Bordisms, Anomalies, and Symmetries

Based on work of the last five year, e.g.:

2107.14227 + 2302.00007 with A. Debray, J. J. Heckman, M. Montero

2212.04503 + to appear with P.-K. Oehlmann, T. Schimannek

2504.02934 with M. Tartaglia

Geometry, Topology & Physics - February 18, 2026

NYU Abu Dhabi

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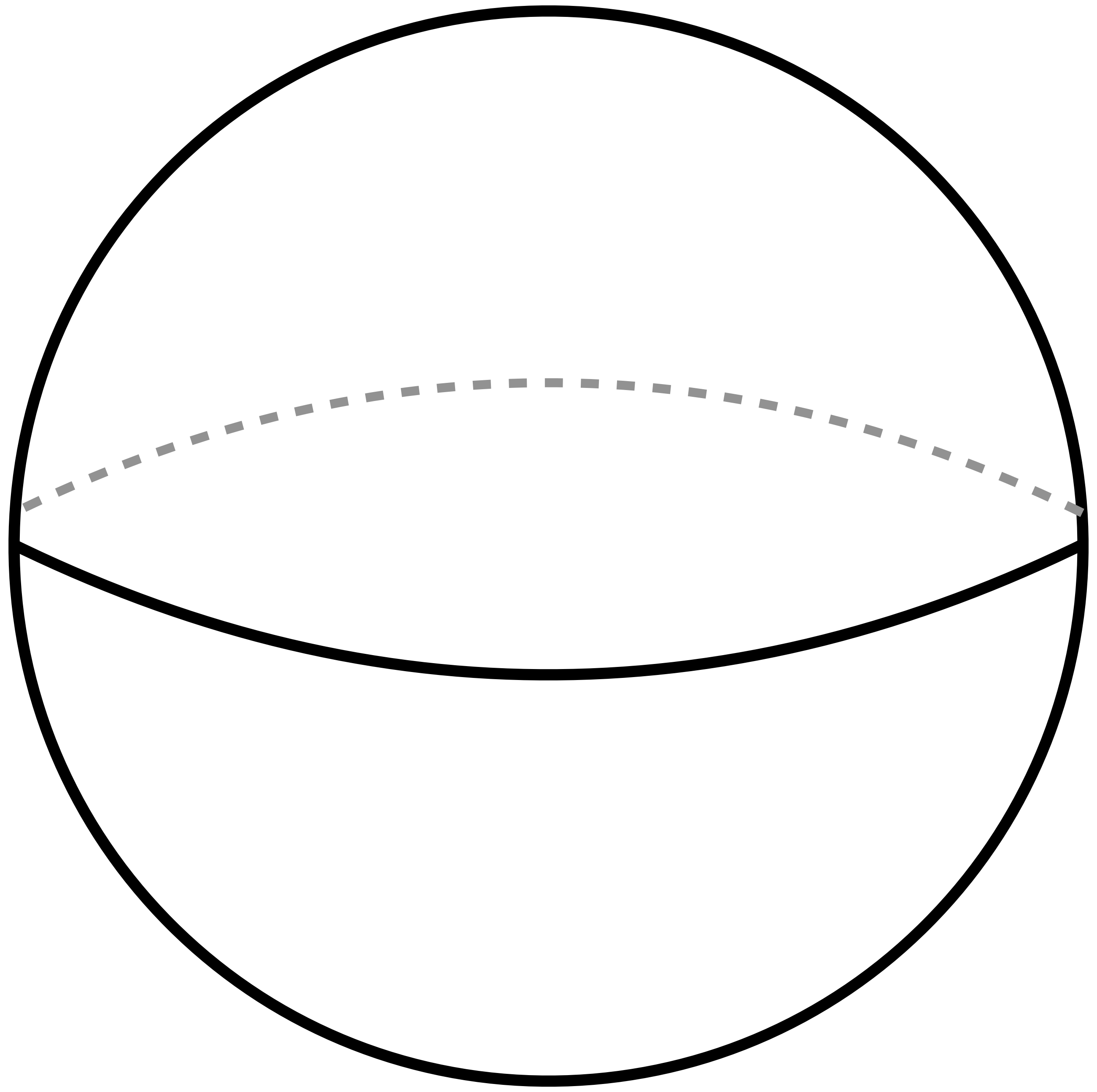
What is a bordism?

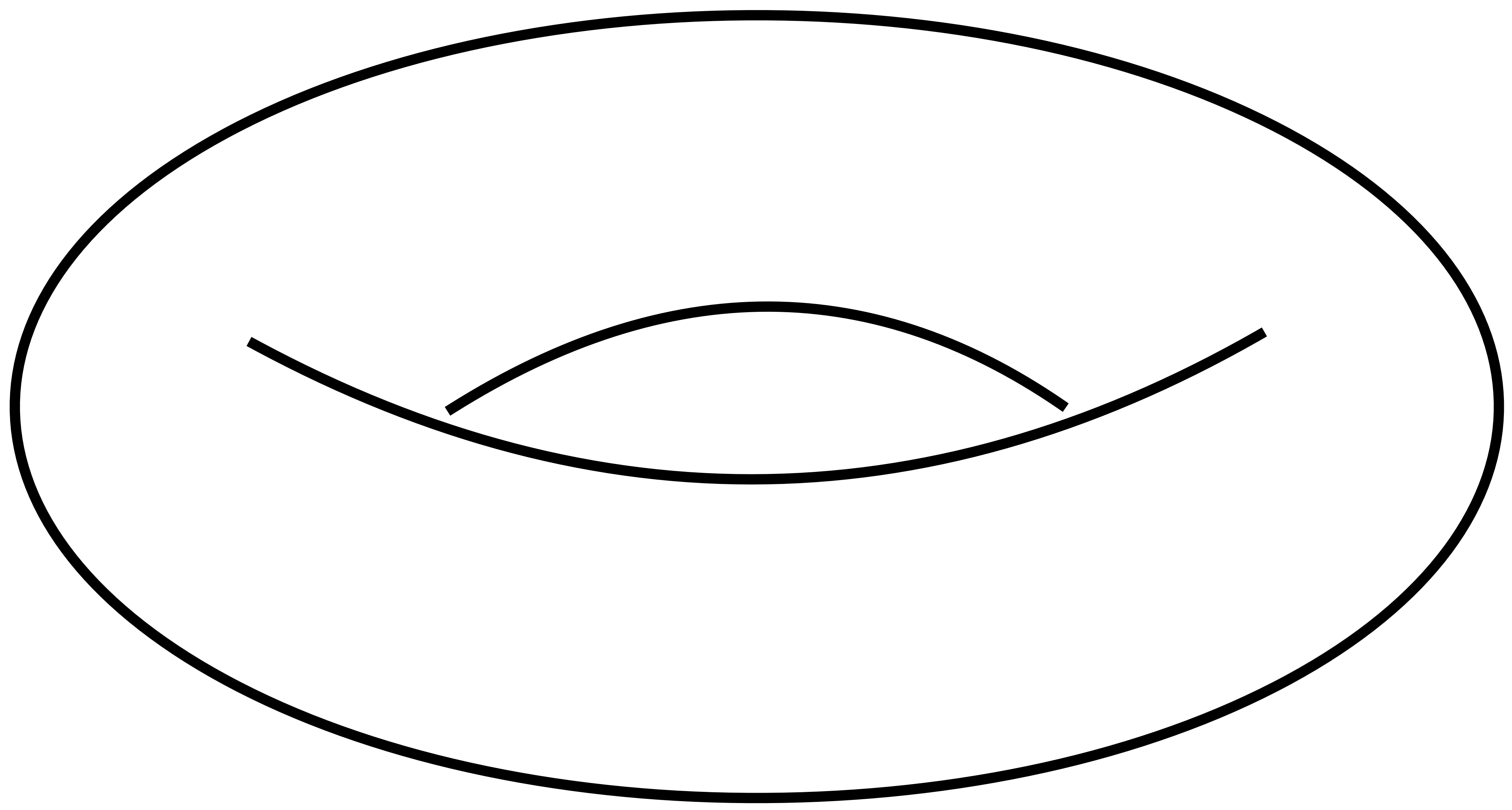
The **answer to the question**:

Can I take this:



What is a bordism?

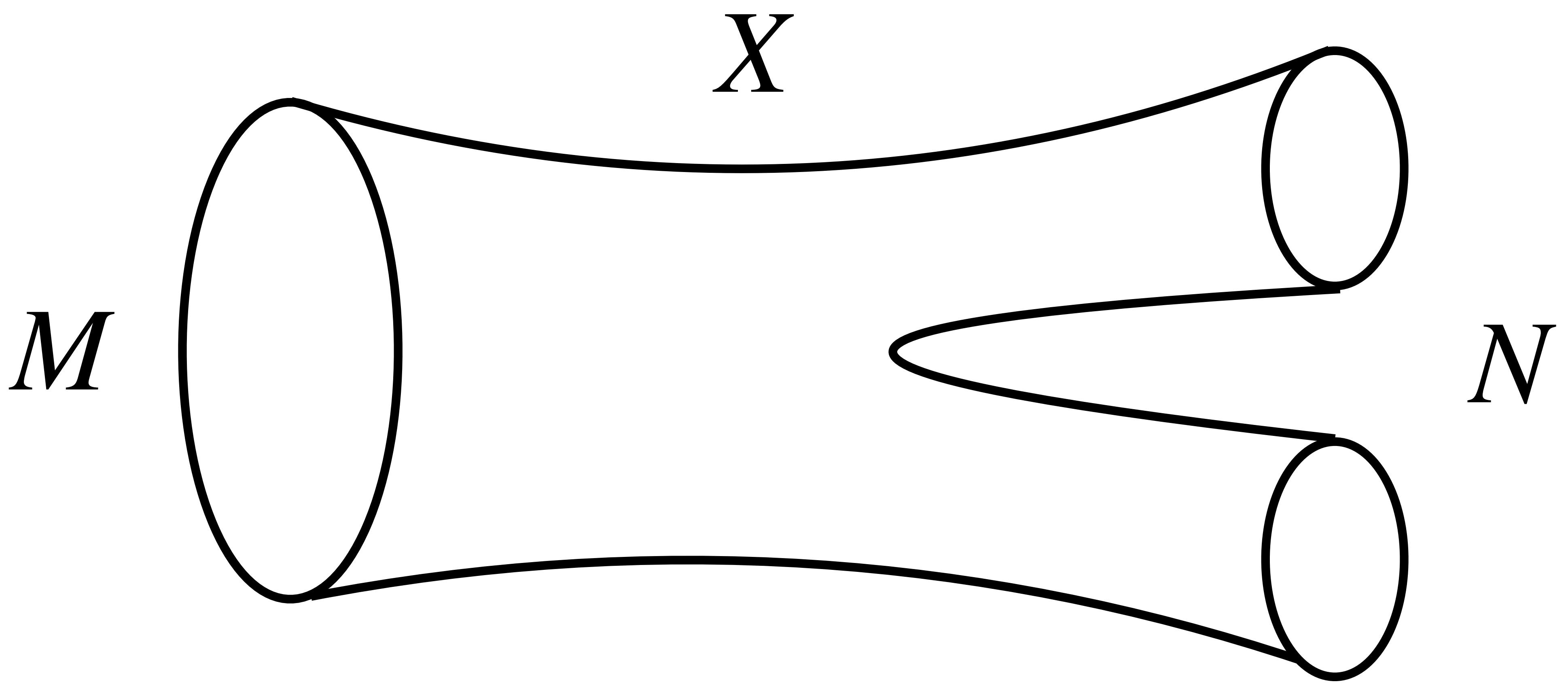
The **answer to the question**:

And deform it into this:  **?**

Yes, if they are bordant (the same element in the bordism group).

What is a bordism?

More generally: compact **d-dimensional** manifolds are **bordant**, if there is a **(d+1)-dimensional** one that ‘**connects**’ them



$$\partial X = M \sqcup -N$$

The deformation classes that cannot be connected form the:

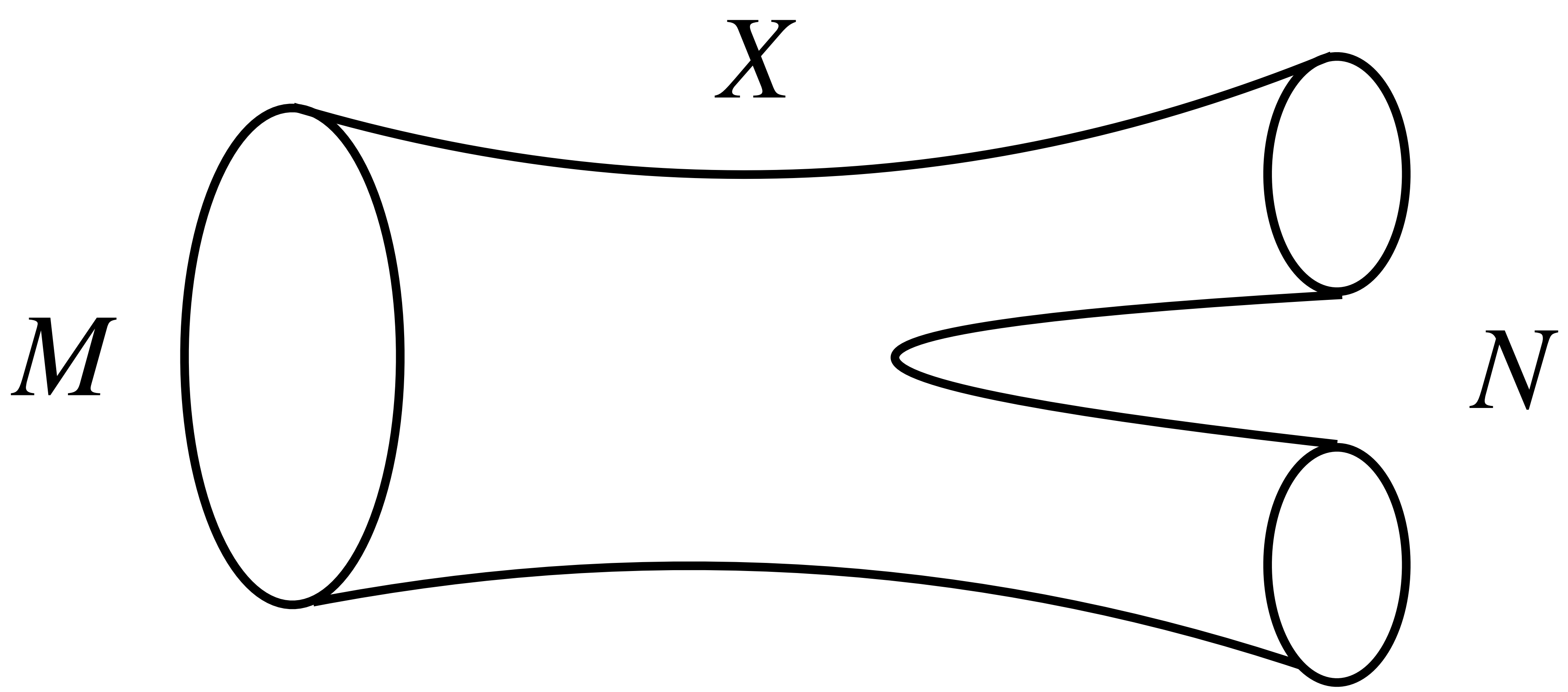
bordism group Ω_d (generalized homology theory)

Add information

One can **restrict deformations** by adding **data:**

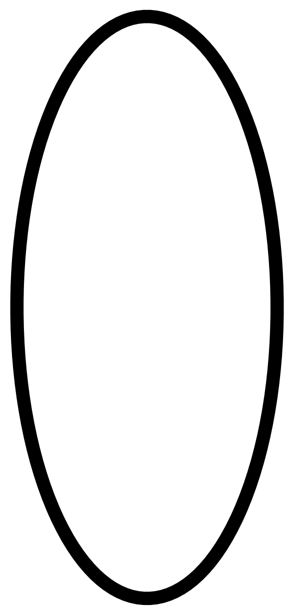
- Requirement on **manifolds**, e.g., Spin, orientation, …
- Inclusion of **gauge fields**
- **Mixtures of the two**

$$\Omega_d^{\xi}(BG)$$

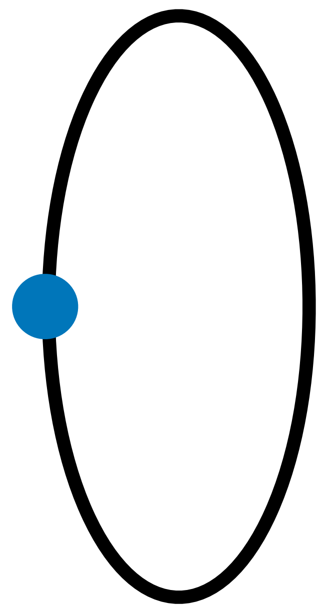


Structure extends and is compatible on boundaries

Examples:

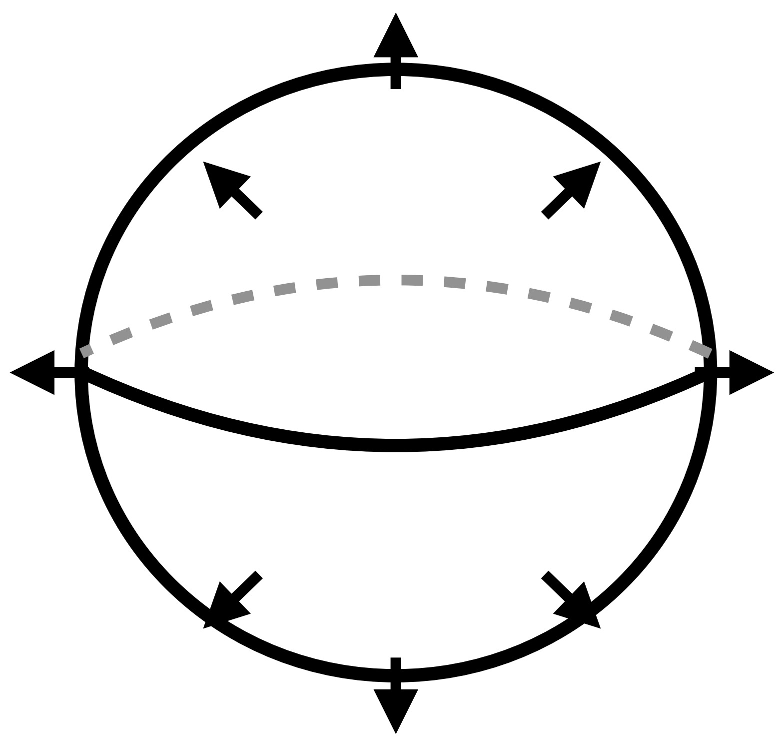


$\psi \to -\psi$



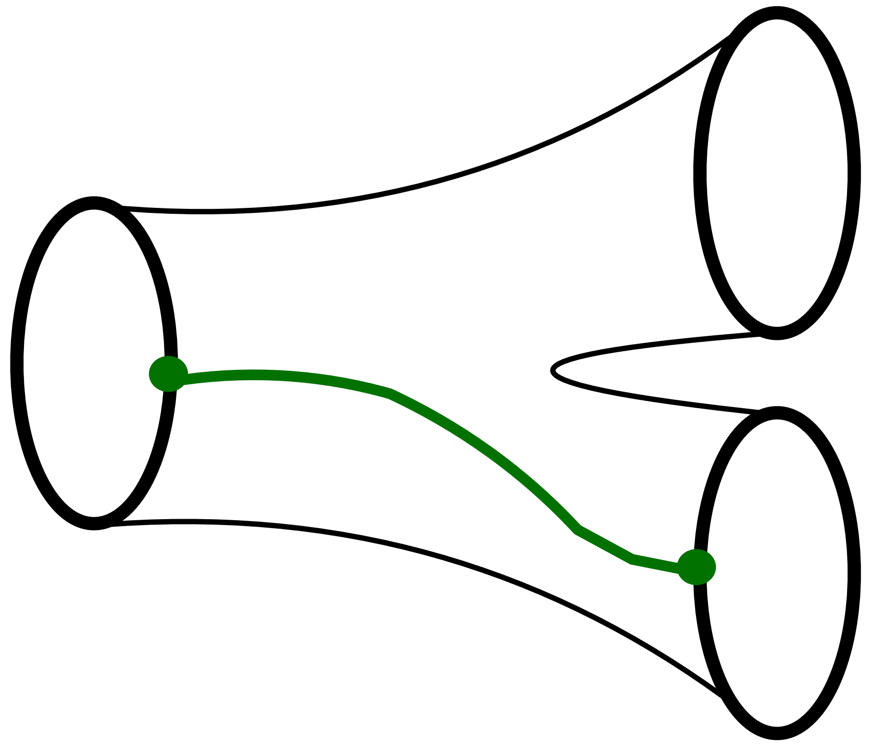
$\psi \to +\psi$

Spin structure $\Omega_1^{\mathrm{Spin}}(\mathrm{pt})$



$\frac{1}{2\pi}\oint F \neq 0$

Magnetic flux $\Omega_2^{\mathrm{SO}}(B\mathrm{U}(1))$

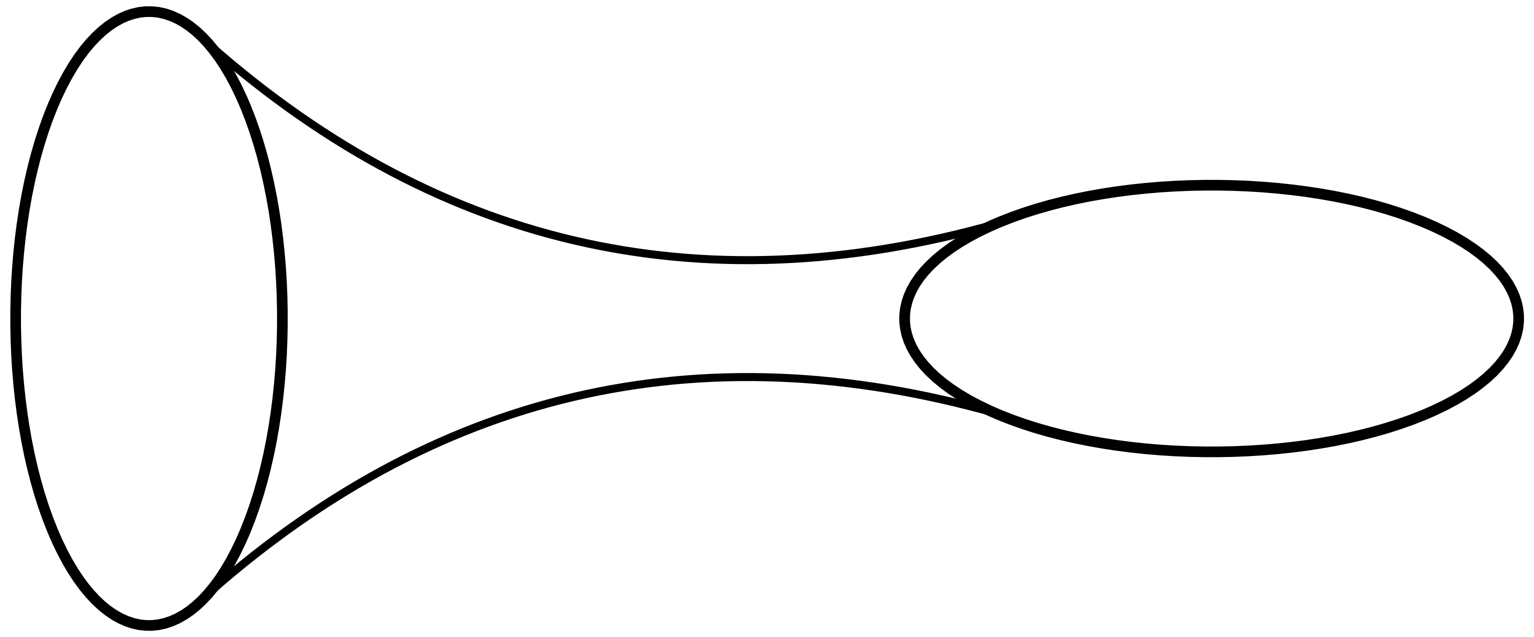


$H^1(X;\mathbb{Z}_n)$

Discrete gauge fields $\Omega_1^{\mathrm{SO}}(B\mathbb{Z}_n)$

Why do we care (in quantum gravity)?

(Here we work in Euclidean signature)

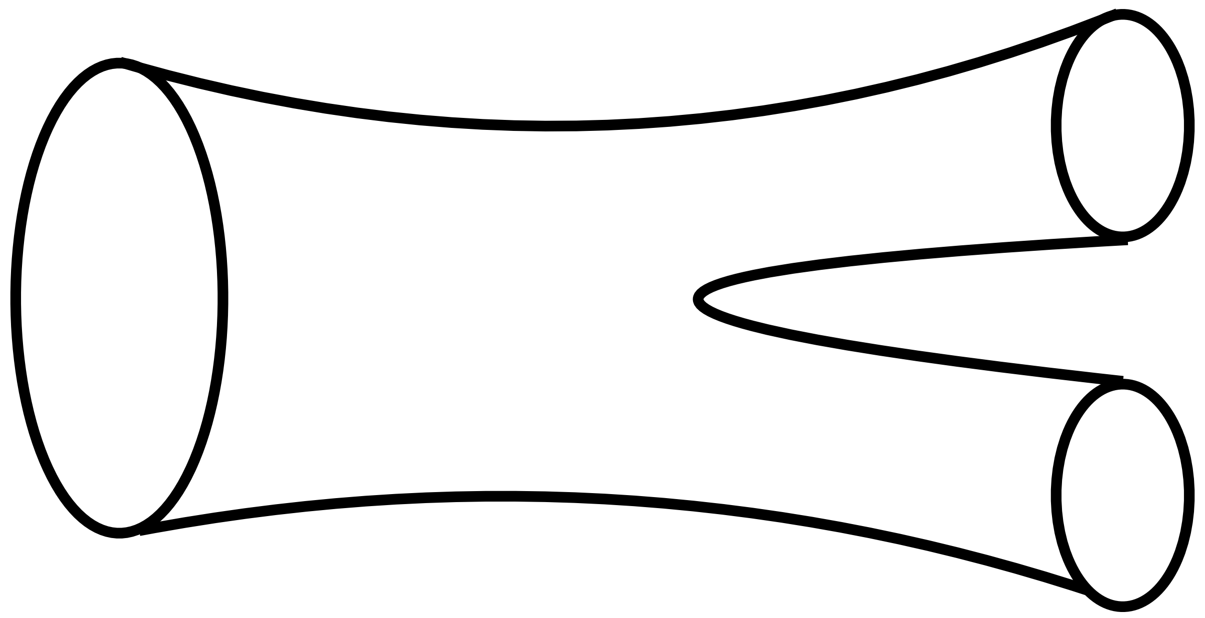


Gravity: continuous dynamics of spacetime

(not so interesting from bordism perspective;
continuous deformation provides bordism)

Quantum Gravity: + discrete changes of topology of spacetime

(Way more interesting from bordism perspective)

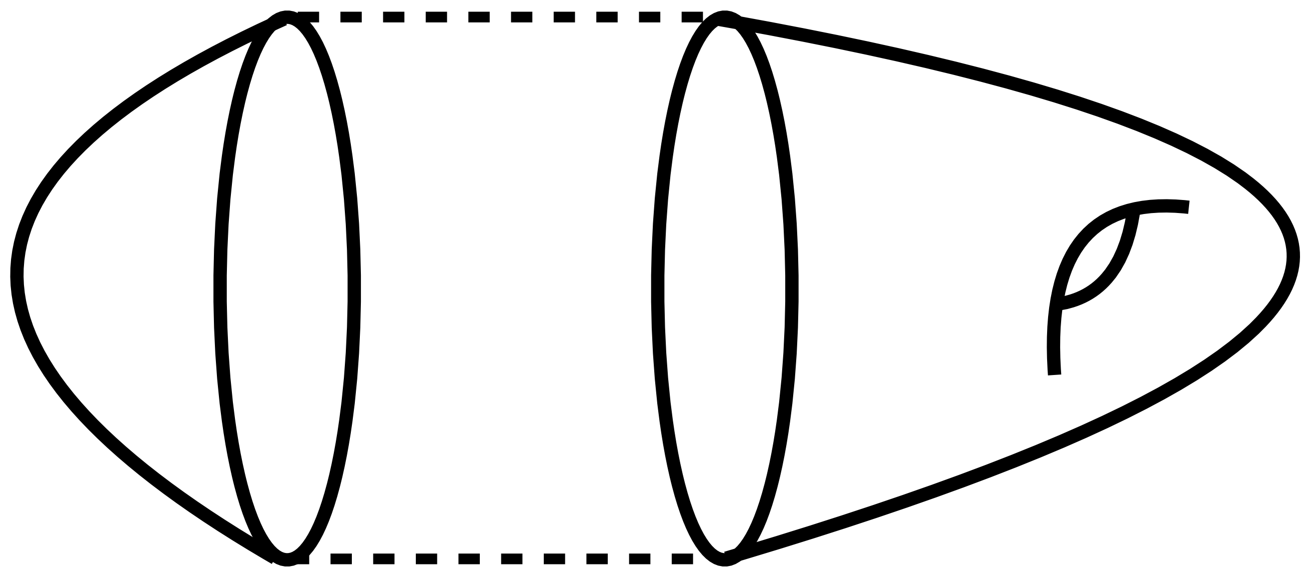


→ **Bordisms** capture some **topological features** of the **low-energy limit** of quantum gravity

Why do we care (in quantum gravity)?

Anomalies

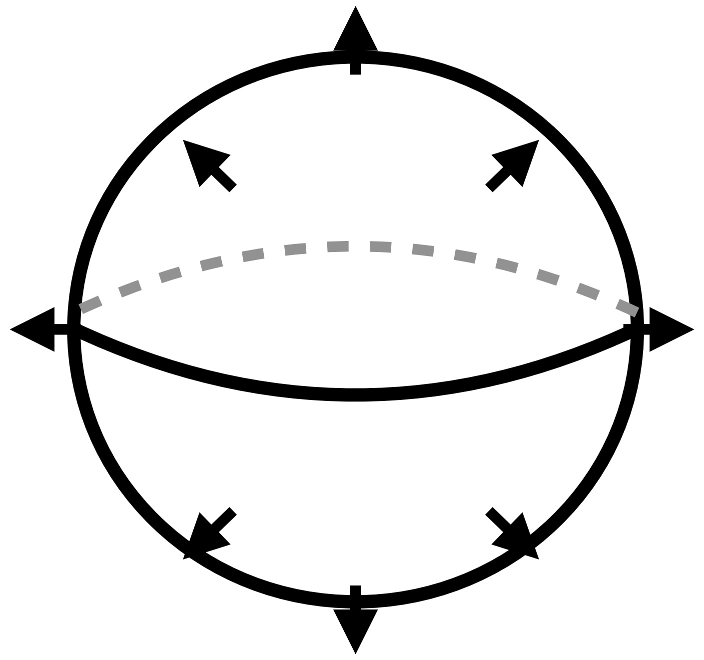
breaking of symmetries via quantum gravity effects



inconsistency for gauge symmetries

Global symmetries

definition of conserved charges associated to symmetries



absent in quantum gravity theories

Anomalies

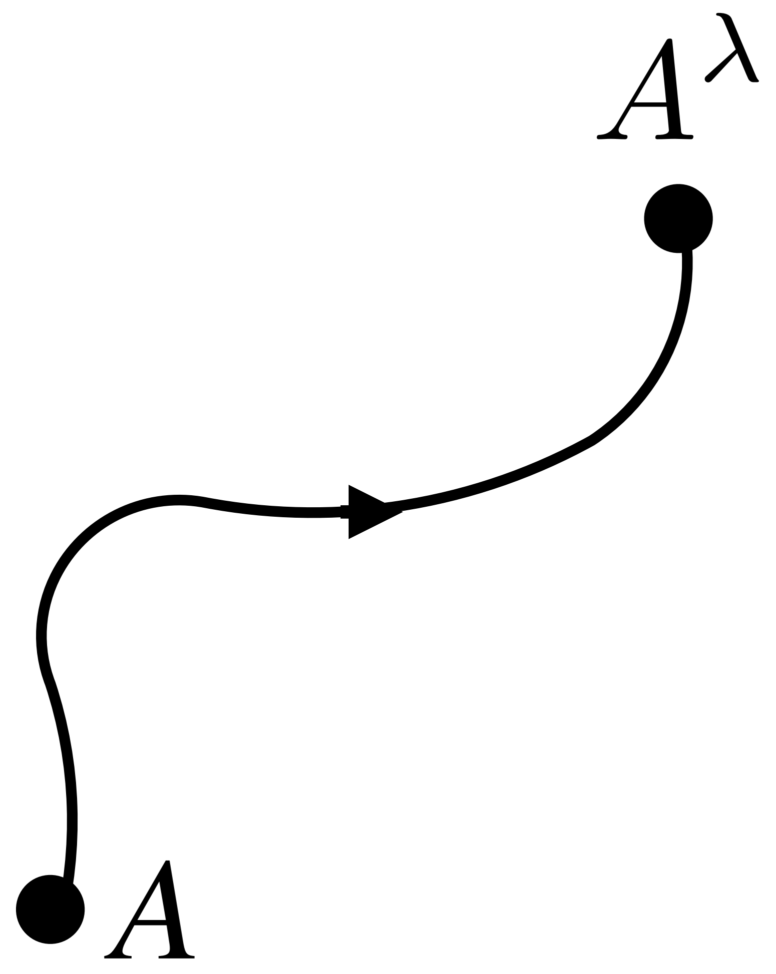
Anomalies

Breaking of symmetry at the quantum level (i.e. partition function)

$$Z(A) \neq Z(A^\lambda)$$

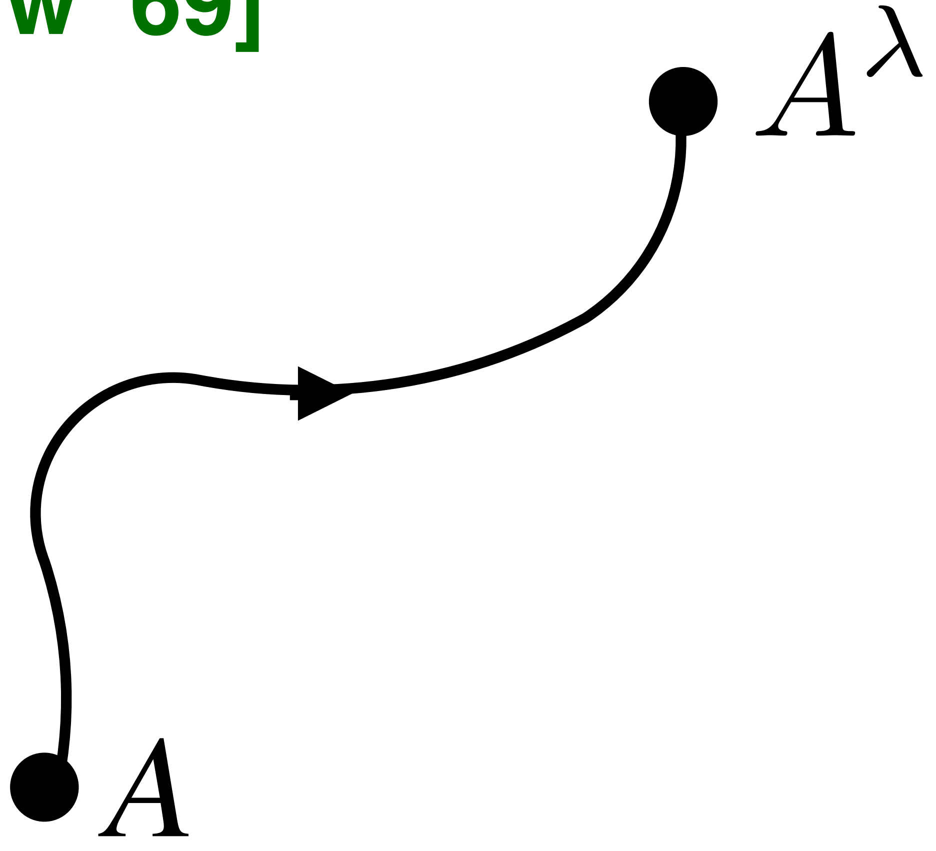
worst that can happen: $Z(A^\lambda) = e^{i\alpha} Z(A)$

