Dubrovin-Frobenius manifolds from classical W-algebras

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Geometric WDVV system

A function $\mathbb{F}(t^1, ..., t^r)$ with particular index, here is r, such that

1. The following $r \times r$-matrix is nondegenerate constant
$$\eta_{ij} = \partial_{t^i}\partial_{t^j}\partial_{t^r}\mathbb{F}(t).$$
2. $\mathbb{F}$ satisfies **the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV)**: $\forall i, j, q, n$
$$\sum_{k,p}\partial_{t^i}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\eta^{kp}\partial_{t^p}\partial_{t^q}\partial_{t^n}\mathbb{F}(t) = \sum_{k,p}\partial_{t^n}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\ \eta^{kp}\ \partial_{t^p}\partial_{t^q}\partial_{t^i}\mathbb{F}(t).$$
3. $\mathbb{F}(t)$ is quasihomogenous, i.e., exists Euler vector field $E = \sum d_i t^i \partial_{t^i}$, $d_r = 1$ such that
$$E\mathbb{F}(t) = (3 - d)\mathbb{F}(t) + \text{ quadratic polynomial.}$$

The numbers d_i's are the **degrees** and d is the **charge**.

Dubrovin-Frobenius manifold

Set $C_{ijk}(t) = \partial_{t^i}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)$ and consider it as a $(0,3)$-tensor on the tangent space $< \partial_{t^1}, ..., \partial_{t^r} >$. Then

- the flat metric $\eta_{ij} = (\partial_{t^i}, \partial_{t^j}) = C_{ijr}$, and
- the product $\partial_{t^i} \circ \partial_{t^j} = \sum_m \eta^{km} C_{mij} \partial_{t^k} = \sum_k C^k_{ij}(t) \partial_{t^k}$

define a family of Frobenius algebras, i.e., it is

1. commutative,
2. associative: as a result of WDVV equations,
3. has identity $e = \partial_{t^r}$,
4. and compatible bilinear form $(\partial_{t^i} \circ \partial_{t^j}, \partial_{t^k}) = (\partial_{t^i}, \partial_{t^j} \circ \partial_{t^k})$.

Examples and classification

- In dimension 2:

$$\begin{aligned}
\mathbb{F}(t_1,t_2) &= \frac{1}{2}t_2^2t_1 + t_1^k,\ k=\frac{3-d}{1-d},\ d\neq -1,1,3\\
\mathbb{F}(t_1,t_2) &= \frac{1}{2}t_2^2t_1 + t_1^2\log t_1,\ d=-1,\\
\mathbb{F}(t_1,t_2) &= \frac{1}{2}t_2^2t_1 + \log t_1,\ d=3\\
\mathbb{F}(t_1,t_2) &= \frac{1}{2}t_2^2t_1 + e^{\frac{2}{k}t_1}, d=1,\ k\neq 0\\
\mathbb{F}(t_1,t_2) &= \frac{1}{2}t_2^2t_1,\ d=1,\ k=0
\end{aligned}$$

Here the potentials are normalized:

$$\eta^{ij}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad E=((1-d)t_1+\frac{k}{2}\delta_{1,d})\partial_{t_1}+t_2\partial_{t_2}$$

Motivation and results

Conjecture on algebraic Dubrovin-Frobenius manifolds

Dubrovin's conjecture: Irreducible semisimple algebraic Dubrovin-Frobenius manifolds with positive degrees and charge are in one-to-one correspondence with quasi-Coxeter conjugacy classes of irreducible finite reflection groups (1998').

A_r	B_r	D_r	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(k)$
1	1	$[r/2]$	3	5	9	2	3	10	m

- Isomonodromic deformation of algebraic Dubrovin-Frobenius manifolds leads to quasi-Coxeter conjugacy classes.

Problem: Existence of these algebraic Dubrovin-Frobenius manifolds.

Classification of polynomials manifolds

Polynomial Dubrovin-Frobenius structures are in one-to-one correspondence with Coxeter conjugacy classes.

1. Dubrovin constructed them on the orbit spaces of reflection representations of Coxeter groups (1992'). (Using the work of Saito, Yano, and Sekiguchi 1980').
2. They exhaust the set of all possible polynomial structures up to an equivalence (Hertling 2002).
3. Dubrovin proved that the isomonodromic deformation leads to Coxeter conjugacy classes.
4. The degrees are $\frac{\eta_i+1}{\eta_r+1}$ and the charge is $\frac{\eta_r-1}{\eta_r+1}$.

