Sign Manifesto

Pierre Deligne and Daniel S. Freed

§1. Standard mathematical conventions

- *We apply the sign rule relentlessly.*

This means that when passing from ordinary algebra to $\mathbb{Z}/2$-graded, or super, algebra we pick up a sign $(-1)^{|a|\,|b|}$ when permuting homogeneous elements a, b of parity $|a|, |b|$. Structure maps (multiplications, Lie brackets, inner products, ...) are even.

For example, consider a graded complex vector space $V = V^0 \oplus V^1$. A hermitian inner product $\langle \cdot, \cdot \rangle$ satisfies, among other properties,

$$\langle v_1, v_2 \rangle = (-1)^{|v_1|\,|v_2|}\overline{\langle v_2, v_1 \rangle}, \qquad v_1, v_2 \in V \text{ homogeneous.} \tag{1}$$

and from the evenness of the inner product it follows that V^0 is orthogonal to V^1. From (1) we deduce that $\langle v, v \rangle$ is real for v even and pure imaginary for v odd. The adjoint T^* of a homogeneous linear operator $T\colon V \to V$ is characterized by

$$\langle Tv_1, v_2 \rangle = (-1)^{|T|\,|v_1|}\langle v_1, T^* v_2 \rangle. \tag{2}$$

Skew-adjoint operators form a super Lie algebra.

- *Symmetry groups act on the left.*

For example, if $\mathfrak{g}$ is a Lie algebra, then an action of $\mathfrak{g}$ on a vector space V is a *homomorphism* $\mathfrak{g} \to \mathrm{End}(V)$. Brackets are preserved. On the other hand, an action of $\mathfrak{g}$ on a manifold M is an *antihomomorphism* $\mathfrak{g} \to \mathrm{VectorFields}(M)$. The reversal of sign comes from the rule $\xi f = \frac{d}{dt}\exp(t\xi)^* f$ and the fact that $f \mapsto u^* f$ is a *right* action of diffeomorphisms (u) on functions (f).

§2. Choices

- *A hermitian inner product on a complex vector space V is conjugate linear in the first variable:*

$$\langle \lambda_1 v_1, \lambda_2 v_2 \rangle = \overline{\lambda_1}\lambda_2 \langle v_1, v_2 \rangle, \qquad \lambda_i \in \mathbb{C}, \quad v_i \in V. \tag{3}$$

- *If $V = V^0 \oplus V^1$ is a super Hilbert space, then*

$$-i\langle v, v \rangle \geq 0, \qquad v \in V^1. \tag{4}$$

- *We pass from self-adjoint operators to skew-adjoint operators using multiplication by $-i$:*

(5) $$T \text{ self-adjoint} \longleftrightarrow -iT \text{ skew-adjoint.}$$

- *The Lorentz metric g on n dimensional Minkowski space has signature $(1, n-1)$:*

(6) $$\text{Signature}(g) = + - - - \cdots$$

- *The quantum hamiltonian $\hat{H}$ is minus the operator which corresponds to infinitesimal time translation:*

(7) $$\hat{H} = -\hat{P}_0.$$

§3. Rationale

The first choice (3) is not the usual one in mathematics, but it has its merits. For example, since linear operators act on the left, it makes sense to have the commuting scalar multiplication act on the right. In mathematics we do follow this convention for modules over noncommutative rings. With right scalar multiplication (3) reads: $\langle v_1\lambda_1, v_2\lambda_2\rangle = \overline{\lambda_1}\langle v_1, v_2\rangle\lambda_2$. We do not adopt this convention for scalar multiplication, but do adopt (3). Physicists like (3) in view of Dirac's notation '$\langle v_1|T|v_2\rangle$' for '$\langle v_1, Tv_2\rangle$'. Comment: In computations it is often more convenient and safer to work with a bilinear form rather than a sesquilinear form, and so to write the sesquilinear inner product as '$\langle \overline{v_1}, v_2\rangle$'.

