Path integral for interacting field

based on S-9

Let's consider an interacting "phi-cubed" QFT:

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi \partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}Z_m m^2\varphi^2 + \tfrac{1}{6}Z_g g\varphi^3 + Y\varphi$$

with fields satisfying:

$$\langle 0|\varphi(x)|0\rangle = 0 \qquad \langle k|\varphi(x)|0\rangle = e^{-ikx}$$

$$\langle 0|0\rangle = 1$$
$$\langle k'|k\rangle = (2\pi)^3 2k^0\delta^3(\mathbf{k}' - \mathbf{k})$$

we want to evaluate the path integral for this theory:

$$Z(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi\, e^{i\int d^4x[\mathcal{L}_0 + \mathcal{L}_1 + J\varphi]}$$

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi \partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}Z_m m^2\varphi^2 + \tfrac{1}{6}Z_g g\varphi^3 + Y\varphi$$

$$Z(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi\, e^{i\int d^4x[\mathcal{L}_0 + \mathcal{L}_1 + J\varphi]}$$

it can be also written as:

$$Z(J) = e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\varphi\, e^{i\int d^4x[\mathcal{L}_0 + J\varphi]}$$
$$\propto e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}\, Z_0(J)\,,$$

epsilon trick leads to additional factor;
to get the correct normalization we require: $Z(0) = 1$

and for the path integral of the free field theory we have found:

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x - x')J(x')\right]$$

$$Z(J) = e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\varphi\, e^{i\int d^4x[\mathcal{L}_0 + J\varphi]}$$
$$\propto e^{i\int d^4x\, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}\, Z_0(J)\,,$$

assumes $\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$

thus in the case of:

$$\mathcal{L} = -\tfrac{1}{2}Z_\varphi \partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}Z_m m^2\varphi^2 + \tfrac{1}{6}Z_g g\varphi^3 + Y\varphi$$

the perturbing lagrangian is:

$$\mathcal{L}_1 = \tfrac{1}{6}Z_g g\varphi^3 + \mathcal{L}_{\rm ct}\,,$$
$$\mathcal{L}_{\rm ct} = -\tfrac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi$$

counterterm lagrangian

in the limit $g \to 0$ we expect $Y \to 0$ and $Z_i \to 1$

we will find $Y = O(g)$ and $Z_i = 1 + O(g^2)$

Let's look at Z(J) (ignoring counterterms for now).
Define:

$$Z_1(J) \propto \exp\left[\frac{i}{6}Z_g g\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right] Z_0(J)$$

$$Z_1(0) = 1$$

exponentials defined by series expansion:

$$Z_1(J) \propto \sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{iZ_g g}{6}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V$$
$$\times \sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y-z)J(z)\right]^P$$

let's look at a term with particular values of P (propagators) and V (vertices):

number of surviving sources, (after taking all derivatives) E (for external) is

$$E = 2P - 3V$$

3V derivatives can act on 2P sources in (2P)! / (2P-3V)! different ways

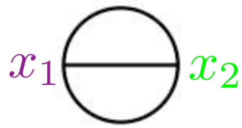
e.g. for V = 2, P = 3 there is 6! different terms

V = 2, E = 0 (P = 3):

$$\frac{1}{2!}\left[\frac{iZ_g g}{6}\int d^4x_1\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)^3\right]\left[\frac{iZ_g g}{6}\int d^4x_2\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right)^3\right]$$

$$\frac{1}{3!}\left[\frac{i}{2}\int d^4y_1\, d^4z_1\, J(y_1)\Delta(y_1-z_1)J(z_1)\right]\left[\frac{i}{2}\int d^4y_2\, d^4z_2\, J(y_2)\Delta(y_2-z_2)J(z_2)\right]\left[\frac{i}{2}\int d^4y_3\, d^4z_3\, J(y_3)\Delta(y_3-z_3)J(z_3)\right]$$

$$\frac{3!\;\;3!\;\;2\;\;2\;\;2}{2!\;\;6\;\;6\;\;3!\;\;2\;\;2\;\;2}$$



$$= \frac{1}{12}\int dx_1 \int dx_2\, (iZ_g g)^2\, \frac{1}{i}\Delta(x_1-x_2)\, \frac{1}{i}\Delta(x_1-x_2)\, \frac{1}{i}\Delta(x_1-x_2)$$