Group	$\eta_1, \dots, \eta_r$
A_n	$1, 2, \dots, r$
B_n	$1, 3, \dots, 2r-1$
D_n	$1, 3, \dots, 2r-3, r-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
H_3	$1, 5, 9$
H_4	$1, 11, 19, 29$
$I_2(K)$	$1, k-1$

Algebraic structures

Reflection groups	$\sharp$ Quasi-Coxeter conjugacy classes	$\sharp$ Dubrovin-Frobenius structures found
A_m	1	1
B_m	1	1
D_{2m}	m	2
D_{2m+1}	m	1
E_6	3	3
E_7	5	3
E_8	9	7
F_4	2	2
H_3	3	3
H_4	10	7
$I_2(k)$	m	m

Construction method

Flat pencil of metrics

Any Dubrovin-Frobenius manifold possesses a **quasihomogeneous flat pencil of metrics** consists of the flat metric $\Omega_1^{ij} = \eta^{ij}$ and the **intersection form**

$$\Omega_2^{ij}(t) := E(dt^i \circ dt^j).$$

This means:

- $\Omega_{(\lambda)}^{ij} := \Omega_2^{ij} + \lambda \Omega_1^{ij}$ defines a flat metric for generic λ,
- The Christoffel symbols of $\Omega_{(\lambda)}^{ij}$ are $\Gamma_{2k}^{ij} + \lambda \Gamma_{1k}^{ij}$.

- Main properties for quasihomogeneity:

$$E = \nabla_2 t^1, \quad e = \nabla_1 t^1$$

$$\mathfrak{L}_E \Omega_2^{ij} = (d-1)\Omega_2, \quad \mathfrak{L}_e \Omega_2^{ij} = \Omega_1^{ij} \text{ and } \mathfrak{L}_e \Omega_1^{ij} = 0$$

Construction from flat pencil of metrics

A contravariant regular quasihomogeneous flat pencil of metrics on a manifold M defines Dubrovin-Frobenius structure on M.

$$\Omega_2^{ij} = (d - 1 + d_i + d_j)\Omega_1^{i\alpha}\Omega_2^{j\beta}\partial_{t^\alpha}\partial_{t^\beta}\mathbb{F}$$

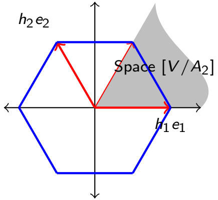
$$E\mathbb{F}(t) = (3 - d)\mathbb{F}(t).$$

Reflection group of type A_2

- The invariant ring is generated by

$$t_1 = \frac{1}{3}(h_1^2 + h_1 h_2 + h_2^2), \quad t_2 = h_1^2 h_2 + h_1 h_2^2.$$

- Flat pencil of metrics:
  1. The intersection form $\langle dt^i | dt^j \rangle = \Omega_2^{ij} = \begin{pmatrix} 2t_1 & 3t_2 \\ 3t_2 & \frac{2}{3} t_1^2 \end{pmatrix}$
  2. Flat metric $\Omega_1^{ij} = \partial_{t_2} \Omega_2^{ij} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$



Hexagon: Symmetry $A_2 \simeq S_3$.

Invariant metric on T^*V

$$\langle dh_i | dh_j \rangle = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\mathbb{F} = \frac{1}{2} t_1 t_2^2 + \frac{3}{8} t_1^4, \quad E[\mathbb{F}] = \frac{1}{3}(2t_1\partial_1 + 3t_2\partial_2)[\mathbb{F}] = (3 - \frac{1}{3})\mathbb{F}$$

Classical W-algebras

The Korteweg–De Vries bihamiltonian structure

$$u_t = 6uu_x - u_{xxx}$$

- Gardner-Zakharov bracket $\{u(x), u(y)\}_1 = \delta'(x-y)$.
  1. Lie algebra structure $\{\mathcal{F}[u(x)], \mathcal{I}[u(y)]\} = \int \frac{\delta \mathcal{F}}{\delta u} \partial_x (\frac{\delta \mathcal{I}}{\delta u}) dx$.
  2. Evolutionary PDE: $u_t = \{u(x), \mathcal{F}[u(y)]\} = \partial_x \frac{\delta \mathcal{F}}{\delta u}$
  3. Hamiltonian $H[u] = \int u^3(x) + \frac{1}{2} u_x^2 dx$ leads to KdV equation.
- Hamiltonian $H[u] = \int u^2(x) dx$ with Magri bracket

$$\{u(x), u(y)\}_2 = -\frac{1}{2}\delta'''(x-y) + 2u(x)\delta'(x-y) + u_x \delta(x-y)$$

  1. They are compatible or form a bihamitonian structure as $\{.,.\}_{(\lambda)} := \{.,.\}_2 + \lambda \{.,.\}_1$ is local Poisson bracket $\forall \lambda$.
  2. Setting $U_m = \int x^{m+1} u dx$ give Virasoro algebra $\{U_m, U_n\}_2 = (m-n)U_{m+n} - \frac{c}{2}(m^3 - m)\delta_{m+n,0}$. (Zomolodchikov, Geravis, Adler, Gelfand, Dickey, Fateev, Lukyanov et al)

Local Lie-Poisson structure

For a simple Lie algebra $\mathfrak{g}$ with bracket $[\cdot,\cdot]$ and Killing form $\langle\cdot|\cdot\rangle$.

