Differential cohomology and applications

Daniel Grady

New York University, Abu Dhabi

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- $\hat{h}^n$ detects refined geometric information which cannot be captured by a topological cohomology theory.
- Every differential cohomology theory comes equipped with a "forgetful" map

  $$\mathcal{I} : \hat{h}^n \to h^n ,$$

  which forgets the geometric information and recovers the underlying topological information.

Why study it?

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- Let G be a Lie group with finitely many connected components. Recall the universal Chern-Weil homomorphism

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- Let $P \to M$ be a principal G-bundle with connection θ. The curvature Ω of the connection θ is a $\mathfrak{g}$ valued 2-form.
- For an invariant polynomial $p \in I^k(\mathfrak{g})$, we can define the differential form $p(\Omega)$ by setting

$$p(\Omega_1 \otimes t_1, \Omega_2 \otimes t_2, \ldots \Omega_k \otimes t_k) = \Omega_1 \wedge \Omega_2 \wedge \ldots \wedge \Omega_k p(t_1, t_2, \ldots, t_k) ,$$

where $t_i \in \mathfrak{g}$ and Ω_i are 2-forms.

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- Define the form $\phi = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$. The differential form

$$Tp(\theta) := \int_0^1 p(\theta \wedge \phi) dt \ ,$$

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satisfies $dTp(\theta) = p(\Omega)$ on P.
- Taking $G = SO(3)$, M an oriented 3-dimensional, and $p = p_1$ to be the first Pontryagin polynomial. Then $p_1(\Omega) = 0$.

- In this case, $Tp_1(\theta)$ is closed. Since M is oriented 3-d, its frame bundle trivializes. Fix a global section $s : M \to F(M)$. The integral

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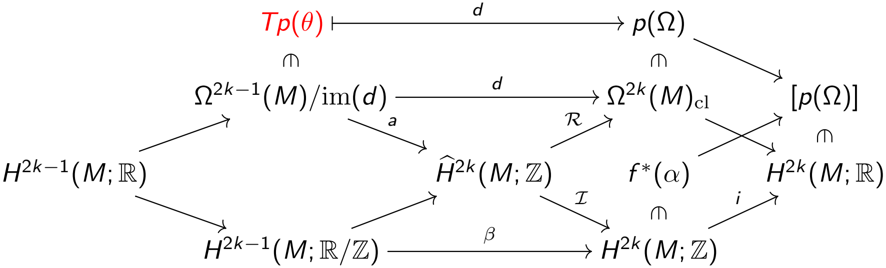
- Cheeger and Simons defined a differential refinement $\widehat{H}^{2k}(M;\mathbb{Z})$ of the ordinary integral cohomology groups $H^{2k}(M;\mathbb{Z})$ which fit into the following commutative diagram.

The diamond diagram

$$
\begin{array}{ccccccc}
 & & \Omega^{2k-1}(M)/\mathrm{im}(d) & \xrightarrow{\quad d \quad} & \Omega^{2k}(M)_{\mathrm{cl}} & & \\
 & \nearrow & & \searrow^{a} \quad \nearrow^{\mathcal{R}} & & \searrow & \\
H^{2k-1}(M;\mathbb{R}) & & & \widehat{H}^{2k}(M;\mathbb{Z}) & & & H^{2k}(M;\mathbb{R}) \\
 & \searrow & & \nearrow \quad \searrow^{\mathcal{I}} & & \nearrow^{i} & \\
 & & H^{2k-1}(M;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\quad \beta \quad} & H^{2k}(M;\mathbb{Z}) & &
\end{array}
$$

The diamond diagram



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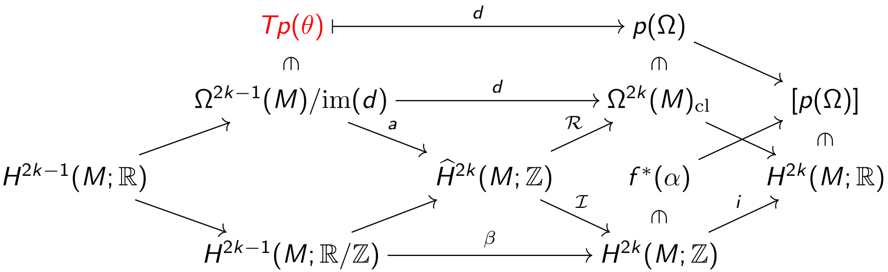

