THE SIMPLICIAL NERVE OF A SIMPLICIAL CATEGORY

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1. Simplicial objects

Let Δ be the following category: the objects are $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$. The morphisms are:

$$Hom_\Delta([m], [n]) = \{f : [m] \to [n] \mid f(i) \leq f(j) \text{ if } i \leq j\}$$

Let $\mathcal{C}$ be a category. A *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. The category of simplicial objects in $\mathcal{C}$ is denoted $\mathcal{C}_\Delta$. We will mostly be concerned with the category of simplicial sets: Set_Δ.

Let ∇ denote the following category: the objects are $[n]$ for $n \geq 1$. The morphisms are

$$Hom_\nabla([m], [n]) = \{f : [m] \to [n] \mid f(0) = 0, f(m) = n, \text{ and } f(i) \leq f(j) \text{ if } i \leq j\}$$

1.1. **Proposition.** *There is an isomorphism of categories* $\nabla \cong \Delta^{op}$.

Proof. We follow the proof given in [Dr]. Given $n \geq 0$, consider the set $Hom_\Delta([n], [1])$. There is a bijection of totally ordered sets

$$Hom_\Delta([n], [1]) \simeq [n + 1]$$
$$\phi \mapsto \sum_{i=0}^{n} \phi(i)$$

Let ϕ_i denote the pre-image of i under this isomorphism.

Given a morphism $f : [m] \to [n]$ in Δ, we have a map

$$f^* : Hom_\Delta([n], [1]) \to Hom_\Delta([m], [1])$$
$$\phi \mapsto \phi \circ f$$

Note that $f^*(\phi_0) = \phi_0$ and $f^*(\phi_n) = \phi_m$. Also, if $i \leq j$ then $f^*(\phi_i) \leq f^*(\phi_j)$. Therefore, we have defined a functor

$$\Delta^{op} \to \nabla$$
$$[n] \mapsto Hom_\Delta([n], [1])$$

The inverse functor is given by

$$\nabla \to \Delta^{op}$$
$$[n] \mapsto Hom_\nabla([n], [1])$$

□

2. Generalities

Let $\mathcal{D}$ be a small category. Let $PSh(\mathcal{D}) = Funct(\mathcal{D}^{op}, Set)$ be the category of presheaves on $\mathcal{D}$. There is a functor

$$\mathcal{D} \to PSh(\mathcal{D})$$

$$x \mapsto h_x = Hom_{\mathcal{D}}(\_, x)$$

The following result is known as the Yoneda Lemma:

2.1. **Theorem.** *Let $\mathcal{D}$ be a small category. For any $x \in \mathcal{D}$ and $F \in PSh(\mathcal{D})$, there is a natural isomorphism*

$$\kappa : Hom_{PSh(\mathcal{D})}(h_x, F) \simeq F(x)$$

Proof. Let $T : h_x \to F$ be a morphism in $PSh(\mathcal{D})$. We can evaluate this natural transformation at the object x:

$$T_x : h_x(x) \to F(x)$$

Note that $\mathrm{id}_x \in h_x(x) = Hom_{\mathcal{D}}(x, x)$. We define

$$\kappa(T) = T_x(\mathrm{id}_x) \in F(x)$$

In the other direction, given $a \in F(x)$, we define

$$T : h_x \to F$$

such that, for any $y \in \mathcal{D}$,

$$T_y : h_x(y) \to F(y)$$

$$f \mapsto F(f)(a)$$

This makes sense because $f : y \to x$, so $F(f) : F(x) \to F(y)$. For any morphism $g : y \to z$ in $\mathcal{D}$, we have the following commutative diagram:

$$\begin{array}{ccc} h_x(z) & \xrightarrow{T_z} & F(z) \\ {\scriptstyle h_x(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_x(y) & \xrightarrow{T_y} & F(y) \end{array}$$

$$\begin{array}{ccc} f & \longmapsto & F(f)(a) \\ \downarrow & & \downarrow \\ f \circ g & \longmapsto & F(f \circ g)(a) = F(g)F(f)(a) \end{array}$$

This shows that T is a natural transformation.

Clearly $\kappa(T) = F(\mathrm{id}_x)(a) = a$, so the composition

$$F(x) \to Hom(h_x, F) \to F(x)$$

is the identity. Finally, let $T : h_x \to F$ and $f : y \to x$ and consider the following diagram:

$$\begin{array}{ccc} h_x(x) & \xrightarrow{T_x} & F(x) \\ {\scriptstyle h_x(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h_x(y) & \xrightarrow{T_y} & F(y) \end{array}$$

$$\begin{array}{ccc} \mathrm{id}_x & \longmapsto & T_x(\mathrm{id}_x) \\ \downarrow & & \downarrow \\ f & \longmapsto & T_y(f) = F(f)(T_x(\mathrm{id}_x))) \end{array}$$

This shows that the composition

$$Hom(h_x, F) \to F(x) \to Hom(h_x, F)$$

is the identity as well. □

2.2. **Remark.** *As a result of the Yoneda Lemma: for any objects $x, y \in \mathcal{D}$, we have*

$$Hom_{PSh(\mathcal{D})}(h_x, h_y) = h_y(x) = Hom_{\mathcal{D}}(x, y)$$

Therefore, the natural functor

$$\mathcal{D} \to PSh(\mathcal{D})$$

is a fully faithful embedding. The functors in its essential image are called representable.

