrix} J_{can} & 0 & 0 & \dots & 0 \\ 0 & J_{can} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J_{can} \end{pmatrix},$$

and define the bundle isomorphism $\tilde{J} : T\tilde{U} \to T\tilde{U}$ by

$$\tilde{J}(\tilde{X})(\tilde{x}) = \tilde{s}(\tilde{x}) \left(\tilde{s}(\tilde{x})^{-1}(\tilde{X}) \cdot J_{can}\right).$$

It satisfies $\tilde{J}^2 = -Id$. In addition, let $\gamma \in \Gamma$, then

$$\begin{aligned}\tilde{J}(\gamma \cdot \tilde{X})(\gamma \cdot \tilde{x}) &= \tilde{s}(\gamma \cdot \tilde{x}) \left(\tilde{s}(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}) \cdot J_{can}\right) \\ &= \gamma \cdot \tilde{s}(\tilde{x}) \cdot g_\gamma^{-1}(\tilde{x}) \left(g_\gamma(\tilde{x}) \cdot \tilde{s}(\tilde{x})^{-1} \cdot \gamma^{-1}(\gamma \cdot \tilde{X}) \cdot J_{can}\right) \\ &= \gamma \cdot \tilde{J}(\tilde{X})(\tilde{x}).\end{aligned}$$

Then $\tilde{J} \in \Omega^1(\tilde{U}, T\tilde{U})$ is a Γ-equivariant form. It follows that $\tilde{J}$ induces a map $J \in \Omega^1(U, TU)$ such that $J^2 = -Id$. Over a non-trivial intersection $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$

$$\begin{aligned}\tilde{J}_\beta(\tilde{X})(\tilde{x}) &= \tilde{s}^\beta(\tilde{x}) \left(\tilde{s}^\beta(\tilde{x})^{-1}(\tilde{X}) \cdot J_{can}\right) \\ &= \tilde{s}^\alpha(\tilde{x}) \cdot g_{\alpha\beta} \left(g_{\alpha\beta}^{-1} \cdot \tilde{s}^\alpha(\tilde{x})^{-1}(\tilde{X}) \cdot J_{can}\right) \\ &= \tilde{J}_\alpha(\tilde{X})(\tilde{x}).\end{aligned}$$

Hence, we have a well-defined global 1-form $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$ such that

$$J^2 = -Id.$$

Definition 3.14. *An almost complex structure on $\mathcal{O}$ is a 1-form*

$$J \in \Omega^1(\mathcal{O}, T\mathcal{O}),$$

such that $J^2 = -Id$.

Then, a $GL_k(\mathbb{C})$-structure $P \curvearrowleft GL_k(\mathbb{C})$ induces an almost complex structure $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$. Conversely, let $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$ be an almost complex structure and $\tilde{J} : T\tilde{U} \to T\tilde{U}$ a local lift. We can always find a $T\tilde{U}$ trivialization of the form $(\tilde{s}^i, \tilde{J}(\tilde{s}^i))_{i=1}^k$. The frame it generates will be denoted by $\tilde{s}_J$. Define the $GL_k(\mathbb{C})$-action of $Z \in GL_k(\mathbb{C})$ by

$$\tilde{s}_J \cdot Z := \tilde{s}_J \cdot Z_{\mathbb{R}}.$$

Over a non-trivial intersection $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$

$$\tilde{s}^i_\beta = \sum_{j=1}^k a_{ij}\tilde{s}^j_\alpha + b_{ij}\tilde{J}_\alpha(\tilde{s}^j_\alpha).$$

Applying $\tilde{J}_\beta$ on both sides yields

$$\tilde{J}_\beta(\tilde{s}^i_\beta) = \sum_{j=1}^k a_{ij}\tilde{J}_\alpha(\tilde{s}^j_\alpha) - b_{ij}\tilde{s}^j_\alpha.$$

Hence, the transition matrix is

$$g_{\alpha\beta} = \begin{pmatrix} a_{11} & b_{11} & \cdots & a_{1k} & b_{1k} \\ -b_{11} & a_{11} & \cdots & -b_{1k} & a_{1k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & b_{k1} & \cdots & a_{kk} & b_{kk} \\ -b_{k1} & a_{k1} & \cdots & -b_{kk} & a_{kk} \end{pmatrix}.$$

The isomorphism

$$g_{\alpha\beta} \cong \begin{pmatrix} a_{11} + ib_{11} & \cdots & a_{1k} + ib_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} + ib_{k1} & \cdots & a_{kk} + ib_{kk} \end{pmatrix},$$

implies that $g_{\alpha\beta} \in GL_k(\mathbb{C})$. It follows that we have a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft GL_k(\mathbb{C}) \\ \big\downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.15. *There is a* 1-1 *correspondence between almost complex structures on* $\mathcal{O}$ *and* $GL_k(\mathbb{C})$*-structures over* $\mathcal{O}$.

3.1.8 $U(k)$-structures

Let $P \curvearrowleft U(n)$ be a $U(n)$-structure. Because of $U(k) < GL_k(\mathbb{C})$, we obtain an almost complex structure $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$. Take a trivialization $(\tilde{s}^i, \tilde{J}(\tilde{s}^i))_{i=1}^k$. It gives the $\mathbb{C}$-linear diffeomorphism $\tilde{U} \times \mathbb{C}^k \to T\tilde{U}$

defined by

$$(\tilde{x}, z_1, \ldots, z_k) \mapsto \sum_{i=1}^{k} \Re(z_1)\tilde{s}^i + \Im(z_1)\tilde{J}(\tilde{s}^i),$$

which induces the complex local frame $\tilde{s} : \tilde{U} \to Fr(\tilde{U})$. That means $\tilde{s}_J(\tilde{x}) : \mathbb{C}^k \overset{\cong}{\to} T_{\tilde{x}}\tilde{U}$ is a $\mathbb{C}$-linear isomorphism of complex vector spaces. The canonical hermitian structure over $\mathbb{C}^k$ is the map $h_{can} : \mathbb{C}^k \oplus \mathbb{C}^k \to \mathbb{C}$ defined by

$$h_{can}\left((z_1, \ldots, z_k), (w_1, \ldots, w_k)\right) = \sum_{i=1}^{k} z_i \overline{w_i}.$$

It is $\mathbb{C}$-linear on the first argument and $h_{can}(z, w) = \overline{h_{can}(w, z)}$. Hence, $h_{can}(z, z) \in \mathbb{R}$ for all $z \in \mathbb{C}^k$ and then it make sense to say that h_{can} is positive definite. Let $\tilde{h} : T\tilde{U} \oplus T\tilde{U} \to \mathbb{C}$ be

$$\tilde{h}(\tilde{X}, \tilde{Y})(\tilde{x}) = h_{can}(\tilde{s}_J(\tilde{x})^{-1}(\tilde{X}), \tilde{s}_J(\tilde{x})^{-1}(\tilde{Y})),$$

and $\gamma \in \Gamma$. Given that $g_\gamma, g_{\alpha\beta} \in U(n)$, then

$$\begin{aligned}
\tilde{h}(\gamma \cdot \tilde{X}, \gamma \cdot \tilde{Y})(\tilde{x}) &= h_{can}(\tilde{s}_J(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{X}), \tilde{s}_J(\gamma \cdot \tilde{x})^{-1}(\gamma \cdot \tilde{Y})) \\
&= h_{can}(g_\gamma(\tilde{x}) \cdot \tilde{s}_J(\tilde{x})^{-1}(\tilde{X}), g_\gamma(\tilde{x}) \cdot \tilde{s}_J(\tilde{x})^{-1}(\tilde{Y})) \\
&= \tilde{h}(\tilde{X}, \tilde{Y})(\tilde{x}),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{h}_\beta(\tilde{X}, \tilde{Y})(\tilde{x}) &= h_{can}(\tilde{s}_J^\beta(\tilde{x})^{-1}(\tilde{X}), \tilde{s}_J^\beta(\tilde{x})^{-1}(\tilde{Y})) \\
&= h_{can}(g_{\alpha\beta}(\tilde{x})^{-1} \cdot \tilde{s}_J^\alpha(\tilde{x})^{-1}(\tilde{X}), g_{\alpha\beta}(\tilde{x})^{-1} \cdot \tilde{s}_J^\alpha(\tilde{x})^{-1}(\tilde{Y})) \\
&= h_{can}(\tilde{s}_J^\alpha(\tilde{x})^{-1}(\tilde{X}), \tilde{s}_J^\alpha(\tilde{x})^{-1}(\tilde{Y})) \\
&= \tilde{h}_\alpha(\tilde{X}, \tilde{Y})(\tilde{x}).
\end{aligned}$$

Thus, we have a well-defined smooth orbifold map $h : T\mathcal{O} \oplus T\mathcal{O} \to \mathbb{C}$.

Definition 3.16. *An almost hermitian structure on $\mathcal{O}$ is a pair (J, h), with $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$ an almost complex structure and $h : T\mathcal{O} \oplus T\mathcal{O} \to \mathbb{C}$ a smooth map such that:*

1. *Is* $\mathbb{C}$*-linear on the first argument.*

2. $h(X,Y) = \overline{h(Y,X)}$.

3. *h is positive definite.*

It follows that a $U(n)$-structure $P \curvearrowleft U(n) \to \mathcal{O}$ induces an almost hermitian structure (J,h) on $\mathcal{O}$. Conversely, take an almost hermitian structure (J,h). As long as we have an almost complex structure $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$, then we have a $GL_k(\mathbb{C})$-principal bundle structure $P \curvearrowleft GL_k(\mathbb{C}) \to \mathcal{O}$. Similarly, as with Riemannian metrics, there is a Gram-Schmidt procedure that guarantees the existence of a complex local frame $(\tilde{\sigma}^i)_{i=1}^k$ such that

$$\tilde{h}(\tilde{\sigma}^i, \tilde{\sigma}^j) = \delta^i_j.$$

It follows that

$$\begin{aligned}\tilde{h}_\alpha(\tilde{\sigma}^i_\alpha, \tilde{\sigma}^j_\alpha) &= \tilde{h}_\beta(\tilde{\sigma}^i_\beta, \tilde{\sigma}^j_\beta) \\ &= \tilde{h}_\alpha(g_{\alpha\beta}\cdot\tilde{\sigma}^i_\alpha, g_{\alpha\beta}\cdot\tilde{\sigma}^j_\alpha),\end{aligned}$$

and then $g_{\alpha\beta} \in U(k)$. Hence, we have a principal bundle structure

$$\begin{array}{c} Fr(\mathcal{O}) \subset P \curvearrowleft U(k) \\ \downarrow \\ \mathcal{O} \end{array}$$

Proposition 3.17. *There is a* 1-1 *correspondence between almost hermitian structures on* $\mathcal{O}$ *and* $U(k)$*-structures over* $\mathcal{O}$*.*

Remark: The group equalities

1. $U(k) = O(2k) \cap GL_k(\mathbb{C})$,

2. $U(k) = Sp_{2k}(\mathbb{R}) \cap GL_k(\mathbb{C})$,

3. $U(k) = O(2k) \cap Sp_{2k}(\mathbb{R}) \cap GL_k(\mathbb{C})$,

implies relations between the geometric structures induced by their corresponding G-structures. The almost hermitian structure (J, h) induces the J-invariant Riemannian structure

$$\langle X, Y\rangle := \Re(h(X, Y)),$$

and the J-invariant almost symplectic structure

$$\omega(X, Y) := -\Im(h(X, Y)).$$

The J-invariant hypothesis stands because

$$h(J(X), J(Y)) = i\bar{i} \cdot h(X, Y) = h(X, Y).$$

Hence, an $U(k)$-structure induces:

1. An almost complex structure $J \in \Omega^1(\mathcal{O}, T\mathcal{O})$.
2. A Riemannian structure $\langle \cdot, \cdot \rangle \in \Sigma^2(T^*\mathcal{O})$.
3. An almost symplectic structure $\omega \in \Omega^2(\mathcal{O})$.

The equation

$$\omega(X, Y) = \langle J(X), Y\rangle,$$

gives the relations between them.

3.1.9 Principal G-bundles vs. G-structures

Remember that a 1-form $\tau \in \Omega^1_{ten}(Fr(\mathcal{O}), \mathbb{R}^n)$ is tensorial if it is G-equivariant and $\ker \tau = T^V P$. The frame bundle $Fr(\mathcal{O})$ has the tautological form $\theta \in \Omega^1_{ten}(Fr(\mathcal{O}), \mathbb{R}^n)$. The G-structure $P \subset Fr(\mathcal{O})$ inherits the tensorial form by

$$\theta_P := \theta|_P \in \Omega^1_{ten}(P, \mathbb{R}^n).$$

Then, every G-structure P possesses a tensorial form θ_P. The existence of that tensorial form allows us to distinguish between principal bundles $P \curvearrowleft G \to \mathcal{O}$ and G-structures $P \subset Fr(\mathcal{O}) \curvearrowleft G \to \mathcal{O}$.

Theorem 3.18. *Let $P \curvearrowleft G \to \mathcal{O}$ be a principal bundle. If there exists a tensorial form $\tau \in \Omega^1_{ten}(P, \mathbb{R}^n)$ then there exists a unique principal bundle embedding $\Phi : P \hookrightarrow Fr(\mathcal{O})$ such that $\Phi^*\theta = \tau$.*

Proof. 1. **Existence.**

Fix a connection $\mathcal{H} \subset TP$. Take the principal bundle structure $\tilde{\mathcal{P}} \to \tilde{U}$. We can define the tensorial form $\tilde{\tau} \in \Omega^1_{ten}(\tilde{\mathcal{P}}, \mathbb{R}^n)$ by

$$\tilde{\tau}(\tilde{Y}) = \tau([\tilde{Y}]).$$

Given that $\tilde{\tau}_{\tilde{p}} : \mathbb{R}^n \to \tilde{\mathcal{H}}_{\tilde{p}}$ is an isomorphism, then the tensorial form defines the trivialization $\tilde{\mathcal{P}} \times \mathbb{R}^n \cong \tilde{\mathcal{H}}$ given by

$$\tilde{\varphi}_1(\tilde{p}, v) = (\tilde{p}, \tilde{\tau}_{\tilde{p}}^{-1}(v)).$$

Furthermore, $d\pi : \tilde{\mathcal{H}} \to \tilde{\pi}^*(T\tilde{U})$ is an isomorphism. It follows that $\tilde{\varphi} : \tilde{\mathcal{P}} \times \mathbb{R}^n \to \tilde{\pi}^*(T\tilde{U})$, defined by

$$\tilde{\varphi}(\tilde{p}, v) = (d\tilde{\pi} \circ \tilde{\varphi}_1)(\tilde{p}, v),$$

is an isomorphism of vector bundles. Hence, a fixed $\tilde{p} \in \tilde{\mathcal{P}}$ defines the isomorphism $\tilde{\varphi}(\tilde{p}, \cdot) : \mathbb{R}^n \to T_{\tilde{\pi}(\tilde{p})}\tilde{U}$, which means it is a frame. A section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$ induces the section $\tilde{\sigma} : \tilde{U} \to Fr(\tilde{U})$ given by

$$\tilde{\sigma}(\tilde{x})(v_1, \ldots, v_n) = \sum_{i=1}^{n} v_i \tilde{\varphi}(\tilde{s}(\tilde{x}), e_i).$$

Define the embedding $\tilde{\Phi} : \tilde{\mathcal{P}} \hookrightarrow Fr(\tilde{U})$ by

$$\tilde{\Phi}(\tilde{p}) = \tilde{\sigma}(\tilde{\pi}(\tilde{p})) \cdot \delta(\tilde{s}(\tilde{\pi}(\tilde{p})), \tilde{p}).$$

It satisfies

$$\begin{aligned}
\tilde{\Phi}(\tilde{p} \cdot g) &= \tilde{\sigma}(\tilde{\pi}(\tilde{p} \cdot g)) \cdot \delta(\tilde{s}(\tilde{\pi}(\tilde{p} \cdot g)), \tilde{p} \cdot g) \\
&= \tilde{\sigma}(\tilde{\pi}(\tilde{p})) \cdot \delta(\tilde{s}(\tilde{\pi}(\tilde{p})), \tilde{p}) \cdot g \\
&= \tilde{\Phi}(\tilde{p}) \cdot g,
\end{aligned}$$

and

$$\begin{aligned}\tilde{\Phi}(\gamma \cdot \tilde{p}) &= \tilde{\sigma}(\gamma \cdot \tilde{\pi}(\tilde{p})) \cdot \delta(\tilde{s}(\gamma \cdot \tilde{\pi}(\tilde{p})), \gamma \cdot \tilde{p}) \\ &= \gamma \cdot \tilde{\sigma}(\tilde{\pi}(\tilde{p})) \cdot g_\gamma^{-1} \cdot \delta(\gamma \cdot \tilde{s}(\tilde{\pi}(\tilde{p})) \cdot g_\gamma^{-1}, \gamma \cdot \tilde{p}) \\ &= \gamma \cdot \tilde{\sigma}(\tilde{\pi}(\tilde{p})) \cdot g_\gamma^{-1} \cdot \delta(\tilde{s}(\tilde{\pi}(\tilde{p})) \cdot g_\gamma^{-1}, \tilde{p}) \\ &= \gamma \cdot \tilde{\sigma}(\tilde{\pi}(\tilde{p})) \cdot g_\gamma^{-1} \cdot g_\gamma \cdot \delta(\tilde{s}(\tilde{\pi}(\tilde{p})), \tilde{p}) \\ &= \gamma \cdot \tilde{\Phi}(\tilde{p}).\end{aligned}$$

Consequently, $\Phi : P \hookrightarrow Fr(\mathcal{O})$ is a G-equivariant embedding.

