A String-ANALOGUE OF $\mathrm{Spin}^{\mathbf{C}}$

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Recollection 1. Recall that $\mathrm{BSpin}^{\mathbf{C}}$ is the fiber of the map $\mathrm{BSO} \to K(\mathbf{Z}, 3)$ detecting $\beta_{\mathbf{Z}}(w_2) \in \mathrm{H}^3(\mathrm{BSO}; \mathbf{Z})$, where $\beta_{\mathbf{Z}} : \mathbf{F}_2 \to \Sigma\mathbf{Z}$ is the Bockstein. Let $\mathrm{bspin}^{\mathbf{C}}$ denote the connective spectrum associated to the infinite loop space $\mathrm{BSpin}^{\mathbf{C}}$, i.e., the fiber of the composite

$$\mathrm{bso} \to \tau_{\leq 2}\mathrm{bso} \simeq \Sigma^2\mathbf{F}_2 \xrightarrow{\beta} \Sigma^3\mathbf{Z}.$$

Let $\mathrm{MSpin}^{\mathbf{C}}$ denote the associated Thom spectrum (so it admits the structure of an $\mathbf{E}_\infty$-ring). A classical theorem of [ABS64] says that there is an orientation $\mathrm{MSpin}^{\mathbf{C}} \to \mathrm{ku}$, and [AHR10] says that this refines to an $\mathbf{E}_\infty$-map $\mathrm{MSpin}^{\mathbf{C}} \to \mathrm{ku}$. There is also a map $\mathrm{bspin} \to \mathrm{bspin}^{\mathbf{C}}$ which sits in a fiber sequence

$$\mathrm{bspin} \to \mathrm{bspin}^{\mathbf{C}} \to \Sigma^2\mathbf{Z},$$

and hence gives a fiber sequence

$$\mathrm{BSpin} \to \mathrm{BSpin}^{\mathbf{C}} \to \mathbf{C}P^\infty. \tag{1}$$

There is an associated $\mathbf{E}_\infty$-map $\mathrm{MSpin} \to \mathrm{ko}$, and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{MSpin} & \longrightarrow & \mathrm{MSpin}^{\mathbf{C}} \\ \downarrow & & \downarrow \\ \mathrm{ko} & \longrightarrow & \mathrm{ku}. \end{array}$$

In fact, even more is true: the obstruction to extending the MSpin-orientation of ko to an $\mathrm{MSpin}^{\mathbf{C}}$-orientation is given by a map $\Sigma^2\mathbf{Z} \to \mathrm{bgl}_1(\mathrm{ko})$, and the composite $S^2 \to \Sigma^2\mathbf{Z} \to \mathrm{bgl}_1(\mathrm{ko})$ detects η. In particular, the composite $S^2 \to \mathbf{C}P^\infty \to \mathrm{BGL}_1(\mathrm{ko})$ detects η; in fact, the Thom spectrum of this map is ku.

Furthermore, there is an equivalence $\mathrm{MSpin}^{\mathbf{C}} \simeq \mathrm{MSpin} \otimes \Sigma^{-2}\mathbf{C}P^\infty$, and the canonical map $\mathrm{MSpin} \otimes \Sigma^{-2}\mathbf{C}P^2 \to \mathrm{MSpin} \otimes \Sigma^{-2}\mathbf{C}P^\infty$ splits in a way which makes the following diagram commutes:

$$\begin{array}{ccc} \mathrm{MSpin} & \longrightarrow & \mathrm{MSpin}^{\mathbf{C}} \simeq \mathrm{MSpin} \otimes \Sigma^{-2}\mathbf{C}P^\infty \\ \Big\downarrow & & \downarrow \\ & & \mathrm{MSpin} \otimes \Sigma^{-2}\mathbf{C}P^2 \\ \Big\downarrow & & \downarrow \\ \mathrm{ko} & \longrightarrow & \mathrm{ku} \simeq \mathrm{ko} \otimes \Sigma^{-2}\mathbf{C}P^2. \end{array}$$

It is useful to view $\Sigma^{-2}\mathbf{C}P^2 = C\eta$ as DA_0, i.e., as a spectrum realizing the double of $A(0)$ (i.e., $\langle \mathrm{Sq}^2 \rangle$).