2.3. **Remark.** *Consider the embedding*

$$\Delta \to PSh(\Delta) = Set_\Delta$$
$$[n] \mapsto h_{[n]} = \Delta^n$$

By the Yoneda Lemma, for any simplicial set K, we have

$$Hom_{Set_\Delta}(\Delta^n, K) = K([n])$$

2.4. **Lemma.** *Let $\mathcal{D}$ be a small category. Then every presheaf on $\mathcal{D}$ is a colimit of representable presheaves.*

Proof. Let $\mathcal{F} : \mathcal{D}^{op} \to Set$ be a presheaf. We define the category $\mathcal{D}/\mathcal{F}$ as follows: an object of $\mathcal{D}/\mathcal{F}$ is a morphism $\phi_d : h_d \to \mathcal{F}$ in $PSh(\mathcal{D})$. A morphism in $\mathcal{D}/\mathcal{F}$ is a diagram

$$\begin{array}{ccc} h_d & \longrightarrow & h_{d'} \\ & {\scriptstyle \phi_d}\searrow \quad \swarrow{\scriptstyle \phi_{d'}} & \\ & \mathcal{F} & \end{array}$$

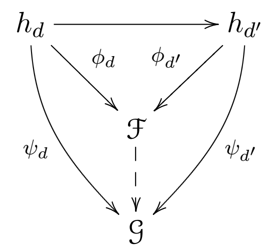
There is a functor

$$\pi : \mathcal{D}/\mathcal{F} \to PSh(\mathcal{D})$$
$$(h_d \to \mathcal{F}) \mapsto h_d$$

We claim that

$$\mathcal{F} \simeq \varinjlim \pi = \varinjlim_{\mathcal{D}/\mathcal{F}} h_d$$

To see this, let $\mathcal{G} \in PSh(\mathcal{D})$, and suppose that we have the following diagram in $PSh(\mathcal{D})$ for all $h_d \to h_{d'}$ in $\mathcal{D}/\mathcal{F}$:



By the Yoneda Lemma, $\phi_d \in \mathcal{F}(d)$ and $\psi_d \in \mathcal{G}(d)$ for all $d \in \mathcal{D}$ such that $\mathcal{F}(d) \neq \emptyset$. The commutativity of the triangles in the above diagram implies that, for any morphism $d \to d'$ in $\mathcal{D}$, $\phi_{d'}$ maps to ϕ_d under the map $\mathcal{F}(d') \to \mathcal{F}(d)$, and $\psi_{d'}$ maps to ψ_d under $\mathcal{G}(d') \to \mathcal{G}(d)$. Therefore, sending ϕ_d to ψ_d defines a morphism of presheaves $\mathcal{F} \to \mathcal{G}$, which is the unique morphism making the diagram commute. □

Next we prove a very general and useful result ([GZ], II.1.3, [Bo] Prop. 4.2):

2.5. **Theorem.** *Let $\mathcal{D}$ be a small category and $\mathcal{C}$ a category with all small colimits. Let $L : PSh(\mathcal{D}) \to \mathcal{C}$ be a functor. Then the following statements are equivalent:*

(1) *L commutes with all small colimits.*

(2) *L has a right adjoint $R : \mathcal{C} \to PSh(\mathcal{D})$.*

Proof. (2) $\implies$ (1). A left adjoint always commutes with all small colimits.

(1) $\implies$ (2). Given $x \in \mathcal{C}$, we with to define a functor $R_x : \mathcal{D}^{op} \to Set$. Given $d \in \mathcal{D}$, let $h_d = Hom_{\mathcal{D}}(\_, d) : \mathcal{D}^{op} \to Set$ and let

$$R_x(d) = Hom_{\mathcal{C}}(L(h_d), x)$$

For any $x \to y$ in $\mathcal{C}$ and $d \to d'$ in $\mathcal{D}$, we get the following commutative diagram:

$$\begin{array}{ccc} Hom(L(h_{d'}), x) & \longrightarrow & Hom(L(h_{d'}), y) \\ \downarrow & & \downarrow \\ Hom(L(h_d), x) & \longrightarrow & Hom(L(h_d), y) \end{array}$$

which shows that

$$R : \mathcal{C} \to PSh(\mathcal{D})$$

$$x \mapsto R_x$$

is a functor. Since every presheaf $\mathcal{F}$ is a colimit of representable presheaves $\mathcal{F}_i$, we conclude that