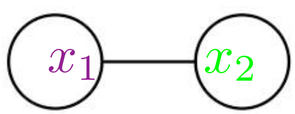
symmetry factor

V = 2, E = 0 (P = 3):

$$\frac{1}{2!}\left[\frac{iZ_g g}{6}\int d^4x_1\left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)^3\right]\left[\frac{iZ_g g}{6}\int d^4x_2\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right)^3\right]$$

$$\frac{1}{3!}\left[\frac{i}{2}\int d^4y_1\, d^4z_1\, J(y_1)\Delta(y_1-z_1)J(z_1)\right]\left[\frac{i}{2}\int d^4y_2\, d^4z_2\, J(y_2)\Delta(y_2-z_2)J(z_2)\right]\left[\frac{i}{2}\int d^4y_3\, d^4z_3\, J(y_3)\Delta(y_3-z_3)J(z_3)\right]$$

$$\frac{3!\;\;3!\;\;3!\;\;2}{2!\;\;6\;\;6\;\;3!\;\;2\;\;2\;\;2}$$



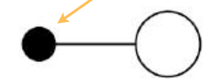
$$= \frac{1}{8}\int dx_1 \int dx_2\, (iZ_g g)^2\, \frac{1}{i}\Delta(x_1-x_1)\, \frac{1}{i}\Delta(x_1-x_2)\, \frac{1}{i}\Delta(x_1-x_1)$$

symmetry factor

Feynman diagrams:

- a line segment stands for a propagator $\frac{1}{i}\Delta(x-y)$
- vertex joining three line segments stands for $iZ_g g\int d^4x$
- a filled circle at one end of a line segment stands for a source $i\int d^4x\, J(x)$

e.g. for V = 1, E = 1



What about those symmetry factors?

symmetry factors are related to symmetries of Feynman diagrams...

Symmetry factors:

we can rearrange three derivatives without changing diagram

we can rearrange three vertices

$$Z_1(J) \propto \sum_{V=0}^{\infty}\frac{1}{\cancel{V!}}\left[\frac{iZ_g g}{\cancel{6}}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V \times \sum_{P=0}^{\infty}\frac{1}{\cancel{P!}}\left[\frac{i}{\cancel{2}}\int d^4y\, d^4z\, J(y)\Delta(y-z)J(z)\right]^P$$

we can rearrange two sources

we can rearrange propagators

this in general results in overcounting of the number of terms that give the same result; this happens when some rearrangement of derivatives gives the same match up to sources as some rearrangement of sources; this is always connected to some symmetry property of the diagram; factor by which we overcounted is the symmetry factor

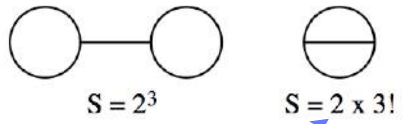


Figure 9.1: All connected diagrams with $E = 0$ and $V = 2$.

the endpoints of each propagator can be swapped
and the effect is duplicated by swapping the two vertices

propagators can be rearranged in 3! ways,
and all these rearrangements can be duplicated by
exchanging the derivatives at the vertices

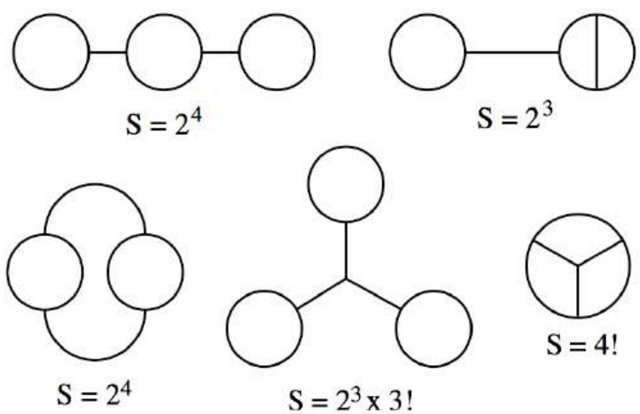


Figure 9.2: All connected diagrams with $E = 0$ and $V = 4$.