We will generalize this picture to tmf.

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Construction 2. A calculation with the Serre spectral sequence for the fibration (1) tells us that $\mathrm{H}^4(\mathrm{BSpin}^{\mathbf{C}};\mathbf{Z}) \cong \mathbf{Z}\{c_1^2\} \oplus \mathbf{Z}\{\frac{c_1^2-p_1}{2}\}$. The class $\frac{c_1^2-p_1}{2}$ defines a map $\mathrm{BSpin}^{\mathbf{C}} \to K(\mathbf{Z},4)$. This map does not refine to a map $\mathrm{bspin}^{\mathbf{C}} \to \Sigma^4\mathbf{Z}$, although the composite $\mathrm{BSpin} \to \mathrm{BSpin}^{\mathbf{C}} \to K(\mathbf{Z},4)$ does refine to a map $\mathrm{bspin} \to \Sigma^4\mathbf{Z}$ of spectra.

There is a cofiber sequence $\Sigma^2\mathrm{ku} \xrightarrow{v_1} \mathrm{ku} \to \mathbf{Z}$, where $v_1 \in \pi_2(\mathrm{ku})$ is the Bott class. The boundary map for this cofiber sequence defines a map $\beta_{\mathrm{ku}} : \mathbf{Z} \to \Sigma^3\mathrm{ku}$ known as the v_1-*Bockstein*; in particular, there is a map $K(\mathbf{Z},n) \to \Omega^\infty\Sigma^{n+3}\mathrm{ku} \simeq \mathrm{B}^n\mathrm{SU}$. It follows that the class $\frac{c_1^2-p_1}{2}$ defines a map $\mathrm{BSpin}^{\mathbf{C}} \to K(\mathbf{Z},4) \to \mathrm{B}^4\mathrm{SU}$. This map *does* refine to a map of spectra $\mathrm{bspin}^{\mathbf{C}} \to \Sigma^7\mathrm{ku}$, because the first k-invariant in the Postnikov tower for $\mathrm{bspin}^{\mathbf{C}}$ is the v_1-Bockstein. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{bspin} & \longrightarrow & \mathrm{bspin}^{\mathbf{C}} \\
{\scriptstyle p_1/2}\downarrow & \overset{(c_1^2-p_1)/2}{\swarrow} & \downarrow \\
\Sigma^4\mathbf{Z} & \xrightarrow[\beta_{\mathrm{ku}}]{} & \Sigma^7\mathrm{ku},
\end{array}$$

where the dotted map exists only on Ω^∞. Let $\mathrm{bstring}^{\mathbf{H}}$ denote the fiber of the map $\mathrm{bspin}^{\mathbf{C}} \to \Sigma^7\mathrm{ku}$, and let $\mathrm{BString}^{\mathbf{H}} = \Omega^\infty\mathrm{bstring}^{\mathbf{H}}$. Let $\mathrm{MString}^{\mathbf{H}}$ denote the associated Thom spectrum (so it is an $\mathbf{E}_\infty$-ring).

Lemma 3. *There is a cofiber sequence*

$$\mathrm{bstring} \to \mathrm{bstring}^{\mathbf{H}} \to \Sigma^2\mathrm{ku},$$

and hence a fiber sequence of infinite loop spaces

$$\mathrm{BString} \to \mathrm{BString}^{\mathbf{H}} \to \mathrm{BU}.$$

Proof. Consider the following square:

$$\begin{array}{ccc}
\mathrm{bspin} & \longrightarrow & \mathrm{bspin}^{\mathbf{C}} \\
\downarrow & & \downarrow \\
\Sigma^4\mathbf{Z} & \xrightarrow[\beta_{\mathrm{ku}}]{} & \Sigma^7\mathrm{ku}.
\end{array}$$