2. $\Phi^*\theta = \tau$.
   Let $Y \in T_pP$. Then

$$\begin{aligned}\Phi^*\theta_p(Y) &= \theta_{\Phi(p)}(d_p\Phi(Y)) \\ &= \Phi(p)^{-1}\left(d_p(\pi \circ \Phi)(Y)\right) \\ &= \Phi(p)^{-1}(d_p\pi(Y)).\end{aligned}$$

   If $h : \pi^*(T\mathcal{O}) \to TP$ is the horizontal lift, then

$$\Phi(p)^{-1}(X) = \tau_p(h_p(X)).$$

   Hence

$$\Phi^*\theta_p(Y) = \tau_p\left(h_p(d_p\pi(Y))\right).$$

   Given that $Y - h_p(d_p\pi(Y)) \in T_p^VP$, then

$$\tau_p(Y - h_p(d_p\pi(Y))) = 0.$$

   We conclude that

$$\begin{aligned}\Phi^*\theta_p(Y) &= \tau_p\left(h_p(d_p\pi(Y))\right) + \tau_p(Y - h_p(d_p\pi(Y))) \\ &= \tau_p(Y),\end{aligned}$$

   which implies $\Phi^*\theta = \tau$.

3. **Uniqueness.**
   Let $\Phi_1, \Phi_2 : P \hookrightarrow Fr(\mathcal{O})$ be two embeddings between principal

bundles such that $\Phi_i^*\theta = \tau$. Then, for all $p \in P$ and $Y \in T_pP$ we have

$$\Phi_1^*\theta_p(Y) = \Phi_2^*\theta_p(Y)$$
$$\Phi_1(p)^{-1}(d_p\pi(Y)) = \Phi_2(p)^{-1}(d_p\pi(Y)).$$

As long as $\pi : P \to \mathcal{O}$ is a submersion, then, for all $X \in T_x\mathcal{O}$, we have that

$$\Phi_1(p)^{-1}(X) = \Phi_2(p)^{-1}(X).$$

Consequently, $\Phi_1(p) = \Phi_2(p)$ and then $\Phi_1 = \Phi_2$.

□

Corollary 3.19. *Let $P \curvearrowleft G \to \mathcal{O}$ be a principal bundle. Then, P is a G-structure if and only if there exists a tensorial form $\theta_P \in \Omega^1(P, \mathbb{R}^n)$.*

It follows that the category of G-structures over $\mathcal{O}$ has as objects the pairs (P, τ) with P a manifold, together with a locally free and proper action $P \curvearrowleft G$ such that $P/G \cong \mathcal{O}$, and a tensorial form $\theta \in \Omega^1(P, \mathbb{R}^n)$.

3.2 Morphisms

The theory of G-structures allows us to study geometric structures through principal subbundles of the frame bundle. In order to preserve the induced geometric structures, we need more than principal bundle morphisms. For, let $f : \mathcal{O}_1 \to \mathcal{O}_2$ be an orbifold diffeomorphism between the orbifold structures $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)_{\alpha \in J}$ and $(\tilde{V}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)_{\alpha \in J}$ (we assume both possesses the same local groups as long as the diffeomorphism f induces isomorphic groups). We can lift $\tilde{f} : \tilde{U} \to \tilde{V}$ to $\tilde{f}_* : Fr(\tilde{U}) \to Fr(\tilde{V})$ by

$$\tilde{f}_*(\tilde{p})(v) := (d_{\tilde{\pi}_1(\tilde{p})}\tilde{f} \circ \tilde{p})(v).$$

As long as $\tilde{f}$ is Γ-equivariant, so is $\tilde{f}_*$. Then, we obtain a well-defined $GL_n(\mathbb{R})$-equivariant diffeomorphism

$$f_* : Fr(\mathcal{O}_1) \to Fr(\mathcal{O}_2),$$

such that

$$\begin{tikzcd} Fr(\mathcal{O}_1) \arrow[r, "f_*"] \arrow[d, "\pi_1"'] & Fr(\mathcal{O}_2) \arrow[d, "\pi_2"] \\ \mathcal{O}_1 \arrow[r, "f"] & \mathcal{O}_2 \end{tikzcd}$$

commutes. Given that the geometric structures induced by the G-structures are defined in terms of the adapted frames (frames that belongs to the G-structure), in order to carry one geometric structure to the other, we need to carry the adapted frames of one G-structure to adapted frames of the other.

3.2.1 Equivalence of G-structures

Definition 3.20. *Let $P_1 \subset Fr(\mathcal{O}_1)$ and $P_2 \subset Fr(\mathcal{O}_2)$ be two G-structures. The diffeomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$ is an equivalence of G-structures if $f_*(P_1) = P_2$.*

Example 3.21. $GL_n^+(\mathbb{R})$**-morphisms**
A system of local frames

$$\tilde{s}_\alpha : \tilde{U}_\alpha \to Fr(\tilde{U}_\alpha),$$

such that $g^1_{\alpha\beta} \in GL_n^+(\mathbb{R})$, induces an orientation on $\mathcal{O}_1$ is induced by . Because $f_* : P_1 \to P_2$ is a diffeomorphism the system of frames induced by $\tilde{f}_{\alpha *} : Fr(\tilde{U}_\alpha) \to Fr(\tilde{V}_\alpha)$ satisfies

$$\delta_2(\tilde{f}_{\alpha *}(\tilde{s}_\alpha), \tilde{f}_{\beta *}(\tilde{s}_\beta)) \in GL_n^k(\mathbb{R}).$$

That means the orbifold structure induced by f_* on $\mathcal{O}_2$ has the same orientation as $\mathcal{O}_1$.

Example 3.22. $SL_n(\mathbb{R})$**-morphisms**
Take $\mu^2_\alpha \in \Omega^n(\tilde{V}_\alpha)$, a local lift of the volume form on $\mathcal{O}_2$ and an adapted frame $\tilde{s}_\alpha \in \tilde{\mathcal{P}}_{1\alpha}$. The local frames $\tilde{f}_{\alpha *}(\tilde{s}_\alpha)$ belongs to $\tilde{\mathcal{P}}_{2\alpha}$. It follows that

$$\mu^2_\alpha(\tilde{X}_1, \ldots, \tilde{X}_n) = \mu_{can}(f_{\alpha *}(\tilde{s}_\alpha)^{-1}(\tilde{X}_1), \ldots, \tilde{s}_\alpha^{-1}(\tilde{X}_n)).$$

Consequently

$$\mu^2_\alpha(\tilde{f}_{\alpha *}(\tilde{s}_\alpha)(e_1), \ldots, \tilde{f}_{\alpha *}(\tilde{s}_\alpha)(e_n)) = 1,$$

and then $f : \mathcal{O}_1 \to \mathcal{O}_2$ is volume-preserving.

Example 3.23. $GL_{k,n-k}(\mathbb{R})$**-morphisms**
Take the two distributions $D_1 \subset T\mathcal{O}_1$ and $D_2 \subset T\mathcal{O}_2$ induced by P_1 and P_2. Let $\tilde{s}_\alpha \in \tilde{\mathcal{P}}_{1\alpha}$ be an adapted frame. Then $(\tilde{s}_\alpha(e_i))_{i=1}^k$ is a trivialization for D_1. Because $\tilde{f}_{\alpha *}(\tilde{s}_\alpha) \in \tilde{\mathcal{P}}_{2\alpha}$, it induces a trivialization $(\tilde{f}_{\alpha *}(\tilde{s}_\alpha))_{i=1}^k$ for D_2. It follows that $df(D_1) = D_2$.

Example 3.24. $O(n)$**-morphisms**
The Riemannian metric $\langle \cdot, \cdot \rangle_i : T\mathcal{O}_i \oplus T\mathcal{O}_i \to \mathbb{R}$ induced by $P_i \subset Fr(\mathcal{O}_i)$ is given by

$$\langle X_i, Y_i \rangle_i = \langle p_i^{-1}(X_i), p_i^{-1}(Y_i) \rangle_{can},$$

with $p_i \in P_i$. Then, for $p \in P_1$ we have that

$$\begin{aligned}\langle X_2, Y_2 \rangle_2 &= \langle f_*(p)^{-1}(X_2), f_*(p)^{-1}(Y_2) \rangle_{can} \\ &= \langle p^{-1}(df^{-1}(X_2)), p^{-1}(df^{-1}(Y_2)) \rangle_{can}.\end{aligned}$$

It follows that for every $X_1, Y_1 \in T\mathcal{O}_1$

$$\begin{aligned}\langle df(X_1), df(Y_1) \rangle_2 &= \langle p^{-1}(X_1), p^{-1}(Y_1) \rangle_{can} \\ &= \langle X_1, Y_1 \rangle_1.\end{aligned}$$

Hence, $f : \mathcal{O}_1 \to \mathcal{O}_2$ is an isometry.

Example 3.25. $Sp_{2k}(\mathbb{R})$**-morphisms**
The symplectic structure $\omega_i \in \Omega^2(\mathcal{O}_i)$ induced by $P_i \subset Fr(\mathcal{O}_i)$ is given by

$$\omega_i(X_i, Y_i) = \omega_{can}(p_i^{-1}(X_i), p_i^{-1}(Y_i)),$$

with $p_i \in P_i$. Then, for $p \in P_1$ we have that

$$\begin{aligned}\omega_2(X_2, Y_2) &= \omega_{can}(f_*(p)^{-1}(X_2), f_*(p)^{-1}(Y_2)) \\ &= \omega_{can}(p^{-1}(df^{-1}(X_2)), p^{-1}(df^{-1}(Y_2))).\end{aligned}$$

It follows that for every $X_1, Y_1 \in T\mathcal{O}_1$

$$\begin{aligned}\omega_2(df(X_1), df(Y_1)) &= \omega_{can}(p^{-1}(X_1), p^{-1}(Y_1)) \\ &= \omega_1(X_1, Y_1).\end{aligned}$$

Hence, $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a symplectomorphism.

Example 3.26. $GL_k(\mathbb{C})$**-morphisms**
Take an adapted frame $\tilde{s}_\alpha \in \tilde{\mathcal{P}}_{1\alpha}$, which means

$$\tilde{s}_\alpha(e_{2i}) = \tilde{J}_1(\tilde{s}_\alpha(e_{2i-1})) \quad i \in \{1, \dots, k\}.$$

Given that $\tilde{f}_{\alpha*}(\tilde{s}_\alpha) \in \tilde{\mathcal{P}}_{2\alpha}$, then

$$\tilde{f}_{\alpha*}\left(\tilde{s}_\alpha(e_{2i})\right) = \tilde{J}_2\left(\tilde{f}_{\alpha*}\left(\tilde{s}_\alpha(e_{2i-1})\right)\right) \quad i \in \{1, \dots, k\}.$$

Consequently

$$\begin{aligned}
\tilde{J}_2(d\tilde{f}_\alpha(\tilde{X})) &= \tilde{f}_{\alpha*}(\tilde{s}_\alpha)\left((\tilde{f}_{\alpha*}(\tilde{s}_\alpha))^{-1}(df_\alpha(\tilde{X}))\cdot J_{can}\right) \\
&= d\tilde{f}_\alpha\left(\tilde{s}_\alpha\left((\tilde{s}_\alpha^{-1}\circ df_\alpha^{-1}\circ df_\alpha)(\tilde{X})\cdot J_{can}\right)\right) \\
&= d\tilde{f}_\alpha(\tilde{J}_1(\tilde{X})).
\end{aligned}$$

If $J_i \in \Omega^1(\mathcal{O}_i, T\mathcal{O}_i)$ are the almost complex structures over $\mathcal{O}_i$, then

$$df \circ J_1 = J_2 \circ df.$$

Example 3.27. $U(k)$**-morphisms**
First of all, by the previous example, an $U(k)$-morphism is a diffeomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that

$$df \circ J_1 = J_2 \circ df.$$

Furthermore, if we have taken adapted frames to the almost complex structure as above then

$$\begin{aligned}
\tilde{h}_2(d\tilde{f}_\alpha(\tilde{X}), d\tilde{f}_\alpha(\tilde{Y})) &= h_{can}((\tilde{f}_{\alpha*}(\tilde{s}_\alpha))^{-1}(d\tilde{f}_\alpha(\tilde{X})), (\tilde{f}_{\alpha*}(\tilde{s}_\alpha))^{-1}(d\tilde{f}_\alpha(\tilde{Y}))) \\
&= h_{can}(\tilde{s}_\alpha^{-1}(\tilde{X}), \tilde{s}_\alpha^{-1}(\tilde{Y})) \\
&= \tilde{h}_1(\tilde{X}, \tilde{Y}).
\end{aligned}$$

Consequently, the diffeomorphism f satisfies

$$h_2(df(X), df(Y)) = h_1(X, Y),$$

with h_1 and h_2 the almost hermitian structures on $\mathcal{O}_1$ and $\mathcal{O}_2$.

Remark: Given that $U(k) = GL_k(\mathbb{C}) \cap Sp_{2k}(\mathbb{R}) \cap O(2k)$, then a $U(n)$-morphism also satisfies

$$\langle df(X), df(Y)\rangle_2 = \langle X, Y\rangle_1,$$
$$\omega_2(df(X), df(Y)) = \omega_1(X, Y),$$

with $\langle \cdot, \cdot \rangle_i \in \Sigma^2(T^*\mathcal{O}_i)$ and $\omega_i \in \Omega^2(\mathcal{O}_i)$ the induced Riemannian and almost symplectic structures (is an isometry and a symplectomorphism too).

Given that a geometric structure over $\mathcal{O}_i$ is codified on the pair (P_i, τ_i), that a diffeomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$ preserves the geometric structures induced, which means $f_* : P_1 \to P_2$ is a G-structure equivalence, could be expressed in terms of the pairs (P_i, τ_i).

Proposition 3.28. *If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a G-structure equivalence then $(f_*)^*\tau_2 = \tau_1$.*

Proof. Let $Y \in T_pP_1$. Then

$$\begin{aligned}
(f_*)^*(\tau_2)_p(Y) &= (\tau_2)_{f_*(p)}(d_pf_*(Y)) \\
&= (f_*(p))^{-1}\left(d_p(\pi_2 \circ f_*)(Y)\right)) \\
&= (d_pf \circ p)^{-1}\left(d_p(f \circ \pi_1)(Y)\right) \\
&= p^{-1}\left(d_p\pi_1(Y)\right) \\
&= (\tau_1)_p(Y).
\end{aligned}$$

It follows that $(f_*)^*\tau_2 = \tau_1$. □

3.2.2 Isomorphisms of Principal G-bundles vs. Equivalence of G-structures

The main difference between equivalences of G-structures and G-principal bundle morphisms is that the differential of a map must induce our morphisms. More precisely, $\phi : P_1 \to P_2$ is an equivalence of G-structures if there exists a diffeomorphism $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that $f_* = \phi$. This condition is crucial because it is the one that guarantees that the diffeomorphism respects the geometric structures induced by the G-structures. If

f is an equivalence of G-structures, then $(f_*)^*\tau_2 = \tau_1$. Let

$$\begin{tikzcd} P_1 \arrow[r, "\phi"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ \mathcal{O}_1 \arrow[r, "f"] & \mathcal{O}_2 \end{tikzcd}$$

be a principal bundle isomorphism. The tautological forms associated to both G-structures characterize when $\phi = f_*$.

Theorem 3.29. *An isomorphism ϕ of principal G-bundles is an equivalence of G-structures if and only if $\phi^*\tau_2 = \tau_1$.*

Proof. ($\Rightarrow$) Already proved.
($\Leftarrow$) Locally, let $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{\phi}$, $\tilde{f}$ and $\tilde{Y} \in T_{\tilde{p}}\tilde{\mathcal{P}}_1$ be local lifts. Then

$$\begin{aligned} \tilde{p}^{-1}(d_{\tilde{p}}\tilde{\pi}_1(\tilde{Y})) &= (\tilde{\tau}_1)_{\tilde{p}}(\tilde{Y}) \\ &= (\tilde{\tau}_2)_{\tilde{\phi}(\tilde{p})}(d_{\tilde{p}}\phi(\tilde{Y})) \\ &= \tilde{\phi}(\tilde{p})^{-1}(d_{\tilde{p}}(\tilde{\pi}_2 \circ \phi)(\tilde{Y})) \\ &= \tilde{\phi}(\tilde{p})^{-1}\left(d_{\tilde{\pi}_1(\tilde{p})}\tilde{f}\left((d_{\tilde{p}}\tilde{\pi}_1)(\tilde{Y})\right)\right). \end{aligned}$$

Given that $\tilde{\pi}_1$ is a submersion, for all $\tilde{X} \in T\tilde{U}_1$ it is true that

$$\tilde{p}^{-1}(\tilde{X}) = \tilde{\phi}(\tilde{p})^{-1}\left(d_{\tilde{\pi}_1(\tilde{p})}\tilde{f}\left(\tilde{X}\right)\right).$$

In addition, $\tilde{p} : \mathbb{R}^n \to T_{\pi_1(\tilde{p})}\tilde{U}_1$ is an isomorphism and then there exists a unique $v \in \mathbb{R}^n$ such that $\tilde{p}(v) = \tilde{X}$. Consequently

$$\tilde{\phi}_{\tilde{p}}(v) = (d\tilde{f} \circ \tilde{p})(v),$$

which means $\tilde{\phi}(\tilde{p}) = \tilde{f}_*(\tilde{p})$. It follows that $\phi = f_*$. $\square$

3.2.3 Characterization of the category of G-structures

Every G-structure over $\mathcal{O}$ is characterized by (P, τ), with $\tau \in \Omega^1_{ten}(P, \mathbb{R}^n)$ a tensorial form . Thus, if we take two G-structures P_i over $\mathcal{O}$, we have two objects defining the geometric structures induced: (P_1, τ_1) and (P_2, τ_2). Furthermore, a G-equivariant diffeomorphism $\phi : P_1 \to P_2$ is a G-structure equivalence if and only if $\tilde{\phi}^*\tau_2 = \tau_1$.

Theorem 3.30. *The category of G-structures over $\mathcal{O}$ with G-structure equivalences as morphisms is isomorphic to the category:*

- ***Objects:*** (P, τ) *with*
  1. *P a manifold with a locally free and proper action $P \curvearrowleft G$.*
  2. *$\tau \in \Omega^1_{ten}(P, \mathbb{R}^n)$ a tensorial form.*
- ***Morphisms:*** *G-equivariant diffeomorphisms $\phi : P_1 \to P_2$ such that $\phi^* \tau_2 = \tau_1$.*

There are no orbifolds involved in this picture. This characterization is fascinating because, even when working with effective orbifolds, G-structure theory happens in the setting of manifolds. The orbifold structure on the base can be recovered from the quotient P/G.