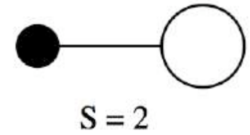


Figure 9.3: All connected diagrams with $E = 1$ and $V = 1$.

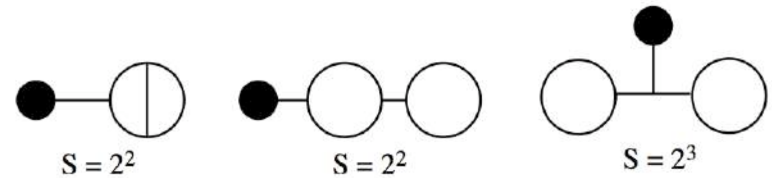


Figure 9.4: All connected diagrams with $E = 1$ and $V = 3$.

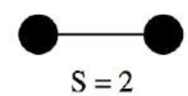


Figure 9.5: All connected diagrams with $E = 2$ and $V = 0$.

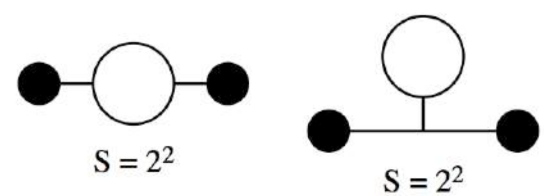


Figure 9.6: All connected diagrams with $E = 2$ and $V = 2$.

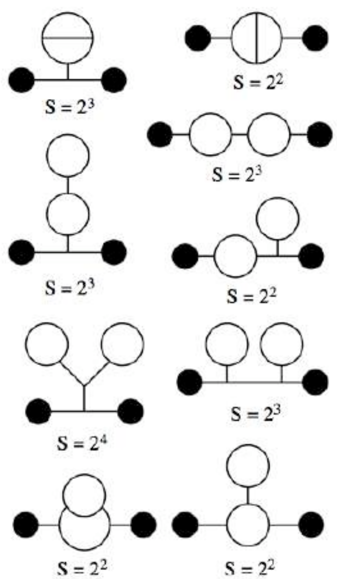


Figure 9.7: All connected diagrams with $E = 2$ and $V = 4$.

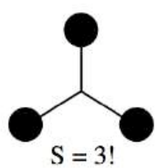


Figure 9.8: All connected diagrams with $E = 3$ and $V = 1$.

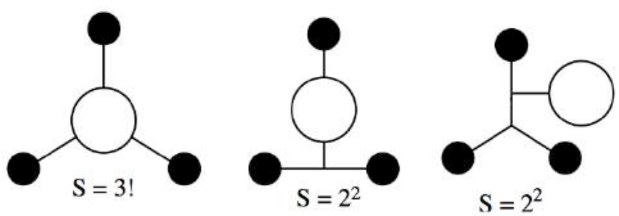


Figure 9.9: All connected diagrams with $E = 3$ and $V = 3$.

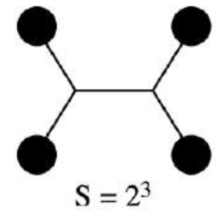


Figure 9.10: All connected diagrams with $E = 4$ and $V = 2$.

