Green-Schwarz anomaly cancellation

Paolo Di Vecchia

Niels Bohr Institutet, Copenhagen and Nordita, Stockholm

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Plan of the talk

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Introduction

- The theory of general relativity for gravity was formulated by Einstein in 1915.
- It is a four-dimensional theory that extends the theory of special relativity.
- While special relativity is invariant under the transformations of the Lorentz group, general relativity is invariant under an arbitrary change of coordinates.
- In the twenties it was proposed by Theodor Kaluza and Oskar Klein to unify electromagnetism with gravity by starting from general relativity in a five-dimensional space-time and compactify the extra-dimension on a small circle.
- In this way one obtains general relativity in four dimensions, a vector gauge field satisfying the Maxwell equations and a scalar.
- This idea of extra dimensions was not pursued in the years after.
- In the sixties and seventies, when I started to work in the physics of the elementary particles, everybody was strictly working in four dimensions.
Also the dual resonance model, being a model for hadrons, was obviously formulated in four dimensions.

and nobody questioned this point until the paper by Lovelace in which he proposed $d = 26$ for avoiding problems with unitarity.

But even then almost nobody took his paper seriously !!

Little by little string theorists started to accept $d = 26$ and $d = 10$.

But at the beginning, this was for them a source of embarrassment.

Only later with the change of perspective from hadron physics to the theory unifying gauge theories with gravity [Scherk and Schwarz, 1975] extra dimensions started to be accepted.

In the same year Cremmer and Scherk compactified the six extra dimensions of the NSR model.

They write: we have proved that this proposal is compatible with everyday experience.

At this point it became clear that the existence of extra dimensions was an important prediction of string theory, rather than a source of embarrassment.
The formulation of supergravity in 1976 and especially of those supergravities with higher amount of supersymmetry, as the $N = 8$ supergravity, opened the possibility of unifying, in a consistent quantum theory, gauge theories with gravity in the framework of ordinary field theory.

Due to the cancellation of bosonic and fermionic contributions, those theories have in fact a much higher degree of convergence.

In particular, it was conjectured that $N = 8$ supergravity was a finite theory of gravity.

It included a $SU(8)$ gauge group that was big enough for particle phenomenology.

This discussion on the finiteness of $N = 8$ supergravity is still going on today, but this is not the subject of this lecture.

For this reason string theory was abandoned.

In 1978 the Lagrangian of a unique and most symmetric supergravity, 11-dimensional supergravity, was written down by Cremmer, Julia and Scherk here in Paris.
By compactifying the seven extra dimensions on various manifolds one obtained a number of four-dimensional theories.

The problem that one soon met was the difficulty of having four-dimensional theories with chiral fermions.

But quarks and leptons are chiral fermions and therefore any phenomenological viable theory must contain chiral fermions.

The difficulty of generating chiral fermions in the Kaluza-Klein reduction of 11-dimensional supergravity led many to go back to string theory.

It was already known that type IIB and type I superstrings had chiral fermions in $d = 10$.

On the other hand, chiral fermions have the tendency to generate gauge and gravitational anomalies.

The question was: are those string theories free from gauge and gravitational anomalies?
The presence of an anomaly means that a current, conserved in the classical theory, is not conserved anymore when we quantize the theory.

Here we want to stress that there are two kinds of anomalies.

Those associated with a current corresponding to a global symmetry as for instance the axial $U(1)$ current in QCD.

This kind of anomaly is not a problem and actually welcome, as in the case of the $U(1)$ current in QCD, in order to reproduce the correct spectrum of pseudoscalar mesons.

Also an anomaly in the axial vector current is essential for getting the decay rate $\pi^0 \rightarrow 2\gamma$ in agreement with the experiments.

These anomalies have an observable and welcome effect.
Other anomalies are those associated to currents coupled to a gauge field and to gravitons.

The lack of conservation of these currents implies that the gauge symmetry or the invariance under arbitrary coordinate transformation is broken at the quantum level.