3.3 Compatible connections

Take a connection form $\omega_P \in \Omega^1(P, \mathfrak{g})$. We can extend it to a connection form $\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ as follows. Let $Y \in T_q Fr(\mathcal{O})$; there exists $g \in G$ such that $p \cdot g = q$. Then

$$\begin{aligned} \omega_P\left(Y_q\right) &= \omega_P\left(R_g^* \circ R_{g^{-1}}^*(Y_q)\right) \\ &= Ad_{g^{-1}}\left(\omega_P\left(R_{g^{-1}}^*(Y_q)\right)\right). \end{aligned}$$

Take the connection $\mathcal{H} = \ker \omega_P$. It follows that

$$\omega_P\left(R_{g^{-1}}^*(Y_q^V + Y_q^{\mathcal{H}})\right) = \omega_P\left(R_{g^{-1}}^*(Y_q^V)\right).$$

Given that $R_{g^{-1}}^*(Y_q^V) \in T_p P$, we can define $\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ by

$$\omega(Y_q) := Ad_{g^{-1}}\left(\omega_P\left(R_{g^{-1}}^*(Y_q^V)\right)\right).$$

Because the extension of ω_P does not depend on the isotropies, its smoothness follows from the local diffeomorphism

$$\delta : Fr(\mathcal{O}) \times_\pi Fr(\mathcal{O}) \to GL_n(\mathbb{R}).$$

The induced connection form satisfies

$$\omega|_P \in \Omega^1(P, \mathfrak{g}).$$

On the other hand, if $\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ satisfies $\omega|_P \in \Omega^1(P, \mathfrak{g})$, then

$$\omega_P := \omega|_P \in \Omega^1(P, \mathfrak{g}),$$

is a connection form on P. Then, we have the 1-1 correspondence between the connection forms

$$\left\{\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R})) \ + \ \omega|_P \in \mathfrak{g}\right\} \overset{1-1}{\longleftrightarrow} \{\omega_P \in \Omega(P, \mathfrak{g})\}.$$

The 1-1 relation

$$(T\mathcal{O}, \nabla) \longleftrightarrow (Fr(\mathcal{O}), \omega_\nabla),$$

allow us to induce a connection ∇_P on $T\mathcal{O}$ from a connection form ω_P on P.

Definition 3.31. *A connection form $\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ is compatible with the G-structure $P \subset Fr(\mathcal{O})$ if*

$$\omega|_P \in \Omega^1(P, \mathfrak{g}).$$

Proposition 3.32. *Take a connection form $\omega_\nabla \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n(\mathbb{R}))$ and ∇ its induced connection on $T\mathcal{O}$. The following statements are equivalent:*

1. *ω_∇ is compatible with P.*

2. *Every connection matrix induced by the local section $\tilde{s}_\bullet : \tilde{U}_\bullet \to \tilde{\mathcal{P}}_\bullet$ and ∇ satisfies $\omega_{\tilde{s}_\bullet} \in \Omega^1(\tilde{U}_\bullet, \mathfrak{g})$.*

3. *Take a path $\eta : I \to \mathcal{O}$. Locally, the parallel transport along $\tilde{\eta} : I \to \tilde{U}$ sends adapted frames to adapted frames. That means for all $\tilde{p}_{t_0} \in \tilde{\mathcal{P}}_{\tilde{\eta}(t_0)}$, there exists $\tilde{p}_{t_1} \in \tilde{\mathcal{P}}_{\tilde{\eta}(t_1)}$ such that*

$$\tilde{T}^{t_1,t_0}_{\tilde{\eta}}(\tilde{p}_{t_0}(e_i)) = \tilde{p}_{t_1}(e_i).$$

Proof. (1) ⇒ (2) Given that

$$\omega_{\tilde{s}_\bullet} = \tilde{s}_\bullet^* \omega_\nabla,$$

it follows that $\omega_{\tilde{s}_\bullet} \in \Omega^1(\tilde{U}_\bullet, \mathfrak{g})$.
(2) ⇒ (1) Take $\tilde{Y} \in T_{\tilde{p}} \tilde{\mathcal{P}}_\bullet$. We can write it as

$$\tilde{Y} = d\tilde{s}_\bullet(d\tilde{\pi}_\bullet(\tilde{Y})) + \tilde{Y} - d\tilde{s}_\bullet(d\tilde{\pi}_\bullet(\tilde{Y})).$$

Because $\tilde{Y} - d\tilde{s}_\bullet(d\tilde{\pi}_\bullet(\tilde{Y})) \in \ker(d\tilde{\pi})$ then it is a vertical vector and

$$\omega_\nabla(\tilde{Y} - d\tilde{s}_\bullet(d\tilde{\pi}_\bullet(\tilde{Y}))) \in \mathfrak{g}.$$

Given that $\tilde{s}_\bullet^* \omega_\nabla(d\tilde{\pi}(\tilde{Y})) = \omega_{\tilde{s}}(d\tilde{\pi}(\tilde{Y})) \in \mathfrak{g}$, we conclude

$$\omega_\nabla(\tilde{Y}) = \tilde{s}_\bullet^* \omega_\nabla(d\tilde{\pi}(\tilde{Y})) + \omega_\nabla(\tilde{Y} - d\tilde{s}_\bullet(d\tilde{\pi}_\bullet(\tilde{Y}))) \in \mathfrak{g},$$

which implies $\omega_\nabla|_P \in \Omega^1(P, \mathfrak{g})$ and then is a connection form on P.
(1) ⇒ (3) Take $\tilde{p}_{t_0} \in \tilde{\mathcal{P}}_{\tilde{\eta}(t_0)}$. Define

$$\tilde{p}_i(t) := \tilde{T}_{\tilde{\eta}}^{t,t_0}(\tilde{p}_{t_0}(e_i)).$$

It induces the frame $\tilde{p} : I \to Fr(\tilde{U})$ defined

$$\tilde{p}(t)(v_1, \ldots, v_n) = \sum_{i=1}^{n} v_i \tilde{p}_i(t).$$

By construction, its differential $\tilde{p}'(t_0) \in \tilde{\mathcal{H}}_{\tilde{p}_{t_0}} \subset T_{\tilde{\eta}(t_0)} Fr(\tilde{U})$ is a horizontal vector. Given that $T_{\tilde{q}} \tilde{\mathcal{P}} = \tilde{\mathcal{H}}_{\tilde{q}} \oplus T_{\tilde{q}}^V \tilde{\mathcal{P}}$, it follows that $\tilde{p} : I \to Fr(\tilde{U})$ must belong to $\tilde{\mathcal{P}}$ in order to have a differential defined on $T\tilde{\mathcal{P}}$. Consequently

$$\tilde{T}_{\tilde{\eta}}^{t_1,t_0}(\tilde{p}_{t_0}(e_i)) = \tilde{p}(t_1)(e_i).$$

(3) ⇒ (1) For all $\tilde{p}_{t_0} \in \tilde{\mathcal{P}}_{\tilde{\eta}(t_0)}$, the parallel translation defines a unique $\tilde{p}(t_1) \in \tilde{\mathcal{P}}_{\tilde{\eta}(t_1)}$ as before. This map defines a G-equivariant diffeomorphism $\tilde{P}_{\tilde{\eta}}^{t_1,t_0} : \tilde{\mathcal{P}}_{\tilde{\eta}(t_0)} \to \tilde{\mathcal{P}}_{\tilde{\eta}(t_1)}$. Let $\tilde{u} : I \to \tilde{\mathcal{P}}$ be a smooth path with $\tilde{u}(0) = \tilde{p}$ and $\tilde{u}'(0) = \tilde{Y}$. There exists $A : I \to GL_n(\mathbb{R})$ such that

$$\tilde{P}_{\tilde{\eta}}^{0,t}(\tilde{u}(t)) = \tilde{p} \cdot A(t).$$

Thus

$$\tilde{u}'(0) = \Psi(\tilde{p}, A'(0)) + \left.\frac{d}{dt}\right|_{t=0} \tilde{P}^{t,0}_{\tilde{\eta}}(\tilde{p}).$$

Because, by definition

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{P}^{t,0}_{\tilde{\eta}}(\tilde{p})$$

is horizontal, then the vertical component is $\tilde{Y}^V = \Psi(\tilde{p}, A'(0))$. In addition

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{P}^{0,t}_{\tilde{\eta}}(\tilde{u}(t)) = \Psi(\tilde{p}, A'(0)) = \tilde{Y}^V.$$

Then

$$\omega_\nabla(\tilde{Y}) = \left.\frac{d}{dt}\right|_{t=0} \tilde{P}^{0,t}_{\tilde{\eta}}(\tilde{u}(t)).$$

Given that $\tilde{P}_{\tilde{\eta}}$ sends adapted frames to adapted frames, the path

$$\tilde{P}^{0,t}_{\tilde{\eta}}(\tilde{u}(t)) \in \tilde{\mathcal{P}}_{\tilde{\eta}(0)}.$$

Hence, $\left.\frac{d}{dt}\right|_{t=0} \tilde{P}^{0,t}_{\tilde{\eta}}(\tilde{u}(t)) \in T^V_{\tilde{p}}\tilde{\mathcal{P}} \cong \mathfrak{g}$ which implies

$$\omega_\nabla|_P \in \Omega^1(P, \mathfrak{g}).$$

$\square$

If $\eta : I \to \mathcal{O}$ is an orbifold path representing a cone field, then the parallel transport $T^{t_1,t_0}_\eta : T_{\eta(t_0)}\mathcal{O} \to T_{\eta(t_1)}\mathcal{O}$ is a homeomorphism. In this setting, statement (3) of the previous proposition says that T^{t_1,t_0}_η is a homeomorphism that preserves the geometric structure induced by the G-structure P.

3.3.1 Compatibility tensor

Denote by $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{R})$. The connection form $\omega \in \Omega^1(Fr(\mathcal{O}), \mathfrak{gl}_n)$ is compatible with P if and only if $\omega|_P \in \Omega^1(P, \mathfrak{g})$. Let $C_\omega \in \Omega^1(P, \mathfrak{gl}_n/\mathfrak{g})$ be defined by

$$C_\omega(Y) = [\omega(Y)]_{\mathfrak{g}}.$$

ω is compatible with P if and only if $C_\omega = 0$. The group $G \curvearrowright \mathfrak{gl}_n/\mathfrak{g}$ acts by

$$g \cdot [\xi]_\mathfrak{g} = [Ad_g(\xi)]_\mathfrak{g},$$

which implies

$$C_\omega(R_g^*(Y)) = [Ad_{g^{-1}}(\omega(Y))]_\mathfrak{g} = g^{-1} \cdot C_\omega(Y),$$

is G-equivariant. Moreover, as long as $T^V P \cong P \times \mathfrak{g}$, then

$$C_\omega(Y^V) = [\omega(Y^V)]_\mathfrak{g} = 0.$$

It follows that $C_\omega \in \Omega^1_{bas}(P, \mathfrak{gl}_n/\mathfrak{g})$ is a basic form. By proposition 2.42

$$\Omega^1_{bas}(P, \mathfrak{gl}_n/\mathfrak{g}) \cong \Omega^1\left(\mathcal{O}, E(P, \mathfrak{gl}_n/\mathfrak{g})\right),$$

so the failure of ∇ being compatible with the G-structure is measured by the 1-form $\nabla P \in \Omega^1\left(\mathcal{O}, E(P, \mathfrak{gl}_n/\mathfrak{g})\right)$ defined by

$$\nabla P(X) = [p, [\omega(Y)]_\mathfrak{g}],$$

with $d\pi(Y) = X$ and $\pi(p) = x$. We want to find an explicit description of the compatibility tensor. For that, we will prove that the fiber bundle $E(P, \mathfrak{gl}_n/\mathfrak{g})$ is the quotient of $E(P, \mathfrak{gl}_n)$ and $E(P, \mathfrak{g})$. Then, we want to find an orbibundle $\mathcal{E}_G$ such that

$$0 \to E(P, \mathfrak{g}) \to E(P, \mathfrak{gl}_n) \to \mathcal{E}_G \to 0$$

is a short exact sequence, which implies $E(P, \mathfrak{gl}_n/\mathfrak{g}) \cong \mathcal{E}_G$. The connection ∇ allow us to write explicitly $\omega(Y) \in \mathfrak{gl}_n$ as derivatives of paths on the vertical components. The path derivatives induce elements on $E(P, \mathfrak{gl}_n)$, and then we can view them as elements on $\mathcal{E}_G$. The vanishing of the resulting expression is the condition that characterizes connections compatible with P.

Take a local section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$, it induces the local structures

$$\begin{array}{ccc} \Gamma \curvearrowright \tilde{U} \times \mathfrak{gl}_n & & \Gamma \curvearrowleft \tilde{U} \times \mathfrak{g} \\ \downarrow & \text{and} & \downarrow \\ \Gamma \curvearrowright \tilde{U} & & \Gamma \curvearrowleft \tilde{U} \end{array}$$

of $E(P,\mathfrak{gl}_n)$ and $E(P,\mathfrak{g})$. With the action $G \curvearrowright \mathfrak{gl}_n/\mathfrak{g}$ already defined, we have the local orbibundle structure

$$\begin{array}{c} \Gamma \curvearrowright \tilde{U} \times \mathfrak{gl}_n/\mathfrak{g} \\ \downarrow \\ \Gamma \curvearrowright \tilde{U} \end{array}$$

of $E(P,\mathfrak{gl}_n/\mathfrak{g})$. Then

$$E(P,\mathfrak{gl}_n/\mathfrak{g}) \cong E(P,\mathfrak{gl}_n)/E(P,\mathfrak{g}).$$

Let $\Phi : E(P,\mathfrak{gl}_n) \to Hom(T\mathcal{O},T\mathcal{O})$ be locally defined by

$$\begin{aligned} E(\tilde{\mathcal{P}},\mathfrak{gl}_n) &\to Hom(T\tilde{U},T\tilde{U}) \\ [\tilde{p},A] &\mapsto \tilde{\Phi}_{[\tilde{p},A]} : T\tilde{U} \to T\tilde{U} \\ &\qquad\qquad\qquad \tilde{X} \mapsto \tilde{p}\circ A\circ \tilde{p}^{-1}(\tilde{X}). \end{aligned}$$

Take $\gamma \in \Gamma$; we have that

$$\begin{aligned} \tilde{\Phi}_{[\gamma\cdot\tilde{p},A]}(\tilde{X}) &= \gamma\cdot\tilde{p}\circ A\circ(\gamma\cdot\tilde{p})^{-1}(\tilde{X}) \\ &= \gamma\cdot\tilde{p}\circ A\circ\tilde{p}^{-1}\circ d\gamma^{-1}(\tilde{X}) \\ &= \gamma\cdot\tilde{\Phi}_{[\tilde{p},A]}(\gamma^{-1}\cdot\tilde{X}) \\ &= \tilde{\Phi}^{\gamma}_{[\tilde{p},A]}(\tilde{X}). \end{aligned}$$

Then, $\tilde{\Phi}$ is a Γ-equivariant map. Every homomorphism arises from a $n\times n$ matrix (the columns are the action coefficients on each basis element). Besides, if $\tilde{\Phi}_{[\tilde{p},A]} = \tilde{\Phi}_{[\tilde{q},B]}$ then

$$\begin{aligned} \tilde{\Phi}_{[\tilde{p},A]} &= \tilde{\Phi}_{[\tilde{p},\delta(\tilde{p},\tilde{q})\cdot B]} \\ \tilde{p}\circ A\circ\tilde{p}^{-1} &= \tilde{p}\circ\delta(\tilde{p},\tilde{q})\circ B\circ\delta(\tilde{p},\tilde{q})^{-1}\circ\tilde{p}^{-1} \\ A &= \delta(\tilde{p},\tilde{q})\circ B\circ\delta(\tilde{p},\tilde{q})^{-1}. \end{aligned}$$

It follows that $[\tilde{p},A] = [\tilde{p},\delta(\tilde{p},\tilde{q})\circ B\circ\delta(\tilde{p},\tilde{q})^{-1}] = [\tilde{p}\cdot\delta(\tilde{p},\tilde{q}),B] = [\tilde{q},B]$. Thus $\tilde{\Phi}$ is a bijective map. Its inverse is given by

$$\begin{aligned} Hom(T\tilde{U},T\tilde{U}) &\to E(\tilde{\mathcal{P}},\mathfrak{gl}_n) \\ &\qquad \tilde{\Phi} \mapsto [\tilde{p},\tilde{p}^{-1}\circ\tilde{\Phi}\circ\tilde{p}], \end{aligned}$$

and

$$\begin{aligned}
[\tilde{p}, \tilde{p}^{-1} \circ \tilde{\Phi}^{\gamma} \circ \tilde{p}] &= [\tilde{p}, \tilde{p}^{-1} \circ d\gamma \circ \tilde{\Phi} \circ d\gamma^{-1} \circ \tilde{p}] \\
&= [\tilde{p}, g_{\gamma} \circ \tilde{p}^{-1} \circ \tilde{\Phi} \circ \tilde{p} \circ g_{\gamma^{-1}}] \\
&= [\gamma \cdot \tilde{p}, \tilde{p}^{-1} \circ \tilde{\Phi} \circ \tilde{p}].
\end{aligned}$$

Then $\Phi : E(P, \mathfrak{gl}_n) \to Hom(T\mathcal{O}, T\mathcal{O})$ is an isomorphism of cone orbibundles. The embedding $\iota : E(P, \mathfrak{g}) \hookrightarrow E(P, \mathfrak{gl}_n)$ induces an embedded orbibundle structure $E(P, \mathfrak{g}) \cong Hom_G(T\mathcal{O}, T\mathcal{O}) \subset Hom(T\mathcal{O}, T\mathcal{O})$.