- Fix a basis $\xi_1, \xi_2, \ldots$ and dual basis $\xi^1, \xi^2, \ldots$

$$[\xi^i, \xi^j] := c^{ij}_k \xi^k, \quad G^{ij} = \langle \xi^i | \xi^j \rangle.$$

- We consider the coordinates $q^i(g) := \langle g - L | \xi^I \rangle$, $L \in \mathfrak{g}$.

The local Lie-Poisson bracket $\mathbb{B}$ has the form

$$\{q^i(x), q^j(y)\}_2 = G^{ij}\delta'(x-y) + c^{ij}_k q^k(x)\delta(x-y).$$

The leading term $c^{ij}_k q$ define the Lie-Poisson structure on $\mathfrak{g}$:

1. The symplectic leaves are the adjoint orbits.
2. The invariant polynomials $P_1, \ldots, P_r$ are global Casimirs.
3. $P_i(q + \lambda\Lambda_1) = \sum_{j\geq 0} \lambda^j P^j_i(q)$ define a completely integrable systems.

Classical W-algebra

Let $\mathcal{Q}$ be a "transverse subspace" to the orbit space of a nilpotent element L_1. Then a local Poisson bracket $\mathbb{B}^Q$ on $\mathcal{Q}$ can be obtained equivalently by using (2009'):

- Drinfeld-Sokolov reduction: Leibniz rule on invariant ring of differential polynomials.
- Dirac reduction: matrices operations.
- Bihamitonian reduction: Solving recursive equation.

- The local Poisson bracket $\mathbb{B}^{\mathcal{Q}}$ is a classical W-algebra.

$$
\begin{aligned}
\{t^1(x), t^1(y)\}_2 &= C\delta'''(x-y) + 2t^1(x)\delta'(x-y) + t^1_x\delta(x-y), \\
\{t^1(x), t^i(y)\}_2 &= (\eta_i + 1)t^i(x)\delta'(x-y) + \eta_i t^i_x\delta(x-y),
\end{aligned}
$$

(De Groot, Hollowood, Miramontes, Burroughs, Feher, O'Raifeartaigh, Ruelle, Tsutsui, Casati, Pedroni et al. from 1977')

Classical W-algebra sl_2

On the loop algebra of sl_2

$$q(x) = \begin{pmatrix} \frac{1}{2}q_1(x) & q_3(x) \\ q_2(x) & -\frac{1}{2}q_1(x) \end{pmatrix}$$

We get the invariant

The Lie-Poisson bracket

$$\begin{aligned} \{q_1(x), q_1(y)\} &= 2\delta'(x-y) \\ \{q_1(x), q_3(y)\} &= 2q_3(x)\delta(x-y) \\ \{q_1(x), q_2(y)\} &= -2q_2(x)\delta(x-y) \\ \{q_3(x), q_2(y)\} &= \delta'(x-y) + q_1(x)\delta(x-y). \end{aligned}$$

$u(x) = q_2(x) - \frac{1}{4}q_1^2(x) + \frac{1}{2}q_1'(x)$ from the gauage fixing

$$\begin{pmatrix} \partial_x & 1 \\ u(x) & \partial_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} \partial_x + \frac{1}{2}q_1 & 1 \\ q_2 & \partial_x - \frac{1}{2}q_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$$

$$\begin{aligned} \{u(x), u(y)\}_2 &= \frac{\partial u(x)}{\partial \partial^m q_i} \partial_x^m \left(\frac{\partial u(y)}{\partial \partial^n q_j} \partial_y^n (\{q_i(x), q_j(y)\}) \right) \\ &= -\frac{1}{2}\delta'''(x-y) + 2u(x)\delta'(x-y) + u_x\delta(x-y) \end{aligned}$$

Geometry of W-algebras

$$\{u^i(x), u^j(y)\}_2 = \sum_{k\geq 0} A_k^{i,j}(u, u_x, u_{xx}, \ldots)\frac{d^k}{dx^k}\delta(x-y).$$

$$\begin{aligned}\{u^i(x), u^j(y)\}_2^{[-1]} &= F_2^{ij}(u(x))\delta(x-y),\\ \{u^i(x), u^j(y)\}_2^{[0]} &= \Omega_2^{ij}(u(x))\delta'(x-y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x-y).\end{aligned}$$