This may imply that the unphysical non-transverse components of the gauge field and of the graviton are not decoupled from the physical spectrum giving rise to zero or negative norm states in the physical spectrum.

This is in contradiction with the positivity of the probability.
A quick look at the abelian axial anomaly

Let us consider a fermion in interaction with an external vector and axial vector fields. The Lagrangian is given by

\[ L = \bar{\Psi} \gamma^\mu D_\mu \Psi \]
\[ D_\mu = \partial_\mu + iV_\mu + iA_\mu \gamma_5 \]

We work in a \( d \) dimensional euclidean space (\( d \) is even) with

\[ \gamma_{d+1} = \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_d \]
\[ \gamma^2_{d+1} = (-1)^{d/2} \]

\( L \) is invariant under \( U(1) \) local vector transformations

\[ \psi(x) \rightarrow e^{-i\Lambda(x)} \psi(x) \; ; \; \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{+i\Lambda(x)} \; ; \; V_\mu \rightarrow V_\mu + \partial_\mu \Lambda(x) \]

and local axial vector transformations

\[ \psi(x) \rightarrow e^{-i\alpha(x)\gamma_5} \psi(x) \; ; \; \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\alpha(x)\gamma_5} \; ; \; A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x) \]

In the quantum theory the vector symmetry is maintained, while the axial one has an anomaly.
This can be seen by computing the functional integral over the fermions:

\[ e^{W(V_\mu, A_\mu)} \equiv \int d\psi d\bar{\psi} \ e^{-\int d^d x \ (\bar{\psi}(\gamma^\mu D_\mu + im)\psi} \]

The functional integral can be computed and one gets:

\[ W(V_\mu, A_\mu) = \text{Tr} \log (\gamma^\mu D_\mu + im) \]

In the quantum theory the vector symmetry can be maintained, but then the axial vector symmetry has an anomaly:

\[ \delta_V W(V_\mu, 0) = 0 ; \quad \delta_A W(V_\mu, 0) \neq 0 \]

In particular, one gets:

\[ \delta_A W(V_\mu, 0) = i \text{Tr} \left[ (\gamma^\mu (\partial_\mu + iV_\mu) + im)^{-1} \gamma^\nu \partial_\nu \alpha \gamma_{d+1} \right] \]
The previous expression can be expanded in powers of the field $V_\mu$ to get ($\mathcal{C} \equiv \gamma^\mu C_\mu$)

$$
\delta W(V_\mu) = i \text{Tr} \left\{ (\partial + im)^{-1} \left[ 1 - i \mathcal{V}(\partial + im)^{-1} \right. \right.
$$

$$
- \left. \mathcal{V}(\partial + im)^{-1} \mathcal{V}(\partial + im)^{-1} + \ldots \right] \partial \alpha \gamma_5 \right\}
$$

If $d = 2$ only the second term is not vanishing and one gets:

$$
\delta_A W(V_\mu) = \text{Tr} \left( (\partial + im)^{-1} \mathcal{V}(\partial + im)^{-1} \partial \alpha \gamma_5 \right)
$$

$$
= i \int d^2x \int d^2y \alpha(x) V_\mu(y) \int \frac{d^2q}{(2\pi)^2} e^{iq \cdot (x-y)} q_\nu \Gamma^{\mu\nu}(q)
$$

where

$$
q_\nu \Gamma^{\mu\nu}(q) = \int \frac{d^2r}{(2\pi)^2} \frac{\text{Tr}[((r+q) - m)\gamma^\mu (r - m) \partial \gamma_5]}{(r^2 + m^2)((r + q)^2 + m^2)}
$$
By suitably regularizing the previous integral we get

$$ q_\nu \Gamma^{\mu\nu} = \frac{1}{\pi} \epsilon^{\mu\nu} q_\nu $$

This implies that:

$$ \delta_A W(V_\mu) = -\frac{1}{2\pi} \int d^2 \mathbf{x} \alpha(\mathbf{x}) \epsilon^{\mu\nu} F_{\mu\nu}(\mathbf{x}) ; \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu $$

that gives the axial anomaly in two dimensions.

