From the Dunkl intertwining operator to simple Hurwitz numbers

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Plan

1. Reflection groups.
2. The Dunkl intertwining operator.
3. Topological expansion & Simple Hurwitz numbers.

Reflection groups : a reminder

Root systems

- $(\mathbb{R}^N, \langle , \rangle)$.
- Reflection orthogonal to $\alpha \neq 0$:

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Definition

- *A* **root system** *R is a collection of vectors in $\mathbb{R}^d \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ for any $\alpha \in R$.*
- *It is reduced if*

$$\mathbb{R}\alpha \cap R = \{\pm\alpha\}, \quad \forall \alpha \in R.$$

Type A :

$$R = \{\pm(e_i - e_j), 1 \leq i < j \leq N\}.$$

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Positive system

1. Pick $v \notin R, v \neq 0$:

$$R_+ := \{\alpha \in R, \langle \alpha, v \rangle > 0\}.$$

2. Splitting : $R = R_+ \cup R_-$.

Example (Type A)

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Simple System of a positive subsystem

Theorem

In any R_+, there exists a unique subset S :

Every positive root is a **positive LC** *of vectors in S.*

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S is the simple system associated to R_+.
$\alpha \in S$ is a simple root.
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Weyl chamber and reflection groups

(R, R_+, S) :

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Another Example

Type B :

1. $R = \{\pm(e_i - e_j), 1 \leq i < j \leq d, \pm e_i, 1 \leq i \leq N\}.$
2. $R_+ = \{(e_i - e_j), 1 \leq i < j \leq N,\ e_i, 1 \leq i \leq N\}.$
3. $$C_B = \{x_1 > \cdots > x_N > 0\}.$$
4. $W = S_N \rtimes (\mathbb{Z}_2)^N.$

Multiplicity function

Definition

A multiplicity function is a map $k : R \to \mathbb{C}$ *s.t.*

$$k(\alpha) = k(w\alpha), \quad w \in W, \alpha \in R.$$

$\Rightarrow$ Takes as many values as the orbit space $|R/W|$.

Examples (Types A et B)

- $|R_A/W_A| = 1 \rightsquigarrow \beta$,
- $|R_B/W_B| = 2 \rightsquigarrow (\beta, \delta)$.

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Dunkl Intertwining operator

Dunkl operators

$\xi \in \mathbb{R}_N \setminus \{0\}$:

$$D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi \rangle \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x \rangle}.$$

$$\boxed{k = 0}$$

$$\Downarrow$$

$$D_\xi(0)f(x) = \partial_\xi f(x).$$

Theorem (Dunkl)

For any **reduced** *root system, the algebra generated by* $\{D_\xi(k, R), \xi \in \mathbb{R}_N \setminus \{0\}\}$ *is commutative.*

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Example

$$R = A_{N-1} :$$

$$D_{e_i - e_j}(k) f(x) = \partial_{e_i - e_j} f(x) + \sum_{1 \le i < j \le N} k(\alpha)(\alpha_i - \alpha_j) \underbrace{\frac{f(\tau_{ij} x) - f(x)}{x_i - x_j}}_{\delta_{i,j}(f)(x)} .$$

Properties

(δ_i) *satisfies the nil-Coxeter relations :*

- $(\delta_i)^2 = 0.$
- $\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1}.$
- $\delta_i \delta_j = \delta_j \delta_i, \quad |i - j| \ge 2.$

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Dunkl Intertwining operator

Seek a (linear) isomorphism V_k acting on polynomials such that

1. $V_k \mathscr{P}_n = \mathscr{P}_n$.
2. $V_k 1 = 1$ (normalization).
3. $D_\xi(k) V_k = V_k \partial_\xi \; (= V_k D_\xi(0))$.

Theorem (Dunkl-Opdam-DeJeu)

If k takes values in

$$M_{reg} := \{k, \cap_\xi Ker[D_\xi(k)] = \mathbb{C}\},$$

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Let U_k be the operator acting on polynomials as :

$$U_k(p)(x) = \frac{1}{n!}\left(\sum_{i=1}^{N} x_i D_{e_i}(k)\right)^n (p), \quad p \in \mathscr{P}_n.$$

Then for any $1 \leq j \leq N$,

$$U_k D_{e_j}(k) = \partial_{e_j} U_k.$$

Formally,

$$V_k = U_k^{-1}.$$

Group algebra calculus

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Construction : Modified de Rham complex

- Polynomial : $p \in \mathscr{P}_n$.
- m- form : $\omega = p dx_{i_1} \wedge \cdots \wedge dx_{i_m}$.
- Action of W on forms :

$$w \cdot \omega := \underbrace{(w \cdot p)}_{p \circ w^{-1}} d(w^{-1}x)_{i_1} \wedge \cdots \wedge d(w^{-1}x)_{i_m}.$$

- Dunkl exterior derivative :

$$d(k)\omega := \sum_{j=1}^{N} \left[D_{e_j}(k) p \right] dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m}.$$

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1. $d(k)$ commutes with the W-action.
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Theorem (Dunkl-De Jeu-Opdam)

V_k exists and is unique if and only if $H^1(d(k)) = 0$.