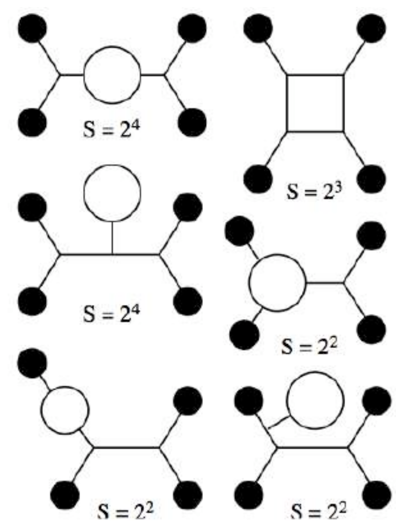


Figure 9.11: All connected diagrams with $E = 4$ and $V = 4$.

All these diagrams are connected, but Z(J) contains also diagrams that are products of several connected diagrams:

e.g. for V = 4, E = 0 (P = 6) in addition to connected diagrams we also have :

and also:

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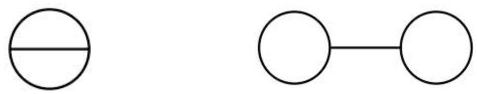

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A general diagram D can be written as:

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}$$

the number of given C in D

particular connected diagram

additional symmetry factor not already accounted for by symmetry factors of connected diagrams; it is nontrivial only if D contains identical C's:

$$S_D = \prod_I n_I!$$

Now $Z_1(J)$ is given by summing all diagrams D:

$$\begin{aligned} Z_1(J) &\propto \sum_{\{n_I\}} D \\ &\propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} \\ &\propto \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} \\ &\propto \prod_I \exp(C_I) \\ &\propto \exp(\textstyle\sum_I C_I) \ . \end{aligned}$$

any D can be labeled by a set of n's

thus we have found that $Z_1(J)$ is given by the exponential of the sum of connected diagrams.

imposing the normalization $Z_1(0) = 1$ means we can omit vacuum diagrams (those with no sources), thus we have:

$$Z_1(J) = \exp[iW_1(J)] \ , \qquad iW_1(J) \equiv \sum_{I \neq \{0\}} C_I \qquad W_1(0) = 0$$

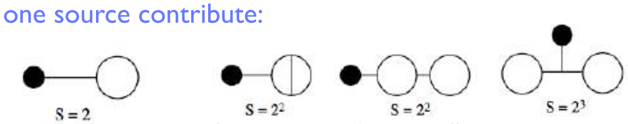
vacuum diagrams are omitted from the sum

If there were no counterterms we would be done: $Z(J) = Z_1(J)$

in that case, the vacuum expectation value of the field is:

$$\begin{aligned} \langle 0|\varphi(x)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x)} Z_1(J)\Big|_{J=0} \\ &= \frac{\delta}{\delta J(x)} W_1(J)\Big|_{J=0} \ . \end{aligned}$$

only diagrams with one source contribute:



(the source is "removed" by the derivative)

and we find:

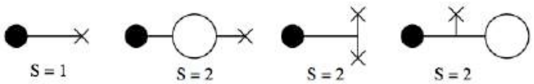
$$\langle 0|\varphi(x)|0\rangle = \tfrac{1}{2} ig \int d^4y \, \tfrac{1}{i}\Delta(x-y)\tfrac{1}{i}\Delta(y-y) + O(g^3)$$

we used $Z_g = 1$ since we know $Z_g = 1 + O(g^2)$

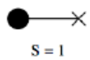
which is not zero, as required for the LSZ; so we need counterterm →

Including $Y\varphi$ term in the interaction lagrangian results in a new type of vertex on which a line segment ends

e.g.



corresponding Feynman rule is: $iY \int d^4y$

at the lowest order of g only  contributes:

$$\langle 0|\varphi(x)|0\rangle = \left(iY + \tfrac{1}{2}(ig)\tfrac{1}{i}\Delta(0)\right) \int d^4y \, \tfrac{1}{i}\Delta(x-y) + O(g^3)$$

in order to satisfy $\langle 0|\varphi(x)|0\rangle = 0$ we have to choose:

$$Y = \tfrac{1}{2} ig\Delta(0) + O(g^3)$$

$$\Delta(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon}$$

Note, $\Delta(0)$ must be purely imaginary so that Y is real; and, in addition, the integral over k is ultraviolet divergent.