Taking vertical and horizontal fibers in all directions produces the following commutative diagram, where each row and column is a fiber sequence:

$$\begin{array}{ccccc}
\Sigma\mathrm{ku} & \longrightarrow & \mathrm{bstring} & \longrightarrow & \mathrm{bstring}^{\mathbf{H}} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma\mathbf{Z} & \longrightarrow & \mathrm{bspin} & \longrightarrow & \mathrm{bspin}^{\mathbf{C}} \\
{\scriptstyle \beta_{\mathrm{ku}}}\downarrow & & \downarrow & & \downarrow \\
\Sigma^4\mathrm{ku} & \longrightarrow & \Sigma^4\mathbf{Z} & \longrightarrow & \Sigma^7\mathrm{ku}.
\end{array}$$

The bottom left vertical map can be identified with the Bockstein $\beta_{\mathrm{ku}} : \Sigma\mathbf{Z} \to \Sigma^4\mathrm{ku}$, which identifies the fiber of the map $\mathrm{bstring} \to \mathrm{bstring}^{\mathbf{H}}$ with $\Sigma\mathrm{ku}$. This gives the desired cofiber sequence. □

Using [Dev20, Proposition 2.1.6], we see:

Corollary 4. *There is an* $\mathbf{E}_\infty$*-map* $\mathrm{BU} \to \mathrm{BGL}_1(\mathrm{MString})$ *whose Thom spectrum is* $\mathrm{MString}^{\mathbf{H}}$.

In fact, this can be used to show that there is an equivalence $\mathrm{MString}^{\mathbf{H}} \simeq \mathrm{MString} \otimes \mathrm{MU}$.

Theorem 5. *There is a map* $\mathrm{MString}^{\mathbf{H}} \to \mathrm{tmf}_1(3)$ *of* $\mathbf{E}_\infty$*-rings such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{MString} & \longrightarrow & \mathrm{MString}^{\mathbf{H}} \\ \downarrow & & \downarrow \\ \mathrm{tmf} & \longrightarrow & \mathrm{tmf}_1(3). \end{array}$$

The obstruction to extending the map $\mathrm{MString}^{\mathbf{H}} \to \mathrm{tmf}_1(3)$ *to a map* $\mathrm{MSpin}^{\mathbf{C}} \to \mathrm{tmf}_1(3)$ *of* $\mathbf{E}_\infty$*-rings is a map* $f : \Sigma^7\mathrm{ku} \to \mathrm{bgl}_1(\mathrm{tmf}_1(3))$ *such that:*

- *the following composite is the Ando-Hopkins-Rezk twisting map:*

$$\Sigma^4\mathbf{Z} \xrightarrow{\beta_{\mathrm{ku}}} \Sigma^7\mathrm{ku} \xrightarrow{f} \mathrm{bgl}_1(\mathrm{tmf}_1(3));$$

- *the bottom class* $S^7 \to \Sigma^7\mathrm{ku} \to \mathrm{bgl}_1(\mathrm{tmf}_1(3))$ *detects an indecomposable* $v_2 \in \pi_7\mathrm{bgl}_1(\mathrm{tmf}_1(3)) \cong \pi_6\mathrm{tmf}_1(3)$.

Proof. All of these results are consequences of (and equivalent to) the following (forthcoming) result of Hahn-Senger: the class $v_2 \in \pi_6(\mathrm{tmf}_1(3))$ extends to a map $\Sigma^7\mathrm{ku} \to \mathrm{bgl}_1(\mathrm{tmf}_1(3))$ such that the composite $\Sigma^4\mathbf{Z} \xrightarrow{\beta_{\mathrm{ku}}} \Sigma^7\mathrm{ku} \xrightarrow{f} \mathrm{bgl}_1(\mathrm{tmf}_1(3))$ is the Ando-Hopkins-Rezk twisting. In particular, the composite $\mathrm{bstring}^{\mathbf{H}} \to \mathrm{bspin}^{\mathbf{C}} \to \Sigma^7\mathrm{ku} \to \mathrm{bgl}_1(\mathrm{tmf}_1(3))$ is null, which implies the desired claim by [AHR10]. □