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Integral representations

- Rank-one case :

$$V_k(p)(x) \propto \int_{-1}^{1} e^{xy}(1-y)^{k-1}(1+y)^k dy, \quad k > 0.$$

- $k \equiv 1$ and W-invariant polynomials.
- Dihedral groups.

Another construction

Set :

$$\gamma := \sum_{\alpha\in R_+} k(\alpha), \quad A_n(p) = \sum_{\alpha\in R+} k(\alpha)p[\sigma_\alpha],\, p \in \mathscr{P}_n.$$

Theorem (H. Youssfi, M. Maslouhi)

- *$k \in M_{reg}$ iff $(n+\gamma) - A_n$ is invertible for all $n \geq 1$.*
- *Set $H_n = [(n+\gamma) - A_n]^{-1}$. Then for any $p \in \mathscr{P}_n$,*

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Expansions & Simple Hurwitz numbers

Generalized Bessel function

- Dunkl theory : Harmonic analysis beyond Lie groups.
- Key role is played by the Dunkl kernel :

$$D_k(x, y) := V_k\left(x \mapsto e^{\langle x,y\rangle}\right)$$

- Invariant theory :

Definition (Generalized Bessel function)

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HCIZ integral

- Symmetric group S_N (Weyl group of $GL(N, \mathbb{C})$).
- One orbit = one multiplicity value k.

Theorem (HCIZ integral)

If $x, y, \in \mathbb{R}^{\mathrm{N}}$ are identified with diagonal Hermitian matrices, then

$$D_1^W(x, y) = \int_{U(N)} e^{tr(xUyU^\star)} \underbrace{dU}_{\text{Haar measure}}$$

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Expansions

1. Character expansion (Fourier analysis) :

$$D_1^W(x,y) = \sum_{\kappa_1 \geq \cdots \geq \kappa_N \in \mathbb{N}^N} \frac{\overbrace{s_\kappa(x)s_\kappa(y)}^{\textit{Schur functions}}}{s_\kappa(1)|\kappa|!}$$

2. String expansion (Trace observables) :

$$D_1^W(x,y) = \sum_{\kappa_1 \geq \cdots \geq \kappa_N \in \mathbb{N}^N} p_\kappa(x)p_\kappa(y) \, |\mathrm{Monotone\ paths\ on\ Cayley}(S_N)|$$

3. Topological expansion : for $z \in \mathbb{C}$,

$$D_1^W(zNx,y) = \sum_{g=0}^{\infty} N^{2-2g} F_{g,N}(z,x,y)$$

$F_{g,N}(z,x,y)$: GFs of monotone double Hurwitz numbers.

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Our approach

Theorem (Deléaval-D-Youssfi)

The following expansion holds

$$D_1^W(x,y) = 1 + \frac{1}{N!}\sum_{n=1}^{\infty}\frac{1}{n}$$

$$\sum_{w_1,\ldots,w_n\in W} C_{n-1}(w_n^{-1}w_{n-1})\ldots C_1(w_2^{-1}w_1)\prod_{j=1}^{n}\langle w_j x, y\rangle$$

where for any $w \in W$,

$$C_n(w) := \sum_{m=0}^{\infty}\frac{c_m(w)}{(n+\gamma)^{m+1}}, \quad \gamma = \frac{N(N-1)}{2}$$

and

$$c_m(w) = \sum_{\sigma_{\alpha_1}\ldots\sigma_{\alpha_m}=w} 1$$

$k \equiv 1$: Simple Hurwitz numbers

- $k \equiv 1$:

$$c_m(w) = |\text{number of factorisations of } w \text{ into } m \text{ reflections}|.$$

- $W = S_N$: simple Hurwitz numbers :
- Counts the number of ramified covering of S^2 with m simple ramifications and a single ramification at ∞ of type w).

Open problem

Find the spectrum of

$$A_n : \mathscr{P}_n \to \mathscr{P}_n?$$

Representation theory of S_N :

Decompose $\mathscr{P}_n$ into irreducibles (Specht modules) ?

Relate generating series of Hurwitz numbers to Schur polynomials ?

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References

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