1. $F_2^{ij}(u)$ define the transverse Poisson structure B_2 on Q of the Lie-Poisson structure, it is known as finite W-algebra.
2. It is an example of polynomial Poisson bracket.
3. The rank is $\dim \mathfrak{g}^f - r$.
4. It admits a dispersionless limit if $\{\cdot,\cdot\}_2^{[-1]} = 0$

Scheme for construction

Scheme for construction

1. Fix a classical W-algebra $\{\cdot,\cdot\}_2$
2. Construct a compatible local Poisson bracket $\{\cdot,\cdot\}_1$.
3. Study the space N defined by Poisson tensors $B_\alpha := F_\alpha^{ij}(q)$

$$N := \{q \in Q : \ker B_1(q) = \ker B_2(q)\}$$

4. The reduction to $\mathfrak{L}(N)$ admits a dispersionless limit. The leading terms are Poisson brackets of hydrodynamic type.

$$\{u^i(x), u^j(y)\}_\alpha^{[0]} = \Omega_\alpha^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_{\alpha,k}(u(x))u_x^k\delta(x-y),$$

5. Nondegeneracy condition leads to a flat pencil of metrics $\Omega_2^{ij}(u)$ and $\Omega_1^{ij}(u)$ (Dubrovin, Novikov 1984')
6. Check quasihomogeneity and construct an associated Dubrovin-Frobenius manifolds.

$D_4(a_1)$

$$\begin{aligned}\mathbb{F} \;=\;& \frac{-Z\,{t_1}^4}{1620}+\frac{Z\,{t_1}^3\,t_2}{810}-\frac{Z\,{t_1}^2\,{t_2}^2}{540}+\frac{Z\,t_1\,{t_2}^3}{810}-\frac{Z\,{t_2}^4}{1620}\\ &-\frac{2\,Z\,{t_1}^2\,t_3}{135}+\frac{2\,Z\,t_1\,t_2\,t_3}{135}-\frac{2\,Z\,{t_2}^2\,t_3}{135}-\frac{4\,Z\,{t_3}^2}{45}\\ &+\frac{91\,{t_1}^5}{103680}-\frac{5\,{t_1}^4\,t_2}{20736}+\frac{{t_1}^3\,{t_2}^2}{10368}+\frac{{t_1}^2\,{t_2}^3}{10368}-\frac{5\,t_1\,{t_2}^4}{20736}\\ &+\frac{91\,{t_2}^5}{103680}+\frac{{t_1}^3\,t_3}{54}-\frac{{t_1}^2\,t_2\,t_3}{36}-\frac{t_1\,{t_2}^2\,t_3}{36}+\frac{{t_2}^3\,t_3}{54}\\ &+\frac{t_1\,{t_3}^2}{24}+\frac{t_2\,{t_3}^2}{24}+\frac{t_1\,t_3\,t_4}{4}-\frac{t_2\,t_3\,t_4}{4}+\frac{t_1\,{t_4}^2}{8}+\frac{t_2\,{t_4}^2}{8}.\end{aligned}$$

Where Z satisfies $Z^2 - t_1^2 - t_1\,t_2 + {t_2}^2 + 12\,t_3 = 0$

$$E[\mathbb{F}] = (t_4\partial_{t_4} + t_3\partial_{t_3} + \frac{1}{2}t_2\partial_{t_2} + \frac{1}{2}t_1\partial_{t_1})\mathbb{F} = \frac{5}{2}\mathbb{F}(t).$$

Pavlyk (2003)