Definition 3.33. *The orbibundle $Hom_G(T\mathcal{O}, T\mathcal{O})$ is called the infinitesimal automorphism bundle associated with P.*

Example 3.34. $Hom_{GL_{k,n-k}(\mathbb{R})}(T\mathcal{O}, T\mathcal{O})$
Take $P \curvearrowleft GL_{k,n-k}(\mathbb{R})$. The Lie algebra $\mathfrak{gl}_{k,n-k}(\mathbb{R})$ is given by the matrices of the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

with $B \in \mathfrak{gl}_k$, $C \in M_{k,n-k}(\mathbb{R})$, and $D \in \mathfrak{gl}_{n-k}$. Hence

$$\mathfrak{gl}_{k,n-k}(\mathbb{R}) = \{\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n \mid \tilde{\phi}(\mathbb{R}^k) \subset \mathbb{R}^k\}.$$

If we take a section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$, it defines the distribution $\tilde{\mathcal{D}}$ by

$$\tilde{s}(\tilde{x})(\mathbb{R}^k) = \tilde{\mathcal{D}}_{\tilde{x}}.$$

The homomorphism induced by $[\tilde{s}(\tilde{x}), A]$ is

$$\tilde{\Phi}_{[\tilde{s}(\tilde{x}),A]}(\tilde{X}) = \tilde{s}(\tilde{x}) \circ A \circ \tilde{s}(\tilde{x})^{-1}(\tilde{X}).$$

If $\tilde{X} \in \tilde{\mathcal{D}}$, then $\tilde{s}(\tilde{x})^{-1}(\tilde{X}) \in \mathbb{R}^k$. Because $A \in \mathfrak{gl}_{k,n-k}$, we have that $A \cdot \tilde{s}(\tilde{x})^{-1}(\tilde{X}) \in \mathbb{R}^k$. Consequently $\tilde{s}(\tilde{x}) \circ A \circ \tilde{s}(\tilde{x})^{-1}(\tilde{X}) \in \tilde{\mathcal{D}}$. It follows that $\tilde{\Phi}_{[\tilde{s},A]} : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$. On the other hand, if $\tilde{\Phi}_{[\tilde{s},A]} : T\tilde{U} \to T\tilde{U}$ is such that $\tilde{\Phi}_{[\tilde{s},A]} : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$, then, for all $\tilde{X} \in \tilde{\mathcal{D}}$

$$A \circ \tilde{s}(\tilde{x})^{-1}(\tilde{X}) \in \mathbb{R}^k,$$

which implies $A(\mathbb{R}^k) \subset \mathbb{R}^k$. We conclude

$$Hom_{GL_{k,n-k}(\mathbb{R})}(T\mathcal{O}, T\mathcal{O}) \cong \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid \Phi(D) \subset D\}.$$

Example 3.35. $Hom_{O(n)}(T\mathcal{O}, T\mathcal{O})$
Take $P \curvearrowleft O(n)$. The Lie algebra $\mathfrak{o}(n)$ is given by

$$\mathfrak{o}(n) = \{A \in GL_n(\mathbb{R}) \mid \langle A \cdot v, w\rangle_{can} + \langle v, A \cdot w\rangle_{can} = 0\}.$$

Take $\tilde{s} : \tilde{U} \to \mathcal{P}$, the metric induced by P is

$$\langle \tilde{X}, \tilde{Y}\rangle = \langle \tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})\rangle_{can}.$$

If $[\tilde{s}, A] \in E(\tilde{\mathcal{P}}, \mathfrak{o}(n))$, we have that

$$\begin{aligned}
&\langle \tilde{\Phi}_{[\tilde{s},A]}(\tilde{X}), \tilde{Y}\rangle + \langle \tilde{X}, \tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y})\rangle \\
=&\langle \tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X})), \tilde{s}^{-1}(\tilde{Y})\rangle_{can} + \langle \tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y}))\rangle_{can} \\
=&\langle A \cdot \tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})\rangle_{can} + \langle \tilde{s}^{-1}(\tilde{X}), A \cdot \tilde{s}^{-1}(\tilde{Y})\rangle_{can} \\
=&0.
\end{aligned}$$

Then

$$Hom_{O(n)}(T\mathcal{O}, T\mathcal{O}) \cong \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid \langle \Phi(X), Y\rangle + \langle X, \Phi(Y)\rangle = 0\}.$$

Example 3.36. $Hom_{Sp_{2k}(\mathbb{R})}(T\mathcal{O}, T\mathcal{O})$
Take $P \curvearrowleft Sp_{2k}(\mathbb{R})$. The Lie algebra $\mathfrak{sp}_2k$ is given by

$$\mathfrak{sp}_{2k} = \{A \in GL_{2k}(\mathbb{R}) \mid \omega_{can}(A \cdot v, w) + \omega_{can}(v, A \cdot w) = 0\}.$$

Take $\tilde{s} : \tilde{U} \to \mathcal{P}$, the almost symplectic structure induced by P is

$$\omega(\tilde{X}, \tilde{Y}) = \omega_{can}(\tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})).$$

If $[\tilde{s}, A] \in E(\tilde{\mathcal{P}}, \mathfrak{sp}_{2k})$, we have that

$$\begin{aligned}
&\omega(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X}), \tilde{Y}) + \omega(\tilde{X}, \tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y})) \\
=&\omega_{can}(\tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X})), \tilde{s}^{-1}(\tilde{Y})) + \omega_{can}(\tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y}))) \\
=&\omega_{can}(A \cdot \tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})) + \omega_{can}(\tilde{s}^{-1}(\tilde{X}), A \cdot \tilde{s}^{-1}(\tilde{Y})) \\
=&0.
\end{aligned}$$

Then

$$\begin{aligned}
&Hom_{Sp_{2k}(\mathbb{R})}(T\mathcal{O}, T\mathcal{O}) \\
&\qquad \cong \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid \omega(\Phi(X), Y) + \omega(X, \Phi(Y)) = 0\}.
\end{aligned}$$

Example 3.37. $Hom_{GL_k(\mathbb{C})}(T\mathcal{O}, T\mathcal{O})$
Take $P \curvearrowleft GL_k(\mathbb{C})$. The Lie algebra $\mathfrak{gl}_k(\mathbb{C})$ is $M_{k\times k}(\mathbb{C})$. A real matrix $A \in M_{2k\times 2k}(\mathbb{R})$ represents a complex matrix if and only if

$$J_{can}A = AJ_{can}.$$

It follows that

$$\mathfrak{gl}_k(\mathbb{C}) = \{A \in M_{2k\times 2k}(\mathbb{R}) \mid J_{can}A = AJ_{can}\}.$$

Take a local section $\tilde{s} : \tilde{U} \to \mathcal{P}$. The almost complex structure induced by P is given by

$$\tilde{J}(\tilde{X}) = \tilde{s}\left(\tilde{s}^{-1}(\tilde{X}) \cdot J_{can}\right).$$

If $[\tilde{s}, A] \in E(\tilde{\mathcal{P}}, \mathfrak{gl}_k(\mathbb{C}))$, we have that

$$\begin{aligned}
\tilde{\Phi}_{[\tilde{s},A]}(\tilde{J}(\tilde{X})) &= \tilde{\Phi}_{[\tilde{s},A]}\left(\tilde{s}\left(\tilde{s}^{-1}(\tilde{X}) \cdot J_{can}\right)\right) \\
&= \tilde{s}\left(A \cdot \tilde{s}^{-1}(\tilde{X}) \cdot J_{can}\right) \\
&= \tilde{s}\left(J_{can} \cdot A \cdot \tilde{s}^{-1}(\tilde{X})\right) \\
&= \tilde{s}\left(J_{can} \cdot \tilde{s}^{-1}\left(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X})\right)\right) \\
&= \tilde{J}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X})).
\end{aligned}$$

Then

$$Hom_{GL_k(\mathbb{C})}(T\mathcal{O}, T\mathcal{O}) \cong \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid \Phi \circ J = J \circ \Phi\}.$$

Example 3.38. $Hom_{U(k)}(T\mathcal{O}, T\mathcal{O})$
Take $P \curvearrowleft U(k)$. The Lie algebra $\mathfrak{u}(k)$ is given by

$$\mathfrak{u}(k) = \{A \in \mathfrak{gl}_k(\mathbb{C}) \mid h_{can}(Az, w) + h_{can}(z, Aw) = 0\}.$$

Let $\tilde{s} : \tilde{U} \to \mathcal{P}$ be a local section and view it as a complex local frame. The almost hermitian structure induced bu P is given by

$$\tilde{h}(\tilde{X}, \tilde{Y}) = h_{can}(\tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})).$$

If $[\tilde{s}, A] \in E(\tilde{\mathcal{P}}, \mathfrak{u}(k))$, we have that

$$\begin{aligned}
&\tilde{h}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X}), \tilde{Y}) + \tilde{h}(\tilde{X}, \tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y})) \\
=&h_{can}(\tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{X})), \tilde{s}^{-1}(\tilde{Y})) + h_{can}(\tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{\Phi}_{[\tilde{s},A]}(\tilde{Y}))) \\
=&h_{can}(A \cdot \tilde{s}^{-1}(\tilde{X}), \tilde{s}^{-1}(\tilde{Y})) + h_{can}(\tilde{s}^{-1}(\tilde{X}), A \cdot \tilde{s}^{-1}(\tilde{Y})) \\
=&0.
\end{aligned}$$

Then

$$\begin{aligned}
&Hom_{U(k)}(T\mathcal{O}, T\mathcal{O}) \\
&\qquad\qquad \cong \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid h(\Phi(X), Y) + h(X, \Phi(Y)) = 0\}.
\end{aligned}$$

We want an orbibundle $\mathcal{E}_G$ such that

$$0 \to Hom_G(T\mathcal{O}, T\mathcal{O}) \to Hom(T\mathcal{O}, T\mathcal{O}) \to \mathcal{E}_G \to 0$$

is a short exact sequence. Besides, the infinitesimal automorphism bundles are described in terms of the vanishing of something (tensorial). Then, we can find out the orbibundle structure of $\mathcal{E}_G$. For example, for $O(n)$-structures a morphism $\Phi \in Hom(T\mathcal{O}, T\mathcal{O})$ is an infinitesimal automorphism if and only if

$$\langle \Phi(X), Y \rangle + \langle X, \Phi(Y) \rangle = 0.$$

Define $F(\Phi) \in \Sigma^2(T^*\mathcal{O})$ by

$$F(\Phi)(X, Y) := \langle \Phi(X), Y \rangle + \langle X, \Phi(Y) \rangle.$$

That means we have a bundle map $F : Hom(T\mathcal{O}, T\mathcal{O}) \to \Sigma^2(T^*\mathcal{O})$ such that $\ker F = Hom_P(T\mathcal{O}, T\mathcal{O})$. To prove the surjectivity of F, take a symmetric 2-tensor $\zeta \in \Sigma^2(T^*\mathcal{O})$ and an adapted frame $\tilde{\sigma}_i$. Define $\zeta_{ii} := \zeta(\tilde{\sigma}_i, \tilde{\sigma}_i) \in C^\infty(\tilde{U})$; we want that

$$\langle \tilde{\Phi}(\tilde{\sigma}_i), \tilde{\sigma}_i \rangle = \frac{1}{2}\zeta_{ii}.$$

That happens precisely if we define

$$\tilde{\Phi}(\tilde{\sigma}_i) = \frac{1}{2}\sum_j \zeta_{jj}\tilde{\sigma}_j.$$

Then $F(\Phi) = \sigma$ and

$$0 \to Hom_{O(n)}(T\mathcal{O}, T\mathcal{O}) \to Hom(T\mathcal{O}, T\mathcal{O}) \to \Sigma^2(T^*\mathcal{O}) \to 0,$$

is the short exact sequence we were looking for. Similar arguments show that

$$0 \to Hom_{Sp_{2k}(\mathbb{R})}(T\mathcal{O}, T\mathcal{O}) \to Hom(T\mathcal{O}, T\mathcal{O}) \to \Lambda^2(\mathcal{O}) \to 0.$$

For almost complex structures, the map F is

$$F(\Phi)(X) = \Phi(J(X)) - J\Phi(X).$$

It satisfies

$$\begin{aligned} F(\Phi)(J(X)) &= -\Phi(X) - J\Phi(J(X)) \\ &= -J(\Phi(J(X)) - J\Phi(X)) \\ &= -JF(\Phi)(X). \end{aligned}$$

Let $Hom_J^A(T\mathcal{O}, T\mathcal{O})$ be the antiholomorphic homomorphisms defined by

$$Hom_J^A(T\mathcal{O}, T\mathcal{O}) = \{\Phi \in Hom(T\mathcal{O}, T\mathcal{O}) \mid \Phi \circ J = -J \circ \Phi\}.$$

Take $\Phi^A \in Hom_J^A(T\mathcal{O}, T\mathcal{O})$, to prove the surjectivity of F, we want $\Phi \in Hom(T\mathcal{O}, T\mathcal{O})$ such that

$$\Phi^A(X) = \Phi(J(X)) - J\Phi(X).$$

Take an adapted frame $(\tilde{\sigma}_i, \tilde{J}(\tilde{\sigma}_i))_{i=1}^k$. Assign an arbitrary value for $\tilde{\Phi}(\tilde{\sigma}_i)$ and define

$$\tilde{\Phi}(\tilde{J}(\tilde{\sigma}_i)) := \tilde{J}\tilde{\Phi}(\tilde{\sigma}_i) - \tilde{\Phi}^A(\tilde{\sigma}_i).$$

Hence, $F(\Phi) = \Phi^A$ and follows that the short exact sequence associated to almost complex structures is

$$0 \to Hom_{GL_k(\mathbb{C})}(T\mathcal{O}, T\mathcal{O}) \to Hom(T\mathcal{O}, T\mathcal{O}) \to Hom_J^A(T\mathcal{O}, T\mathcal{O}) \to 0.$$

It is not necessary to do this for distributions because we already have the condition the homomorphism Φ must satisfy: $\Phi(D) \subset D$. For almost hermitian structures, as long as $U(k) = Sp_{2k}(\mathbb{R}) \cap O(2k) \cap GL_k(\mathbb{C})$, the compatibility conditions obtained from the other geometric structures characterize its compatibility.

Take a cone field $X \in \mathfrak{X}(\mathcal{O})$. We want to know the homomorphism induced by $[p, \omega(Y)]$. As long as Y and p must satisfy $d\pi(Y) = X$ and $\pi(p) = x$, we can take a local section $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$, a lift $\tilde{X}$ of X and obtain $d\tilde{s}(\tilde{X}) \in T\tilde{\mathcal{P}}$. Because $\tilde{X}$ is Γ-invariant, so is $d\tilde{s}(\tilde{X}) \in T\tilde{\mathcal{P}}$. Then $[p, \omega(Y)] = [s(x), \omega(ds(X))]$. But $s_*\omega(X) = \omega_s(X)$ is the connection matrix over the frame $s : U \to P$. Take the local frame $\tilde{\sigma}_i := \tilde{s}(e_i)$, then

$$\tilde{\Phi}_{[p,\omega(Y)]}(\tilde{\sigma}_i) = \sum_k \omega_s(X)_{ik} \tilde{\sigma}_k.$$

But

$$\nabla_X \tilde{\sigma}_i = \sum_k \omega_s(X)_{ik} \tilde{\sigma}_k,$$

and then

$$\tilde{\Phi}_{[p,\omega(Y)]}(\tilde{\sigma}_i) = \nabla_X \tilde{\sigma}_i.$$

However, it is a homomorphism, and then it is not true that

$$\tilde{\Phi}_{[p,\omega(Y)]}(\tilde{Y}) \neq \nabla_X \tilde{Y},$$

for all $\tilde{Y}$. If $\tilde{Y} = \sum a^i \tilde{\sigma}_i$, we have that

$$\tilde{\Phi}_{[p,\omega(Y)]}(\tilde{Y}) = \sum_i a^i \nabla_X \tilde{\sigma}_i.$$

Given that $a^i \nabla_X \tilde{\sigma}_i = \nabla_X a^i \tilde{\sigma}_i - da^i(X)\tilde{\sigma}_i$, we obtain

$$(\nabla_X P)(\tilde{Y}) = \nabla_X \tilde{Y} - \sum_i da^i(X)\tilde{\sigma}_i.$$

Theorem 3.39. *Let $G < GL_n(\mathbb{R})$ be one of the following groups*

$$GL_{k,n-k}(\mathbb{R}), O(n), Sp_{2k}(\mathbb{R}), GL_k(\mathbb{C}), U(k).$$

A connection ∇ is compatible with a G-structure if and only if

1. $\nabla_X(D) \subset D$ *when* $G = GL_{k,n-k}(\mathbb{R})$*.*

2. $\mathcal{L}_X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ *when* $G = O(n)$*.*

3. $\mathcal{L}_X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$ *when* $G = Sp_{2k}(\mathbb{R})$*.*

4. $\nabla_X(JY) = J\nabla_X Y$ *when* $G = GL_k(\mathbb{C})$*.*

5. $\mathcal{L}_X(h(Y, Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$ *when* $G = U(k)$*.*

Proof. Take an adapted frame $\tilde{\sigma}_i$. Let $\tilde{Y} = \sum_i a^i \tilde{\sigma}_i$ and $\tilde{Z} = \sum_i b^i \tilde{\sigma}_i$. The connection is compatible with a G-structure if and only if $\nabla_X P = 0$. The short exact sequence

$$0 \to Hom_G(T\mathcal{O}, T\mathcal{O}) \xrightarrow{\iota} Hom(T\mathcal{O}, T\mathcal{O}) \xrightarrow{F} \mathcal{E}_G \to 0,$$

characterizes the compatibility of the connection because $\nabla_X P = 0$ if and only if $F(\nabla_X P) = 0$.