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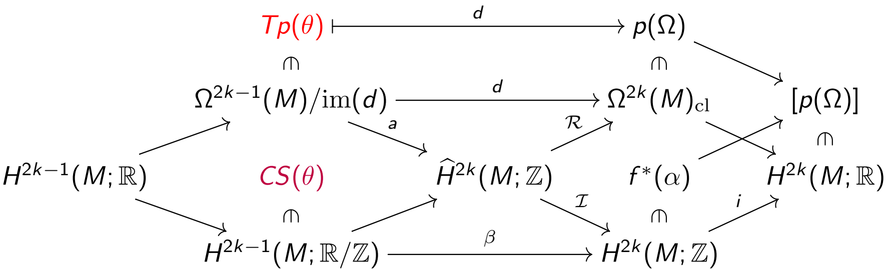
- When $p(\Omega) = 0$, we have a secondary characteristic class



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- The need for the differential refinement $\widehat{H}^n(M;\mathbb{Z})$ came from the *specific* differential from representative $p(\Omega)$ for the cohomology class $[p(\Omega)]$.

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- In physics applications, a variety of differential forms appear and these forms are organized in various ways.
- For example, in type II string theory, the Ramond-Ramond fields arise as differential forms. They are usually combined together in a way that mixes the degrees. In type IIA one considers the formal expression

$$F = F_0 + F_2 + F_4 + \dots \, ,$$

where F_{2i} has degree $2i$.

- In type IIB, one considers the similar expression

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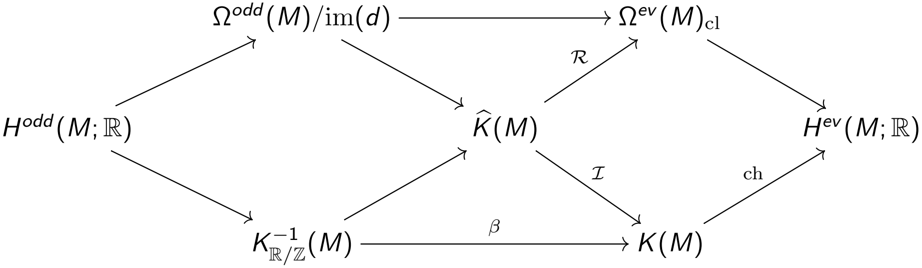
- Our current refinement only works for a single form of fixed degree. We can find other cohomology theories which recover these types of expressions as the "underlying" differential form representative.
- For example, complex K-theory admits a differential refinement $\widehat{K}$. This theory fits into the diagram



- Since this refines even forms, one can consider applications to type IIA string theory.

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- A key feature of differential cohomology is the existence of the diamond diagram.
- If we take the existence of the diamond diagram as an axiom for differential refinements, we can ask for an analogue of Brown representability in this setting.
- Let's first recall the classical statement of Brown representability.

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What is a *topological* cohomology theory?

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What is a *topological* cohomology theory? These are characterized by the Eilenberg-Steenrod axioms.

Definition

A (reduced) cohomology theory h^* is a functor $\mathcal{T}op_+ \to \mathcal{G}r\mathcal{A}b$ satisfying the following

- **Homotopy**: For $f : X \to Y$ a basepoint preserving homotopy equivalence, the induced map $f^* : h^*(Y) \to h^*(X)$ is an isomorphism.
- **Suspension**: For the based suspension ΣX, we have a canonical isomorphism

$$\sigma^* : h^{*+1}(\Sigma X) \cong h^*(X)$$

Definition (Cont.)

- **Exactness**: For the sequence of based spaces $f : X \to Y \to \mathrm{cone}(f)$, with $\mathrm{cone}(f)$ the mapping cone, we have an induced long exact sequence

$$\ldots \to h^*(\mathrm{cone}(f)) \to h^*(Y) \to h^*(X) \xrightarrow{\sigma^*} h^{*+1}(\mathrm{cone}(f)) \to \ldots$$

- **Additivity**: For a Wedge product $X = \bigvee_i X_i$, we have an isomorphism

$$h^*(X) \cong \bigoplus_i h^*(X_i)\ .$$

- It turns out that all cohomology theories are representable by objects called *spectra*

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- E is called a spectrum (or Ω-spectrum) if the adjoint of σ (under the bijection $\hom(\Sigma E_n, E_{n+1}) \cong \hom(E_n, \Omega E_{n+1})$) is an equivalence. That is,

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satisfy the Eilenberg-Steenrod axioms. Surprisingly, the converse is also true!