$F_4(a_2)$

$$\begin{aligned}
\mathbb{F} \;=\;& \frac{9\,Z^2\,{t_1}^5}{44800} + \frac{3\,Z^2\,{t_1}^4\,t_2}{89600} - \frac{3\,Z^2\,{t_1}^3\,{t_2}^2}{89600} - \frac{3\,Z^2\,{t_1}^2\,{t_2}^3}{640000} + \frac{153\,Z^2\,t_1\,{t_2}^4}{89600000} + \frac{1107\,Z^2\,{t_2}^5}{4480000000}\\
+& \frac{81\,Z^2\,{t_1}^2\,t_3}{2800} + \frac{243\,Z^2\,t_1\,t_2\,t_3}{14000} + \frac{729\,Z^2\,{t_2}^2\,t_3}{280000} + \frac{409\,Z\,{t_1}^6}{2419200} - \frac{191\,Z\,{t_1}^5\,t_2}{1344000}\\
+& \frac{187\,Z\,{t_1}^4\,{t_2}^2}{5376000} + \frac{67\,Z\,{t_1}^3\,{t_2}^3}{13440000} - \frac{319\,Z\,{t_1}^2\,{t_2}^4}{179200000} + \frac{529\,Z\,t_1\,{t_2}^5}{13440000000} + \frac{1247\,Z\,{t_2}^6}{38400000000}\\
+& \frac{27\,Z\,{t_1}^3\,t_3}{560} - \frac{117\,Z\,{t_1}^2\,t_2\,t_3}{5600} + \frac{9\,Z\,t_1\,{t_2}^2\,t_3}{56000} + \frac{369\,Z\,{t_2}^3\,t_3}{560000} + \frac{243\,Z\,{t_3}^2}{70}\\
-& \frac{29459\,{t_1}^6\,t_2}{580608000} + \frac{6089\,{t_1}^5\,{t_2}^2}{276480000} - \frac{254609\,{t_1}^4\,{t_2}^3}{34836480000} + \frac{152263\,{t_1}^3\,{t_2}^4}{116121600000}\\
-& \frac{300457\,{t_1}^2\,{t_2}^5}{5806080000000} - \frac{1973651\,t_1\,{t_2}^6}{174182400000000} + \frac{292289\,{t_2}^7}{193536000000000} + \frac{17\,{t_1}^4\,t_3}{44800}\\
-& \frac{2647\,{t_1}^3\,t_2\,t_3}{336000} + \frac{6059\,{t_1}^2\,{t_2}^2\,t_3}{2240000} - \frac{18223\,t_1\,{t_2}^3\,t_3}{33600000} + \frac{11443\,{t_1}^7}{174182400} + \frac{60131\,{t_2}^4\,t_3}{1344000000}\\
+& \frac{t_1\,{t_3}^2}{20} + \frac{3\,t_2\,{t_3}^2}{200} - 2\,t_1\,t_3\,t_4 + \frac{3\,t_2\,t_3\,t_4}{5} + 2\,t_1\,{t_4}^2.
\end{aligned}$$

$$Z^3 - Z\left(\frac{{t_1}^2}{48} + \frac{t_1\,t_2}{80} + \frac{3\,{t_2}^2}{1600}\right) - \frac{{t_1}^3}{96} + \frac{13\,{t_1}^2\,t_2}{2880} - \frac{t_1\,{t_2}^2}{28800} - \frac{41\,{t_2}^3}{288000} - \frac{3\,t_3}{2} = 0 \;\textbf{(2007')}$$

$E_8(a_1)$

A potential consist of 303 monomial(2010') simplified by J. Sekiguchi (2020').

$$
\begin{aligned}
F \quad = \quad & \frac{27574725136653658656973151610282411395268792171888640000000 t_1^{25}}{262919712977108888812209859761} - \frac{32940189785485281510076190264\cdots}{273615218\cdots} \\
& - \frac{1945051421153779792882712347780124853351219200000\sqrt{\frac{288230}{3289}}\, t_2 t_1^{21}}{970315921277476588361547} + \frac{60257490748451656689759529090\cdots}{515111706363\cdots} \\
& + \frac{51276965550447936573186450702116293967872000000 t_8^2 t_1^{19}}{849124434117393738404 1} + \frac{860966420036743658768452651790400225280\cdots}{1084508261782846942941 9\cdots} \\
& + \frac{2375079779411706644878490073904552345600000\sqrt{\frac{19}{19499}}\, t_4 t_1^{18}}{380778653527744909437} - \frac{115890833656029279684298134137539133 4\cdots}{51327530099351194 6\cdots} \\
& + \frac{73410938799576400480453955749278834176819 2000 t_2^2 t_1^{17}}{694592785228196748593 7} - \frac{27453660677782683650021654528000000000 0\sqrt{\cdots}}{57221326755129537} \\
& - \frac{5452777995210797138612019621629054156800000\sqrt{\frac{2}{134589}}\, t_8^3 t_1^{16}}{2384044104068998371} + \frac{20559901137566712646108812966625280\cdots}{305692228025763579} \\
& + \frac{87138535415522897242687767838720000\sqrt{\frac{1263994}{21}}\, t_3 t_8 t_1^{16}}{35723692959645537} + \frac{10638293512640789914383391129600000000\sqrt{\cdots}}{1438080300679496373 9} \\
& - \frac{26661242077408439262555938492514304000\sqrt{\frac{288230}{3289}}\, t_2 t_8^2 t_1^{15}}{240252506611604487} + \frac{23723957365073674821358425133788692480\cdots}{119672416629972075 87} \\
- \quad & \frac{308009}{289}\sqrt{\frac{1885}{6752823}}\, t_3 t_5 t_6 + t_3 t_4 t_7 + t_2 t_5 t_7 - \frac{791095425835\sqrt{\frac{26354185}{483}}\, t_2 t_3^3 t_8}{5474043792} \\
& + \frac{64900572109494168980582719029248 0\sqrt{\frac{12710}{55913}}\, t_2 t_4 t_1^{14}}{\cdots} + \frac{963430128982101679678408885774214758400 0\sqrt{\frac{\cdots}{3913 4\cdots}}}{\cdots}
\end{aligned}
$$