In conclusion, classically $\delta_A S = 0$, while quantum mechanically $\delta_A W \neq 0$.

The next contribution in the expansion is a term with three propagators that gives the axial anomaly in $d = 4$ and so on.
Few words on forms

- For treating the gauge and gravitational anomalies it is very convenient to use a formalism based on forms.
- Instead of working with the Yang-Mills potential $A_\mu$ and field strength $F_{\mu\nu}$, it is more convenient to introduce the following one- and two-forms:

\[
A \equiv A_\mu dx^\mu \quad ; \quad F \equiv F_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2} \quad ; \quad F = dA + A \wedge A
\]

and work with them.
- $A$ and $F$ are matrices corresponding to a representation of the gauge group.
- $A$ and $F$ transform as follows under a gauge transformation:

\[
\delta A = d\Lambda + [A, \Lambda] \quad ; \quad \delta F = [F, \Lambda]
\]

- In the following we will consider only the fundamental and the adjoint representation of the group and we denote with $\text{tr}$ the trace in the fundamental and with $\text{Tr}$ the trace in the adjoint.
Analogously in the case of gravity, instead of working with the spin connection $\omega_{\mu ab}$, one introduces the one-form $\omega$ and the two-curvature form $R$:

$$\omega = \omega_\mu dx^\mu \; ; \; \; R = R_{\mu \nu} \frac{dx^\mu \wedge dx^\nu}{2} = d\omega + \omega \wedge \omega$$

They are also matrices transforming according to the fundamental representation of the Lorentz group $SO(9, 1)$.

The wedge products $\omega \wedge \omega$ and $A \wedge A$ represent also the product of matrices.

In the following, for the sake of simplicity, we omit to explicitly write the $\wedge$ symbol.

$\omega$ and $R$ transform as follows under a local Lorentz rotation:

$$\delta \omega = d\Theta + [\omega, \Theta] \; ; \; \; \delta R = [R, \Theta]$$

This description of gravity is different from the usual one where one introduces the metric, the Christoffel symbols and the curvature tensor $R_{\mu \rho,\nu \sigma}$ and from it one computes the Ricci tensor $R_{\mu \nu}$. $R_{\mu \nu}$ above is not the Ricci tensor!
Simple calculations with forms

\[ F \equiv F_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2} = dA + A^2 \]

\[ = \partial_\mu A_\nu dx^\mu \wedge dx^\nu + A_\mu A_\nu dx^\mu \wedge dx^\nu \]

\[ = \frac{dx^\mu \wedge dx^\nu}{2} \left[ \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \right] \]

Remember that

\[ dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \]
Anomaly cancellation in type IIB superstring theory

- Type IIB superstring is a chiral theory because both the two gravitinos and dilatinos are chiral fermions in ten dimensions.
- It has also a chiral boson because it contains in the spectrum a four-index R-R potential $C_4$ with self-dual field strength:

$$\tilde{F}_5 = \ast \tilde{F}_5 \quad ; \quad F_5 = dC_4 - \frac{1}{2} C_2 \wedge F_3 + \frac{1}{2} B_2 \wedge F_3$$

Remember that it contains also a R-R $C_0$ and $C_2$ with field strength $F_3 = dC_2$ and a NS-NS $B_2$ with field strength $H_3 = dB_2$.

- The contribution of these chiral fields to the gauge and gravitational anomalies were computed by Alvarez-Gaumé and Witten in 1984.
- In type IIB there is no gauge field and therefore no gauge anomaly.
They actually computed an auxiliary quantity $A_{12}$ (it is a 12-form) for the different chiral fields.

The anomaly can then be computed from $A_{12}$ following the following procedure.

From $A_{12}$ one determines first $A_{11}$ such that:

$$A_{12} = dA_{11}$$

$A_{11}$ is a 11-form.