1. In $GL_{k,n-k}(\mathbb{R})$ structures, the compatibility condition means the homomorphism $\nabla_X P$ satisfies

$$\nabla_X P(D) \subset (D).$$

In addition, $\nabla_X P\tilde{\sigma}_i = \nabla_X \tilde{\sigma}_i$ and, as long as $(\tilde{\sigma}_i)_{i=1}^k$ generates the distribution $\tilde{\mathcal{D}}$, we get

$$\nabla_X(D) \subset D.$$

2. Take an $O(n)$-structure P. We have

$$\begin{aligned} F(\nabla_X P)(\tilde{Y}, \tilde{Z}) &= \langle \nabla_X P(\tilde{Y}), \tilde{Z}\rangle + \langle \tilde{Y}, \nabla_X P(\tilde{Z})\rangle \\ &= \langle \nabla_X \tilde{Y} - \sum_i da^i(X)\tilde{\sigma}_i, \tilde{Z}\rangle \\ &\qquad + \langle \tilde{Y}, \nabla_X \tilde{Z} - \sum_i db^i(X)\tilde{\sigma}_i\rangle. \end{aligned}$$

Moreover

$$\begin{aligned} \langle \sum_i da^i(X)\tilde{\sigma}_i, \tilde{Z}\rangle &= \langle \sum_i da^i(X)\tilde{\sigma}_i, \sum_j b^j \tilde{\sigma}_j\rangle \\ &= \sum_k da^k(X) b^k, \end{aligned}$$

and

$$\langle\tilde{Y},\sum_i db^i(X)\tilde{\sigma}_i\rangle=\sum_k db^k(X)a^k.$$

Hence

$$F(\nabla_X P)(\tilde{Y},\tilde{Z})=\langle\nabla_X\tilde{Y},\tilde{Z}\rangle+\langle\tilde{Y},\nabla_X\tilde{Z}\rangle-\sum_k da^k(X)b^k+db^k(X)a^k.$$

Finally, because

$$\begin{aligned}\mathcal{L}_X(\langle\tilde{Y},\tilde{Z}\rangle)&=\mathcal{L}_X(\langle\sum_i a^i\tilde{\sigma}_i,\sum_j b^j\tilde{\sigma}_j\rangle)\\&=\mathcal{L}_X(\sum_k a^kb^k)\\&=\sum_k da^k(X)b^k+a^kdb^k(X),\end{aligned}$$

we get that

$$F(\nabla_X P)(\tilde{Y},\tilde{Z})=\langle\nabla_X\tilde{Y},\tilde{Z}\rangle+\langle\tilde{Y},\nabla_X\tilde{Z}\rangle-\mathcal{L}_X(\langle\tilde{Y},\tilde{Z}\rangle).$$

3. Take an $Sp_{2k}(\mathbb{R})$-structure P. The adapted frame is given by $(\tilde{\sigma}_i^1,\tilde{\sigma}_i^2)_{i=1}^k$ and

$$\begin{aligned}F(\nabla_X P)(\tilde{Y},\tilde{Z})&=\omega(\nabla_X P(\tilde{Y}),\tilde{Z})+\omega(\tilde{Y},\nabla_X P(\tilde{Z}))\\&=\omega(\nabla_X\tilde{Y}-\sum_i da_1^i(X)\tilde{\sigma}_i^1+da_2^i(X)\tilde{\sigma}_i^2,\tilde{Z})\\&\qquad+\omega(\tilde{Y},\nabla_X\tilde{Z}-\sum_i db_1^i(X)\tilde{\sigma}_i^1+db_2^i(X)\tilde{\sigma}_i^2).\end{aligned}$$

Moreover

$$\begin{aligned}&\omega(\sum_i da_1^i(X)\tilde{\sigma}_i^1+da_2^i(X)\tilde{\sigma}_i^2,\tilde{Z})\\&=\omega(\sum_i da_1^i(X)\tilde{\sigma}_i^1+da_2^i(X)\tilde{\sigma}_i^2,\sum_i b_1^i\tilde{\sigma}_i^1+b_2^i\tilde{\sigma}_i^2)\\&=\sum_k da_1^k(X)b_2^k-da_2^k(X)b_1^k,\end{aligned}$$

and

$$\omega(\tilde{Y}, \sum_i db^i(X)\tilde{\sigma}_i) = \sum_k a_2^k db_1^k(X) - a_1^k db_2^k(X).$$

Hence

$$\begin{aligned} F(\nabla_X P)(\tilde{Y}, \tilde{Z}) &= \omega(\nabla_X \tilde{Y}, \tilde{Z}) + \omega(\tilde{Y}, \nabla_X \tilde{Z}) \\ &\quad - \sum_k da_1^k(X) b_2^k + a_2^k db_1^k(X) - da_2^k(X) b_1^k - a_1^k db_2^k(X). \end{aligned}$$

Finally, because

$$\begin{aligned} \mathcal{L}_X(\omega(\tilde{Y}, \tilde{Z})) &= \mathcal{L}_X(\sum_k a_1^k b_2^k - a_2^k b_1^k) \\ &= \sum_k da_1^k(X) b_2^k + a_1^k db_2^k(X) - da_2^k(X) b_1^k - a_1^k db_2^k(X), \end{aligned}$$

we get that

$$F(\nabla_X P)(\tilde{Y}, \tilde{Z}) = \omega(\nabla_X \tilde{Y}, \tilde{Z}) + \omega(\tilde{Y}, \nabla_X \tilde{Z}) - \mathcal{L}_X(\omega(\tilde{Y}, \tilde{Z})).$$

4. Take a $GL_k(\mathbb{C})$-structure P. The adapted (real) frame is given by $(\tilde{\sigma}_i, \tilde{J}(\tilde{\sigma}_i))_{i=1}^k$ and $\tilde{Y} = \sum_j a^j \tilde{\sigma}_j + b^j \tilde{J}(\tilde{\sigma}_j)$. We have that

$$F(\nabla_X P)(\tilde{Y}) = \nabla_X P(\tilde{J}(\tilde{Y})) - \tilde{J}(\nabla_X P(\tilde{Y})).$$

The vanishing of $F(\nabla_X P)$ for the adapted frames means

$$\nabla_X(\tilde{J}(\tilde{\sigma}_i)) = \tilde{J}(\nabla_X(\tilde{\sigma}_i)).$$

Because $\tilde{J}$ is an isomorphism, we have that

$$\nabla_X(\tilde{J}(\tilde{Y})) = \tilde{J}(\nabla_X(\tilde{Y})).$$

5. Take a $U(k)$-structure P, the almost complex structure J induced by $U(k)$ and the hermitian structure

$$h(X, Y) = \langle X, Y \rangle - i\omega(X, Y),$$

with $\langle \cdot\cdot \rangle$ and $\omega(\cdot, \cdot)$ the induced Riemannian and almost symplectic structures. By the previous items we have that

- $\nabla_X J(Y) = J(\nabla_X Y)$.
- $\mathcal{L}_X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.
- $\mathcal{L}_X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$.

Because the Lie derivative, on the almost complex setting, splits by

$$\mathcal{L}(u + iv) = \mathcal{L}(u) + i\mathcal{L}(v),$$

then

$$\begin{aligned}
\mathcal{L}(h(X, Y)) &= \mathcal{L}(\langle X, Y\rangle - i\omega(X, Y)) \\
&= \mathcal{L}(\langle X, Y\rangle) - i\mathcal{L}(\omega(X, Y)) \\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle - i(\omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)) \\
&= h(\nabla_X Y, Z) + h(Y, \nabla_X Z).
\end{aligned}$$

□

3.3.2 The space of compatible connections

Fix a connection form $\omega \in \Omega^1(P, \mathfrak{g})$. If $\omega_2 \in \Omega^1(P, \mathfrak{g})$ is another connection form, then $\eta := \omega_2 - \omega \in \Omega^1_{bas}(P, \mathfrak{g})$ is a basic form. In addition, if $\eta \in \Omega^1_{bas}(P, \mathfrak{g})$, it follows that $\omega_2 := \eta + \omega \in \Omega_1(P, \mathfrak{g})$ is a connection form.

Definition 3.40. *The space of compatible connections with a fixed G-structure P is given by*

$$Con(P) = \{\omega \in \Omega^1(P, \mathfrak{g}) \mid \omega(\Psi(\cdot, \xi)) = \xi \quad and \quad R_g^*\omega = Ad_{g^{-1}}\omega\}.$$

Proposition 3.41. *A connection form $\omega \in \Omega^1(P, \mathfrak{g})$ induces a bijection between*

$$\Omega^1_{bas}(P, \mathfrak{g}) \xleftrightarrow{1-1} Con(P).$$

That means the space of compatible connections is an affine space modeled on $\Omega^1_{bas}(P, \mathfrak{g})$. On the other hand, basic forms are isomorphic to

$$\Omega^1_{bas}(P, \mathfrak{g}) \cong \Omega^1(\mathcal{O}, Hom_P(T\mathcal{O}, T\mathcal{O})).$$

Fix ∇^ω, the 1-1 correspondence between connection forms ω and connections ∇^ω in $T\mathcal{O}$, together with the previous proposition, gives us the bijection

$$\Omega^1(\mathcal{O}, Hom_P(T\mathcal{O}, T\mathcal{O})) \overset{1-1}{\longleftrightarrow} \{\text{Connections } \nabla_P \text{ compatible with } P\}.$$

Explicitly, if $\eta \in \Omega^1(\mathcal{O}, Hom_P(T\mathcal{O}, T\mathcal{O}))$, the bijection is given by

$$\nabla^\eta_X Y := \nabla^\omega_X Y + \eta(X, Y).$$

Hence, the space of connections ∇ compatible with P is an affine space modeled on $\Omega^1(\mathcal{O}, Hom_P(T\mathcal{O}, T\mathcal{O}))$.

3.4 Integrability

Take a G-structure $P \curvearrowleft G \to \mathcal{O}$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ an orbifold atlas that belongs to the same orbifold structure as P/G. Fix an orbifold chart around $\pi(p) = x \in \mathcal{O}$ and denote it by $(\tilde{U}_x, \Gamma_x, \tilde{\phi}_x)$. Because they belong to the same orbifold structure, there exists a slice $S_p \subset P$, a diffeomorphism $\tilde{f}_{xp} : \tilde{U}_x \to S_p$ and an isomorphism $\theta_{xp} : \Gamma_x \to G_p$ such that $\tilde{f}_{xp}$ is θ_{xp}-equivariant. It follows that

$$\begin{tikzcd}
Fr(\tilde{U}_x) \arrow[r, "\tilde{f}_{xp*}"] \arrow[d, "\tilde{\pi}_x"'] & Fr(S_p) \arrow[d, "\tilde{\pi}_p"] \\
\tilde{U}_x \arrow[r, "\tilde{f}_{xp}"] & S_p
\end{tikzcd}$$

commutes and induces the orbifold commutative diagram

$$\begin{tikzcd}
Fr(U_x) \arrow[r, "f_{xp*}"] \arrow[d, "\pi_x"'] & Fr(S_p)/G_p \arrow[d, "\pi_p"] \\
U_x \arrow[r, "f_{xp}"] & S_p/G_p
\end{tikzcd}$$

where the horizontal arrows are diffeomorphisms. But $G_p < GL_n(\mathbb{R})$ and follows that the G_p-action is by linear transformations. Take the chart $\tilde{\varphi}_p : S_p \to \mathbb{R}^n$ given by the manifold structure of S_p. It defines the commutative diagram

$$\begin{tikzcd} Fr(S_p) \arrow[r, "\tilde{\varphi}_{p*}"] \arrow[d, "\tilde{\pi}_p"'] & Fr(\mathbb{R}^n) \arrow[d, "pr"] \\ S_p \arrow[r, "\tilde{\varphi}_p"] & \mathbb{R}^n \end{tikzcd}$$

where all maps are G_p-equivariant. Define $f_x := \varphi_p \circ f_{xp}$, then

$$\begin{tikzcd} Fr(U_x) \arrow[r, "f_{x*}"] \arrow[d, "\tilde{\pi}_x"'] & Fr(\mathbb{R}^n)/G_p \arrow[d, "pr"] \\ U_x \arrow[r, "f_x"] & \mathbb{R}^n/G_p \end{tikzcd}$$

is a commutative diagram such that the horizontal lines are embeddings. Besides, we have a canonical structure $\mathbb{R}^n_{can}$ such that $f_{x*}(q) \in Fr(\mathbb{R}^n_{can})$ if and only if $q \in P$. In addition, a local section $\tilde{s} : \tilde{U}_x \to Fr(\tilde{U}_x)$ generates the local G-principal bundle structure $Fr(\tilde{U}_x)_G$ defined by

$$Fr(\tilde{U}_x)_G := \tilde{s}(\tilde{U}_x) \cdot G.$$

Then, $Fr(\tilde{U}_x)_G = \tilde{\mathcal{P}}$ if and only if $\tilde{s} : \tilde{U}_x \to \tilde{\mathcal{P}}_x$ is an adapted local section.

Proposition 3.42. *A local section $\tilde{s} : \tilde{U}_x \to Fr(\tilde{U}_x)$ is an adapted frame if and only if*

$$\begin{tikzcd} Fr(U_x)_G \arrow[r, "f_{xp*}"] \arrow[d, "\tilde{\pi}_x"'] & Fr(\mathbb{R}^n_{can})/G_p \arrow[d, "\tilde{\pi}_p"] \\ U_x \arrow[r, "f_{xp}"] & \mathbb{R}^n_{can}/G_p \end{tikzcd}$$

is an equivalence of G-structures.

Be aware that in the orbifold setting, there is no canonical local model. Instead, it depends on $\Gamma_{\pi(p)}$. The *canonical local model around* $p \in P$ is the principal bundle structure $Fr(\mathbb{R}^n_{can}) \curvearrowleft G_p \to \mathbb{R}^n/G_p$.

Definition 3.43. *A G-structure P is called integrable if there exists an orbifold atlas $(\tilde{U}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ such that for all $x \in \mathcal{O}$, $\pi(p) = x$ and all orbifold chart U_x around x*

$$\begin{tikzcd} Fr(U_x)_G \arrow[r, "f_{xp*}"] \arrow[d, "\tilde{\pi}_x"'] & Fr(\mathbb{R}^n_{can})/G_p \arrow[d, "\tilde{\pi}_p"] \\ U_x \arrow[r, "f_{xp}"] & \mathbb{R}^n_{can}/G_p \end{tikzcd}$$

is an equivalence of G-structures.

By the previous proposition, P is integrable if and only if we can find an orbifold atlas such that every induced local frame $\tilde{s} : \tilde{U} \to Fr(\tilde{U})$ is an adapted frame $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$. The change of coordinates between the adapted frames is given by elements of G. Hence, the change of coordinates of the orbifold structure $\mathcal{O}$ belongs to G. Whether a G-structure is integrable or not is a central question in G-structure theory. All $GL_n^+(\mathbb{R})$-structures are integrable by definition. Furthermore, every $SL_n(\mathbb{R})$-structure is integrable too because the difference between an adapted and non-adapted frame is given by the multiplication of a smooth function $\tilde{f} : \tilde{U} \to \mathbb{R}$ and then they belong to the same orbifold structure.

Example 3.44. $GL_{k,n-k}(\mathbb{R})$**-structures**
If $P \curvearrowleft GL_{k,n-k}(\mathbb{R})$ is an integrable structure, then there exist orbifold charts $(\tilde{U}^k_\alpha \times \tilde{U}^{n-k}_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$ such that the local frames $(\tilde{s}^i_\alpha)^k_{i=1}$ generates the distribution $\tilde{\mathcal{D}}_\alpha$. Take the orbifold $\mathcal{O}_D$ given by the atlas $(\tilde{U}^k_\alpha, \Gamma_\alpha, \tilde{\phi}_\alpha)$. The Γ_α-equivariant embeddings $\tilde{\psi}_\alpha : \tilde{U}^k_\alpha \hookrightarrow \tilde{U}^k_\alpha \times \tilde{U}^{n-k}_\alpha$ defines an embedding $\mathcal{O}_D \hookrightarrow \mathcal{O}$. By construction $T\mathcal{O}_D = D$ and then an integrable distribution is a foliation. On the other hand, if we have a foliation, we have a $GL_{k,n-k}(\mathbb{R})$-structure. If we complete the local frames given by $\mathcal{O}_D \hookrightarrow \mathcal{O}$, we obtain an orbifold atlas adapted to D. Hence, a $GL_{k,n-k}(\mathbb{R})$-structure is integrable if and only if it is a foliation.

Example 3.45. $O(n)$**-structures**
Take the Riemannian structure $\langle \cdot, \cdot \rangle$ induced by P and a local frame $\tilde{s} : \tilde{U} \to \tilde{\mathcal{P}}$. Define

$$g_{ij}(\tilde{x}) := \langle \tilde{s}(\tilde{x})(e_i), \tilde{s}(\tilde{x})(e_j) \rangle,$$

we have that

$$\langle \tilde{s}(\tilde{x})(v), \tilde{s}(\tilde{x})(w)\rangle = (v_1, \ldots, v_n)\cdot \begin{pmatrix} g_{11}(\tilde{x}) & \cdots & g_{1n}(\tilde{x}) \\ \vdots & \ddots & \vdots \\ g_{n1}(\tilde{x}) & \cdots & g_{nn}(\tilde{x}) \end{pmatrix} \cdot (w_1, \ldots, w_n)^T.$$

If $\tilde{s}$ is an adapted frame then $g_{ij}(\tilde{x}) = \delta^i_j$ for all $\tilde{x} \in \tilde{U}$. Hence, if P is integrable, for all $x \in \mathcal{O}$, there exists an orbifold chart U_x such that

$$\langle \cdot, \cdot \rangle = d\tilde{x}_1^2 + \ldots + d\tilde{x}_n^2, \tag{3.4.1}$$

with $(\tilde{x}_1, \ldots, \tilde{x}_n)$ the coordinate functions associated to $\tilde{U}_x$. Conversely, take a local frame $\tilde{s}$, if the Riemannian structure is given by (3.4.1), then $\langle \tilde{s}(\tilde{x})(e_i), \tilde{s}(\tilde{x})(e_j)\rangle = \delta^i_j$ and it follows that $\tilde{s}$ is an adapted frame.

Example 3.46. $Sp_{2k}(\mathbb{R})$**-structures**
As in Riemannian structures, an almost symplectic structure is integrable if and only if there exists an orbifold atlas such that the almost symplectic form ω is locally given by

$$\omega = d\tilde{x}_1 \wedge d\tilde{y}_1 + \ldots + d\tilde{x}_k \wedge d\tilde{y}_k.$$

The coordinates such that ω has this form are called Darboux coordinates.