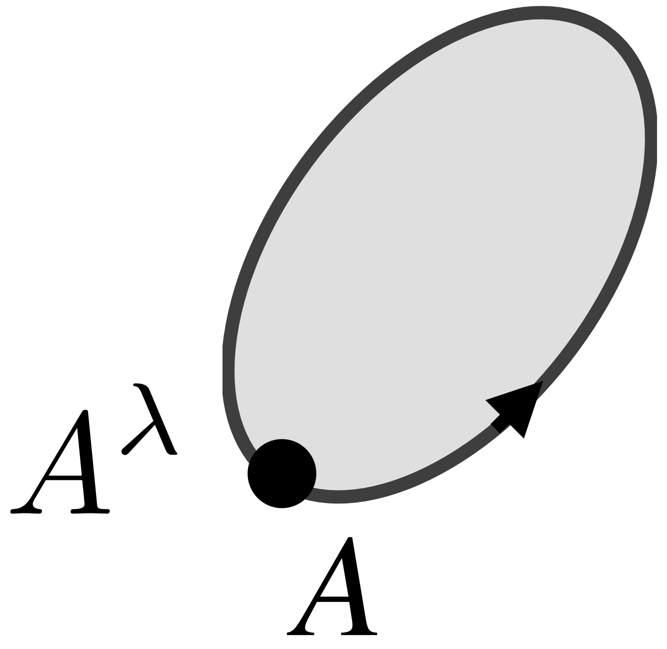
- **Couple symmetry to background connection**
- **Move in configuration space**
- **Calculate partition function**



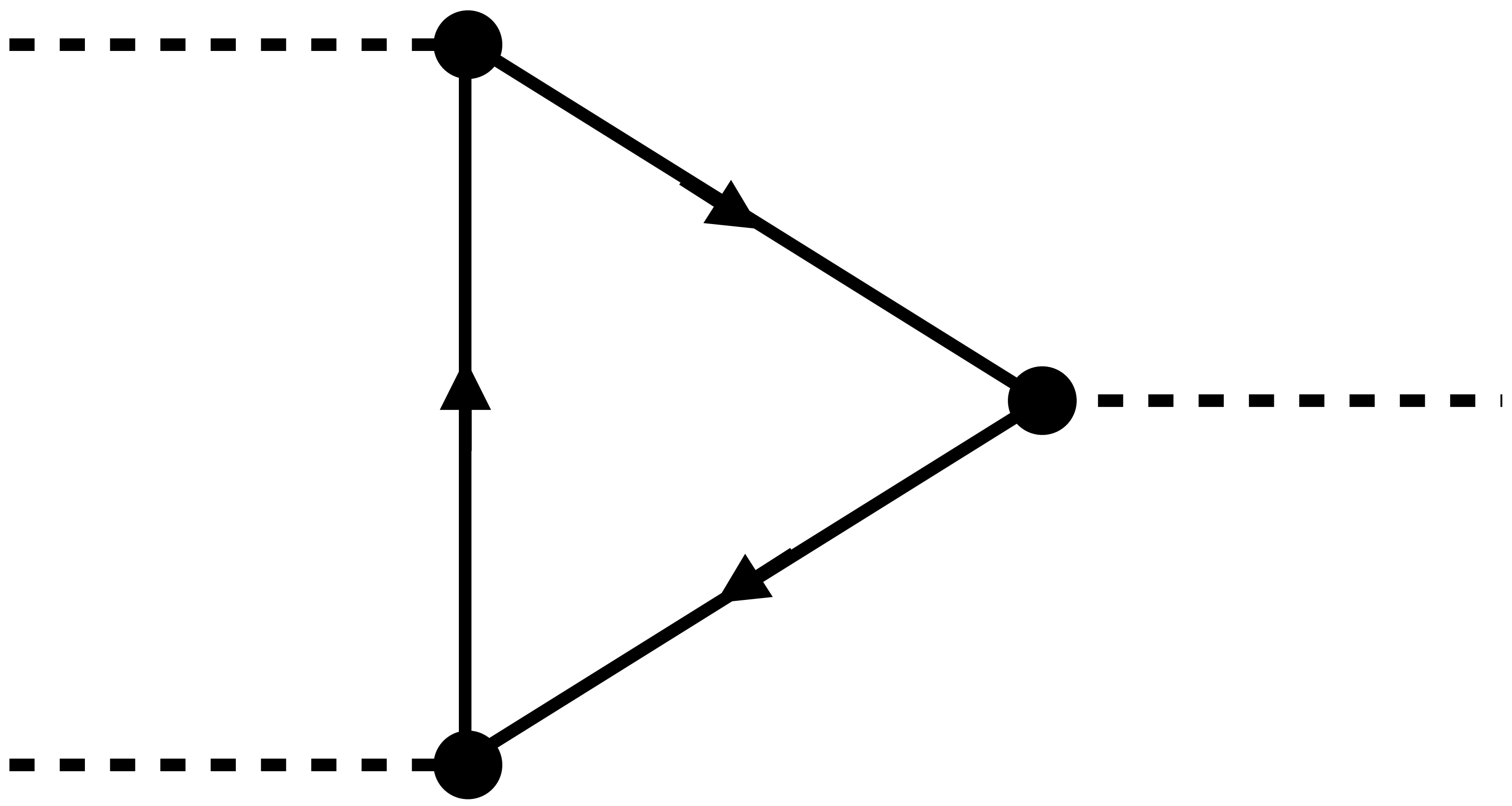
Perturbative anomalies

[Adler ’69, Bell, Jackiw ’69]



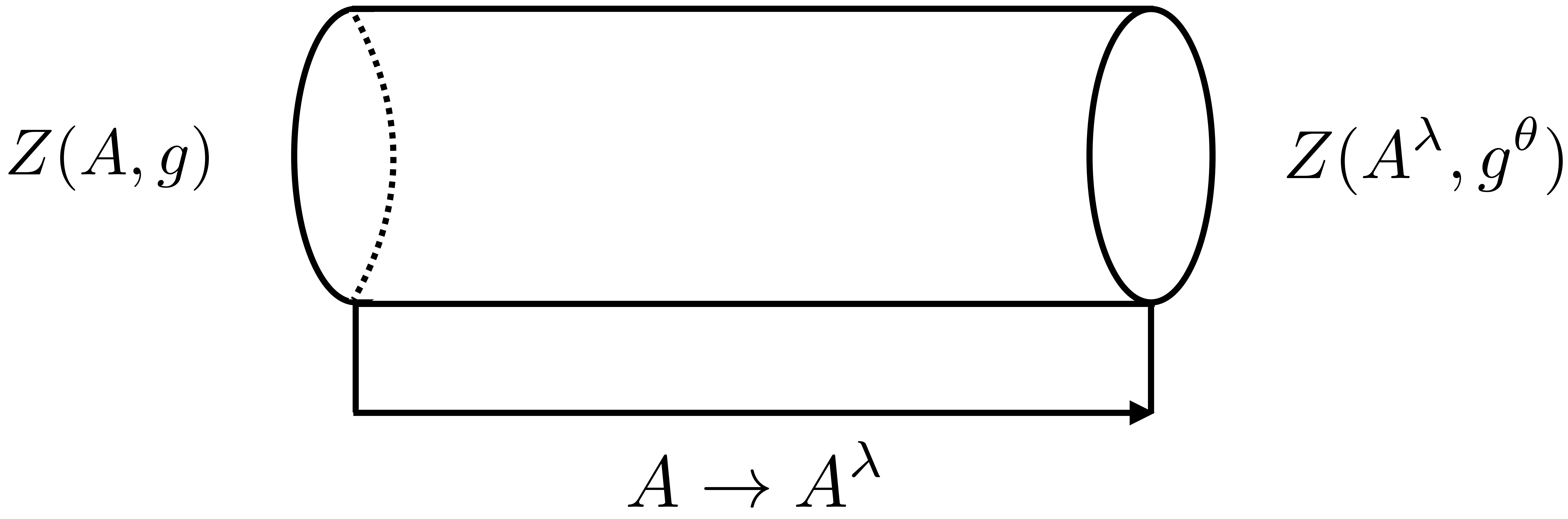


- **Small variations** (contractible paths): **perturbative anomalies**
- **Symmetry** needs to be **continuous**



- **1-loop** amplitude (exact)
- In terms of field strength
- Anomaly polynomial

Geometrize

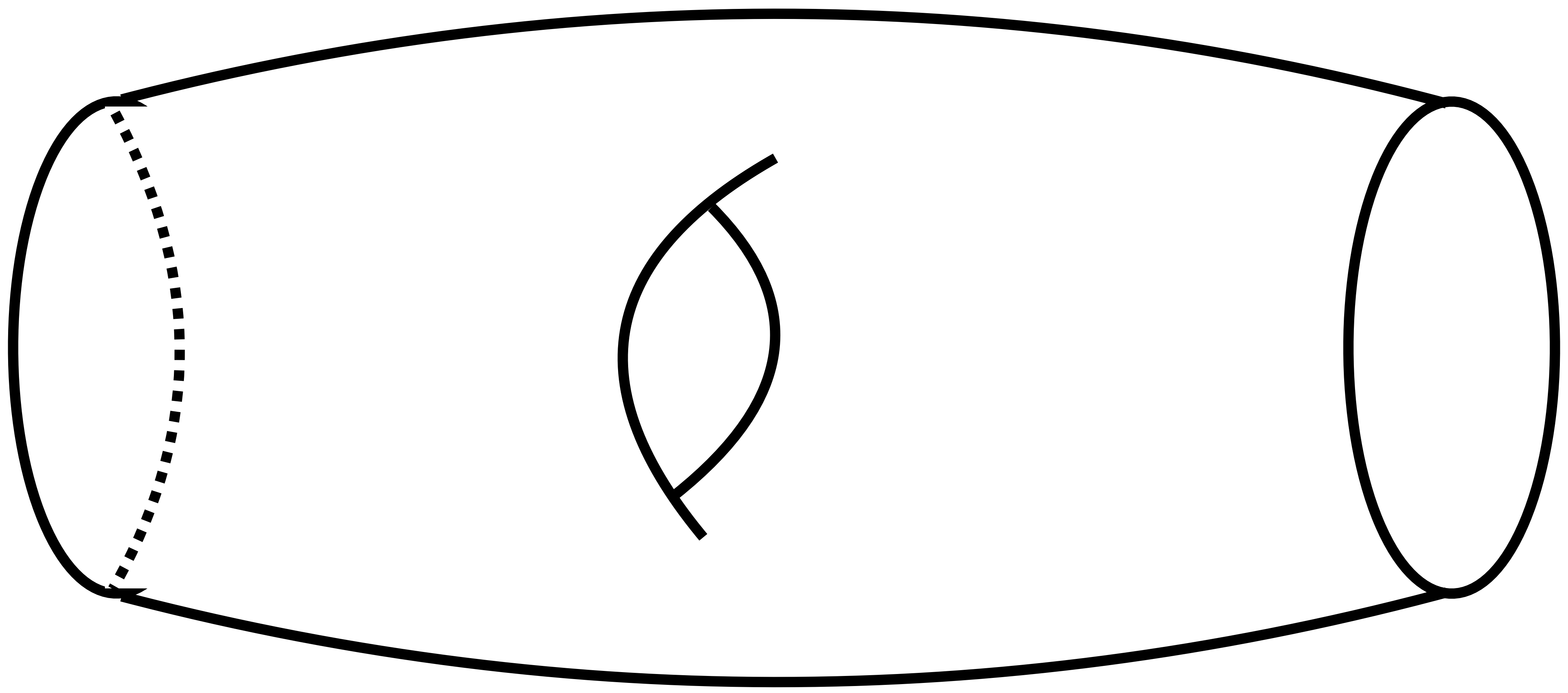


- **Large variations** (non-contractible paths): **global anomalies**
- Spans a **(d+1)-dimensional manifold**

➜ gluing the ends: **mapping tori** [Witten '82]

Quantum Gravity anomalies

[Dai, Freed ’94], [Witten ’15], [Yonekura ’16], see also [Montero, Garcia-Etxebarria ’18]

$Z(A,g)$  $Z(A^{\lambda},g^{\theta})$

Changes in spacetime topology during interpolation

All possible interpolations by **classification of deformation classes**

conserving the structure
(e.g. orientation, spin, …)

$$\Omega_{D+1}^{\mathrm{QG}}$$

Anomaly field theory

e.g. [Freed, Teleman '14]

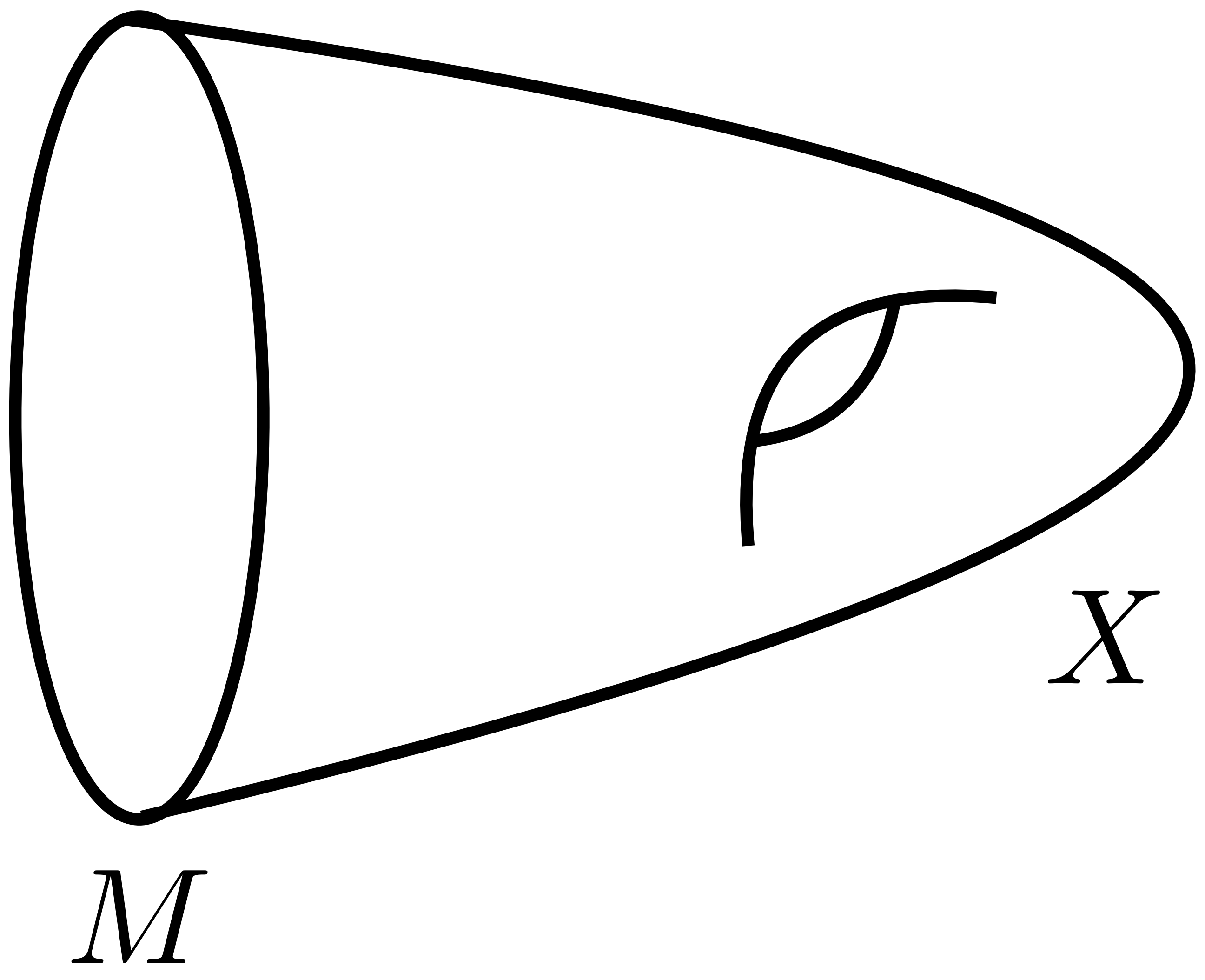
There is a **(d+1)-dimensional invertible field theory** $\mathcal{A}$ such that:

$$\frac{Z[M]}{|Z[M]|} = e^{2\pi i \mathcal{A}[X]}\,, \qquad M = \partial X$$

Physical data:

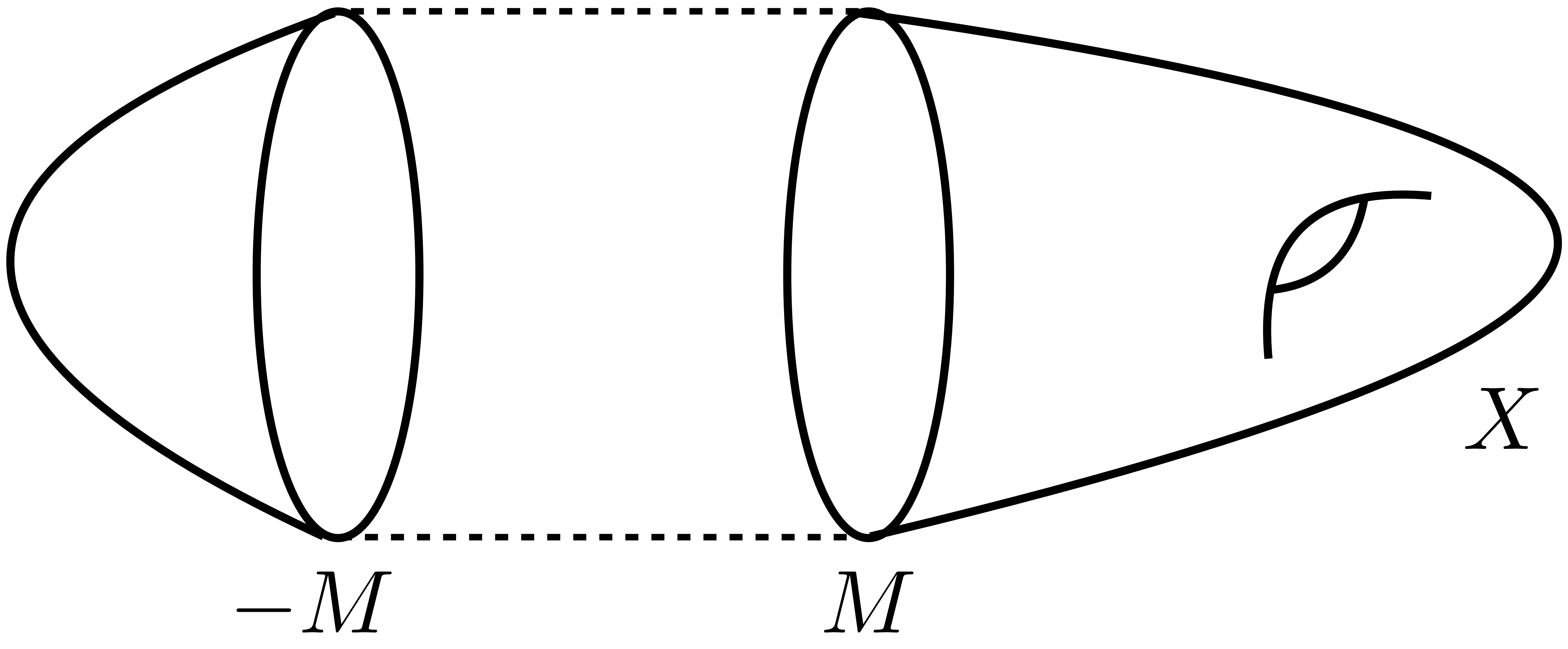
- Spin structure
- Gauge fields
- …

extends $M \to X$



Anomaly field theory

No anomaly if it **does not depend on extension** $M \to X$



$$e^{2\pi i \mathcal{A}[Y]} = 1$$

for **all closed** Y

- ➜ classified by **bordism groups** $\Omega^{\xi}_{D+1}(BG)$
- ➜ **specific theory** encoded in realization of $\mathcal{A}$

Discrete gauge anomalies

e.g. [Hsieh '18], [Garcia-Etxebarria '18], [Monnier, Moore '18]

Consider theory with **fermions** in **D dimension** with **gauge group** $\mathbb{Z}_n$

$$\Omega^{\mathrm{Spin}}_{D+1}(B\mathbb{Z}_n)$$

Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p\left(B\mathbb{Z}_n; \Omega^{\mathrm{Spin}}_q(\mathrm{pt})\right)$$

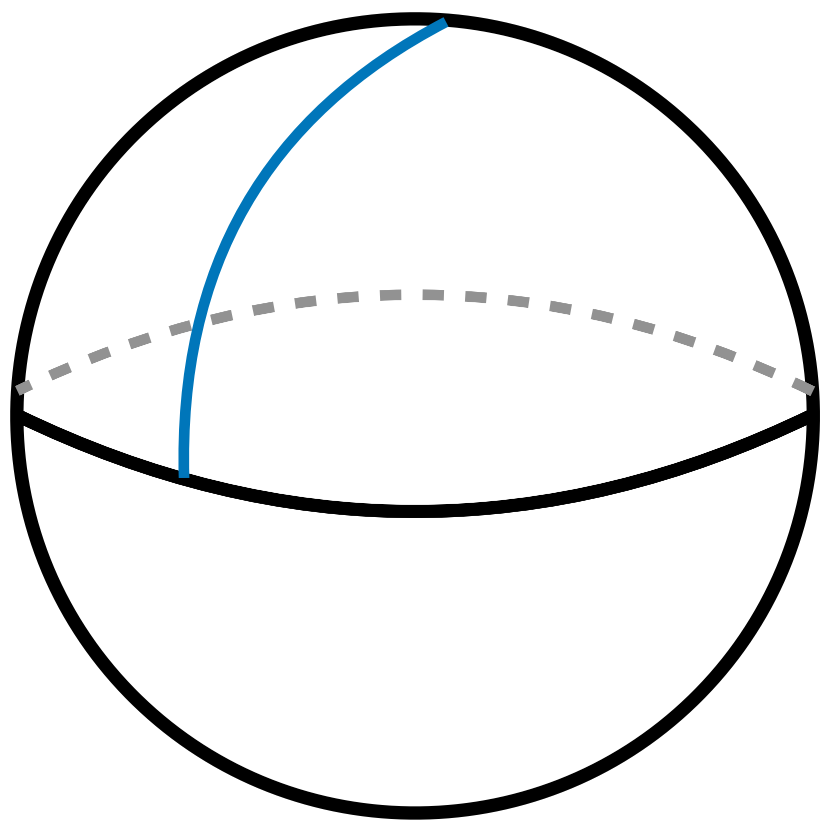
(challenge: differentials and extensions)

q \ p	0	1	2	3	4
4	$\mathbb{Z}$	$\mathbb{Z}_n$		$\mathbb{Z}_n$	
3					
2	$\mathbb{Z}_2$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$	$\mathbb{Z}_{(n,2)}$
0	$\mathbb{Z}$	$\mathbb{Z}_n$		$\mathbb{Z}_n$	

Discrete gauge anomalies

Example: $D = 6\,, \quad n = 3$

$$\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}_3) = \mathbb{Z}_9$$ generated by



$L_3^7 \simeq S^7/\mathbb{Z}_3$
with discrete gauge bundle

$$\mathcal{A} = (n_1 + n_2)(\eta_1^{\mathrm{D}} - \eta_0^{\mathrm{D}})$$

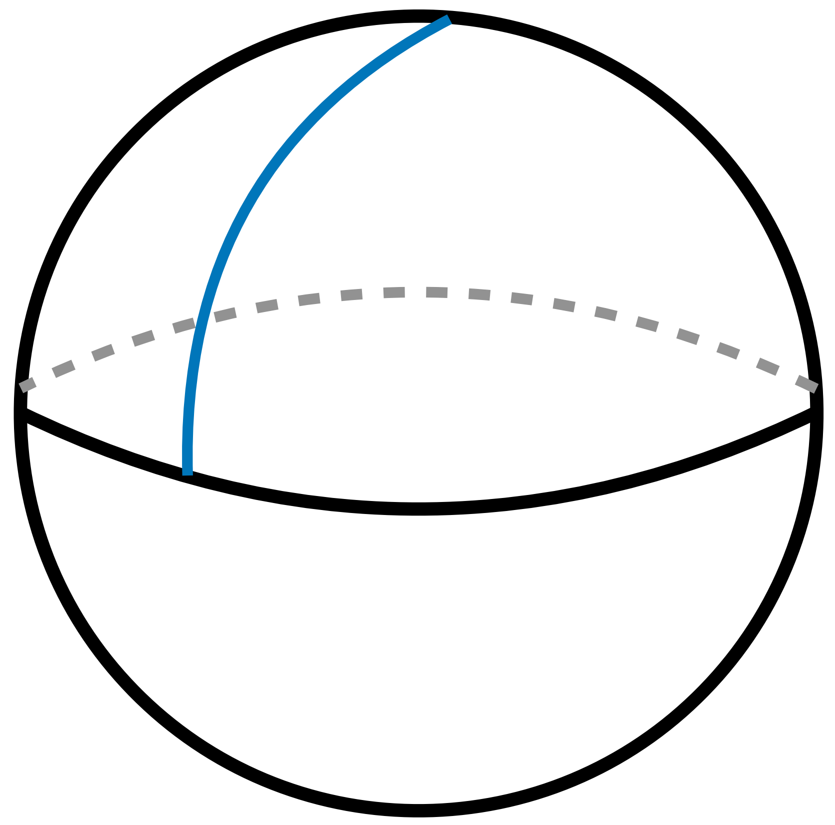
number of charged chiral fermions

η-invariant

subtraction of perturbative gravitational makes $\mathcal{A}$ topological

Discrete gauge anomalies

$$(\eta_1^{\mathrm{D}} - \eta_0^{\mathrm{D}})[L_3^7] = \frac{1}{9}$$



$L_3^7 \simeq S^7/\mathbb{Z}_3$

→ **Discrete anomaly condition:** $(n_1 + n_2) \in 9\mathbb{Z}$

But can one relax the condition?

Discrete anomaly cancellation

[Monnier, Moore '18], [MD, Oehlmann, Schimannek '22 + to appear], [MD, Tartaglia '25]

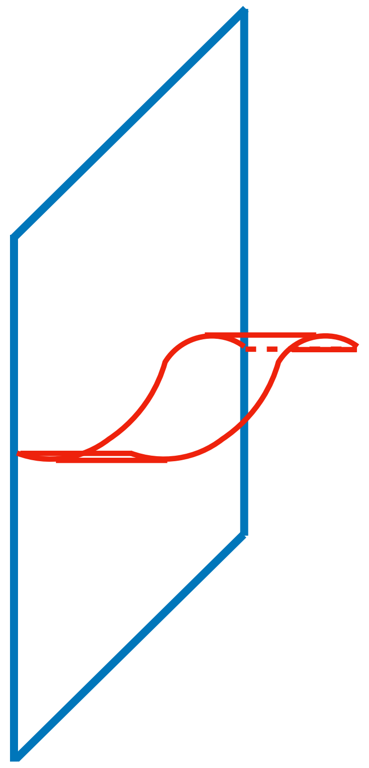
Six-dimensional **supergravity** has **chiral 2-form fields**:

- **self-dual in gravity multiplet**
- **anti-self-dual in tensor multiplets**

$$*H = \pm H$$

Can be described as **boundary degrees of freedom** of 3-form C

[Hsieh, Tachikawa, Yonekura '20], see also [Witten '96],[Freed, Moore, Segal '06], [Belov, Moore '06]



with **Chern-Simons-like** term in the bulk

$$\sim C \wedge dC$$

Self-duality

[Hsieh, Tachikawa, Yonekura ’20], see also [Witten ’96],[Freed, Moore, Segal ’06], [Belov, Moore ’06]

Chern-Simons term is **not integer quantized** ➡ **extra data**

‘Wu structure’

➡ automatic for Spin-7-manifolds: $\nu_4 = w_4 = 0 \in H^4(X_7; \mathbb{Z}_2)$

$$\frac{1}{2} C \wedge dC$$

Definition requires **integer lift**: $w_4^{\mathbb{Z}} = \left(\frac{1}{2} + k\right) p_1$

(analogue in 3d is the Spin structure, necessary to have half-integer Chern Simons level)

Differential Cohomology

[Cheeger, Simons '85], [Hopkins, Singer '02] see also [Hsieh, Tachikawa, Yonekura '20], [Moore, Saxena '25]

$$C \longrightarrow \check{C} \sim (A_C, F_C, N_C)$$

$\in H^4(X;\mathbb{Z})$

$$\frac{1}{2} C \wedge dC \longrightarrow \mathcal{Q}(\check{C})$$

quadratic refinement (of differential cohomology pairing)

$$(\check{A}, \check{B}) \sim \mathcal{Q}(\check{A} + \check{B}) - \mathcal{Q}(\check{A}) - \mathcal{Q}(\check{B}) + \mathcal{Q}(0)$$

Anomaly

So far **only gravitational contributions of self-dual field:**

$$\mathcal{A}^{B}_{\mathrm{grav}} \sim \mathcal{Q}(0)$$

reproduces known answer (slightly different interpretation)

[Alvarez-Gaume, Witten ’83]

We **need gauge contribution**

➛ Couple field to **discrete gauge background** $\check{X}$

$$B \wedge X \longrightarrow (\check{C}, \check{X})$$

relation to **twisted tangential structures**: [Sati, Schreiber, Stasheff ’09], [Sati ’11], ...

Discrete gauge anomaly contribution

[MD, Oehlmann, Schimannek ’22 + to appear], [MD, Tartaglia ’25]

Now there is a **contribution to the discrete gauge anomaly**

$$\mathcal{A}^B_{\text{gauge}} \sim \mathcal{Q}(\check{X}) - \mathcal{Q}(0)$$

Depends on:

- **Choice of quadratic refinement** $\mathcal{Q}$
- **Choice of gauge theory class** $\check{X}$

➡ **Non-perturbative anomaly cancellation** if: $\left(\mathcal{A}^F + \mathcal{A}^B_{\text{gauge}}\right)[L^7_3] \in \mathbb{Z}$

Discrete gauge anomaly contribution

[MD, Oehlmann, Schimannek '22 + to appear], [MD, Tartaglia '25]

For **self-dual 2-form** (similar constraints for other cyclic groups):

$$n_1 + n_2 = 3\tilde{k} \mod 9$$

choice of quadratic refinement

Note that this is asymmetric, i.e., allowed spectra: $(\tilde{k} = 2)$

$$n_1 + n_2 = 6\,, 15\,, 24\,, \ldots$$

→ Precisely, what is **found in F-theory:** $n_1 + n_2 = 186\,, 195\,, \ldots$

[MD, Oehlmann, Schimannek '22 + to appear]

Choice of quadratic refinement

[MD, Oehlmann, Schimannek: to appear]

Choice of quadratic refinement **seems to be fixed in F-theory** models!

Can be understood via **unHiggsing** (can all?):

$$\mathbb{Z}_n \hookrightarrow \mathrm{U}(1)$$

In combination with **anomaly interplay:**

[Davighi, Lohitsiri '20]

$$\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}_3) \simeq \mathbb{Z}_9 \longrightarrow \mathbb{Z}^{\oplus 4} \simeq \Omega_8^{\mathrm{Spin}}(B\mathrm{U}(1))$$

(mixed) gravitational fixes quadratic refinement

The strings

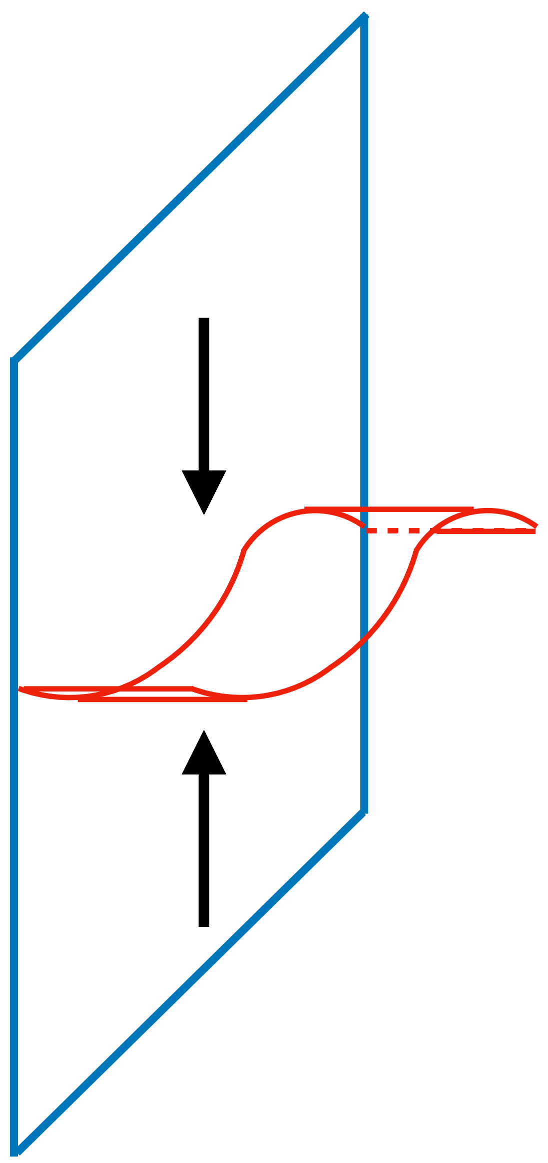
[MD, Oehlmann, Schimannek ’22 + to appear], [MD, Tartaglia ’25]