From a mathematical point of view it is more natural to quantize with *skew*-adjoint operators, since they form a Lie algebra. We use (5) to convert to *self*-adjoint operators, whose real eigenvalues correspond to physical measurements.

The sign choice in (6) leads to the usual bosonic lagrangian (20) with a plus sign in front of the kinetic energy.

In (7) we assume that time translation is a symmetry of a quantum theory, so that the infinitesimal generator is represented by a self-adjoint operator $\hat{P}_0$ on the quantum (super)Hilbert space. The minus sign gives the standard answer for the hamiltonian of a classical free particle.

§4. Notation

Throughout $i = \sqrt{-1}$.

Let M denote n dimensional affine Minkowski space with associated vector space of translations V and future timelike cone $C \subset V$. We fix linear coordinates $x^0, \dots, x^{n-1}$ with respect to which the metric is

(8) $$g = g_{\mu\nu}\, dx^\mu \otimes dx^n = (dx^0)^2 - \cdots - (dx^{n-1})^2,$$

and the cone is

(9) $$C = \{x : \langle x, x\rangle \geq 0 \text{ and } x^0 \geq 0\}.$$

Let $\{e_\mu\}$ be the corresponding basis of V and ∂_μ the corresponding vector field on M. The standard density on M is

(10) $$|d^n x| = |dx^0 \dots dx^{n-1}|.$$

Let S be a real spin representation. Fix a basis $\{f^a\}$ of S and dual basis $\{f_a\}$ of S^*. Then there are symmetric pairings

$$\begin{aligned} \Gamma&\colon S^* \otimes S^* \longrightarrow V \\ \tilde{\Gamma}&\colon S \otimes S \longrightarrow V. \end{aligned} \tag{11}$$

We write

$$\begin{aligned} \Gamma(f_a, f_b) &= \Gamma^{\mu}_{ab} e_{\mu}, \\ \tilde{\Gamma}(f^a, f^b) &= \tilde{\Gamma}^{\mu ab} e_{\mu}, \end{aligned} \tag{12}$$

where as usual we sum over repeated indices if one is upstairs and the other is downstairs. We raise and lower indices using the metric. The pairings (11) are assumed to satisfy the Clifford relation

$$\tilde{\Gamma}^{\mu ab}\Gamma^{\nu}_{bc} + \tilde{\Gamma}^{\nu ab}\Gamma^{\mu}_{bc} = 2g^{\mu\nu}\delta^a_c \tag{13}$$

and the positivity condition

$$\Gamma(s^*, s^*) \in C \qquad \text{for all } s^* \in S^*. \tag{14}$$

For $v \in C^o$, the form $\langle v, \Gamma(s^*, s^*)\rangle$ is then positive definite. From (13) and (14) it follows that

$$\tilde{\Gamma}(s, s) \in C \qquad \text{for all } s \in S. \tag{15}$$

In a classical field theory we work with a space of fields $\mathcal{F}$, where $f \in \mathcal{F}$ is some sort of function on M. An infinitesimal symmetry is a vector field ξ on $\mathcal{F}$ which preserves the lagrangian in a certain sense. Corresponding to ξ is a Noether current J_ξ, which is a twisted $(n-1)$-form on M. The Noether charge Q_ξ is the integral of J_ξ over a time slice. We usually consider the current and charge only on the space of classical solutions $\mathcal{M}$, which carries a closed 2-form ω. The infinitesimal symmetry and Noether charge are related by

$$dQ_\xi = -\iota(\xi)\omega. \tag{16}$$

For $(\mathcal{M}, \omega)$ symplectic, (16) can be rewritten

$$\xi f = \{Q_\xi, f\} \tag{17}$$

for f a function on $\mathcal{M}$.

Quantization is, in principle, a map

$$Q \longmapsto \hat{Q} \tag{18}$$

from functions on $\mathcal{M}$ to operators on a complex Hilbert space $\mathcal{H}$. We assume

$$\hat{\bar{Q}} = \hat{Q}^*, \tag{19}$$

so that real functions map to self-adjoint operators.