Example 3.47. $GL_k(\mathbb{C})$**-structures**
Take two adapted frames $(\tilde{\sigma}^i_\alpha, \tilde{J}(\tilde{\sigma}^i_\alpha))^k_{i=1}$ and $(\tilde{\sigma}^i_\beta, \tilde{J}(\tilde{\sigma}^i_\beta))^k_{i=1}$. Then, the transition matrix

$$\tilde{\sigma}^\alpha \cdot g_{\alpha\beta} = \tilde{\sigma}^\beta,$$

satisfies $g_{\alpha\beta} \in GL_k(\mathbb{C})$. Hence, $\tilde{\psi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ is a holomorphic function (it comes from the (complex) linear map given by the multiplication of a complex matrix varying smoothly on $\tilde{U}_\alpha \cap \tilde{U}_\beta$). Consequently, if P is integrable, then $\mathcal{O}$ has a complex structure, which means an almost complex structure J such that the transition functions are holomorphic functions with respect to J.

Integrability can be thought of as extending the local properties induced by the standard geometric structure $\mathbb{R}^n_{can}(\mathbb{C}^k_{can})$ to the whole orbifold $\mathcal{O}$. Then, we are interested in which properties characterizes $Fr(\mathbb{R}^n_{can})/G_p$. First of all, the vector and principal bundles $T\mathbb{R}^n_{can}$ and $Fr(\mathbb{R}^n_{can})$ are canonically trivializable by

$$(v, w) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \left.\frac{d}{dt}\right|_{t=0} v + tw \in T_v\mathbb{R}^n,$$

and

$$(v, g) \in \mathbb{R}^n \times G \mapsto g^{-1} \in Fr_v(\mathbb{R}^n),$$

with

$$g^{-1} \cdot \left.\frac{d}{dt}\right|_{t=0} v + tw = g^{-1} \cdot w.$$

The canonical tautological form is defined by

$$\theta^{can}_{(v,g)}(Y) = g^{-1} \cdot u,$$

with

$$d_{(v,g)}\pi(Y) = \left.\frac{d}{dt}\right|_{t=0} v + tu.$$

Define the distribution $\zeta_i \in TFr(\mathbb{R}^n)$ by

$$\zeta_i(v, g) := \left.\frac{d}{dt}\right|_{t=0} (v + tg \cdot e_i, g).$$

It satisfies

$$\theta^{can}(\zeta_i) = e_i,$$

which implies $(\zeta_i)_{i=1}^n$ is a (canonical) horizontal subbundle of $TFr(\mathbb{R}^n)$. Furthermore, because every canonical vector field on $T\mathbb{R}^n$ commutes, then

$$[\zeta_i, \zeta_j] = 0 \quad \text{and} \quad [\zeta_i, \Psi(\xi)] = 0,$$

for all $\xi \in \mathfrak{g}$. Because $G_p \curvearrowright \mathbb{R}^n$ acts by fixed linear transformations, all constructions above are G_p-equivariant. Hence, if a G-structure is integrable, for every $x \in \mathcal{O}$ the equivalence

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\tilde{\pi}_x"'] & Fr(\mathbb{R}^n_{can})/G_p \arrow[d, "\tilde{\pi}_p"] \\ U_x \arrow[r, "f_{xp}"] & \mathbb{R}^n_{can}/G_p \end{tikzcd}$$

induces vector fields $\zeta_i \in TP|_{U_x}$ such that

$$\theta(\zeta_i) = e_i \quad , \quad [\zeta_i, \zeta_j] = 0 \quad \text{and} \quad [\zeta_i, \Psi(\xi)] = 0,$$

for all $\xi \in \mathfrak{g}$.

Proposition 3.48. *A G-structure (P, θ) is integrable if and only if for every $p \in P$ there exist an open set $P|_U$ together with local vector fields $\zeta_i \in \mathfrak{X}(P|_U)$ such that*

$$\theta(\zeta_i) = e_i \quad , \quad [\zeta_i, \zeta_j] = 0 \quad \textit{and} \quad [\zeta_i, \Psi(\xi)] = 0,$$

for all $\xi \in \mathfrak{g}$.

Proof. ($\Rightarrow$) Already done.
($\Leftarrow$) Take the distribution $\zeta_i \in TP|_{U_x}$. By Frobenius theorem, the vector fields ζ_i defines a foliation $Z \subset P$, which means, $TZ = span(\zeta_1, \ldots, \zeta_n)$. It follows that there exists an orbifold chart $\tilde{V}_x$ around x, and a local section $\tilde{\sigma} : \tilde{V}_x \to \tilde{Z} \subset \tilde{\mathcal{P}}$ such that

$$\frac{\partial \tilde{\sigma}}{\partial x_i} = \tilde{\zeta}_i.$$

Define $\tilde{\varphi} : \tilde{V}_x \times G \to \tilde{\mathcal{P}}$ by $\tilde{\varphi}(\tilde{x}, g) = \tilde{\sigma}(\tilde{x}) \cdot g$. It is a principal bundle isomorphism. Take a basis $(\frac{\partial}{\partial x_i})_{i=1}^n$ for the horizontal distribution, we have that

$$d_{(\tilde{x},g)}\tilde{\varphi}\left(\frac{\partial}{\partial x_i}, 0\right) = R_g^*\left(\frac{\partial \tilde{\sigma}}{\partial x_i}\right) = R_g^*\left(\tilde{\zeta}_i\right).$$

It follows that

$$(\tilde{\varphi}^*\theta)_{(\tilde{x},g)}\left(\frac{\partial}{\partial x_i}, 0\right) = \theta_{\tilde{\varphi}(\tilde{x},g)}\left(R_g^*\tilde{\zeta}_i\right) = g^{-1} \cdot e_i = \theta^{can}_{(\tilde{x},g)}(\frac{\partial}{\partial x_i}).$$

Hence, $\tilde{\varphi}^*\theta = \theta^{can}$ which implies $\tilde{\varphi}$ is a G-structure equivalence. $\square$

The proposition characterizes the integrability problem in terms of the manifold P instead of the orbifold P/G because all the informations relies on the existence of local vector field on P satisfying conditions expressed only in terms of the geometry of P.

3.4.1 Affine structure of the compatible connections

Fix a connection $\mathcal{H} = \ker \omega$. Let $\theta \in \Omega^1(P, \mathbb{R}^n)$ be the tautological form, $\Psi : P \times \mathfrak{g} \to TP$ the infinitesimal action and $Y_v = \theta_{\mathcal{H}}^{-1}(v)$, where $v \in \mathbb{R}^n$. They induce the isomorphism $\phi_{\mathcal{H}} : P \times (\mathbb{R}^n \oplus \mathfrak{g}) \to TP$ defined by

$$\phi_{\mathcal{H}}(p, v + \xi) = (Y_v)_p + \Psi(p, \xi).$$

If we have taken another connection $\mathcal{H}' = \ker \omega'$, the isomorphism becomes

$$\phi_{\mathcal{H}'}(p, v + \xi) = (Y'_v)_p + \Psi(p, \xi),$$

with $Y'_v = \theta_{\mathcal{H}'}^{-1}(v)$. Given that $d_p\pi(Y'_v) = p(v) = d_p\pi(Y_v)$, we get

$$Y'_v - Y_v \in T^V P \cong P \times \mathfrak{g},$$

and we will identify them. We can compare the two isomorphisms with

$$\phi_{\mathcal{H}}^{-1} \circ \phi_{\mathcal{H}'} : P \times (\mathbb{R}^n \oplus \mathfrak{g}) \overset{\cong}{\to} P \times (\mathbb{R}^n \oplus \mathfrak{g}),$$

given by

$$(\phi_{\mathcal{H}}^{-1} \circ \phi_{\mathcal{H}'})(p, v + \xi) = \big(p, v + (\xi + Y'_v - Y_v)\big)\,.$$

Define $S_{\mathcal{H}',\mathcal{H}} : P \times \mathbb{R}^n \to P \times \mathfrak{g}$ by

$$S_{\mathcal{H}',\mathcal{H}}(p, v) = (p, Y'_v - Y_v). \tag{3.4.2}$$

Denote $S_{\mathcal{H}'_p,\mathcal{H}_p}(\cdot) = S_{\mathcal{H}',\mathcal{H}}(p, \cdot)$. $S_{\mathcal{H}',\mathcal{H}}$ is a homomorphism of vector bundles such that

$$(\phi_{\mathcal{H}}^{-1} \circ \phi_{\mathcal{H}'})(p, v + \xi) = \Big(p, v + (\xi + S_{\mathcal{H}'_p,\mathcal{H}_p}(v))\Big)\,.$$

The 1-1 correspondence

$$\Omega^1_{bas}(P, \mathfrak{g}) \longleftrightarrow Con(P),$$

gives for every $\omega' \in Con(P)$ an element $\eta \in \Omega^1_{bas}(P, \mathfrak{g})$ such that

$$\omega' = \eta + \omega.$$

Because η is horizontal, we have that

$$\eta|_{\mathcal{H}} : \mathcal{H} \to P \times \mathfrak{g},$$

is a well-defined homomorphism of vector bundles. Moreover, the tautological form $\theta \in \Omega^1(P, \mathbb{R}^n)$ induces the isomorphism

$$\theta_{\mathcal{H}} : \mathcal{H} \xrightarrow{\cong} P \times \mathbb{R}^n.$$

It follows that every basic form η defines the homomorphism of vector bundles

$$\eta \circ \theta_{\mathcal{H}}^{-1} : P \times \mathbb{R}^n \to P \times \mathfrak{g}.$$

The homomorphism $S_{\mathcal{H}',\mathcal{H}}$ defined on (3.4.2) is related to the previous homomorphism by

$$S_{\mathcal{H}',\mathcal{H}} = -\eta \circ \theta_{\mathcal{H}}^{-1}.$$

Thus, the difference between the two connections is identified with a homomorphism $S_{\mathcal{H}',\mathcal{H}} : P \times \mathbb{R}^n \to P \times \mathfrak{g}$.

Define $G \curvearrowright Hom(\mathbb{R}^n, \mathfrak{g})$ by

$$(g \cdot \phi)(v) = Ad_g \phi(g^{-1} v),$$

and

$$C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G = \{\eta \in C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g})) \mid \eta(p \cdot g) = g^{-1} \cdot \eta(p)\}.$$

Proposition 3.49. *A connection $\mathcal{H} = \ker \omega$ induces the bijection*

$$C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G \xleftrightarrow{F_\omega} Con(P),$$

defined by

$$F_\omega(\eta)(Y) = \eta(\theta(Y)) + \omega(Y).$$

Proof. We want to recover the previous homomorphism, which means

$$F_\omega^{-1}(\omega') = -S_{\mathcal{H}',\mathcal{H}}.$$

Besides

$$\omega'(\theta_{\mathcal{H}}^{-1}(v)) = (\omega + \eta)(\theta_{\mathcal{H}}^{-1}(v)) = \eta(\theta_{\mathcal{H}}^{-1}(v)) = -S_{\mathcal{H}',\mathcal{H}}(v),$$

and then

$$F_\omega^{-1}(\omega') = \omega' \circ \theta_{\mathcal{H}}^{-1}.$$

Proving that F_ω is a bijection requires that F_ω^{-1} is its inverse on both sides. Firstly

$$\begin{aligned}(F_\omega^{-1} \circ F_\omega)(\eta)(p)(v) &= F_\omega(\eta)(\theta_{\mathcal{H}_p}^{-1}(v)) \\ &= \eta_p(\theta \circ \theta_{\mathcal{H}_p}^{-1}(v)) + \omega(\theta_{\mathcal{H}_p}^{-1}(v)) \\ &= \eta_p(v),\end{aligned}$$

which implies $F_\omega^{-1} \circ F_\omega = Id$. Secondly

$$\begin{aligned}(F_\omega \circ F_\omega^{-1})(\omega')(Y) &= F_\omega^{-1}(\omega')(\theta(Y)) + \omega(Y) \\ &= \omega'(\theta_{\mathcal{H}}^{-1} \circ \theta(Y)) + \omega(Y^V) \\ &= \omega'(Y^{\mathcal{H}}) + \omega'(Y^V) \\ &= \omega'(Y),\end{aligned}$$

which implies $F_\omega \circ F_\omega^{-1} = Id$. Then F_ω is a bijection.

Take $\eta \in C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G$, then $F_\omega(\eta) \in Con(P)$ because

1. $F_\omega(\eta)(\Psi(\xi)) = \xi$.

$$F_\omega(\eta)(\Psi(\xi)) = \eta(\theta(\Psi(\xi))) + \omega(\Psi(\xi)) = \omega(\Psi(\xi)) = \xi.$$

2. $R_g^* F_\omega(\eta) = Ad_{g^{-1}} F_\omega(\eta)$.

$$\begin{aligned}R_g^*(F_\omega(\eta))_p(Y) &= (F_\omega(\eta))_{p\cdot g}(R_g^*(Y)) \\ &= \eta_{p\cdot g}(\theta_p(R_g^*(Y))) + \omega_p(R_g^*(Y)) \\ &= \eta_{p\cdot g}(g^{-1} \cdot \theta_p(Y)) + Ad_{g^{-1}} \cdot \omega_p(Y) \\ &= Ad_{g^{-1}} \cdot \eta_p(g \cdot g^{-1} \cdot \theta_p(Y)) + Ad_{g^{-1}} \cdot \omega_p(Y) \\ &= Ad_{g^{-1}} \cdot (\eta_p(\theta_p(Y)) + \omega_p(Y)) \\ &= Ad_{g^{-1}} \cdot (F_\omega(\eta)_p(Y)).\end{aligned}$$

Conversely, if $\omega' \in Con(P)$, it follows that

$$\begin{aligned} F_\omega^{-1}(\omega')(p\cdot g)(v) &= \omega'_{p\cdot g}(\theta^{-1}_{\mathcal{H}_{p\cdot g}}(v)) \\ &= \omega'_{p\cdot g}\left(R_g^*\left(\theta^{-1}_{\mathcal{H}_p}(g\cdot v)\right)\right) \\ &= Ad_{g^{-1}}\cdot F_\omega(\omega')(p)(g\cdot v), \end{aligned}$$

which implies $F_\omega^{-1}(\omega') \in C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G$. □

3.4.2 First order integrability obstruction: intrinsic torsion

Two G-structures (P_1, θ_1) and (P_2, θ_2) are equivalent if and only if there exists a G-equivariant diffeomorphism $f : P_1 \to P_2$ such that

$$f^*\theta_2 = \theta_1.$$

An integrable G-structure (θ, P) is locally equivalent to the canonical G-structure

$$\begin{array}{c} (\theta_{can},\ Fr(\mathbb{R}^n_{can})/G_p \curvearrowleft G) \\ \big\downarrow \\ \mathbb{R}^n_{can}/G_p \end{array}$$

so $f^*\theta_{can} = \theta$. Then, a necessary condition for the integrability of a G-structure is

$$f^* d\theta_{can} = d\theta. \tag{3.4.3}$$

We want an explicit description of $d\theta$. Take a connection $\mathcal{H} = \ker\omega$. It induces the trivialization

$$\begin{aligned} \varphi_{\mathcal{H}} : P \times (\mathbb{R}^n \oplus \mathfrak{g}) &\to TP \\ (p, v+\xi) &\mapsto \theta^{-1}_{\mathcal{H}_p}(v) + \Psi(p,\xi). \end{aligned}$$

Fix two basis $(\xi_j)_{j=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$ and $(e_i)_{i=1}^n$ of $\mathbb{R}^n$. They induce the covectors $\theta^i, \omega^j \in \Omega^1(P)$ characterized by

$$\theta = \sum_i \theta^i e_i \quad \text{and} \quad \omega = \sum_j \omega^j \xi_j.$$

Given that $\varphi_{\mathcal{H}}$ is an isomorphism, the covectors $(\theta^i, \omega^j)_{i,j}$ form a basis of T^*P. It follows that

$$d_p\theta^k = \sum_{i,j,l,m} A^k_{ij}(p)\theta^i \wedge \omega^j + B^k_{lj}(p)\omega^l \wedge \omega^j + C^k_{im}(p)\theta^i \wedge \theta^m,$$

which implies

$$d\theta = A\,(\theta \wedge \omega) + B\,(\omega \wedge \omega) + C\,(\theta \wedge \theta)\,,$$

with

$$\begin{aligned} A &: P \to Hom(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n), \\ B &: P \to Hom(\Lambda^2\mathfrak{g}, \mathbb{R}^n), \\ C &: P \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n). \end{aligned}$$

Because θ is horizontal and ω vertical, we can find the coefficients A, B and C values using vertical and horizontal vectors. Every horizontal vector has the form $Y_v(p) = \theta^{-1}_{\mathcal{H}_p}(v)$, and every vertical vector will be identified by the equation $\Psi(p, \xi) = \tilde{\xi}_p$.

1. $A : P \to Hom(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n)$.
   Given that $\theta(\tilde{\xi}) = 0$, and by Cartan's magic formula, we have that

   $$\mathcal{L}_{\tilde{\xi}}\theta = d\iota_{\tilde{\xi}}\theta + \iota_{\tilde{\xi}}d\theta = \iota_{\tilde{\xi}}d\theta.$$

   Then
   $$d_p\theta(Y_v, \tilde{\xi}) = -d_p\theta(\tilde{\xi}, Y_v) = -\mathcal{L}_{\tilde{\xi}}\theta(Y_v).$$

   Besides,

   $$\mathcal{L}_{\tilde{\xi}}\theta = \left.\frac{d}{dt}\right|_{t=0} R^*_{\exp(t\xi)}\theta = \left.\frac{d}{dt}\right|_{t=0} \exp(-t\xi)\cdot\theta = -\xi\cdot\theta.$$

   Consequently
   $$d_p\theta(Y_v, \tilde{\xi}) = \xi\cdot\theta(Y_v) = \xi\cdot v.$$

   It follows that
   $$A(p)(v \otimes \xi) = \xi(v).$$

2. $B : P \to Hom(\Lambda^2 \mathfrak{g}, \mathbb{R}^n)$

$$d_p\theta(\tilde{\xi}_1, \tilde{\xi}_2) = \tilde{\xi}_1\theta(\tilde{\xi}_2) - \tilde{\xi}_2\theta(\tilde{\xi}_1) - \theta([\tilde{\xi}_1, \tilde{\xi}_2]) = 0,$$

which implies $B(p) = 0$.