The coupling to $\check{X}$ makes **2-form sensitive to gauge transformations**

➔ **Anomaly inflow onto string**

chiral degrees of freedom

charged under discrete symmetry



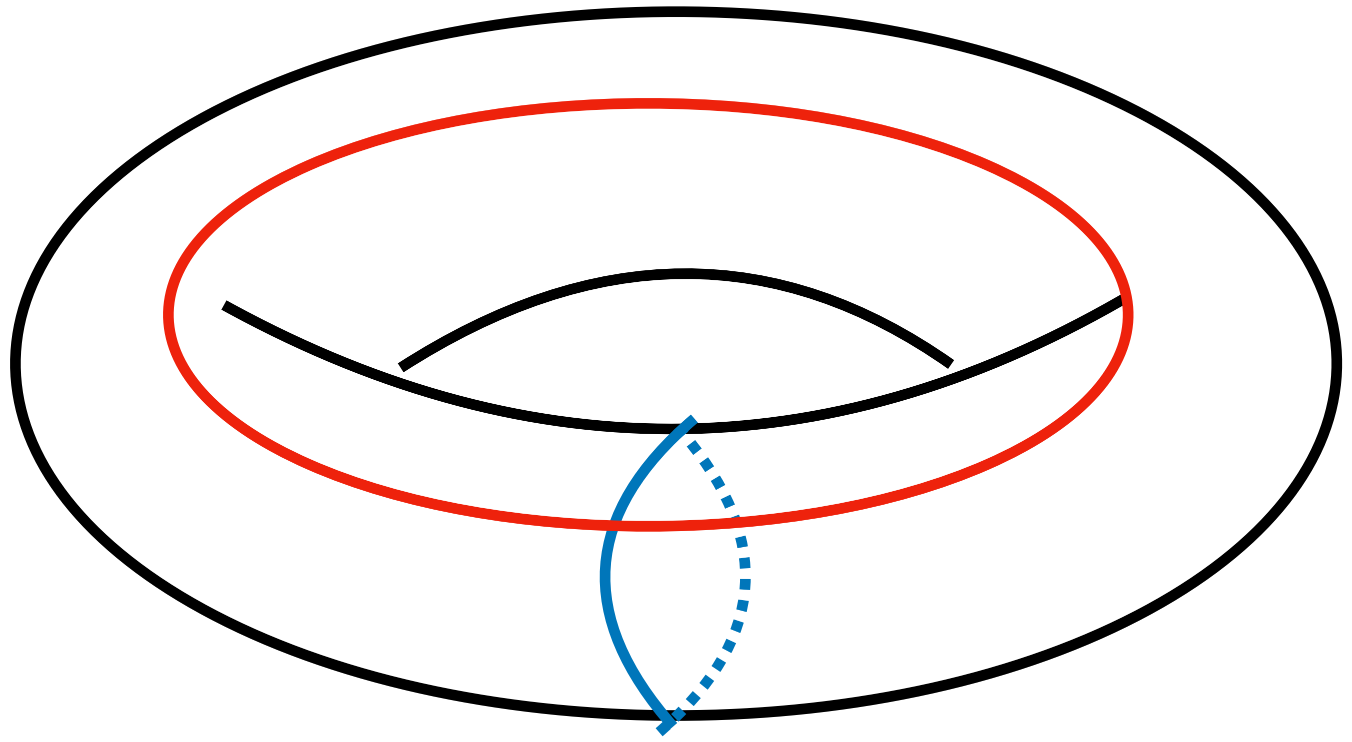
➔ Can be tested: **topological string partition function**

$$Z(\tau, m, \lambda)$$

Topological string

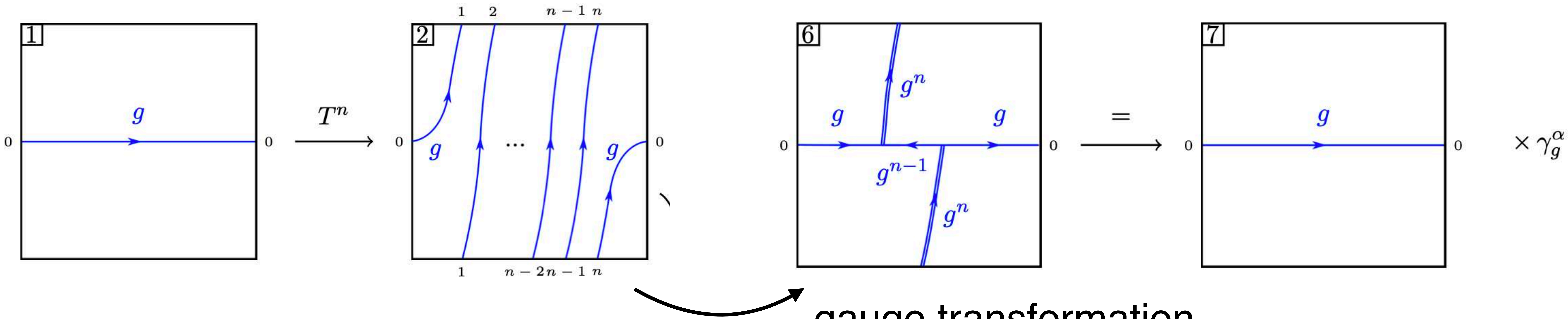
[MD, Oehlmann, Schimannek '22]

6d theory



on torus with **discrete Wilson lines**

4d theory with twisted-twined genera capturing **modular properties**



gauge transformation

Topological string

[MD, Oehlmann, Schimannek '22]

Knows about **discrete anomaly** of **worldsheet degrees of freedom**

$$\mathcal{Q}(\check{X}) \longleftrightarrow Z(\tau, m, \lambda)$$

matches!

➙ Relation to **geometry of F-theory** compactification

(Remember: self-dual strings from wrapped D3-branes)

multi-section geometries

e.g.: [Cota, Klemm, Schimannek '19], [Knapp, Scheidegger, Schimannek '21],[Pioline, Schimannek '25]

Duality anomaly for type IIB

[Debray, MD, Heckman, Montero '21]

Type IIB supergravity has **(gauged) duality:**

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{Z}) & \xrightarrow{\text{orientation reversal of worldsheet}} & \mathrm{GL}(2,\mathbb{Z}) \\ \downarrow{\scriptstyle\text{fermions}} & & \downarrow \\ \mathrm{Mp}(2,\mathbb{Z}) & \longrightarrow & \boxed{\mathrm{GL}^+(2,\mathbb{Z})} \end{array}$$

orientation reversal of worldsheet

fermions

[Pantev, Sharpe '16]

[Tachikawa, Yonekura '18]

$\rightarrow$ $\Omega_{11}^{\text{Spin-GL}^+(2,\mathbb{Z})}(\text{pt})$

The bordism group

$$\Omega_{11}^{\text{Spin-GL}^+(2,\mathbb{Z})} = (\mathbb{Z}_2)^{\oplus 9} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{27}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

$$\simeq \Omega_{11}^{\text{Spin}}(B\mathbb{Z}_3)$$

Anomaly theory (more complicated):

[Hsieh, Tachikawa, Yonekura '20], see also [Freed, Moore, Segal '06], [Belov, Moore '06]

$$\mathcal{A}(X) = \underbrace{\eta_1^{\text{RS}}(X) - 2\eta_1^{\text{D}}(X) - \eta_{-3}^{\text{D}}(X)}_{\text{fermions}} - \underbrace{\tfrac{1}{8}\eta_-^{\text{sig}}(X) + \text{Arf}(X) - \tilde{\mathcal{Q}}(\breve{c})}_{\text{4-form}}$$

Involves **chiral degrees of freedom:** dilatini, gravitini, self-dual 4-form

The duality anomaly

$\mathbb{Z}_{27}$	L_3^{11}	→	$\mathcal{A}(X) = \frac{1}{3}$
$\mathbb{Z}_3$	$\mathbb{HP}^2 \times L_3^3$	→	$\mathcal{A}(X) = \frac{1}{3}$
$\mathbb{Z}_8$	Q_4^{11}	→	$\mathcal{A}(X) = \frac{k}{4}$
$\mathbb{Z}_2$	$\mathbb{HP}^2 \times L_4^3$	→	$\mathcal{A}(X) = \frac{1}{2}$

→ **Duality is anomalous**

But can be canceled → Transformation of 4-form under duality

$$\tilde{\mathcal{Q}}(\breve{c}_0) \text{ with } \breve{c}_0 = \left(\lambda_1 \beta(a)^2 + \lambda_2 \frac{(p_1)_3}{2}\right) \cup a + \frac{\lambda_3}{2}[(p_1)_4 - \mathcal{P}(w_2)] \cup b + \kappa \beta(b)^2 \cup b$$

$\lambda_i \in \{-1, +1\}\,, \;\; \kappa \in \mathbb{Z} \bmod 4$ (other physical systems (S-folds) suggest $\lambda_{1,3} = 1$)

Discrete 10d landscape vs Topological Swampland

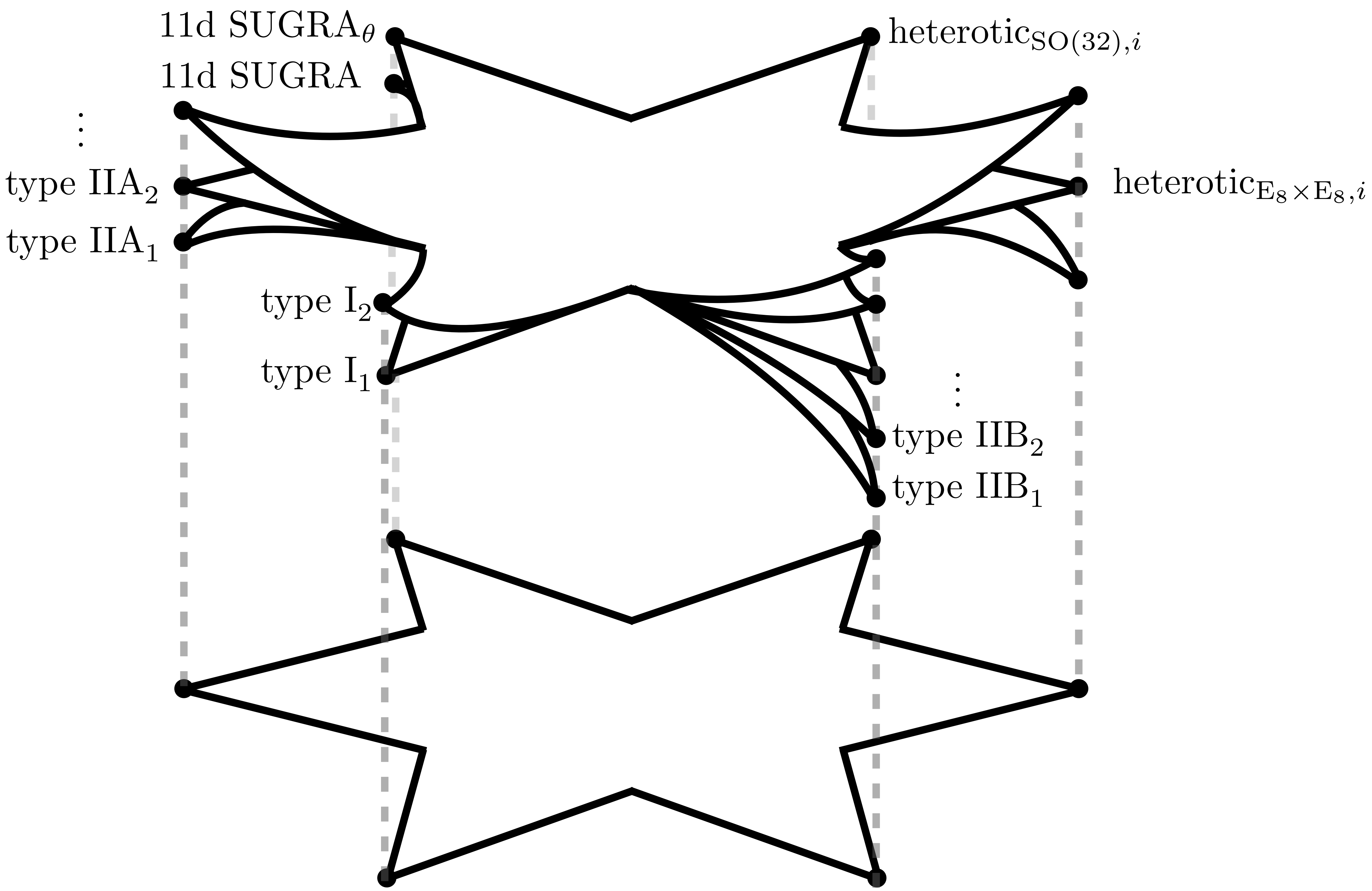
Actually there is other ways to cancel anomaly!

Two possibilities:

- **Topological GS in the Swampland ➔ Why?**
- **Alternative consistent UV completions ➔ Discrete Landscape**

Domain walls connecting the different possibilities (cobordism conjecture)

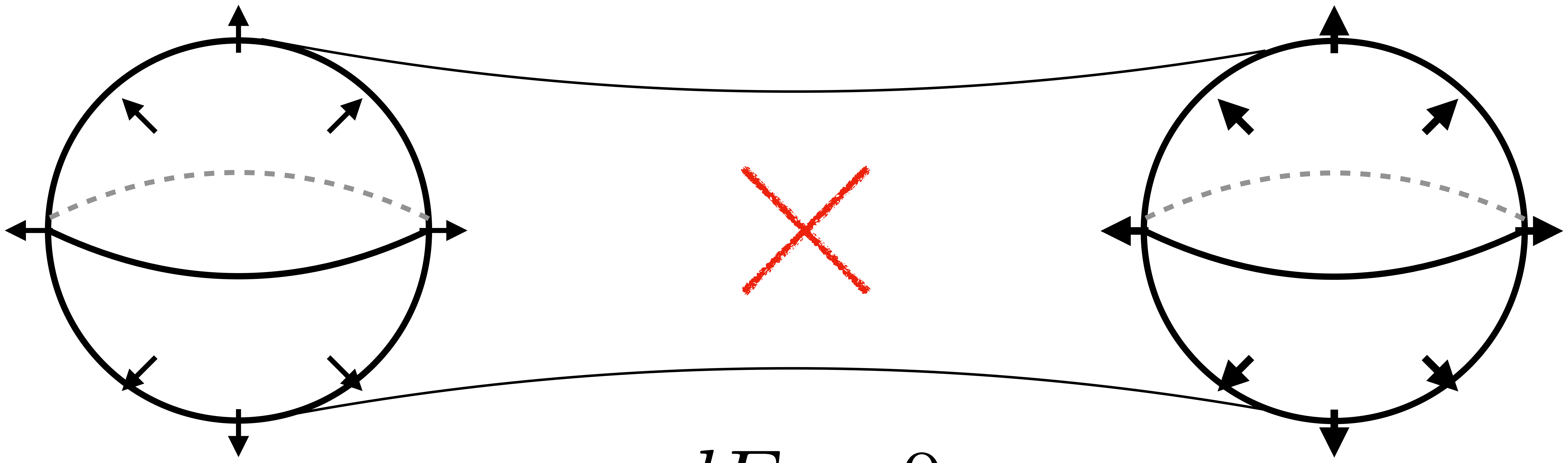
[McNamara, Vafa ’19], [Montero, Vafa ’20]



Global Charges

Deformation classes

Sectors are 'disconnected'



$$dF = 0$$

$\frac{1}{2\pi}\oint F = 1$

(d-3)-form symmetry

$\frac{1}{2\pi}\oint F = 2$

Conserved quantity that labels sectors of theory: Global Charge

➔ associated generalized symmetries

Cobordism Conjecture

[McNamara, Vafa '19]

To avoid global symmetries:

$$\Omega_d^{\mathrm{QG}} = 0\,, \quad 0 < d < D$$

Interesting situation for:

$$\Omega_d^{\mathrm{IR}} \neq 0\,, \quad 0 < d < D$$

➞ **Include new symmetry-breaking objects**

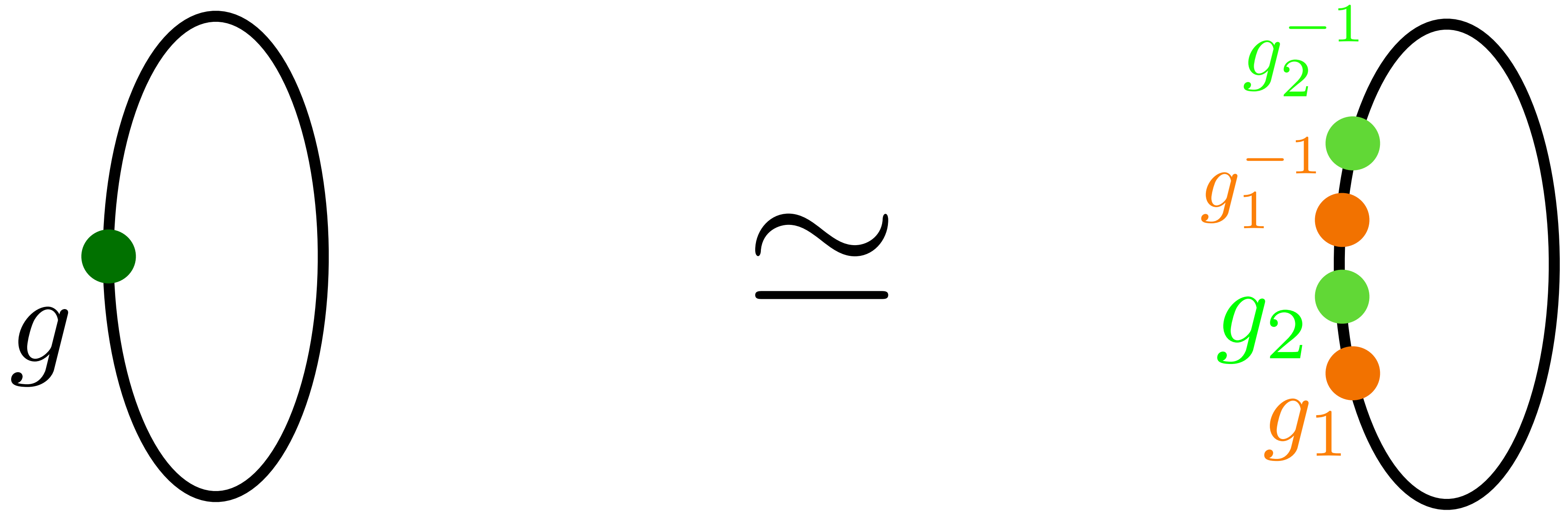
(Alternative: gauge the would be global symmetry)

[MD, Minasian, Novicic - to appear]

Example: type IIB supergravity

[Debray, MD, Heckman, Montero '23]