Distinguished nilpotent elements of semisimple type

Transverse subspace to a nilpotent orbit

Fix a nilpotent element L_1 in a simple Lie algebra $\mathfrak{g}$ of rank r and a sl_2-triple

$$[h, L_1] = L_1,\ [h, f] = -f,\ [L_1, f] = 2h$$

1. The set of weights $\eta_1, \ldots, \eta_n$:

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathcal{V}_i, \quad \dim \mathcal{V}_i = 2\eta_i + 1.$$

2. Dynkin grading: $\mathfrak{g} = \oplus_{i=-\eta_r}^{\eta_r} \mathfrak{g}_i; \qquad \mathfrak{g}_i := \{g \in \mathfrak{g} : \mathrm{ad}_h g = ig\}$
3. A transverse subspace Q to the adjoint orbit of L_1

$$Q := L_1 + \mathfrak{g}^f, \qquad \mathfrak{g}^f := \{g \in \mathfrak{g} : [f, g] = 0\}$$

4. The affine loop space $\mathcal{Q} = L_1 + \mathfrak{L}(\mathfrak{g}^f)$

Drinfeld-Sokolov reduction-gauge action

For each operator of the form

$$\mathcal{L} = \partial_x + b + L_1 \qquad \text{where } b \in \mathfrak{L}(\mathfrak{b}), \quad \mathfrak{b} := \bigoplus_{i \leq 0} \mathfrak{g}_i.$$

there is a unique element $\mathrm{w} \in \mathfrak{L}(\mathfrak{n})$, $\mathfrak{n} := \bigoplus_{i<0} \mathfrak{g}_i$ such that

$$(\exp \mathrm{ad}\,\mathrm{w})\mathcal{L} = \partial_x + q + L_1, \quad q \in \mathfrak{L}(\mathfrak{g}^f)$$

- Coordinates of q and w are differential polynomials in the coordinates of b.

Distinguished nilpotent elements of semisimple type

Assume L_1 is a distinguished nilpotent element of semisimple type:

1. $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_1$.
2. There is $K_1 \in \mathfrak{g}_{-\eta_r}$ such that $\Lambda_1 := L_1 + K_1$ is regular semisimple.

Associated structure:

- The opposite Cartan subalgebra $\mathfrak{h}' = \mathfrak{g}^{\Lambda_1}$.
- $w := \exp \frac{2\pi \mathbf{i}}{\eta_r+1}\mathrm{ad}_h$ acts on $\mathfrak{h}'$ as a representative of regular cuspidal conjugacy class in the associated Weyl group.

$$\mathfrak{h}' =< \Lambda_1, \dots, \Lambda_r >, \ w(\Lambda_i) = e^{\frac{2i\pi\eta_i}{\eta_r+1}}\Lambda_i$$

- We call the numbers $\eta_1, \dots, \eta_r$ the exponents of L_1.