Theorem (Brown)

For every cohomology theory h^*, there is a spectrum E such that

$$h^n(X) := \pi_0 Map(X, E_n) .$$

- Spectra can be organized into a category $\mathcal{S}p$. The objects are spectra and the morphisms are levelwise maps $f_n : E_n \to G_n$ which commute with the suspension σ.

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- The morphisms can be organized into a topological space $Map(E, G)$. Even better, $Map(E, G)$ is an infinite loop space and there is a *spectrum* of maps $F(E, G)$ whose infinite loop space is $Map(E, G)$.
- The existence of the mapping spaces allows us to talk about the *homotopy theory* of spectra.

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- To arrive at a differential version of Brown representability, we need to somehow combine differential geometry and stable homotopy theory.
- Since we're not yet sure what the correct generalization is, we take a cue from Grothendieck. Any good notion of "smooth spectra" should come from the "functor of points" perspective.
- More specifically, we should consider sheaves of spectra on the site of smooth manifolds (which encodes our differential geometry). We then look for a convenient subcategory which characterizes differential refinements.

Definition

A *sheaf of spectra* is a functor

$$\mathcal{E} : \mathcal{M}an^{op} \to \mathcal{S}p$$

which satisfies descent with respect to good open covers of manifolds – i.e. it glues up to higher homotopy coherence on such covers.

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- We can now state the "representability theorem" for differential cohomology theories. This is due to Bunke, Nikolaus and Völkl.

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- For each sheaf of spectra $\mathcal{E}$, we can evaluate on the point manifold $* \in \mathcal{M}\mathrm{an}$ to obtain an ordinary spectrum E. This operation is functorial and so we get a functor

$$\Gamma : \mathcal{S}h_\infty(\mathcal{M}an;\mathcal{S}p) \to \mathcal{S}p$$

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- We also have a functor $\delta : \mathcal{S}p \to \mathcal{S}h_\infty(\mathcal{M}an; \mathcal{S}p)$, which sends each ordinary spectrum E to the the sheafification of the constant presheaf $M \mapsto E$ for all $M \in \mathcal{M}an$.

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- More interestingly, we have a functor

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which associates to each sheaf of spectra $\mathcal{E}$, the spectrum which is the homotopy colimit over the diagram

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- These functors can be organized into a (homotopy) quadruple adjunction

$$\mathcal{S}p_\infty(\mathcal{M}an; \mathcal{S}p) \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\delta} \\ \xrightarrow{\Gamma} \\ \longleftarrow \end{array} \mathcal{S}p$$

This structure is called *cohesion* and was introduced by Urs Schreiber.

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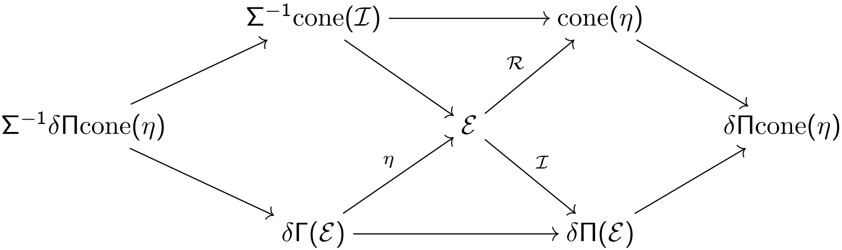

- This is a homotopy commutative diagram in $Sh_\infty(\mathcal{M}an; \mathcal{S}p)$. The left and right squares are homotopy pullback squares.

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- Let $\Omega^{e/o}$ be the sheaf of chain complexes which associates to each smooth manifold M the chain complex

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- Consider the *truncated complex* $\tau_{\geq 0}\Omega^{e/o}$

$$\ldots \longrightarrow \Omega^{odd}(M) \xrightarrow{d} \underset{\uparrow \text{deg zero}}{\Omega^{even}(M)} \longrightarrow 0 \longrightarrow .$$

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- The graded ring $\mathbb{R}[u, u^{-1}]$ gives rise to a complex

$$\Big(0 \longrightarrow \mathbb{R} \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow 0 \longrightarrow \dots \Big)$$

- It follows from the Poincaré lemma that the chain map $i : \delta(\mathbb{R}[u, u^{-1}]) \to \Omega^{e/o}$, defined levelwise by