Let P_μ be infinitesimal translation in the Poincaré algebra and Q_a the odd generator of the supersymmetry algebra.[1] Let $\hat{P}_\mu, \hat{Q}_a$ be the corresponding quantum operators.

[1] The notational conflict between the supersymmetry generator and the Noether charge is too ingrained to correct.

§5. Consequences of §2 on other signs

- *The kinetic lagrangian for a scalar field* $\phi\colon M \to \mathbb{R}$ *is*

$$L = \frac{1}{2}|d\phi|^2\,|d^n x| = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,|d^n x|. \tag{20}$$

The sign of this term is a consequence of (6); it is the main rationale for preferring (6) over the other choice.

- *Suppose* $V = V^0 \oplus V^1$ *is a graded hermitian vector space and* T *an odd skew-adjoint operator. Then*

$$i[T,T] \geq 0. \tag{21}$$

Observe from (2), (3) and (4) that an odd skew-adjoint operator has eigenvalues on the line $i^{-1/2}\mathbb{R} \subset \mathbb{C}$.

- *The bracket in the supersymmetry algebra is*

$$[Q_a, Q_b] = -2\Gamma^{\mu}_{ab}P_\mu. \tag{22}$$

Because we use left group actions, upon quantization we expect a homomorphism from the supersymmetry algebra to *skew*-adjoint operators. Using (5) we see that the sign in (22) leads to

$$[-i\hat{Q}_a, -i\hat{Q}_b] = -2\Gamma^{\mu}_{ab}(-i\hat{P}_\mu). \tag{23}$$

Setting $a = b$ we see from (21) that $-2\Gamma^{\mu}_{aa}\hat{P}_\mu > 0$ for all a. From (14) we see that $\Gamma^{\mu}_{aa}P_\mu$ has nonnegative norm in V. Except possibly in dimension 2, the positive cone generated by $\{\Gamma^{\mu}_{aa}P_\mu\}_a$ includes P_0, and so the sign choice in (22) renders the hamiltonian nonnegative (rather than nonpositive), in view of (7).

- *The vector field* $\hat{\xi}_{P_\mu}$ *on* $\mathcal{F}$ *corresponding to infinitesimal translation* P_μ *is*

$$\hat{\xi}_{P_\mu}f = -\partial_\mu f, \qquad f \in \mathcal{F}. \tag{24}$$

This follows since a diffeomorphism $\varphi\colon M \to M$ acts on functions by $(\varphi^{-1})^*$.

- *If* $\mathfrak{g}$ *is a Lie algebra of infinitesimal symmetries, then the vector fields* ξ_λ $(\lambda \in \mathfrak{g})$ *on the space of fields* $\mathcal{F}$ *satisfy*

$$[\xi_{\lambda_1}, \xi_{\lambda_2}] = -\xi_{[\lambda_1,\lambda_2]}. \tag{25}$$

- *The Noether currents satisfy*[2]

$$\{j_{\lambda_1}, j_{\lambda_2}\} = -j_{[\lambda_1,\lambda_2]}. \tag{26}$$

- *The Noether charges satisfy*

$$\{Q_{\lambda_1}, Q_{\lambda_2}\} = -Q_{[\lambda_1,\lambda_2]}. \tag{27}$$

- *The quantum operators satisfy*

$$[-i\hat{Q}_{\lambda_1}, -i\hat{Q}_{\lambda_2}] = -i\hat{Q}_{[\lambda_1,\lambda_2]}. \tag{28}$$

[2] The bracketing operation on Noether currents is defined in [**I-Classical Fields**, §2.6].

Equations (25)–(27) follow from the fact that $\mathfrak{g} \to \text{VectorFields}(\mathcal{F})$ is an antihomorphism and the standard equations for Poisson brackets which follow from (16). Equation (28) says that $\mathfrak{g} \to \text{End}(\mathcal{H})$ is a homomorphism to skew-adjoint operators, where we use (5).