From $A_{11}$ one determines $A_{10}^1$ such that:

$$\delta A_{11} = dA_{10}^1$$

$A_{10}^1$ gives the anomaly in ten dimensions that is linear in the parameters of the gauge transformation (this is why the index 1).

$\delta$ means gauge and gravitational variations:

$$\delta A = d\Lambda + [A, \Lambda] ; \quad \delta F = [F, \Lambda]$$
$$\delta \omega = d\Theta + [\omega, \Theta] ; \quad \delta R = [R, \Theta]$$
In the following we write the contributions of the various chiral fields.

The contribution to the anomaly of the gravitino is given by:

\[
A_{12}^{3}(R) = -\frac{11 \cdot 9 \cdot 5}{2^6} \left[ \frac{\text{tr}(R^6)}{5670} + \frac{\text{tr}(R^2)\text{tr}(R^4)}{4320} + \frac{(\text{tr}(R^2))^3}{10368} \right] \\
+ \frac{\text{tr}(R^2)}{3 \cdot 2^7} \left[ \text{tr}(R^4) + \frac{1}{4} (\text{tr}(R^2))^2 \right]
\]

That of a spin \(\frac{1}{2}\) is equal to

\[
A_{12}^{1}(R, F) = -\frac{1}{720} \text{Tr}(F^6) + \frac{1}{24 \cdot 48} \text{Tr}(F^4)\text{tr}(R^2) \\
- \frac{1}{2^8} \text{Tr}(F^2) \left[ \frac{1}{45} \text{tr}(R^4) + \frac{1}{36} (\text{tr}(R^2))^2 \right] \\
+ \frac{n}{2^6} \left[ \frac{1}{5670} \text{tr}(R^6) + \frac{1}{4320} \text{tr}(R^2)\text{tr}(R^4) + \frac{1}{10368} (\text{tr}(R^2))^3 \right]
\]

\(n\) is the number of generators of the gauge group.
Finally that of the antisymmetric tensor is equal to

\[ A_{12}^A(R) = \frac{1}{2835} \left[ \frac{496}{128} \text{tr}(R^6) - \frac{588}{4 \cdot 64} \text{tr}(R^2) \text{tr}(R^4) + \frac{140}{8 \cdot 64} \left( \text{tr}(R^2) \right)^3 \right] \]

The anomaly in type IIB is proportional to the following combination:

\[ A_{12}^{TOT} \sim A_{12}^3(R) - A_{12}^1(R, 0) + A_{12}^A(R) \]

Remember that type IIB superstring is a closed string theory without gauge fields.

The relative minus sign between gravitino and dilatino is due to their opposite chirality.

Using the previous formulas it is easy to check that \( A_{12}^{TOT} = 0 \)

No gravitational anomaly in type IIB superstring !!
Anomaly cancellation in type I superstring

In type I theory the anomaly is given by:

\[ A_{12} \sim A_{12}^3(R) - A_{12}^1(R) - A_{12}^2(R, F) \]

We have no contribution from the antisymmetric tensor (absent in the spectrum of type I), but we have the contribution of the Weyl-Majorana gauginos.

Using the previous formulas we get:

\[ A_{12} \sim \frac{n - 496}{2^6} \left[ \frac{\text{tr}(R^6)}{5670} + \frac{\text{tr}(R^4)\text{tr}(R^2)}{4320} + \frac{(\text{tr}(R^2))^3}{10368} \right] \]

\[ + \frac{\text{tr}(R^2)}{3 \cdot 2^7} \left[ \text{tr}(R^4) + \frac{1}{4} (\text{tr}(R^2))^2 \right] + \frac{1}{24 \cdot 48} \text{Tr}(F^4)\text{tr}(R^2) \]

\[ - \frac{1}{2^8} \text{Tr}(F^2) \left[ \frac{1}{45} \text{tr}(R^4) + \frac{1}{36} (R^2)^2 \right] - \frac{1}{720} \text{Tr}(F^6) \]
The first term can be cancelled only if the gauge group has 496 generators.