Has a **duality symmetry**: $\mathrm{SL}(2,\mathbb{Z})$

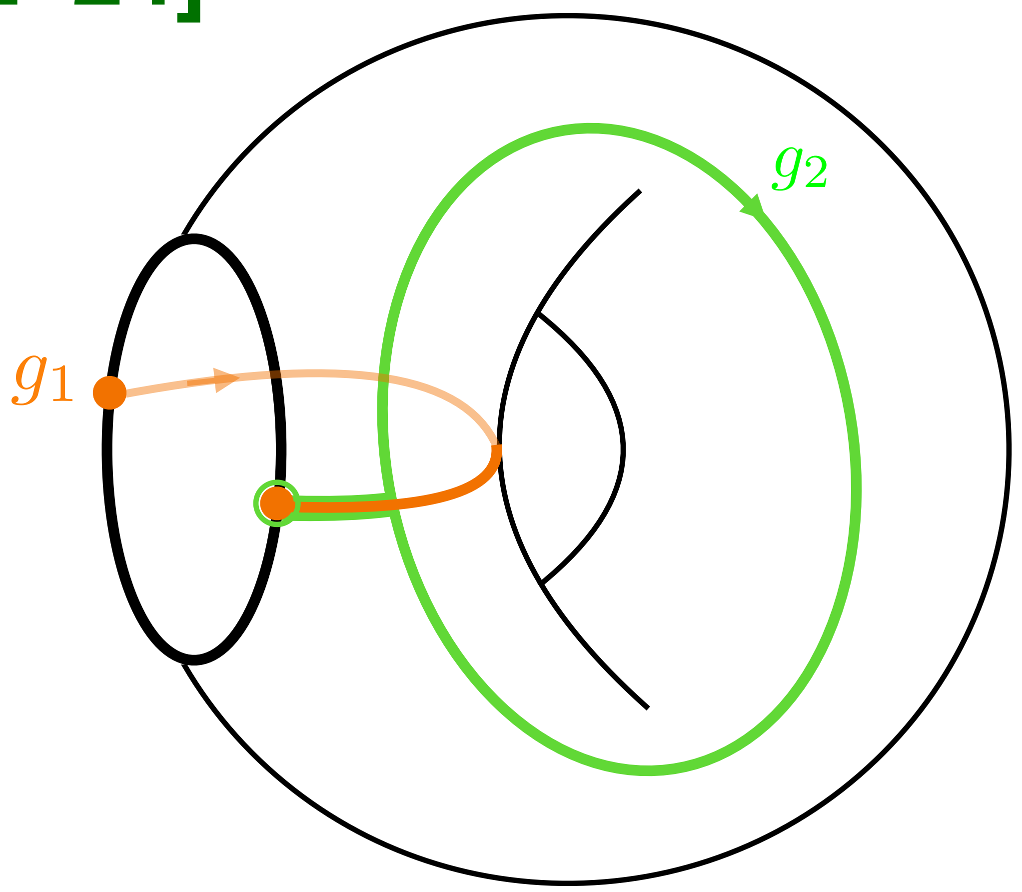


But there are **gravitational solitons:**

see also [McNamara '21], [Ruiz '24]

for

$$g = g_1 g_2 g_1^{-1} g_2^{-1} = [g_1, g_2]$$



$\mathrm{Ab}(\mathrm{SL}(2,\mathbb{Z}))$

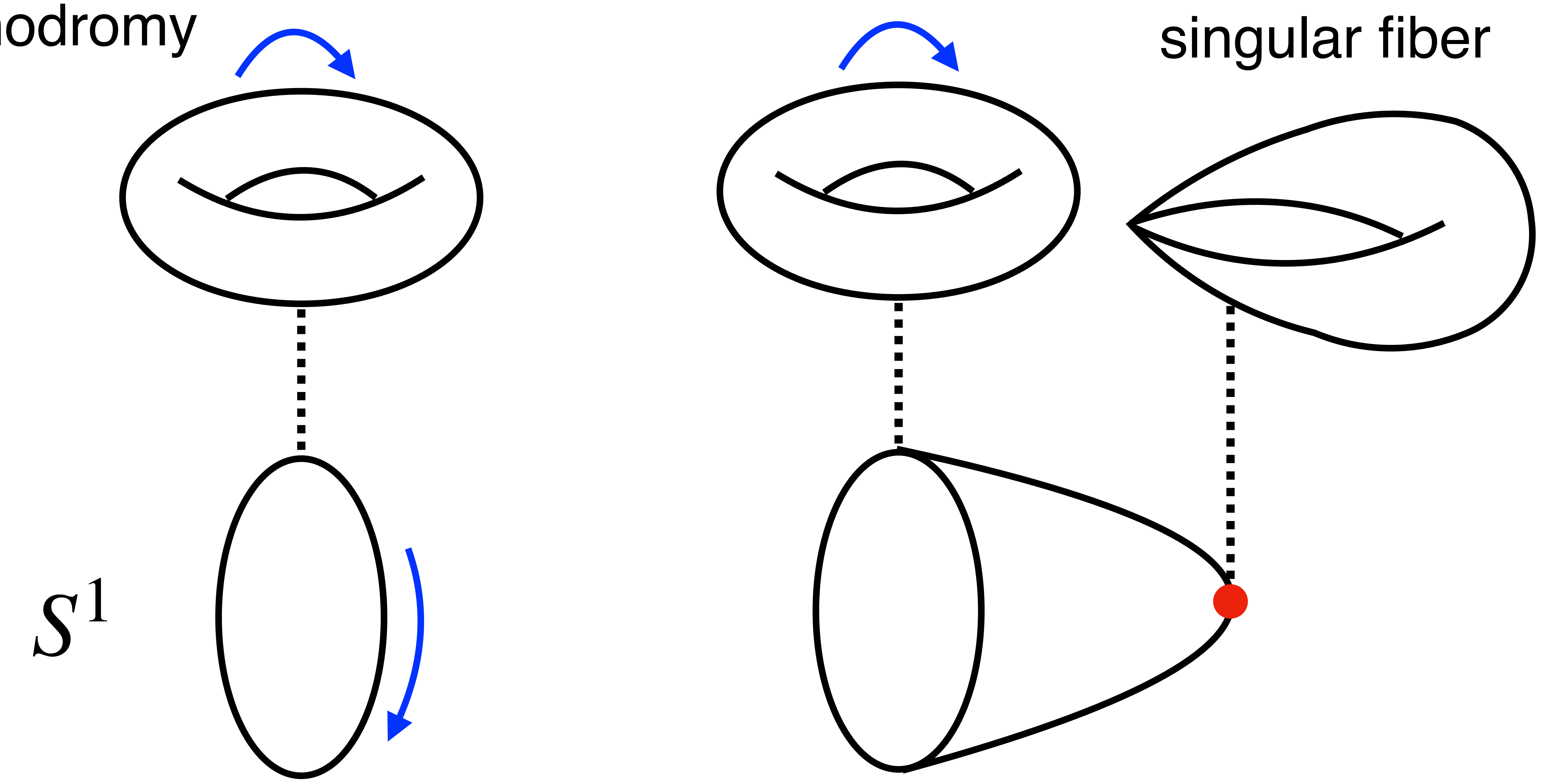
$$\Omega_1^{\mathrm{Spin}}(B\mathrm{SL}(2,\mathbb{Z})) = \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$$

(bordism groups are Abelian)

Example: type IIB supergravity

[Debray, MD, Heckman, Montero '23]

The **defects from F-theory**:



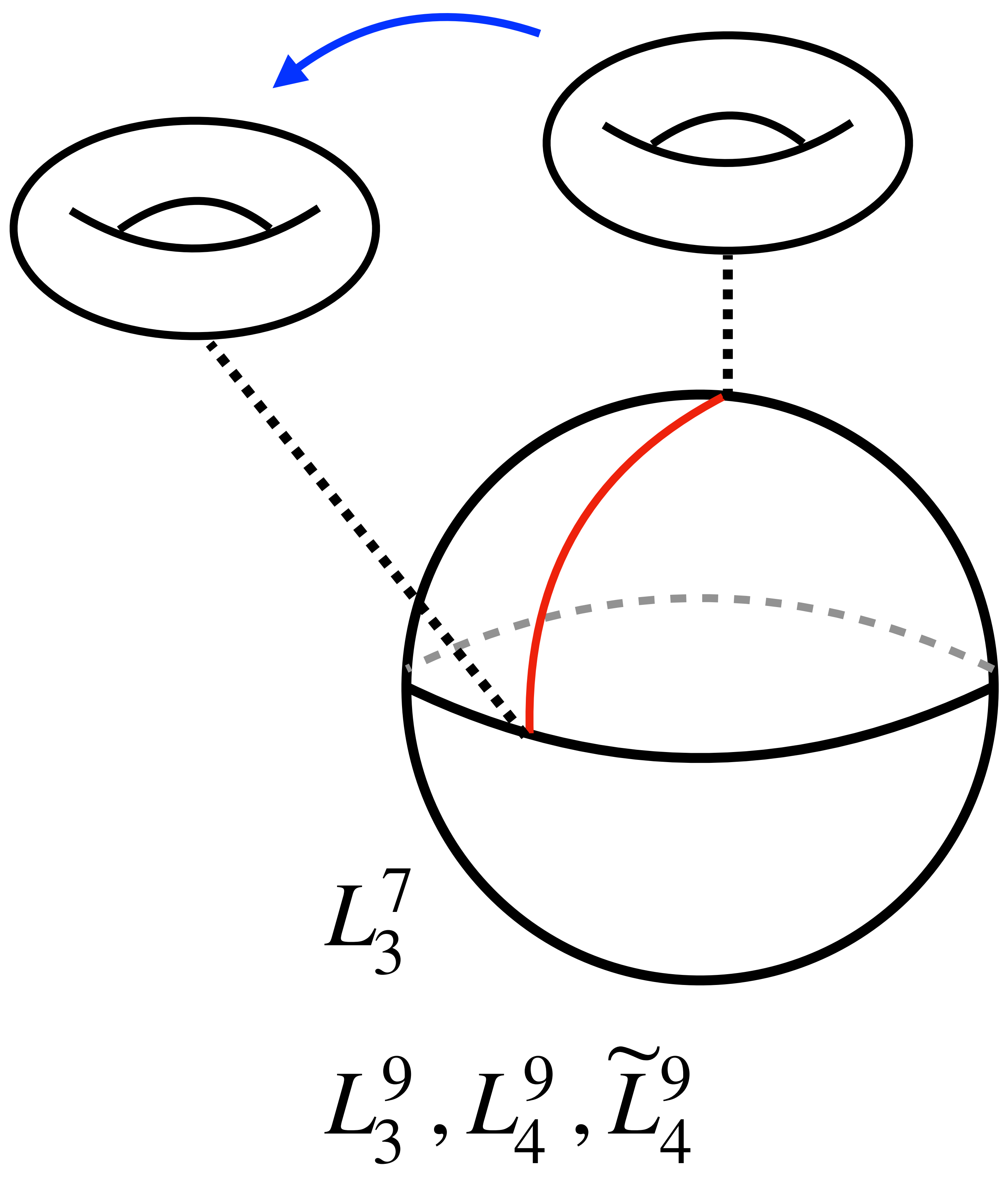
defects are [p,q]-7-brane stacks

See also [MD, Heckman '20]

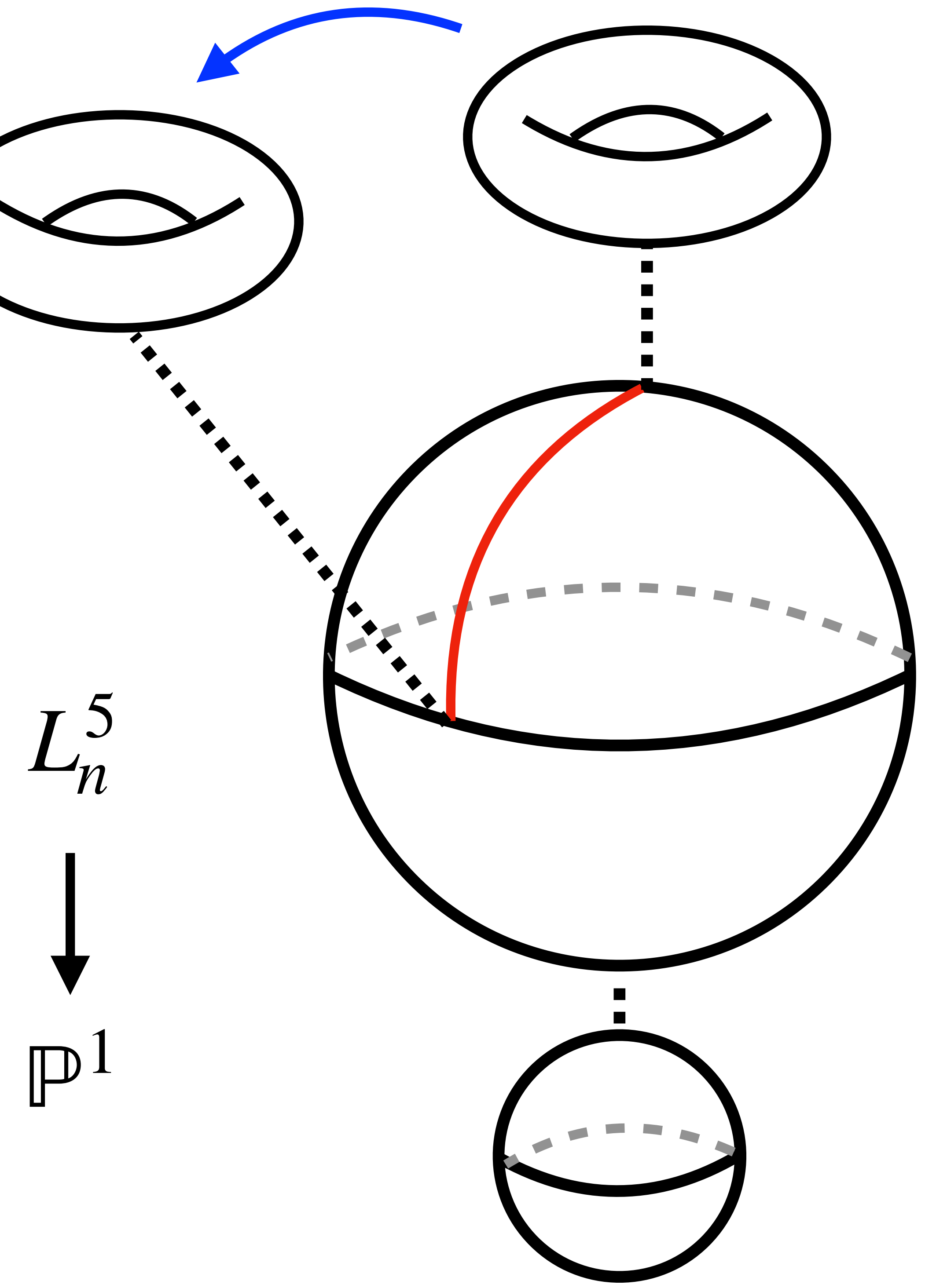
Example: type IIB supergravity

[Debray, MD, Heckman, Montero '23]

Many other interesting configurations



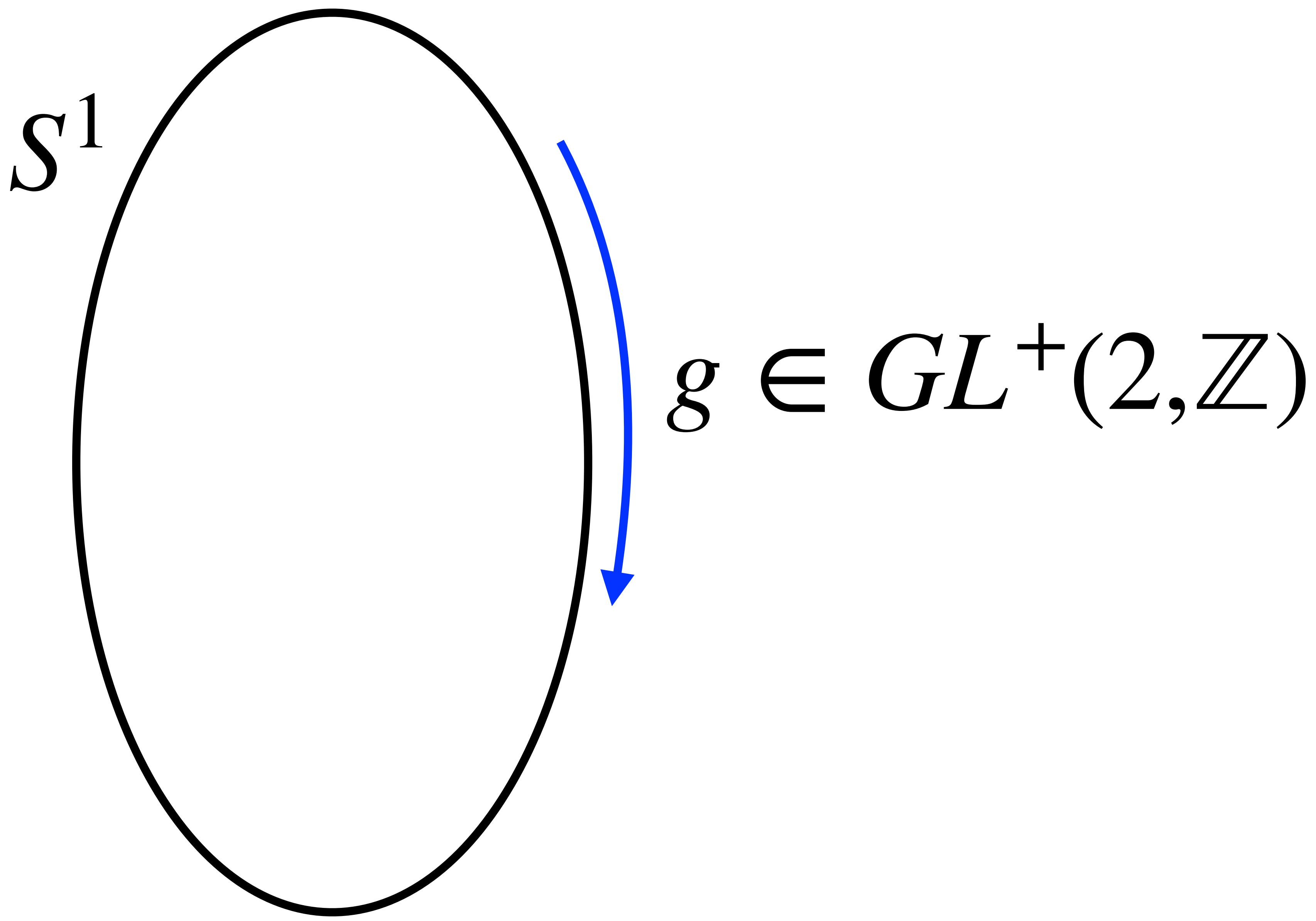
- S-folds
- topological twists
- ...