to make sense out of it, we introduce an ultraviolet cutoff $\Lambda \gg m$

and in order to keep Lorentz-transformation properties of the propagator we make the replacement:

$$\Delta(x-y) \to \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \left(\frac{\Lambda^2}{k^2 + \Lambda^2 - i\epsilon} \right)^2$$

the integral is now convergent:

$$\Delta(0) = \frac{i}{16\pi^2} \Lambda^2$$

we will do this type of calculations later...

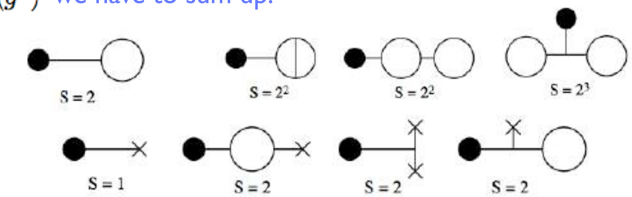
and indeed, $\Delta(0)$ is purely imaginary.

after choosing Y so that $\langle 0|\varphi(x)|0\rangle = 0$ we can take the limit $\Lambda \to \infty$

Y becomes infinite

... we repeat the procedure at every order in g

e.g. at $O(g^3)$ we have to sum up:



and add to Y whatever $O(g^3)$ term is needed to maintain $\langle 0|\varphi(x)|0\rangle = 0$ ...

this way we can determine the value of Y order by order in powers of g.

Adjusting Y so that $\langle 0|\varphi(x)|0\rangle = 0$ means that the sum of all connected diagrams with a single source is zero!

In addition, the same infinite set of diagrams with source replaced by ANY subdiagram is zero as well.

Rule: ignore any diagram that, when a single line is cut, fall into two parts, one of which has no sources. = tadpoles

all that is left with up to 4 sources and 4 vertices is:

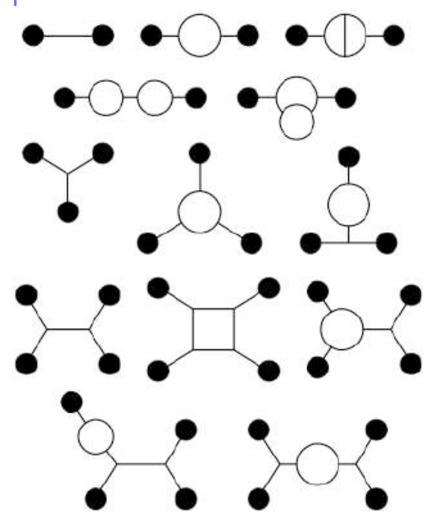


Figure 9.13: All connected diagrams without tadpoles with $E \le 4$ and $V \le 4$.

finally, let's take a look at the other two counterterms:

$$\mathcal{L}_1 = \tfrac{1}{6} Z_g g \varphi^3 + \mathcal{L}_{\rm ct}\ ,$$

$$\mathcal{L}_{\rm ct} = -\tfrac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi$$

we get

$$A = Z_\varphi - 1\ , \qquad B = Z_m - 1$$

$$Z(J) = \exp\left[-\frac{i}{2}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(-A\partial_x^2 + Bm^2\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_1(J)$$

we used integration by parts

it results in a new vertex at which two lines meet, the corresponding vertex factor or the Feynman rule is $(-i)\int d^4x\,(-A\partial_x^2 + Bm^2)$

for every diagram with a propagator there is additional one with this vertex

Summary:

we have calculated $Z(J)$ in φ^3 theory and expressed it as

$$Z(J) = \exp[iW(J)]$$

where W is the sum of all connected diagrams with no tadpoles and at least two sources!