$$\begin{aligned}
Hom_{\mathcal{C}}(L(\mathcal{F}), x) &= Hom_{\mathcal{C}}(L(\varinjlim_i \mathcal{F}_i), x) \\
&= Hom_{\mathcal{C}}(\varinjlim_i L(\mathcal{F}_i), x) \\
&= \varprojlim_i Hom_{\mathcal{C}}(L(\mathcal{F}_i), x) \\
&\simeq \varprojlim_i Hom_{PSh(\mathcal{D})}(\mathcal{F}_i, R_x) \\
&= Hom_{PSh(\mathcal{D})}(\varinjlim_i \mathcal{F}_i, R_x) \\
&= Hom_{PSh(\mathcal{D})}(\mathcal{F}, R_x)
\end{aligned}$$

□

2.6. **Corollary.** *Let $\mathcal{D}$ be a small category and $\mathcal{C}$ a category with all small colimits. Then there is an equivalence between $Funct(\mathcal{D}, \mathcal{C})$ and the full subcategory of $Funct(PSh(\mathcal{D}), \mathcal{C})$ consisting of functors that commute with all small colimits. In particular, for any functor $F : \mathcal{D} \to \mathcal{C}$, there exists an adjoint pair of functors (unique up to canonical isomorphism):*

$$PSh(\mathcal{D}) \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{C}$$

such that $F(d) = L(h_d)$ for any $d \in \mathcal{D}$.

Proof. We use the formula $L(h_d) = F(d)$ to define L on representable presheaves. Now assume that L commutes with all small colimits, and use this to define L for all presheaves. By the previous theorem, L has a right adjoint. □

3. The geometric realization of a simplicial set

Consider the functor $\Delta \to Top$ which sends $[n]$ to the topological n-simplex $|\Delta^n|$. By Corollary 2.6, we have a adjoint pair

$$Set_{\Delta} \underset{Sing}{\overset{|\cdot|}{\rightleftarrows}} Top$$

where

$$Sing(Y)_n = Hom_{Set_{\Delta}}(\Delta^n, Y) = Hom_{Top}(|\Delta^n|, Y)$$

for any topological space Y.

4. The nerve of a category

Consider the functor $\Delta \to Cat$ which sends the poset $[n]$ to the corresponding category $\mathbf{n}$. By Corollary 2.6, we have a adjoint pair

$$Set_\Delta \underset{N}{\overset{P}{\rightleftarrows}} Cat$$

where $N(\mathcal{C})$ denotes the *nerve* of the category $\mathcal{C}$:

$$N(\mathcal{C})_n = Hom_{Set_\Delta}(\Delta^n, N(\mathcal{C})) = Hom_{Cat}(\mathbf{n}, \mathcal{C})$$

Given $X \in Set_\Delta$, the category $P(X)$ is called the *Poincaré category* of X.

5. The simplicial nerve of a simplicial category

The construction of the simplicial nerve of a simplicial category was implicit in work by Boardman and Vogt in the 1970's, and was made explicit by J.-M Cordier and T. Porter in the 1980's. Consider the functor ([Lu], 1.1.5):

$$\mathfrak{C} : \Delta \to Cat_\Delta$$

$$[n] \mapsto \mathfrak{C}[n]$$

The set of objects of $\mathfrak{C}[n]$ is $\{0, 1, \ldots, n\}$. Given $0 \leq i, j, \leq n$, the simplicial set of morphisms from i to j is

$$Hom_{\mathfrak{C}[n]}(i, j) = N(P_{i,j})$$

where $P_{i,j}$ is the following poset:

$$P_{i,j} = \{I \subset \mathbb{Z} \mid i, j \in I, \text{ and } k \in I \implies i \leq k \leq j\}$$

If $j \geq i$, we can identify $P_{i,j}$ with the set of subsets of $\{i+1, \ldots, j-1\}$, with the poset structure induced by inclusions of subsets. This implies that

$$|P_{i,j}| = \begin{cases} 0 & i > j \\ 1 & i = j, j = i+1 \\ 2^{j-i-1} & i < j \end{cases}$$

By Corollary 2.6, we have a adjoint pair

$$Set_\Delta \underset{\mathfrak{N}}{\overset{\mathfrak{C}}{\rightleftarrows}} Cat_\Delta$$

where the *simplicial nerve* of a simplicial category is defined by:

$$\mathfrak{N}(\mathcal{C})_n = Hom_{Set_\Delta}([n], \mathfrak{N}(\mathcal{C})) = Hom_{Cat_\Delta}(\mathfrak{C}[n], \mathcal{C})$$

Given $K \in Set_\Delta$, the simplicial category $\mathfrak{C}(K)$ is called the *simplicial Poincaré category* of K.

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