Until now, we have that

$$d\theta = C(\theta \wedge \theta) - \omega \wedge \theta,$$

with

$$(\omega \wedge \theta)(X, Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X).$$

In order to find C, we need to calculate $d\theta$ on two horizontal vectors. The operator

$$\begin{aligned} D_\omega : \Omega^k_{bas}(P, \mathbb{R}^n) &\to \Omega^{k+1}_{bas}(P, \mathbb{R}^n) \\ &\eta \mapsto D_\omega\eta(Y_0, \dots, Y_k) = d\eta(Y_0^{\mathcal{H}}, \dots, Y_k^{\mathcal{H}}), \end{aligned}$$

is an exterior derivative on the algebra of basic forms. Let ∇ be the connection induced by ω. Define $d_\nabla : \Omega^k(\mathcal{O}, T\mathcal{O}) \to \Omega^{k+1}(\mathcal{O}, T\mathcal{O})$ by

$$\begin{aligned} (d_\nabla\omega)(Y_0, \dots, Y_k) = &\sum_i (-1)^i \nabla_{Y_i}\omega(Y_0, \dots, \hat{Y}_i, \dots, Y_k) \\ &+ \sum_{i<j} (-1)^{i+j}\omega([Y_i, Y_j], Y_0, \dots, \hat{Y}_i, \dots, \hat{Y}_j, \dots, Y_k). \end{aligned}$$

Lemma 3.50. *The following diagram commutes*

$$\begin{tikzcd} \Omega^k_{bas}(P, \mathbb{R}^n) \arrow[r, "D_\omega"] \arrow[d] & \Omega^{k+1}_{bas}(P, \mathbb{R}^n) \arrow[d] \\ \Omega^k(\mathcal{O}, T\mathcal{O}) \arrow[r, "d_\nabla"] & \Omega^{k+1}(\mathcal{O}, T\mathcal{O}) \end{tikzcd}$$

Proof. Take $k = 0$, $s : \mathcal{O} \to T\mathcal{O}$ a cone field and $f_s : P \to \mathbb{R}^n$ defined by

$$f_s(p) = p^{-1}(s(\pi(p))).$$

Given that $f_s(p \cdot g) = g^{-1} \cdot f_s(p)$, then $f_s \in C^\infty(P, \mathbb{R}^n)^G$. Let $Y \in TP$, $X = d\pi(Y) \in \mathfrak{X}(\mathcal{O})$ and $h : \pi^*(T\mathcal{O}) \to TP$ the horizontal lift. It follows that

$$D_\omega f_s(Y) = df_s(Y^{\mathcal{H}}) = df_s(h(X)) = \nabla_X s.$$

On the other hand

$$(d_\nabla s)X = \nabla_X s.$$

Then, for $k = 0$, the diagram commutes. Both maps D_ω and d_∇ are $\mathbb{R}$-linear and satisfy Leibniz. Furthermore, $\Omega^k_{bas}(P, \mathbb{R}^n)$ is a $\Omega^\bullet(P)$-module and $\Omega^k(\mathcal{O}, T\mathcal{O})$ a $\Omega^\bullet(\mathcal{O})$-module. Hence, we can extend this diagram for an arbitrary $k > 0$. □

We have that

$$C(p)(u \wedge v) = D_\omega \theta(Y_u, Y_v) = d\theta(Y_u, Y_v).$$

The tautological form comes from the identity morphism

$$Id \in \Omega^1(\mathcal{O}, T\mathcal{O}).$$

Take $p(u) = d\pi(Y_u) = X_u$. The previous lemma tell us that taking $D_\omega \theta$ corresponds to

$$d_\nabla Id(p(u), p(v)) = \nabla_{p(u)} p(v) - \nabla_{p(u)} p(v) - [p(u), p(v)].$$

Definition 3.51. *The 2-form $T_\nabla \in \Omega^2(\mathcal{O}, T\mathcal{O})$ defined by*

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

is called the torsion of the connection ∇.

3. $C : P \to Hom(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n)$

$$C(p)(u \wedge v) = p^{-1} \left(T_\nabla \left(p(u), p(v) \right) \right),$$

and is called *the torsion of ω.*

The coefficient $C_{\mathcal{H}}$ depends on the choice of $\mathcal{H} = \ker \omega$. Take another connection $\mathcal{H}' = \ker \omega'$ with torsion $C_{\mathcal{H}'}$ and let $Y'_u = \theta^{-1}_{\mathcal{H}'}(u)$. Then

$$\begin{aligned} C_{\mathcal{H}'}(u,v) - C_{\mathcal{H}}(u,v) &= d\theta(Y'_u, Y'_v) - d\theta(Y_u, Y_v) \\ &= d\theta(Y'_u - Y_u, Y'_v) + d\theta(Y_u, Y'_v - Y_v). \end{aligned}$$

But $Y'_u - Y_u \in T^V P$ and $Y'_v - Y_v \in T^V P$. Using the homomorphism $S_{\mathcal{H}',\mathcal{H}}$, and the expression for coefficient A, we have that

$$C_{\mathcal{H}'}(u,v) - C_{\mathcal{H}}(u,v) = -S_{\mathcal{H}',\mathcal{H}}(u)(v) + S_{\mathcal{H}',\mathcal{H}}(v)(u).$$

Definition 3.52. *The linear model for the torsion*

$$\partial : Hom(\mathbb{R}^n, \mathfrak{g}) \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n),$$

is defined by

$$\partial S(u \wedge v) = S(u)(v) - S(v)(u).$$

It follows that the torsions are related by

$$C_{\mathcal{H}'_p} + \partial S_{\mathcal{H}'_p,\mathcal{H}_p} = C_{\mathcal{H}_p}. \tag{3.4.4}$$

A fixed connection $\mathcal{H} = \ker \omega$ induces the homomorphism $S_{\mathcal{H}',\mathcal{H}}$ that comes from the 1-1 correspondence

$$\begin{aligned} C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G &\longleftrightarrow Con(P) \\ \eta &\mapsto \omega'(Y) := \eta(\theta^{-1}_{\mathcal{H}}(Y)) + \omega(Y). \end{aligned}$$

The torsion corresponds to

$$\begin{aligned} C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G &\to P \times Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \\ \eta &\mapsto (p, \partial S_{\mathcal{H}'_p,\mathcal{H}_p}). \end{aligned}$$

Hence, the image of ∂ gives the torsion of all the possible compatible connections.

Definition 3.53. *The intrinsic torsion of P is denoted by*

$$\overline{C_P} : P \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)/Im\ \partial,$$

and defined by

$$\overline{C_P}(p) = [C_{\mathcal{H}_p}].$$

Theorem 3.54. *If $f : P \to Q$ is an equivalence of G-structures then $\overline{C_Q} \circ f = \overline{C_P}$.*

Proof. Let $\theta_P \in \Omega^1(P, \mathbb{R}^n)$, $\theta_Q \in \Omega^1(Q, \mathbb{R}^n)$ be the tautological forms and take a connection $\mathcal{H}_Q = \ker \omega_Q$. Its pullback $\omega_P := f^*\omega_Q$ is a connection $\mathcal{H}_P = \ker \omega_P$ too. Take the coframes induced by the connections ω_P and ω_Q. The differential of the tautological forms are

$$d\theta_P = C_P(\theta_P \wedge \theta_P) - \omega_P \wedge \theta_P,$$

and

$$d\theta_Q = C_Q(\theta_Q \wedge \theta_Q) - \omega_Q \wedge \theta_Q.$$

Besides, f is an equivalence, and then $f^* d\theta_Q = d\theta_P$, which implies

$$C_Q \circ f = C_P.$$

Take $\eta_Q \in C^\infty(Q, Hom(\mathbb{R}^n, \mathfrak{g}))^G$ and define

$$\eta_P := \eta_Q \circ f \in C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g})).$$

Given that

$$\eta_P(p \cdot g) = (\eta_Q \circ f)(p \cdot g) = \eta_Q(f(p) \cdot g) = g^{-1} \cdot \eta_P(p),$$

we have that $\eta_P \in C^\infty(P, Hom(\mathbb{R}^n, \mathfrak{g}))^G$. Thus

$$S^P_{\mathcal{H}'_p, \mathcal{H}_p} = S^Q_{\mathcal{H}'_{f(p)}, \mathcal{H}_{f(p)}}.$$

Consequently

$$(\overline{C_Q} \circ f)(p) = [(C_Q \circ f)(p)] = [C_P(p)] = \overline{C_P}(p).$$

□

Let us see what happens on the canonical G-structures $Fr(\mathbb{R}^n_{can})/G_p$. By proposition (3.32), the connection matrix $\omega_{can} \in \Omega^1(\mathbb{R}^n, \mathfrak{gl}_n)$ defined by

$$\omega_{can}\left(\left.\frac{d}{dt}\right|_{t=0} v + tu\right) = 0,$$

induces a connection compatible with the canonical G-structure. Let ∇_{can} be the connection that comes from ω_{can}. The torsion of ∇_{can} is

$$T_{\nabla_{can}} = 0.$$

Hence, the torsion $C_{can} : Fr(\mathbb{R}^n_{can})/G_p \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ of ω_{can} is

$$C_{can}(p)(u \wedge v) = p^{-1}(T_{\nabla_{can}}(p(u), p(v))) = 0.$$

Consequently, its intrinsic torsion

$$\overline{C_{can}} : Fr(\mathbb{R}^n_{can})/G_p \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)/Im\ \partial$$

satisfies

$$\overline{C_{can}} = 0.$$

Corollary 3.55. *If a G-structure P is integrable then $\overline{C_P} = 0$.*

Take $p \in P$ and fix a connection $\mathcal{H} = \ker \omega$. They give the bijection

$$\begin{aligned} Hom(\mathbb{R}^n, \mathfrak{g}) &\longleftrightarrow Con_p(P) \\ S_{\mathcal{H}_p,\mathcal{H}'_p} &\mapsto \omega'_p(\cdot) := S_{\mathcal{H}_p,\mathcal{H}'_p}(\theta_{\mathcal{H}_p}(\cdot)) + \omega_p(\cdot) \end{aligned}$$

By equation (3.4.4), if $C_{\mathcal{H}_p}$ is the torsion of ω and $C_{\mathcal{H}'_p}$ of ω', we have that

$$\begin{aligned} \partial : Hom(\mathbb{R}^n, \mathfrak{g}) &\to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \\ S_{\mathcal{H}_p,\mathcal{H}'_p} &\mapsto S_{\mathcal{H}_p,\mathcal{H}'_p}(\cdot)(*) - S_{\mathcal{H}_p,\mathcal{H}'_p}(*)(\cdot) \\ &= C_{\mathcal{H}'_p} - C_{\mathcal{H}_p}. \end{aligned}$$

Definition 3.56. *The first prolongation of $\mathfrak{g}$ is the Lie algebra*

$$\mathfrak{g}^{(1)} := \ker \partial.$$

Definition 3.57. *The torsion space of $\mathfrak{g}$ is given by*

$$\mathcal{T}(\mathfrak{g}) = Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)/Im\ \partial.$$

The torsion and intrinsic torsion information is codified on $\mathfrak{g}^{(1)}$ and $\mathcal{T}(\mathfrak{g})$.

Theorem 3.58. *Take a G-structure P.*

1. *If $\mathcal{T}(\mathfrak{g}) = 0$, then there exists a compatible connection ∇ with zero torsion.*

2. *If $\mathfrak{g}^{(1)} = 0$, then two compatible connections ∇_1 and ∇_2 with equal torsion are equal.*

Proof.

1. Condition $\mathcal{T}(\mathfrak{g}) = 0$ means ∂ is surjective. Take a connection $\omega \in \Omega^1(P, \mathfrak{g})$ with torsion $C_{\mathcal{H}}$. Given that ∂ is surjective, there exists $S_{\mathcal{H}_p,\mathcal{H}'_p} \in Hom(\mathbb{R}^n, \mathfrak{g})$ such that

$$\partial S_{\mathcal{H}_p,\mathcal{H}'_p} = -C_{\mathcal{H}_p}.$$

Take the connection ω' induced by the homomorphisms $S_{\mathcal{H}_p,\mathcal{H}'_p}$. Its torsion satisfies

$$C_{\mathcal{H}'_p} = \partial S_{\mathcal{H}_p,\mathcal{H}'_p} + C_{\mathcal{H}_p} = 0.$$

The connection ∇' induced by ω' is a compatible connection with zero torsion.

2. Take two connections $\omega, \omega' \in \Omega^1(P, \mathfrak{g})$ with

$$\omega' = S_{\mathcal{H},\mathcal{H}'}(\theta_{\mathcal{H}}(\cdot)) + \omega(\cdot).$$

Let ∇, ∇' be the affine compatible connections induced by the connection 1-forms. Condition $\mathfrak{g}^{(1)} = 0$ means $\ker \partial = 0$. Moreover, as long as $C_{\mathcal{H}} = C_{\mathcal{H}'}$, we have that

$$0 = C_{\mathcal{H}'_p} - C_{\mathcal{H}_p} = \partial S_{\mathcal{H}_p,\mathcal{H}'_p}.$$

Then $S_{\mathcal{H}_p,\mathcal{H}'_p} = 0$ for all $p \in P$. It follows that $\omega = \omega'$, which implies $\nabla = \nabla'$.

□

3.4.3 $O(n)$-structures

Theorem 3.59. ***Fundamental theorem of Riemannian geometry*** *Every Riemannian structure over an effective orbifold admits a unique compatible connection with zero torsion.*

Proof. By theorem 3.58, if $\mathcal{T}(\mathfrak{o}(n)) = 0$ and $\mathfrak{o}(n)^{(1)} = 0$, then there exists a unique compatible connection with zero torsion. Take

$$\partial : Hom(\mathbb{R}^n, \mathfrak{o}(n)) \to Hom(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n).$$

1. $\mathfrak{o}(n)^{(1)} = 0$.
   Take $\phi \in \mathfrak{o}(n)^{(1)}$, i.e., $\phi \in \ker \partial$. Hence, for all $u, v \in \mathbb{R}^n$

   $$\phi(u)(v) = \phi(v)(u).$$

   Besides, as long as $\phi(\cdot) \in \mathfrak{o}(n)$, we have that

   $$\langle \phi(u)v, w \rangle_{can} = -\langle v, \phi(u)w \rangle_{can}.$$

   Consequently

   $$\begin{aligned}
   \langle \phi(u)v, w \rangle_{can} &= -\langle v, \phi(u)w \rangle_{can} \\
   &= -\langle v, \phi(w)u \rangle_{can} \\
   &= \langle \phi(w)u, v \rangle_{can} \\
   &= \langle \phi(u)w, v \rangle_{can} \\
   &= -\langle w, \phi(u)v \rangle_{can},
   \end{aligned}$$

   and then $\langle \phi(u)v, w \rangle_{can} = 0$ for all $u, v, w \in \mathbb{R}^n$. It follows that $\phi = 0$.

2. $\mathcal{T}(\mathfrak{o}(n)) = 0$
   The previous item shows that ∂ is injective. Given that

   $$\dim(Hom(\mathbb{R}^n, \mathfrak{o}(n))) = \frac{n^2(n-1)}{2} = \dim Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n),$$

   $\partial : Hom(\mathbb{R}^n, \mathfrak{o}(n)) \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ is an isomorphism.

$\square$

3.4.4 $Sp_{2k}(\mathbb{R})$-structures

For simplicity, take $n = 2k$.

Lemma 3.60. *The sequence*

$$\Sigma^3((\mathbb{R}^n)^*) \xrightarrow{\iota_w} Hom(\mathbb{R}^n, \mathfrak{sl}_n) \xrightarrow{\partial} Hom(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) \xrightarrow{\partial_\omega} \Lambda^3((\mathbb{R}^n)^*)$$

is an exact sequence. In particular

$$\mathfrak{sl}_n^{(1)} \cong \Sigma^3((\mathbb{R}^n)^*),$$

and

$$\mathcal{T}(\mathfrak{sp}_n) \cong \Lambda^3((\mathbb{R}^n)^*).$$

Proof. 1. $\iota_w : \Sigma^3((\mathbb{R}^n)^*) \to Hom(\mathbb{R}^n, \mathfrak{sl}_n)$.

Let $\phi \in \mathfrak{sp}_n^{(1)}$. For all $v, u \in \mathbb{R}^n$

$$\phi(u)(v) = \phi(v)(u).$$

Besides

$$\omega_{can}(\phi(u)v, w) = -\omega_{can}(v, \phi(u)w),$$

and then, if $\sigma(u, v, w) = \omega_{can}(\phi(u)v, w)$, we have that $\sigma \in \Sigma^3(\mathbb{R}^n)$. Given that ω_{can} is non-degenerate, the relation

$$\sigma(u, v, w) = \omega_{can}(\phi(u)v, w),$$

is a bilateral relation between ϕ and σ. That gives us the isomorphism

$$\begin{aligned} \iota_w : \Sigma^3(\mathbb{R}^n) &\to \mathfrak{sp}_n^{(1)} \\ \sigma &\mapsto \phi. \end{aligned}$$

2. $\partial_\omega : Hom(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) \to \Lambda^3((\mathbb{R}^n)^*)$.