Weights and exponents

	W(L_1)		
$Z_r(a_s)$	$\eta_1 \leq \ldots \leq \eta_r$	$\eta_{r+1} \leq \ldots \leq \eta_n$	$[\mu_1, \ldots, \mu_r]$
$A_r(a_0)$	$1, 2, \ldots, r$	-	$[0^r]$
$B_r(a_0)$	$1, 3, \ldots, 2r-1$	-	$[0^r]$
$B_{2m}(a_m)$	$1, 1, 3, 3, \ldots, 2m-1, 2m-1$	$1, 2, \cdots, m-1; m-1,$ $m; m, m+1, \ldots, 2m-2$	$[0^{m+1}, 1^{m-1}]$
$C_r(a_0)$	$1, 3, \ldots, 2r-1$	-	$[0^r]$
$D_r(a_0)$	$1, 3, \ldots, r-1; r-1, r-3, \ldots, 2r-3$	-	$[0^r]$
$D_{2m}(a_{m-1})$	$1, 1, 3, 3, \ldots, 2m-1, 2m-1$	$1, 2, \cdots, 2m-2$	$[0^{m+1}, 1^{m-1}]$
$E_6(a_0)$	$1, 4, 5, 7, 8, 11$	-	$[0^n]$
$E_6(a_1)$	$1, 2, 4, 5, 7, 8$	$3, 5$	$[0^{n-1}, 1]$
$E_6(a_3)$	$1, 1, 2, 4, 5, 5$	$1, 2, 2, 3, 3, 4$	$[0^3, 1^3]$
$E_7(a_0)$	$1, 5, 7, 9, 11, 13, 17$	-	$[0^7]$
$E_7(a_1)$	$1, 3, 5, 7, 9, 11, 13$	$5, 8$	$[0^6, 1]$
$E_7(a_5)$	$1, 1, 1, 3, 5, 5, 5$	$1, 1, 1, 2, 2, 2, 2,$ $3, 3, 3, 3, 4, 4, 4$	$[0^2, 1^3, 2^2]$
$E_8(a_0)$	$1, 7, 11, 13, 17, 19, 23, 29$	-	$[0^8]$
$E_8(a_1)$	$1, 5, 7, 11, 13, 17, 19, 23$	$9, 14$	$[0^7, 1]$
$E_8(a_2)$	$1, 3, 7, 9, 11, 13, 17, 19$	$5, 8, 11, 14$	$[0^6, 1^2]$
$E_8(a_4)$	$1, 2, 4, 7, 8, 11, 13, 14$	$3, 5, 5, 7, 7, 9, 9, 11$	$[0^5, 1^3]$
$E_8(a_5)$	$1, 1, 5, 5, 7, 7, 11, 11$	$1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 9, 10$	$[0^3, 1^4, 2]$
$E_8(a_6)$	$1, 1, 3, 3, 7, 7, 9, 9$	$1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 8$	$[0^2, 1^4, 2^2]$
$E_8(a_7)$	$1, 1, 1, 1, 5, 5, 5, 5$	$1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2$ $3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4$	$[0, 1^2, 2^2, 3^2, 4]$
$F_4(a_0)$	1,5,7,11	-	$[0^4]$
$F_4(a_1)$	1,3,5,7	2,5	$[0^3, 1]$
$F_4(a_2)$	1,1,5,5	1,2,3,4	$[0^2, 1^2]$
$F_4(a_3)$	1,1,3,3	1,1,1,1,2,2,2,2	$[0, 1^2, 2]$
$G_2(a_0)$	$1, 5$	-	$[0^2]$
	E(L_1)	$\bar{\text{E}}(L_1)$	

Table 1: Exponents and weights of distinguished nilpotent elements of semisimple type

Integrable systems

$$Q = L_1 + \sum z^i X_i, \;\; X_i \in \mathfrak{g}^f$$

$$\mathrm{ad}_h X_i = -\eta_i X_i, \deg z^i := \eta_i + 1.$$

- Consider Drinfeld-Sokolov bihamiltonian structure.

$$\{.,.\}_1 := \mathfrak{L}_{\partial_{u^r}} \{.,.\}_2, \quad \deg u^r = (\max_i \eta_i) + 1.$$

- Using argument shift method:

$$\overline{P}_i(q + \lambda K_1) = \sum_{j=0}^{\mu_i} \lambda^j \overline{P}_i^j(q), \;\; q \in Q.$$

1. $\overline{P}_i^0$ are Casimirs of B_2 and $\overline{P}_i^{\mu_i}$ are Casimirs of B_1.
2. $\overline{P}_i^j$ define a completely integrable system.

Special coordinates

There exist quasihomogeneous change of coordinates

$$t^i = \begin{cases} z^1, & \text{i=1}, \\ z^i + \text{non linear terms}, & \text{i=2,...,r}, \\ z^i, & \text{i=r+1,...,n}. \end{cases}$$

where $t^1, \ldots, t^r$ form a **complete set of Casimirs** for $\{.,.\}_1^{[-1]}$ and they are in involution with respect to $\{.,.\}_2^{[-1]}$.

- The restriction of the adjoint quotient map to Q

$$\begin{aligned} \Phi(t_1, \ldots, t_n) &:= (\overline{P}_1, \ldots, \overline{P}_r), \quad s = \frac{1}{2}(3r - n) \\ &= (t_1, \ldots, t_s, \overline{P}_{s+1}, \ldots, \overline{P}_r). \end{aligned}$$

For subregular nilpotent elements (n=r+2), it is a semiuniversal deformation of simple hypersurface singularity of the same type as $\mathfrak{g}$ (Brieskorn, Slodowy 1980') .

Space of common equilibrium points

The space of common equilibrium points N has the following equivalent definitions.

$$\begin{aligned} N &:= \{q \in Q : \ker B_1(q) = \ker B_2(q)\} \\ &= \{t : F_2^{i\beta}(t) = 0;\ i = 1, \dots, r,\ \beta = r+1, \dots, n\}, \\ &= \{t : \partial_{t^\beta} \overline{P}_j^0(t) = 0; j = r-s+1, \dots, r,\ \ \beta = r+1, \dots, n\}. \end{aligned}$$

Moreover, $(t^1, \dots, t^r)$ provide local coordinates around generic points of N.