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathbb{R} & \longrightarrow & 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \Omega^{even} & \xrightarrow{d} & \Omega^{odd} & \xrightarrow{d} & \Omega^{even} & \longrightarrow & \dots \end{array}$$

  is a quasi-isomorphism of sheaves of complexes.
- We then consider the composite map

$$\boxed{c : \delta(K) \xrightarrow{\delta(\mathrm{ch})} H(\delta(\mathbb{R}[u, u^{-1}])) \xrightarrow{H(i)\ \simeq} H(\Omega^{e/o})}\ .$$

- Define the sheaf of spectra $\widehat{K}$ as the homotopy pullback in $\mathcal{S}h_\infty(\mathcal{M}an; \mathcal{S}p)$

$$\begin{array}{ccc} \widehat{K} & \longrightarrow & H(\tau_{\geq 0}\Omega^{e/o}) \\ \downarrow & & \downarrow \\ \delta(K) & \xrightarrow{c} & H(\Omega^{e/o}) \end{array}$$

- Then $\widehat{K}(M) = \pi_0 Map(M; \widehat{K}_0)$.

The Atiyah-Hirzebruch spectral sequence and apps

- For an topological cohomology theory h^*, Atiyah and Hirzebruch constructed a spectral sequence of the form

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- Let's first recall the classical construction by Atiyah and Hirzebruch.

The Atiyah-Hirzebruch spectral sequence

Suppose X is a finite dimensional CW-complex. Then we have a CW-filtration on X

$$X = \lim \left\{ F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \dots \right\} .$$

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$$X = \lim \left\{ F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \dots \right\} .$$

The successive quotients are given as the wedge products of spheres

$$\boxed{F_n/F_{n-1} \simeq \bigvee_{\sigma \in C(n)} S^n} .$$

The filtration gives rise to an exact couple

$$\begin{array}{ccc} \prod_{p,q} h^{p+q}(F_p) & \longrightarrow & \prod_{p,q} h^{p+q}(F_p) \\ & \nwarrow \quad \swarrow_{\partial} & \\ & \prod_{p,q} h^{p+q}(F_p, F_{p-1}) & \end{array}$$

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$$\begin{aligned} E_1^{p,q} &= h^{p+q}(F_p, F_{p-1}) \\ &\cong \bigoplus_{\sigma \in C(n)} \widetilde{h}^{p+q}(S^p) \\ &\cong \bigoplus_{\sigma \in C(p)} \widetilde{h}^{q}(S^0) \cong C^p(X; h^q(*)) \end{aligned}$$

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- Considering the case for complex K-theory, we have $K^q(*) \cong \mathbb{Z}$, q even and $K^q(*) \cong 0$ if q odd. So in this case,

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- Obstructions are in the differentials of the spectral sequence, for example on the E_3-page, Atiyah showed that

$$\boxed{d_3 = Sq^3_{\mathbb{Z}} : H^p(X;\mathbb{Z}) \to H^{p+3}(X;\mathbb{Z})}$$

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- If $\eta \in H^p(X; \mathbb{Z})$ is such that $Sq^3_{\mathbb{Z}}(\eta) \neq 0$, then η cannot represent a K-theory class.

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- More precisely, we consider the simplicial object

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- We can filter the realization of the simplicial object $C(\{U_\alpha\})$ (again taken in $\mathcal{S}h_\infty(\mathcal{M}an)$) by skeleta and take

$$\boxed{F_n = |sk_k C(\{U_\alpha\})|}$$

so that $\lim F_n \simeq M$ (by descent).

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- In the case of K-theory we have the following.
- (Grady, Sati). This gives a spectral sequence with E_2-page

$$E_2^{p,q} = H^p(X; U(1)), \quad q < 0, \ q \text{ odd}$$
$$E_2^{0,0} = \Omega^{\mathrm{ev}}_{\mathrm{cl}}(X)$$
$$E_2^{p,q} = H^p(X; \mathbb{Z}), \quad q > 0, \ q \text{ even}$$

Theorem (Grady, Sati)

The differential on the E_3-page of the spectral sequence is given by

$$d_3 = \widehat{Sq}^3 : H^p(M; U(1)) \to H^{p+3}(M; U(1)) \, .$$

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Conjecture: For M admitting $Spin^c$-structure, we have

$$\boxed{d_{2k}(\omega) = n_{2k}\left[\hat{A}(M) \wedge e^{c_1/2} \wedge \omega\right]_{2k}} \ ,$$

where n_{2k} is an integer related to solving Steenrod's problem for representability of a cycle $c : \Delta^{2k} \to M$ by a smooth manifold.

Thank you!