A new 7-brane

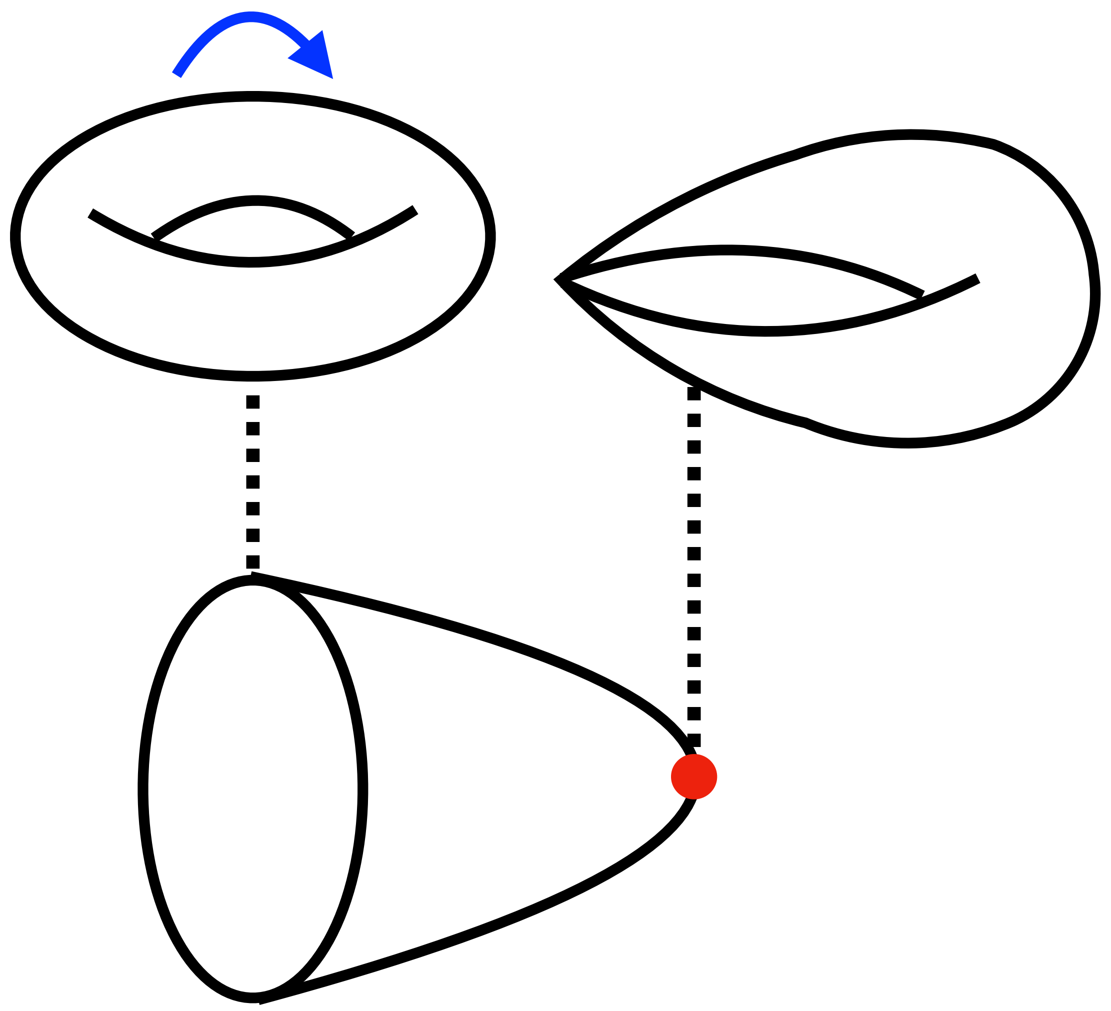
[MD, Heckman, Montero, Torres ’22 +’23]

$$\Omega_1^{Spin-GL^+(2,\mathbb{Z})}(pt) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$



taken care of by [p,q]-7-branes

(F-theory) see also [MD, Heckman ’20]

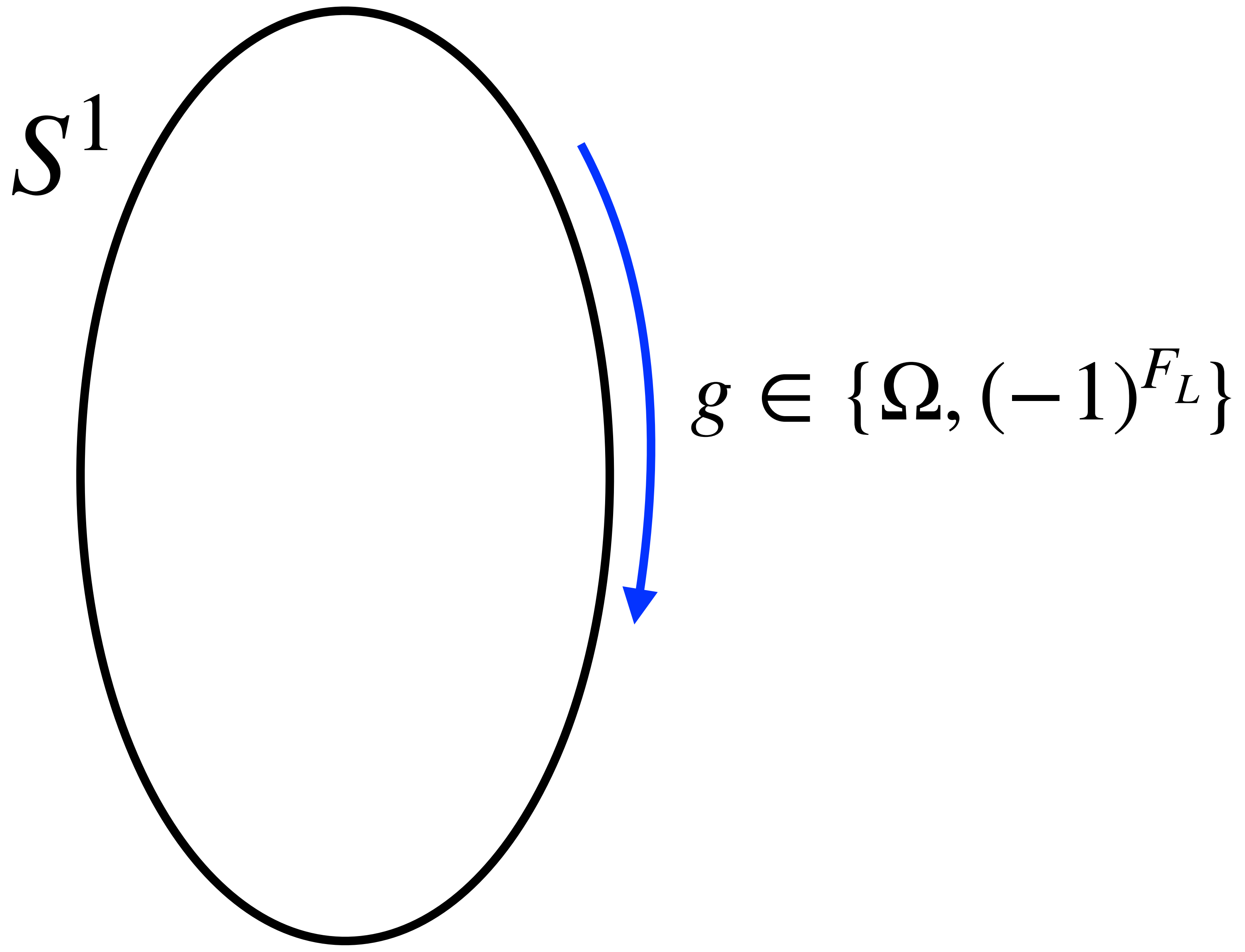


A new 7-brane

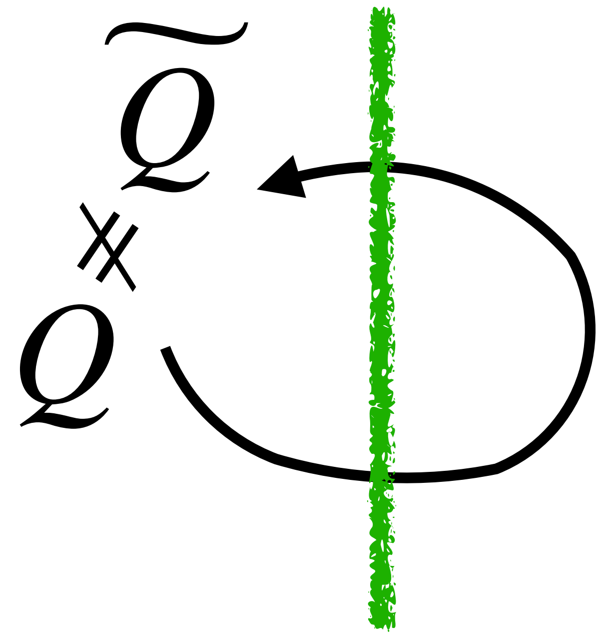
$$\Omega_1^{Spin-GL^+(2,\mathbb{Z})}(pt) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

new 'reflection' 7-brane

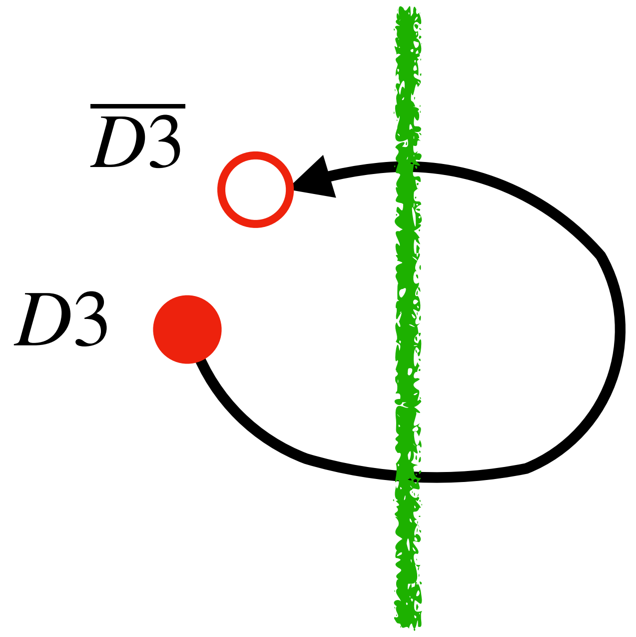
hints in [Distler, Freed, Moore '09]



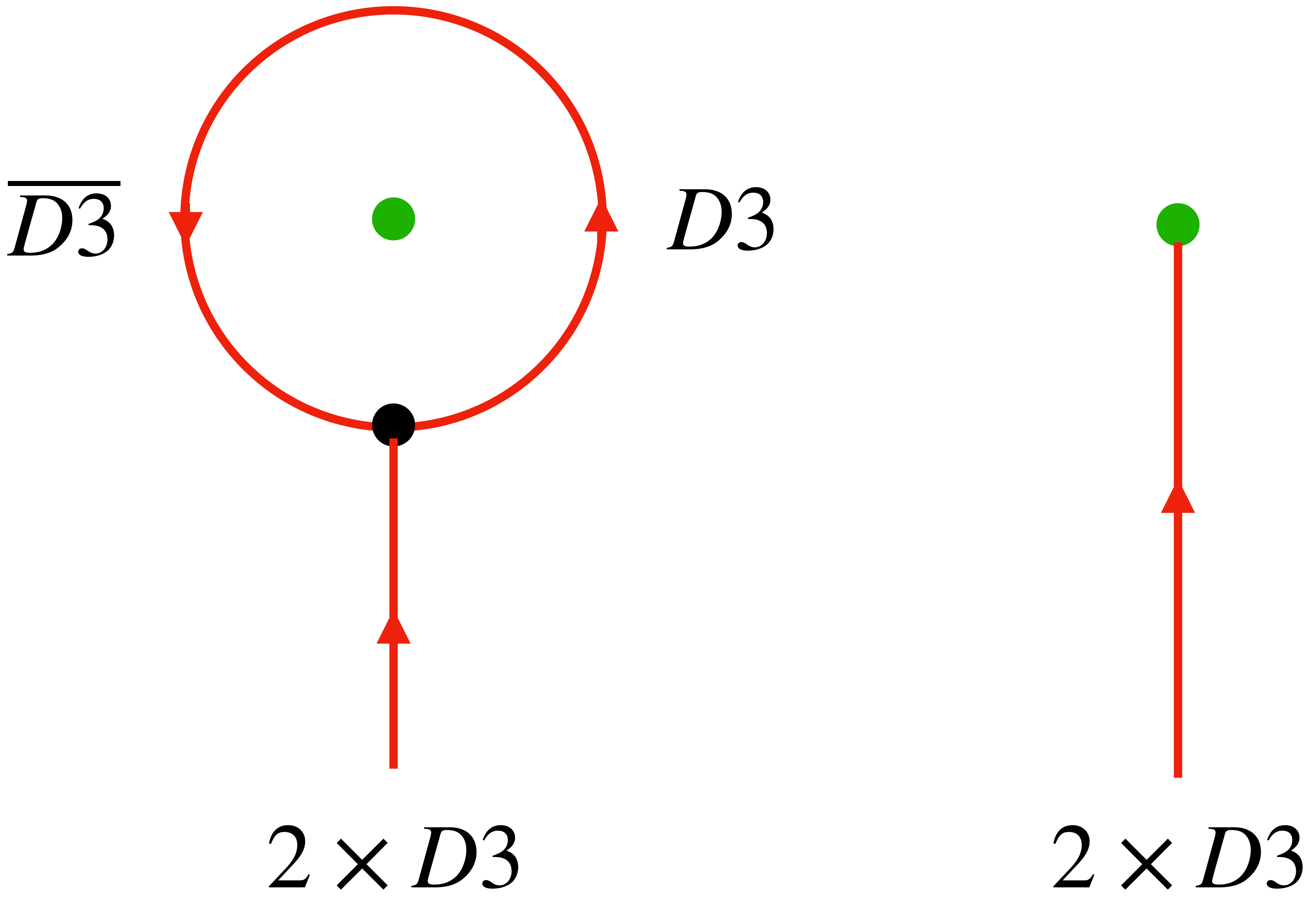
Breaks supersymmetry



Alice string for D3 branes



D3-branes can end on R7-brane

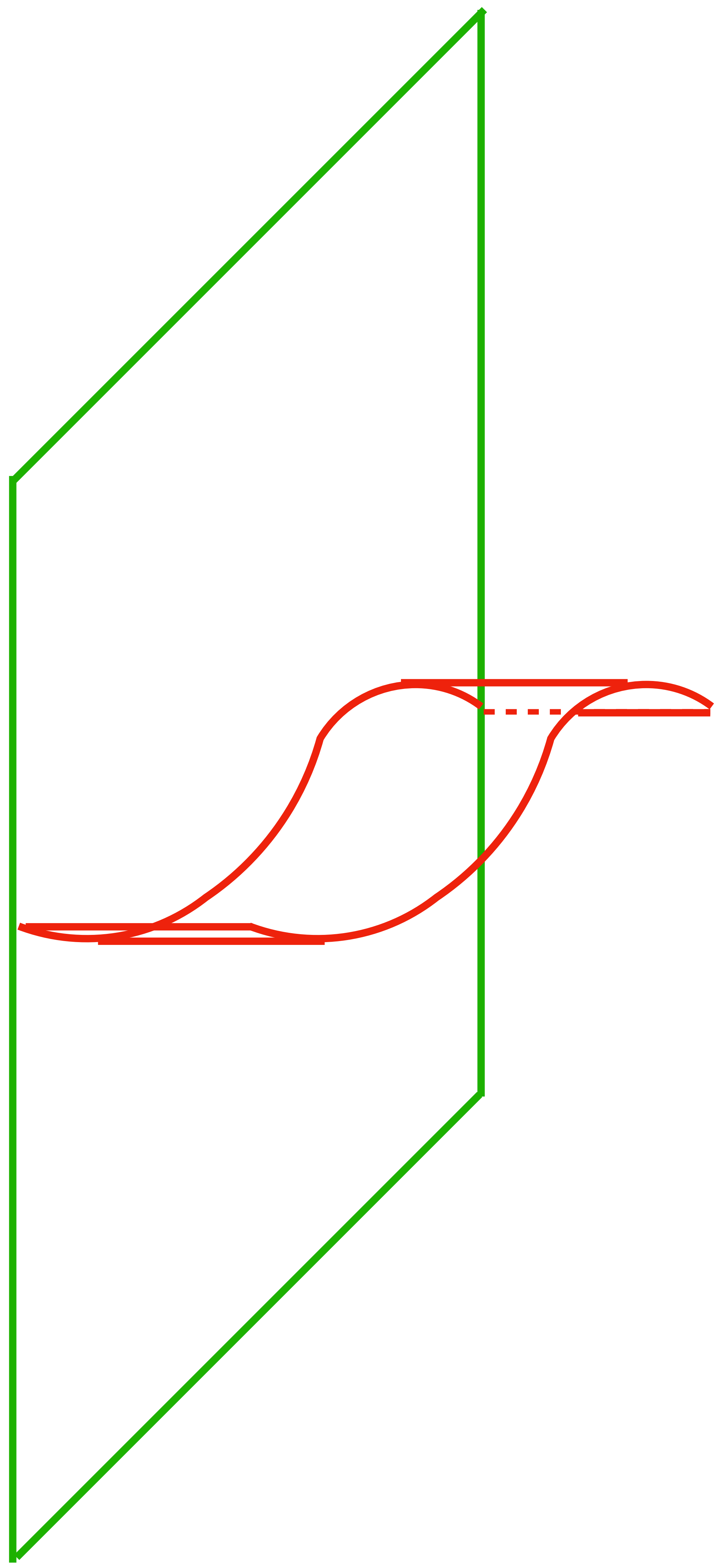


D3-branes end because of $C_4 \rightarrow -C_4$ transformation

(at least in pairs)

➡ something should absorb charge

3-form fields



D3-brane creates 3d worldvolume in R7-brane

➞ flux on transverse S^4

suggests $F_4 = dC_3$ (odd under reflections)

➞ massless 3-form on R7-brane

(potentially interesting behavior under S-duality;
interacting non-supersymmetric CFT in 8d???)