Local Poisson bracket on $\mathfrak{L}(N)$

On the loop space $\mathcal{N} = \mathfrak{L}(N)$

$$\{.,.\}_2^{\mathcal{Q}},\ \{.,.\}_1^{\mathcal{Q}} \xrightarrow{Dirac\ \ Reductions} \{.,.\}_2^{\mathcal{N}}, \{.,.\}_1^{\mathcal{N}}$$

- $\{.,.\}_2^{\mathcal{N}}$ is an algebraic classical W-algebra:

$$\{t^u(x), t^v(y)\}_\alpha^{\mathcal{N}} = \{t^u(x), t^v(y)\}_\alpha^{\mathcal{Q}},\ \ u, v = 1, \dots, r$$

with $t^k, k > r$ are solutions of the polynomial equations defining N.

- The leading terms of $\{.,.\}_2^{\mathcal{N}}, \{.,.\}_1^{\mathcal{N}}$ define bihamiltonian structure of hydrodynamic type

$$\{t^u(x), t^v(y)\}_\alpha^{[0]} = \Omega_\alpha^{uv}(t(x))\delta'(x-y) + \Gamma_{\alpha k}^{uv}(t(x))t_x^k\delta(x-y).$$

Algebraic Frobenius manifold on N

- From construction: $\partial_{t^r}\Omega_2^{uv}(t) = \Omega_1^{uv}(t)$ and $\partial_{t^r}\Omega_1^{uv}(t) = 0$.
- nondegenerate: from Leibniz rule of Drinfeld-Sokolov reduction.

$$\Omega_1^{u,r-u}(t) = \langle \Lambda_u | \Lambda_{r-u} \rangle = \eta_r + 1, \quad \det \Omega_1^{uv}(t) = (\eta_r + 1)^r$$

- quasihomogenius: follows from definition of classical W-algebra:

$$\Omega_2^{1v}(t) = (\eta_v + 1)t^v$$

- regular: from the structure of the exponents of L_1.

It leads to algebraic Dubrovin-Frobenius manifold structure with charge $\frac{\eta_r - 1}{\eta_r + 1}$ and degrees $\frac{\eta_i + 1}{\eta_r + 1}$

In Progress: Key challenges and next steps

Regular nilpotent elements

- L_1 is regular if $\dim \mathfrak{g}^f = rank\ \mathfrak{g} = r$.
- $Q = N$ and the resulting Dubrovin-Frobenius manifolds is polynomial of the same type as $\mathfrak{g}$.

Things obtained in this case:

1. The manifolds are semisimple.
2. The central invariants of the Drinfeld-Sokolov bihamiltonian structures are calculated.
3. For simply laced Lie algebra $\mathfrak{g}$, the bihamiltonian structure is of topological type.
4. Relation to Coxeter conjugacy classes through isomonodromic deformation are obtained.

Algebraic non-polynomial structures

Reflection groups	$\sharp$ non-Coxeter conjugacy classes	$\sharp$ Dubrovin-Frobenius structures found
$D_{2m}, m \geq 2$	$m-1$	1
$D_{2m+1}, m \geq 2$	$m-1$	0
E_6	2	2
E_7	4	2
E_8	8	6
F_4	1	1
H_3	2	2
H_4	9	6
$I_2(k)$	m	m

- Third column is associated to regular quasi-Coxeter conjugacy classes.
- Algebraic non-polynomials Dubrovin-Frobenius manifolds for the groups of type H are founded by the work of Dubrovin, Mazzocco, Sekiguchi, Mano, Kato, Douvropoulos, Feigin, Wright et al.

Subregular nilpotent element in A_r

- $\dim \mathfrak{g}^{L_1} = \mathit{rank}\ \mathfrak{g} + 2$, Thus $\dim Q = r + 2$.
- There exists opposite Cartan subalgebra $\mathfrak{h}'$.
- The reduction of Drinfeld-Sokolov bihamiltonian structure fails to define flat pencil of metrics.

We construct a bihamiltonian structure associated to classical W-algebra:

1. Logarithmic Dubrovin-Frobenius manifold.
2. Conjugacy class of partition $[r]$ in S_{r+1}.
3. Orbit spaces of the natural representation of S_r.
4. In low dimension, the bihamiltonian structure is of topological type.

- Example in dimension 4:

$$-\frac{it_4 t_1^4}{12\sqrt{3}} - \frac{1}{2} i\sqrt{3} t_2 t_4 t_1^2 + t_3 t_4 t_1 + \frac{3t_2^4}{8} + \frac{1}{2} t_2 t_3^2 - \frac{3}{2} i\sqrt{3} t_2^2 t_4 + \frac{3}{2} t_4^2 \log(t_4)$$

Thank you!