This selects groups as $SO(32)$ and $E_8 \times E_8$ or also $E_8 \times (U(1))^{248}$.

Since type I theory has a $SO(N)$ gauge group, here we consider only $SO(32)$.

Actually also $SP(N)$ is allowed in type I, but it will be anomalous.

For the sake of simplicity, we restrict our analysis only to the gauge anomaly that is given by only one term:

$$A_{12} \sim -\frac{1}{720} \text{Tr}(F^6)$$

The trace is in the adjoint representation of $SO(32)$.

In $SO(32)$ we have the following identity:

$$\text{Tr}(F^6) = 15\text{tr}(F^2) \cdot \text{tr}(F^4)$$

The last two traces are in the fundamental representation of $SO(32)$.
We get:

\[ A_{12} \sim -\frac{1}{48} \text{tr}(F^2) \cdot \text{tr}(F^4) \]

\[ A_{11} \] can easily be computed from the observation that

\[ d\omega_3 \equiv d\text{tr} \left( AF - \frac{1}{3} A^3 \right) = \text{tr}(F^2) \]

\( \omega_3 \) is called the Chern-Simon three-form.

Since \( d\text{tr}(F^4) = 0 \) one gets:

\[ A_{11} \sim -\omega_3 \cdot \text{tr}(F^4) \]

Using then the following identity:

\[ \delta \omega_3 = \text{tr}(d\wedge dA) \]

one gets:

\[ \delta A_{11} = -\delta \omega_3 \text{tr}(F^4) = -\text{tr}(d\wedge dA) \cdot \text{tr}(F^4) \]

\[ = -d\left( \text{tr}(\wedge dA) \cdot \text{tr}(F^4) \right) \equiv dA_{10}^1 \]
Few details on the Chern-Simon 3-form

The Chern-Simon 3-form is given by:

\[ \omega_3 \equiv \text{tr} \left( AF - \frac{1}{3} A^3 \right) = \text{tr} \left( \frac{1}{2} A_\mu F_{\nu\rho} - \frac{1}{3} A_\mu A_\nu A_\rho \right) dx^\mu \wedge dx^\nu \wedge dx^\rho \]

From it one gets:

\[ d\omega_3 = \text{tr} \left( dAF - AdF - \frac{1}{3} dAA^2 + \frac{1}{3} AdAA - \frac{1}{3} A^2 dA \right) \]
\[ = \text{tr} \left( F^2 - A^2 F - A(dAA - AdA) - A^2 dA \right) = \text{tr} \left( F^2 \right) \]

It is also easy to show that

\[ \delta \omega_3 = \text{tr} \left( d\wedge dA \right) \]

where

\[ \delta A = d\Lambda + [A, \Lambda] ; \quad \delta F = [F, \Lambda] \]
The anomaly in ten dimensions is then given by:

$$A_{10}^1 \sim -\text{tr}(\Lambda dA) \cdot \text{tr}(F^4)$$

The previous calculation shows that there is an anomaly.

In the previous considerations, however, we have used the formulas computed by Alvarez-Gaumé and Witten that are valid in the limiting field theory with very specific couplings.

The limiting field theory is type I supergravity.

In type I supergravity there are additional couplings involving the Chern-Simon 3-form.

Because of this, the two-form potential $C_2$ is changed by a gauge transformation.

This gives an additional possibility to cancel the anomaly.