- *The self-adjoint quantum operators $\hat{Q}_1, \hat{Q}_2$ which correspond to classical functions Q_1, Q_2 on $\mathcal{M}$ satisfy*

(29) $$[\hat{Q}_1, \hat{Q}_2] = -i\hbar\{Q_1, Q_2\}\hat{} \quad \textit{modulo } O(\hbar^2).$$

To the extent that (29) holds exactly, it says that the map

(30) $$Q \longmapsto \frac{-i}{\hbar}\hat{Q}$$

to skew-adjoint operators is an antihomorphism. The sign in (30) is dictated by (5). The desire to have an *anti*homomorphism is dictated by (27) and (28), and this determines the sign in (29).

- *The Schrödinger equation for the evolution of a state ψ is*

(31) $$\frac{\partial\psi}{\partial t} = \frac{-i}{\hbar}\hat{H}\psi.$$

Evolution through time t for a static hamiltonian $\hat{H}$ is $e^{-it\hat{H}/\hbar}$.

The sign follows from (7) and (5).

- *Let η_1^a, η_2^b be odd parameters and $\hat{\xi}_i$ the even vector field on $\mathcal{F}$ corresponding to $\eta_i^a Q_a$. Then*

(32) $$[\hat{\xi}_1, \hat{\xi}_2]f = 2\eta_1^a\eta_2^b\Gamma^\mu_{ab}\partial_\mu f, \qquad f \in \mathcal{F}.$$

To see this, observe from (22) that in the abstract supersymmetry algebra we have

(33) $$[\eta_1^a Q_a, \eta_2^b Q_b] = 2\eta_1^a\eta_2^b\Gamma^\mu_{ab}P_\mu.$$

Then (32) follows from (25) and (24); the minus signs in these two equations cancel.

- *If ψ_1, ψ_2 are complex classical odd quantities, then complex conjugation satisfies*

(34) $$\overline{\psi_1\psi_2} = \overline{\psi_1}\;\overline{\psi_2}.$$

This is a consequence of the sign rule if we assume that $\psi \mapsto \overline{\psi}$ is a $*$-*operation* and ψ_1 (super)commutes with ψ_2. (A $*$ operation satisfies $(ab)^* = (-1)^{|a|\,|b|}b^*a^*$.) Notice that the classical statement (34) is consistent with the quantum statement (19), since the adjoint operation on linear operators is also a $*$ operation. Notice that the product of real commuting odd quantities is real.

- *The kinetic lagrangian for a spinor field $\psi\colon M \to S$ is*

(35) $$L = \frac{1}{2}\psi \not{D}\psi\;|d^n x| = \frac{1}{2}\tilde{\Gamma}^{\mu ab}\psi_a\partial_\mu\psi_b\;|d^n x|.$$

- *The kinetic lagrangian for a dual spinor field $\lambda\colon M \to S^*$ is*

(36) $$L = \frac{1}{2}\lambda\not{D}\lambda\;|d^n x| = \frac{1}{2}\Gamma^\mu_{ab}\lambda^a\partial_\mu\lambda^b\;|d^n x|.$$

The spinor fields are odd. In view of (34), the lagrangians (35) and (36) are real, as they must be in Minkowski space. It is easiest to check the sign in classical mechanics ($n = 1$). Then from (35) we deduce[3] the classical Poisson bracket

$$\{\psi, \psi\} = -1. \tag{37}$$

Upon quantization we know the corresponding operators satisfy (29). The sign in (37) is compatible with (21), and this means that the sign in (35) is correct.

- *The energy-momentum tensor is minus the Noether current of P_μ. The supercurrent is minus the Noether current of Q_a.*

This is a definition and follows by superPoincaré invariance from the definition (7) of the hamiltonian. It means that the charges computed from the energy-momentum tensor are energy and *minus* momentum.

§6. Differential forms

- *When computing with differential forms on superspace, we use a bigraded point of view.*[4]

Objects have a "cohomological" degree, corresponding to a classical (that is, non-super) degree, and a parity. The permutation of objects of parity p_1, p_2 and cohomological degree d_1, d_2 introduces two signs: a classical sign $(-1)^{d_1 d_2}$ and an additional factor $(-1)^{p_1 p_2}$.