Example: Maximal supergravity

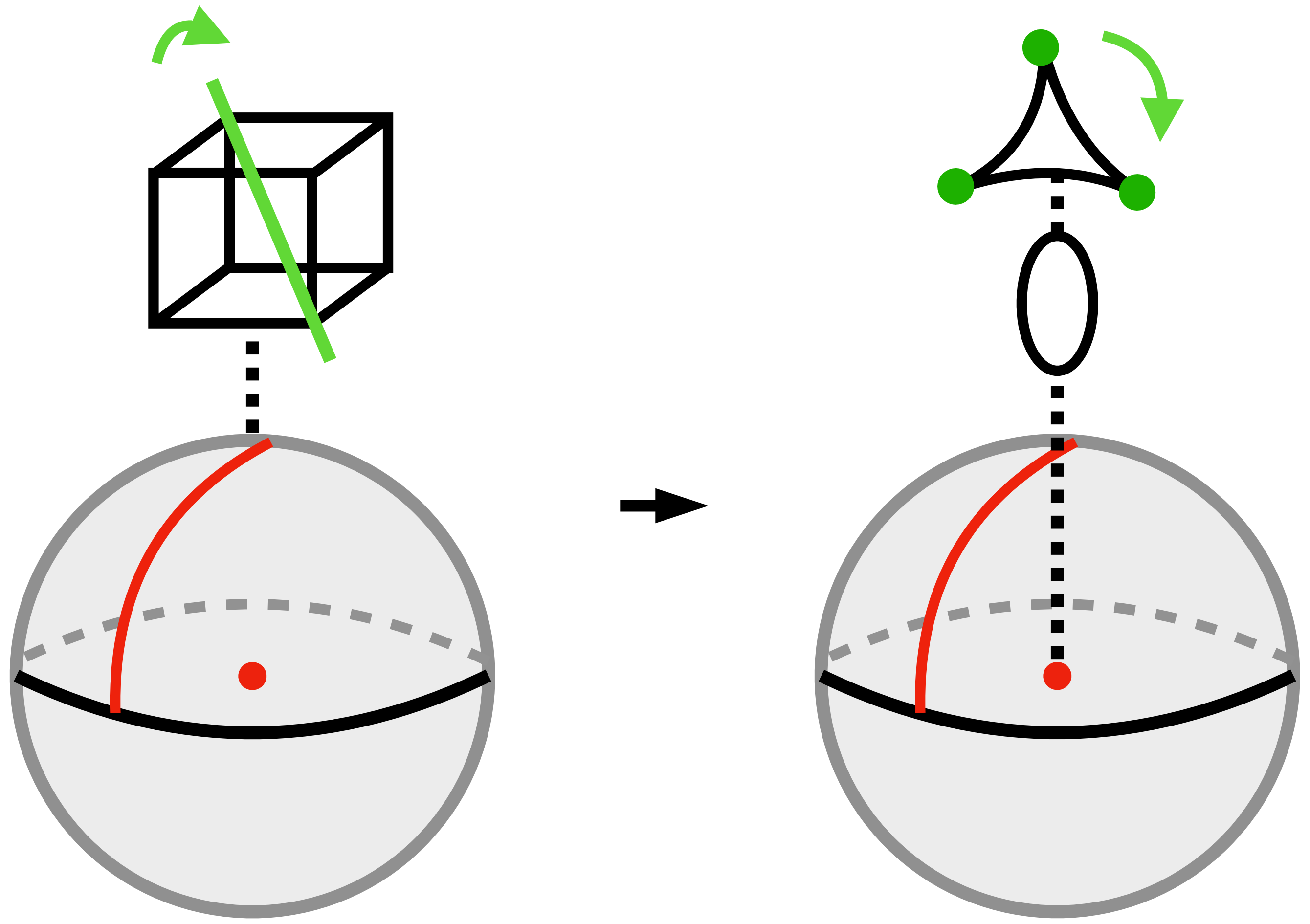
[Braeger, Debray, MD, Heckman, Montero ’25]

Has a **U-duality**:

D	$G_U^{D\mathrm{d}}$
10	1
9	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{SO}(5,5,\mathbb{Z})$
5	$\mathrm{E}_{6(6)}(\mathbb{Z})$
4	$\mathrm{E}_{7(7)}(\mathbb{Z})$
3	$\mathrm{E}_{8(8)}(\mathbb{Z})$

e.g. [Obers, Pioline ’98]

$$\Omega_d^{\mathrm{Spin}}\big(B(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z}))\big)$$



many interesting defects

importance of **non-geometric defects**

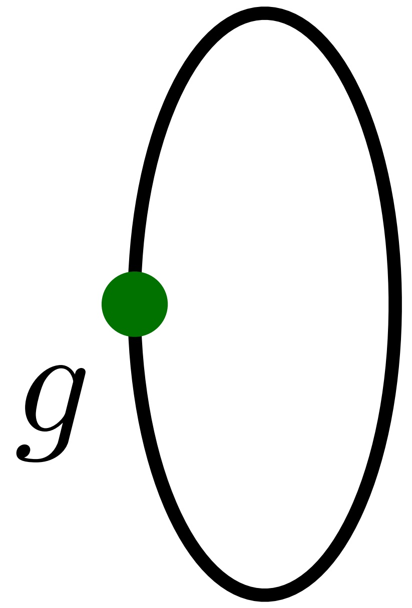
Example: Maximal supergravity

[Braeger, Debray, MD, Heckman, Montero ’25]

Has a **U-duality**: think about **M-theory on torus**

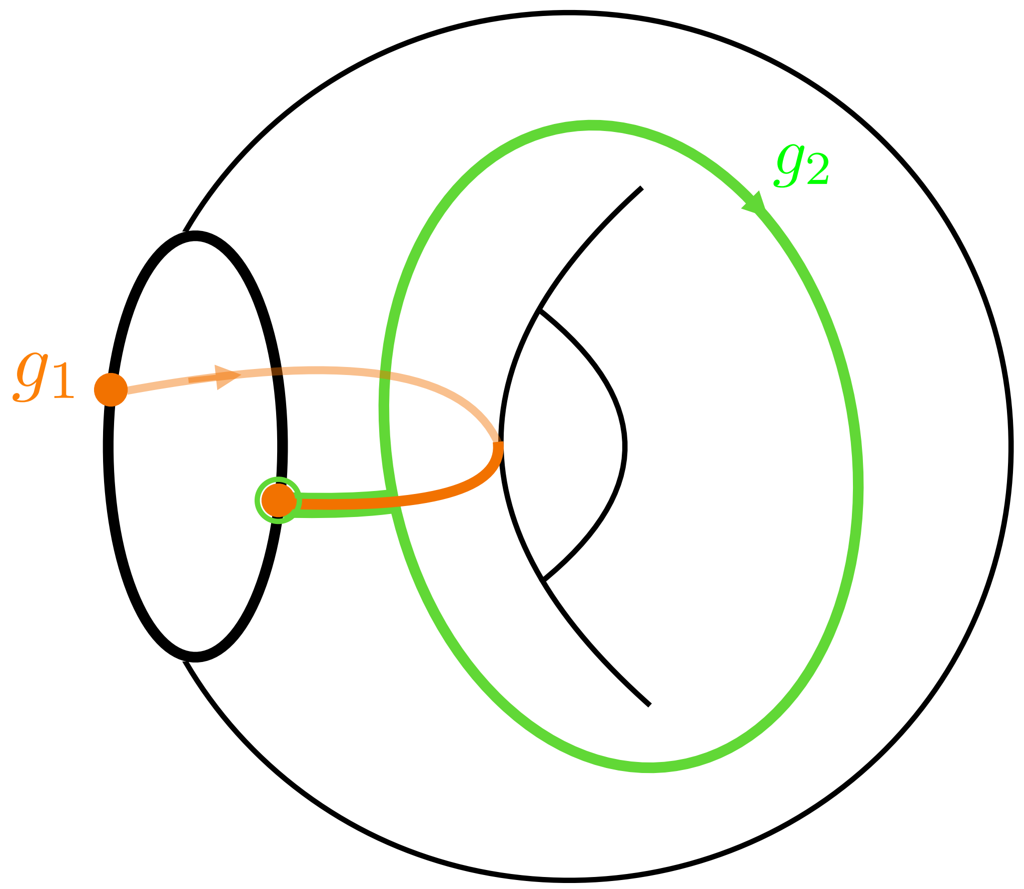
D	$G_U^{D\text{d}}$
10	1
9	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{SO}(5,5,\mathbb{Z})$
5	$\mathrm{E}_{6(6)}(\mathbb{Z})$
4	$\mathrm{E}_{7(7)}(\mathbb{Z})$
3	$\mathrm{E}_{8(8)}(\mathbb{Z})$

e.g. [Obers, Pioline ’98]



(rows 7–3) perfect: $\mathrm{Ab}(G_U) = 0$

$$\Omega_1^{\mathrm{Spin}}(BG_U) = \mathbb{Z}_2$$

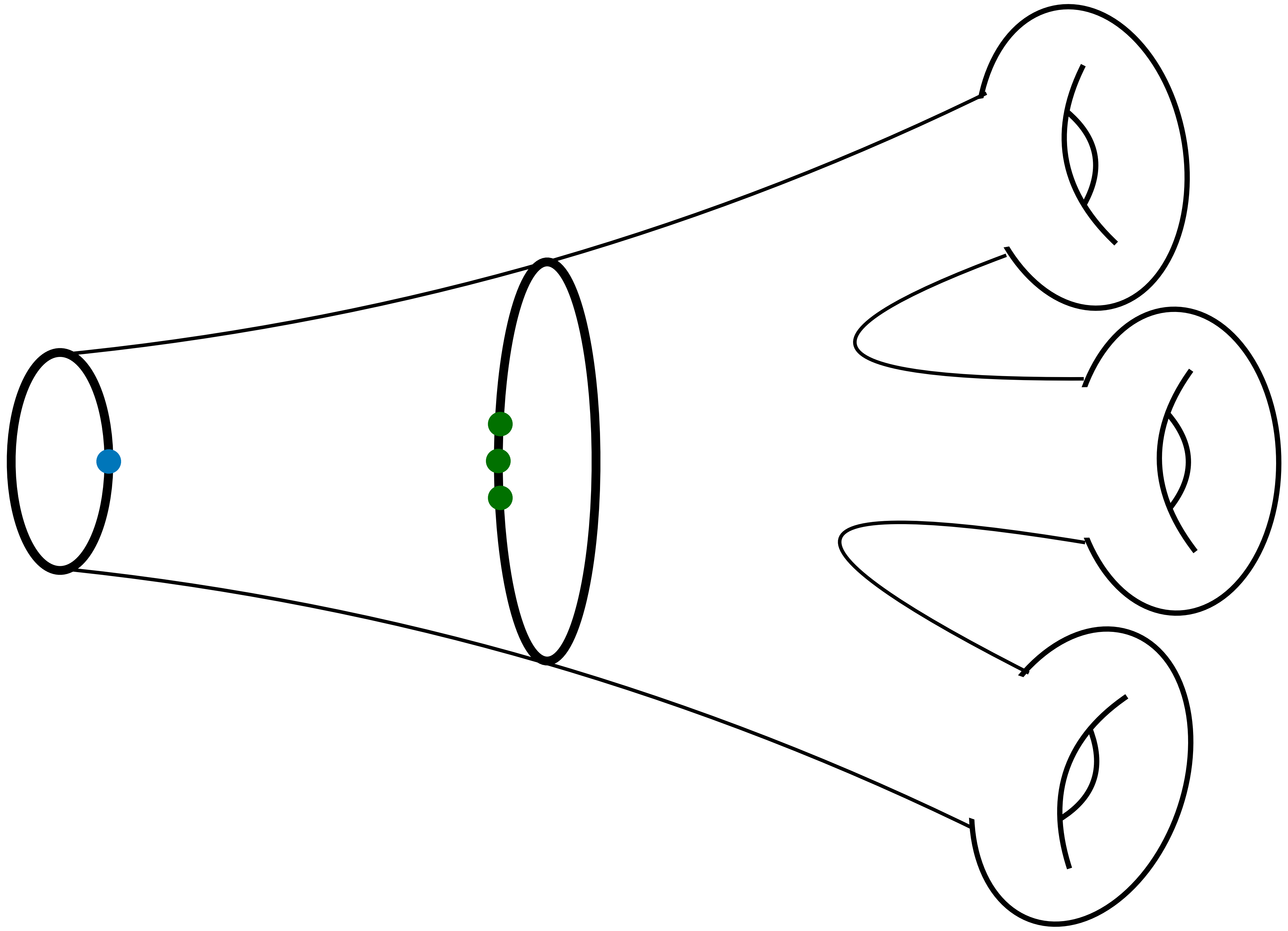


Example: Maximal supergravity

[Chakrabhavi, Debray, MD, Heckman ’25]

Again the **action different on bosons and fermions:**

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{G}_U \longrightarrow G_U \longrightarrow 0$$



$$\Omega_1^{\text{Spin-}\widetilde{G}_U}(\text{pt}) = 0$$

Example: Maximal supergravity

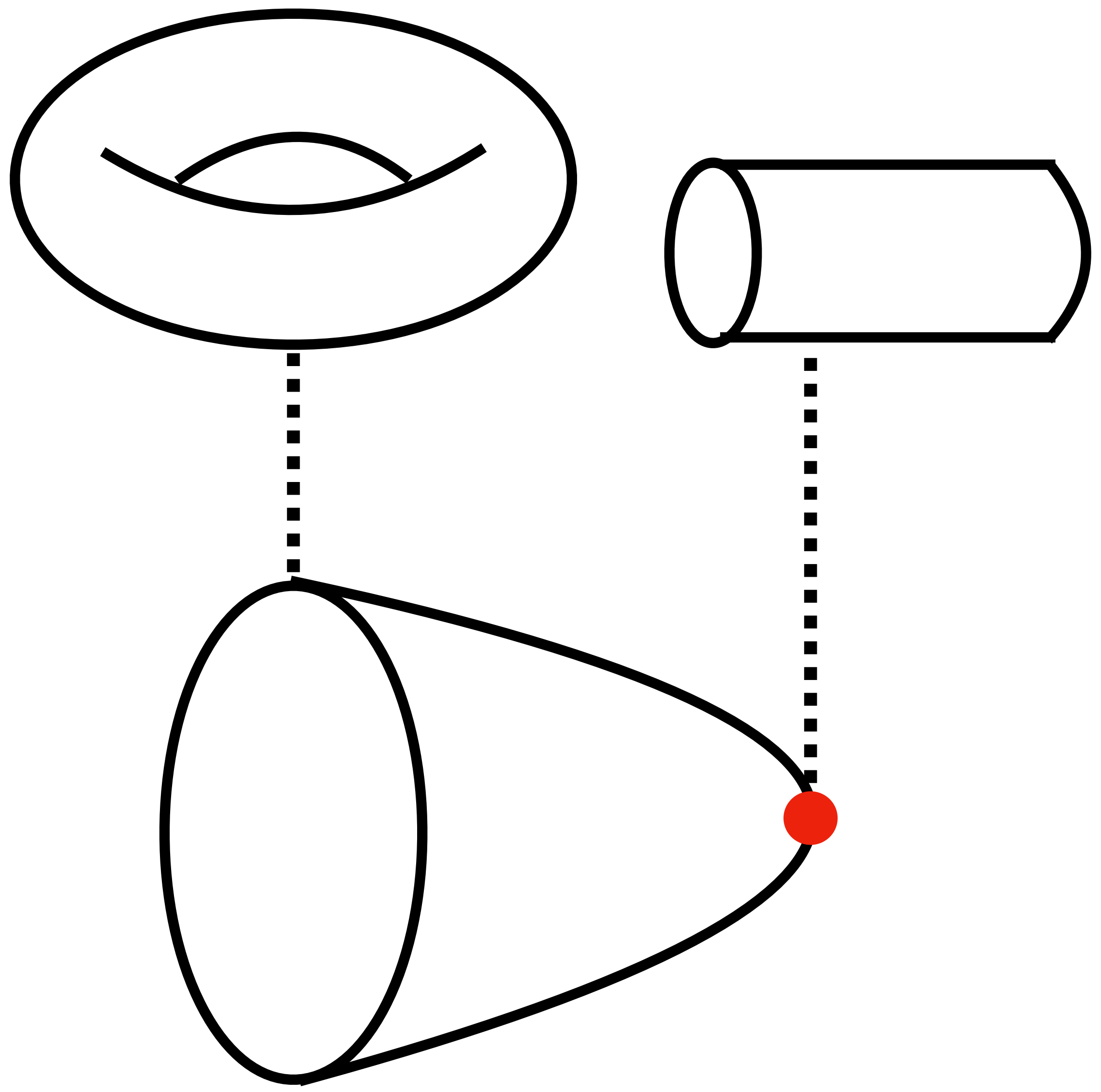
[Chakrabhavi, Debray, MD, Heckman '25]

But there is also a **reflection**:

$$\widetilde{G}_U \rtimes \mathbb{Z}_2 \qquad\qquad \Omega_1^{\text{Spin-}(\widetilde{G}_U \rtimes \mathbb{Z}_2)}(\text{pt}) = \mathbb{Z}_2$$

always **survives**

M-theory makes sense **on un-oriented** spaces



need to include **new reflection branes**

also [MD, Heckman, Montero, Torres '22 + '23] in IIB

Conclusion

Bordisms are a **very efficient** tool to learn about **quantum gravity**