Remark 6. Theorem 5 is the main mathematical reason for believing that $\mathrm{String}^{\mathbf{H}}$-structures are the appropriate generalization of $\mathrm{Spin}^{\mathbf{C}}$-structures. Indeed, $\mathrm{tmf}_1(3)_{(2)} = \mathrm{BP}\langle 2\rangle$ is to $\mathrm{tmf}_{(2)}$ as $\mathrm{ku}_{(2)} = \mathrm{BP}\langle 1\rangle$ is to $\mathrm{ko}_{(2)}$; so Theorem 5 tells us that $\mathrm{MString}^{\mathbf{H}}$ is to MString as $\mathrm{MSpin}^{\mathbf{C}}$ is to MSpin. Although there are notions of "String^c-structures" in the literature, it does not seem to us that these notions are related in any way to the Witten genus.

Remark 7. It is natural to ask for the *geometric* interpretation of a $\mathrm{String}^{\mathbf{H}}$-structure. To understand this, recall that the cofiber sequence

$$\Sigma^4\mathrm{ku} \to \Sigma^4\mathbf{Z} \to \Sigma^7\mathrm{ku}$$

defines a fiber sequence of infinite loop spaces

$$\mathrm{BSU} \simeq \Omega^\infty\Sigma^4\mathrm{ku} \to K(\mathbf{Z}, 4) \to \mathrm{B}^4\mathrm{SU} \simeq \Omega^\infty\Sigma^7\mathrm{ku}.$$

The map $\mathrm{BSU} \to K(\mathbf{Z}, 4)$ classifies $c_2 \in \mathrm{H}^4(\mathrm{BSU}; \mathbf{Z})$. Then, there is a Cartesian square

$$\begin{array}{ccc} \mathrm{BString}^{\mathbf{H}} & \longrightarrow & \mathrm{BSU} \\ \downarrow & & \downarrow c_2 \\ \mathrm{BSpin}^{\mathbf{C}} & \xrightarrow[\frac{c_1^2 - p_1}{2}]{} & K(\mathbf{Z}, 4). \end{array}$$

It follows that a $\mathrm{String}^{\mathbf{H}}$-structure on a manifold M is the data of a $\mathrm{Spin}^{\mathbf{C}}$-structure on M (which gives the associated $\mathrm{Spin}^{\mathbf{C}}$-line bundle $\mathcal{L}$) along with the data of a virtual SU-bundle ξ such that $c_2(\xi) = \frac{c_1(\mathcal{L})^2 - p_1(M)}{2}$. The SU-bundle ξ on a $\mathrm{String}^{\mathbf{H}}$-manifold plays the role of the complex $\mathrm{Spin}^{\mathbf{C}}$-line bundle $\mathcal{L}$ on a $\mathrm{Spin}^{\mathbf{C}}$-manifold N (indeed, $\mathcal{L}$ is a witness to $\beta w_2(N) = 0$, in the sense that $c_1(\mathcal{L}) \equiv w_2(N) \pmod 2$ in $\mathrm{H}^2(N; \mathbf{Z})$).

Note that $p_1 = c_1^2 - 2c_2 \in \mathrm{H}^4(\mathrm{BU};\mathbf{Z})$, so that $2c_2 = p_1 - c_1^2$. In particular, if M is a stably almost complex manifold, then taking $\xi = T_M$ shows that M admits the structure of a $\mathrm{String}^{\mathbf{H}}$-manifold. This is simply another way of saying that the map $\mathrm{BU} \to \mathrm{BSpin}^{\mathbf{C}}$ lifts to a map $\mathrm{BU} \to \mathrm{BString}^{\mathbf{H}}$ (which Thomifies to a map $\mathrm{MU} \to \mathrm{MString}^{\mathbf{H}}$).

References

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