Let us then look at type I supergravity.
The action of type I supergravity is given by:

\[ S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left[ R + 4 \partial_\mu \phi \partial^\mu \phi \right. \\
- \left. \frac{1}{12} (F_3)_{\mu_1\mu_2\mu_3} (F_3)^{\mu_1\mu_2\mu_3} \right] - \frac{1}{2g_{10}^2} \int d^{10}x e^{-\phi} \text{tr} \left( F_{2\mu_1\mu_2} F^{\mu_1\mu_2}_2 \right) \]

Local supersymmetry requires the presence of the Chern-Simon form \( \omega_3 \):

\[ \tilde{F}_3 = F_3 - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3 ; \quad F_3 = dC_2 ; \quad C_2 \equiv C_{2\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2} \]

\[ F_3 = \partial_\rho C_{2\mu\nu} \frac{dx^\rho \wedge dx^\mu \wedge dx^\nu}{6} \]

The previous Lagrangian that has been determined by requiring local supersymmetry, contains two arbitrary coupling constants: the gravitational one \( \kappa_{10} \) and the gauge one \( g_{10} \).
Since $\omega_3$ transforms as follows under a local gauge transformation:

$$\delta \omega_3 = \text{tr}(d \wedge dA)$$

the previous action is invariant under gauge transformations if $C_2$ also transforms as follows:

$$\delta C_2 = \frac{\kappa_{10}^2}{g_{10}^2} \text{tr}(dA) \implies \delta \tilde{F}_3 = 0$$

The fact that $C_2$ transforms in such an unconventional way under a gauge transformation allows one to add a local counter-term whose gauge variation cancels the anomaly.
The local counter-term is equal to:

\[ W \sim \frac{g_{10}^2}{\kappa_{10}^2} C_2 \wedge tr(F^4) \]

By performing on it a gauge transformation we get:

\[ \delta W \sim \frac{g_{10}^2}{\kappa_{10}^2} \delta C_2 \wedge tr(F^4) = tr(\Lambda dA) \wedge tr(F^4) \]

One finds that

\[ l_{10}^1 + \delta W = 0 \]

No anomaly in type I supergravity.

The cancellation of the anomaly occurs more naturally in type I string theory.

Because of chiral fermions, one gets an anomaly due to the one-loop hexagonal planar diagram, that is equal to the one that we have computed before in field theory.
But in string theory one has also a non-planar one-loop diagram. This contribution is also anomalous and such to cancel the anomaly that comes from the planar loop.

In particular, the anomalous contribution in the non-planar loop comes from a closed string "tree diagram" where the field $C_2$ is exchanged.

Although the anomaly has also been cancelled in the limiting type I supergravity by adding by hand a local counter-term, this cancellation appears more natural in string theory.

In a way this cancellation is due to string effects.

This follows from the fact that in string theory $\kappa_{10}$ and $g_{10}$ are given by:

\[
\kappa_{10}^2 \sim (2\pi)^7 (\alpha')^4 \quad ; \quad g_{10}^2 \sim (2\pi)^7 (\alpha')^3 \quad \Rightarrow \quad \frac{\kappa_{10}^2}{g_{10}^2} \sim \alpha'
\]

This means that ($c$ is a numerical constant )

\[
\tilde{F}_3 = F_3 - c\alpha' \omega_3
\]
Conclusions

- The fact that type I string theory has chiral fermions and is free from gauge and gravitational anomalies generated in 1984 a lot of enthusiasm among particle physicists.
- A lot of them started to work again in string theory.
- Two new heterotic string theories with gauge groups $E_8 \times E_8$ and $SO(32)$ were constructed.
- They are called heterotic because the left movers are as in superstring, while the right movers are as in the bosonic string with $16 = 26 - 10$ directions compactified on a 16-torus.
- They are closed string theories, but, unlike type II theories, contain gauge bosons and their interaction unifies, in a consistent quantum theory, gauge theories with gravity.
- They are free from gauge and gravitational anomalies as it was anticipated by the anomaly calculation.
- All five consistent superstring theories are 10-dimensional and supersymmetric.
In order to be consistent with experiments, we have to compactify six of the ten dimensions on a compact manifold.

It was found that, by compactifying the six extra dimensions on a Calabi-Yau manifold, one is left with a $\mathcal{N} = 1$ four-dimensional supersymmetric theory that can have chiral fermions.

In other words, one started to try to connect string theory with experiments.

This is still going on today.