- *On $\mathbb{R}^{p|q}$ with coordinates $t^1, \dots t^p, \theta^1, \dots \theta^q$ we have the following table of parities and cohomological degrees:*

quantity	type	parity ($\mathbb{Z}/2\mathbb{Z}$)	coh deg ($\mathbb{Z}$)
t^μ	even coordinate	0	0
θ^a	odd coordinate	1	0
$\partial/\partial t^\mu$, $\iota(\partial/\partial t^\mu)$	even vector field	0	-1
$\partial/\partial\theta^a$, $\iota(\partial/\partial\theta^a)$	odd vector field	1	-1
Lie(ξ)	Lie derivative	$p(\xi)$	0
$dt^\mu, \epsilon(dt^\mu)$	even 1-form	0	1
$d\theta^a, \epsilon(d\theta^a)$	odd 1-form	1	1
$d^pt\, d^q\theta$	berezinian	q (mod 2)	p
$\lvert d^pt\rvert\, d^q\theta$	density	q (mod 2)	0
$\iota(\xi_r)\dots\iota(\xi_1)\, d^pt\, d^q\theta$	integral form	$\sum p(\xi_i) + q$ (mod 2)	$p - r$
$\iota(\xi_r)\dots\iota(\xi_1)\, \lvert d^pt\rvert\, d^q\theta$	integral density	$\sum p(\xi_i) + q$ (mod 2)	$-r$

- *For X a (super)manifold the canonical pairing of vectors and 1-forms is written with the vector on the left.*

Therefore, by the sign rule, for a tangent vector ξ and a 1-form α there is a sign[5] when passing from the canonical pairing $\iota(\xi)\alpha$ to $(-1)^{p(\xi)p(\alpha)}\alpha(\xi)$, where p is the parity.

[3] See Problem FP2 of [**I-Homework**] for a derivation.
[4] See the appendix to Chapter 1 of [**I-Supersymmetry**].
[5] See [**I-Supersymmetry**, §3.3].

- *For integration over odd variables we have*

$$\int d\theta\, \theta = 1$$

$$\int d\theta^2 d\theta^1\, \theta^1\theta^2 = \int d\theta^2 \left(\int d\theta^1\, \theta^1\right)\theta^2 = 1.$$

- *For any vector field ξ we have the Cartan formula*

(38) $$\mathrm{Lie}(\xi) = [d, \iota(\xi)].$$

Both sides of (38) act on differential or integral forms.

§7. Miscellaneous signs

- *Let X be a smooth manifold, ξ a vector field on X, φ_t the one-parameter group of diffeomorphisms generated, and T a tensor field. Then*

(39) $$\mathrm{Lie}(\xi)T = \frac{d}{dt}\Big|_{t=0} \varphi_t^* T = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* T.$$

- *On a Kähler manifold X the Riemannian metric g, Kähler form ω, complex structure J, and a local Kähler potential K are related by the equations*

(40) $$\begin{aligned} \omega(\xi_1, \xi_2) &= g(J\xi_1, \xi_2) \\ \omega &= i\,\partial\bar{\partial}K. \end{aligned}$$

- *Suppose f is a (suitable) function on a vector space V of dimension n. Then f and its Fourier transform $\hat{f}$ on V^* are related by*

(41) $$\begin{aligned} \hat{f}(k) &= \frac{1}{(2\pi)^{n/2}} \int_V e^{-i\langle k,x\rangle} f(x)\, |d^n x|, \\ f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{V^*} e^{+i\langle k,x\rangle} \hat{f}(k)\, |d^n k|. \end{aligned}$$

- *Suppose $\mathfrak{g}$ is the Lie algebra of a real Lie group G. Then the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ carries a conjugation whose set of real points is $\mathfrak{g}$.*

For example, starting with the unitary group $G = U(n)$, we obtain the conjugation $A \mapsto -A^*$ on complex $n \times n$ matrices. (A^* is the conjugate transpose of A.)