Take an homomorphism $\phi \in Hom(\mathbb{R}^n, \mathfrak{sp}_n)$ and let $\Phi \in Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ be

$$\partial\phi = \Phi.$$

We have that

$$\omega_{can}(\Phi(u, v), w) = \omega_{can}(\phi(u)v, w) - \omega_{can}(\phi(v)u, w),$$

which implies

$$\begin{aligned}
&\omega_{can}(\Phi(u,v),w)+\omega_{can}(\Phi(w,u),v)\\
=&\omega_{can}(\phi(w)u,v)-\omega_{can}(\phi(v)u,w)\\
=&\omega_{can}(\phi(w)v,u)-\omega_{can}(\phi(v)w,u)\\
=&-\omega_{can}(\Phi(v,w),u).
\end{aligned}$$

Define $\partial_\omega : Hom(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)\to\Lambda^3((\mathbb{R}^n)^*)$ by

$$\partial_\omega(\Phi):=\omega_{can}(\Phi(u,v),w)+\omega_{can}(\Phi(w,u),v)+\omega_{can}(\Phi(v,w),u).$$

It follows that $\partial(Hom(\mathbb{R}^n,\mathfrak{sp}_n))\subset\ker\partial_\omega$. It is a surjective map because, as long as ω_{can} is non-degenerate, for all $\eta\in\Lambda^3((\mathbb{R}^n)^*)$ there exists $T_\eta\in Hom(\Lambda^2\mathbb{R}^n,\mathbb{R}^n)$ such that

$$\omega_{can}(T_\eta(u,v),w)=\frac{1}{3}\eta(u,v,w).$$

Consequently

$$\begin{aligned}
&\partial_\omega(T_\eta)(u,v,w)\\
=&\omega_{can}(T_\eta(u,v),w)+\omega_{can}(T_\eta(w,u),v)+\omega_{can}(T_\eta(v,w),u)\\
=&\frac{1}{3}\left(\eta(u,v,w)+\eta(w,u,v)+\eta(v,w,u)\right)\\
=&\eta(u,v,w).
\end{aligned}$$

We already show $\partial(Hom(\mathbb{R}^n,\mathfrak{sp}_n))\subset\ker\partial_\omega$. Their equality follows because they have equal dimensions. For, firstly

$$\begin{aligned}
\dim\partial(Hom(\mathbb{R}^n,\mathfrak{sp}_n))&=\dim Hom(\mathbb{R}^n,\mathfrak{sp}_n)-\dim\ker\partial\\
&=\dim Hom(\mathbb{R}^n,\mathfrak{sp}_n)-\dim\Sigma^3((\mathbb{R}^n)^*),
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{sp}_n&\overset{\cong}{\to}\Sigma^2((\mathbb{R}^n)^*)\\
\xi&\mapsto\omega_{can}(\xi(u),v).
\end{aligned}$$

Then

$$\begin{aligned}\dim \partial(Hom(\mathbb{R}^n, \mathfrak{sp}_n)) &= n \cdot \frac{n(n+1)}{2} - \frac{n(n+1)(n+2)}{6} \\ &= \frac{n(n^2-1)}{3}.\end{aligned}$$

On the other hand

$$\begin{aligned}\dim \ker \partial_\omega &= \dim Hom(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) - \dim \Lambda^3((\mathbb{R}^n)^*) \\ &= \frac{n(n-1)}{2} \cdot n - \frac{n(n-1)(n-2)}{6} \\ &= \frac{n(n^2-1)}{3}.\end{aligned}$$

It follows that $\partial(Hom(\mathbb{R}^n, \mathfrak{sp}_n)) = \ker \partial_\omega$.

□

Hence, if an $Sp_n(\mathbb{R})$-structure is integrable, then there exists a connection $\mathcal{H} = \ker \omega$, with torsion $C_\mathcal{H}$, such that

$$\partial_\omega(C_\mathcal{H}) = 0.$$

Besides

$$\begin{aligned}\varphi : E(P, Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)) &\to \Omega^2(\mathcal{O}, T\mathcal{O}) \\ [p, \Phi] &\mapsto p\left(\Phi\left(p^{-1}(X) \wedge p^{-1}(Y)\right)\right),\end{aligned}$$

is an isomorphism such that $\varphi([p, C_{\mathcal{H}_p}]) = T_\nabla$, with ∇ the connection induced by $\mathcal{H}$. Also

$$\begin{aligned}E(P, \Lambda^3((\mathbb{R}^n)^*)) &\stackrel{\cong}{\to} \Omega^3(\mathcal{O}) \\ [p, \eta] &\mapsto \eta\left(p^{-1}(X) \wedge p^{-1}(Y) \wedge p^{-1}(Z)\right),\end{aligned}$$

are isomorphic. The map ∂_ω descends to

$$\begin{aligned}\partial_\omega : E(P, Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)) &\to E(P, \Lambda^3((\mathbb{R}^n)^*)) \\ [p, \Phi] &\mapsto [p, \partial_\omega(\Phi)].\end{aligned}$$

The induced map $\partial_\omega : \Omega^2(\mathcal{O}, T\mathcal{O}) \to \Omega^3(\mathcal{O})$ is

$$\partial_\omega(T_\nabla)(X,Y,Z) = \omega(T_\nabla(X,Y),Z) + \omega(T_\nabla(Z,X),Y) + \omega(T_\nabla(Y,Z),X).$$

Thus, if the $Sp_n(\mathbb{R})$-structure is integrable, there exists a compatible affine connection ∇ such that

$$\partial_\omega(T_\nabla) = 0.$$

We have

$$\begin{aligned}\partial_\omega(T_\nabla)(X,Y,Z) = {} & \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) - \omega([X,Y],Z)\\ & + \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y) - \omega([Z,X],Y)\\ & + \omega(\nabla_Y Z, X) + \omega(Z, \nabla_Y X) - \omega([Y,Z],X).\end{aligned}$$

Given that ∇ is a compatible connection we get

$$\begin{aligned}\partial_\omega(T_\nabla)(X,Y,Z) = {} & X\omega(Y,Z) - \omega([X,Y],Z)\\ & + Z\omega(X,Y) - \omega([Z,X],Y)\\ & + Y\omega(Z,X) - \omega([Y,Z],X).\end{aligned}$$

On the other hand

$$\begin{aligned}d\omega(X,Y,Z) = {} & X\omega(Y,Z) - Y\omega(X,Z) + Z\omega(X,Y)\\ & - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X),\end{aligned}$$

and then

$$\partial_\omega(T_\nabla) = d\omega.$$

Theorem 3.61. *If an almost symplectic structure $\omega \in \Omega^2(\mathcal{O})$ is integrable then $d\omega = 0$.*

3.4.5 $GL_{k,n-k}(\mathbb{R})$-structures

Lemma 3.62. *The sequence*

$$\begin{aligned} 0 \to \mathfrak{gl}^{(1)}_{k,n-k} \to Hom(\mathbb{R}^n, \mathfrak{gl}_{k,n-k}) \xrightarrow{\partial} Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \\ \xrightarrow{\partial_D} Hom(\Lambda^2\mathbb{R}^k, \mathbb{R}^n/\mathbb{R}^k) \to 0,\end{aligned}$$

is an exact sequence.

Proof. 1. $\partial_D : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \to Hom(\Lambda^2\mathbb{R}^k, \mathbb{R}^n/\mathbb{R}^k)$.

Take $\phi \in Hom(\mathbb{R}^n, \mathfrak{gl}_{k,n-k})$. Then, for all $u \in \mathbb{R}^n$, we have that

$$\phi(u)(\mathbb{R}^k) \subset \mathbb{R}^k.$$

If $\Phi = \partial\phi$, then for all $u, v \in \mathbb{R}^k$ we get

$$\Phi(u,v) = \phi(u,v) - \phi(v,u) \in \mathbb{R}^k.$$

Define

$$\begin{aligned} \partial_D : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) &\to Hom(\Lambda^2\mathbb{R}^k, \mathbb{R}^n/\mathbb{R}^k) \\ \Phi &\mapsto \Phi|_{\mathbb{R}^k} \bmod \mathbb{R}^k, \end{aligned}$$

it follows that $\partial_D \circ \partial = 0$ and then $Im\ \partial \subset \ker \partial_D$. For the other inclusion, take $\Phi \in \ker \partial_D$. Given that $\Phi(u,\cdot) : \mathbb{R}^k \to \mathbb{R}^k$, the matrix induced by $\Phi(u,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} | & | & & | \\ \Phi(u,e_1) & \Phi(u,e_2) & \dots & \Phi(u,e_n) \\ | & | & & | \end{pmatrix}$$

belongs to $\mathfrak{gl}_{k,n-k}$. Define $\phi \in Hom(\mathbb{R}^n, \mathfrak{gl}_{k,n-k}$ by

$$\phi(u) = \frac{1}{2}\Phi(u,\cdot).$$

We have that

$$\begin{aligned} \partial\phi(u,v) &= \phi(u,v) - \phi(v,u) \\ &= \frac{1}{2}\left(\Phi(u,v) - \Phi(v,u)\right) \\ &= \Phi(u,v), \end{aligned}$$

which implies $\ker \partial_D = Im\ \partial$. Clearly ∂_D is a surjective map.

$\square$

It follows that

$$\mathcal{T}(\mathfrak{gl}_{k,n-k}) \cong Hom(\Lambda^2\mathbb{R}^k, \mathbb{R}^n/\mathbb{R}^k),$$

and if a $GL_{k,n-k}(\mathbb{R})$-structure is integrable, then there exists a connection $\mathcal{H} = \ker\omega$ such that

$$\partial_D(C_{\mathcal{H}}) = 0.$$

Let $D \subset T\mathcal{O}$ be the distribution induced by the $GL_{k,n-k}(\mathbb{R})$-structure P. By similar arguments as the ones used on almost symplectic structures, the map ∂_D descends to

$$\begin{aligned}\partial_D : \Omega^2(\mathcal{O}, T\mathcal{O}) &\to \Omega^2(D, T\mathcal{O}/D)\\ T_\nabla &\mapsto T_\nabla|_D \bmod D.\end{aligned}$$

If P is integrable, then there exists a compatible affine connection ∇ such that

$$\partial_D(T_\nabla) = 0.$$

Explicitly, if $X, Y \in D$, then

$$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0 \bmod D,$$

if and only if

$$\nabla_X Y - \nabla_Y X - [X,Y] \in D.$$

Given that ∇ is a compatible connection and $X, Y \in D$, we have that

$$\nabla_X Y \in D \quad \text{and} \quad \nabla_Y X \in D.$$

Then, $\partial_D(T_\nabla) = 0$ if and only if every $X, Y \in D$ satisfies

$$[X,Y] \in D.$$

Definition 3.63. *A distribution $D \subset T\mathcal{O}$ is called involutive if*

$$[Sec(D), Sec(D)] \in Sec(D).$$

Theorem 3.64. *If a distribution $D \subset T\mathcal{O}$ is integrable, then it is involutive.*

3.4.6 $GL_k(\mathbb{C})$-structures

Let $n = 2k$.

Lemma 3.65. *The sequence*

$$Hom(\mathbb{R}^n, \mathfrak{gl}_k(\mathbb{C})) \xrightarrow{\partial} Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{N_J} Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \xrightarrow{\partial_J} Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$$

is an exact sequence.

Proof. 1. $N_J : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$.

Let $\phi \in Hom(\mathbb{R}^n, \mathfrak{gl}_k(\mathbb{C}))$. It can be though of as an element of $\mathbb{R}^* \otimes \mathbb{R}^*$ by

$$\phi(u, v) := \phi(u)(v).$$

Given that $\phi(\cdot) \in \mathfrak{gl}_k(\mathbb{C})$, then

$$\phi(u, Jv) = J\phi(u, v). \tag{3.4.5}$$

Take $\Phi \in Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ defined by $\Phi = \partial\phi$. We have that

$$\begin{aligned}\Phi(u, v) &= \phi(u, v) - \phi(v, u)\\ &= -J(\phi(u, Jv) - \phi(v, Ju)),\end{aligned}$$

and

$$\phi(u, Jv) - \phi(v, Ju) = \Phi(u, Jv) + \phi(Jv, u) - \Phi(v, Ju) - \phi(Ju, v).$$

Then

$$\begin{aligned}\Phi(u, v) &= -J\Phi(u, Jv) + J\Phi(v, Ju) + \phi(Ju, Jv) - \phi(Jv, Ju)\\ &= -J\Phi(Ju, v) - J\Phi(u, Jv) + \Phi(Ju, Jv).\end{aligned}$$

Consequently, if we define

$$\begin{aligned}N_J : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) &\to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)\\ \Phi &\mapsto \Phi(u, v) + J\Phi(Ju, v) + J\Phi(u, Jv) - \Phi(Ju, Jv),\end{aligned}$$

we have that $N_J \circ \partial = 0$. If $\ker N_J \subset Im\ \partial$, it follows that the sequence is exact on N_J. For that, take $\Phi \in \ker N_J$. We want an element $\phi \in Hom(\mathbb{R}^n, \mathfrak{gl}_k(\mathbb{C}))$ such that

$$\Phi(u,v) = \phi(u,v) - \phi(v,u). \tag{3.4.6}$$

The way we obtain N_J from ϕ uses the homomorphisms

$$\{\Phi(u,v), J\Phi(Ju,v), J\Phi(u,Jv), \Phi(Ju,Jv)\},$$

where Φ was $\partial\phi$. Hence, it is fair to ask for ϕ to be a linear combination of these homomorphisms, which means

$$\phi(u,v) = a\Phi(u,v) + bJ\Phi(Ju,v) + cJ\Phi(u,Jv), \tag{3.4.7}$$

with $a, b, c \in \mathbb{R}^n$ (the element $\Phi(Ju,Jv)$ does not appear since $\Phi \in \ker N_J$). Replacing the equation (3.4.7) on equation (3.4.6) we get

$$(2a-1)\Phi(u,v) + (b+c)\left(J\Phi(Ju,v) + J\Phi(u,Jv)\right) = 0.$$

Moreover, ϕ must satisfy equation (3.4.5), replacing (3.4.7) we have

$$(a-b+c)J\Phi(u,v) - (a-b+c)\Phi(u,Jv) = 0.$$

Thus, $a = \dfrac{1}{2}$, $b = \dfrac{1}{4}$ and $c = -\dfrac{1}{4}$. It follows that $Im\ \partial = \ker N_J$.

2. $\partial_J : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$.

   Take $\eta \in Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ equals to $\eta = N_J(\Phi)$. It satisfies

$$\eta(Ju,v) = \Phi(Ju,v) - J\Phi(u,v) + J\Phi(Ju,Jv) + \Phi(u,Jv),$$

and

$$\eta(u,Jv) = \Phi(u,Jv) + J\Phi(Ju,Jv) - J\Phi(u,v) + \Phi(Ju,v).$$

Then

$$\eta(Ju,v) + \eta(u,Jv) + 2J\eta(u,v) = 0.$$

Define

$$\begin{aligned}\partial_J : Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) &\to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)\\ \eta &\mapsto \eta(Ju, v) + \eta(u, Jv) + 2J\eta(u, v),\end{aligned}$$

we have that $\partial_J \circ N_J = 0$, and then $Im\ N_J \subset \ker \partial_J$. Let $Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) := \{\eta \mid \eta(Ju, v) = \eta(u, Jv) = -J\eta(u, v)\}.$$

We will show that $\ker \partial_J = Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$.

- $Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \subset \ker \partial_J$.

  If $\eta \in Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ then

  $$\begin{aligned}\partial_J(\eta)(u, v) &= \eta(Ju, v) + \eta(u, Jv) + 2J\eta(u, v)\\ &= -2J\eta(u, v) + 2J\eta(u, v) = 0.\end{aligned}$$

- $\ker \partial_J \subset Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$.

  If $\eta \in \ker \partial_J$ then

  $$\eta(Ju, v) + \eta(u, Jv) + 2J\eta(u, v) = 0. \tag{3.4.8}$$

  Hence, replacing u by Ju we get

  $$-\eta(u, v) + \eta(Ju, Jv) + 2J\eta(Ju, v) = 0,$$

  and replacing v by Jv

  $$\eta(Ju, Jv) - \eta(u, v) + 2J\eta(u, Jv) = 0.$$

  Subtracting the two equations we get

  $$\eta(Ju, v) = \eta(u, Jv).$$

  Replacing this expression on equation (3.4.8) we obtain

  $$\eta(Ju, v) = -J\eta(u, v),$$

  and then $\eta \in Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$.

The relation $\ker \partial_J \subset Im\ N_J$ stands because if $\eta \in Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$, then

$$\begin{aligned} N_J\left(\frac{1}{2}\eta\right)(u,v) &= \frac{1}{2}\left(\eta(u,v) + J\eta(Ju,v) + J\eta(u,Jv) - \eta(Ju,Jv)\right) \\ &= \eta(u,v). \end{aligned}$$

□

By the lemma

$$\begin{aligned} Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)/Im\partial &\xrightarrow{\cong} Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n) \\ [\Phi] &\mapsto N_J(\Phi), \end{aligned}$$

and then

$$\mathcal{T}(\mathfrak{gl}_k(\mathbb{C})) \cong Hom_J(\Lambda^2\mathbb{R}^n, \mathbb{R}^n).$$

It follows that if a $GL_k(\mathbb{C})$-structure P is integrable, then there exists a connection $\mathcal{H} = \ker \omega$ such that

$$N_J(C_{\mathcal{H}}) = 0.$$

By the same arguments used on $Sp_k(\mathbb{R})$ structures, we have an induced map

$$\begin{aligned} N_J : \Omega^2(\mathcal{O}, T\mathcal{O}) &\to \Omega^2(\mathcal{O}, T\mathcal{O}) \\ T_\nabla &\mapsto T_\nabla(X,Y) + JT_\nabla(JX,Y) + JT_\nabla(X,JY) - T_\nabla(JX,JY). \end{aligned}$$

Then, if P is integrable, there exists a compatible affine connection ∇ such that

$$N_J(T_\nabla) = 0.$$

Explicitly, using the compatibility of ∇, we obtain

$$\begin{aligned} N_J(T_\nabla)(X,Y) &= T_\nabla(X,Y) + JT_\nabla(JX,Y) + JT_\nabla(X,JY) - T_\nabla(JX,JY) \\ &= -[X,Y] - J([JX,Y] + [X,JY]) + [JX,JY]. \end{aligned}$$

Definition 3.66. *The Nijenhuis tensor is the* 2*-form* $N_J \in \Omega^2(\mathcal{O}, T\mathcal{O})$ *defined by*

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY].$$

Theorem 3.67. *If an almost complex structure* $J \in Hom(T\mathcal{O}, T\mathcal{O})$ *is integrable, its Nijenhuis tensor* $N_J = 0$ *vanishes.*

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